ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 30, nº 1 (1979), p. 27-50
<http://www.numdam.org/item?id=AIHPA_1979__30_1_27_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXX, nº 1, 1979, p. 27-50. *Physique théorique.*

On the geometrical structure of shock waves in general relativity

by

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ABSTRACT. — A systematic and geometrical analysis of shock structures in a Riemannian manifold is developed. The jump, the infinitesimal jump and the covariant derivative jump of a tensor are defined globally. By means of derivation laws induced on the shock hypersurface, physically significant operators are defined. As physical applications, the charged fluid electromagnetic and gravitational interacting fields are considered.

INTRODUCTION

Several authors have developed the shock waves from different points of view, under both mathematical and physical aspects.

In General Relativity shock waves assume a peculiar theoretical role. In fact they constitute one of the few strictly covariant signals occurring in the space-time manifolds, where the usual way, to describe waves (as plane waves, Fourier series, etc.) are globally meaningless. Of course shock may be considered as a mathematical abstraction that approximates more realistic physical phenomena.

A very large bibliography on shock waves in General Relativity is quoted in [9].

We refer chiefly to Lichnerowicz's researches [1], [2], [3], which range over this topic and employ refined mathematical techniques.

We believe that a deep understanding of shock waves in General Relativity requires an adequate geometrical analysis.

In fact Hadamard's formulas have not a tensorial character and their application to the complex entities occurring in General Relativity leads to results that could seem involved, if the geometrical structures utilized are not emphasized.

Our purpose is, following Lichnerowicz's approach, to develop a systematic geometric theory of tensor jumps in a Riemannian manifold and to apply it to General Relativity. We get a global theory, expressed by an intrinsical language, adequate for a geometrical point of view. Care is devoted to distinguish the role played by different structures, as the differential structure, the metric, the connection, etc. The case which requires distributional techniques will be treated in a subsequent work.

We consider a C^∞ manifold M and an embedded hypersurface Σ (1), first we define the jump $[t]$ of a tensors t across Σ. By means of Lie derivatives we define the higher jumps $\varepsilon^k t$, which involve only the manifold structure: so we get a first generalization of Hadamard's formulas (which are local and hold for functions). As a particular case, we consider the jump of Riemannian metric $\varepsilon^k g$. To describe the jump of the Riemannian connection we get a veritable tensor $[\Gamma]^k$, which is directly expressed by means of $\varepsilon^k g$. The jump of the covariant derivative $[\nabla^k t]$ is obtained by means of $\varepsilon^k g$ and $[\Gamma]^k$: this is a second generalization of Hadamard's formulas (which operate only on functions by partial derivatives). In this way we get a global expression of the jump [R] of the curvature tensor. Particular interest have several derivation laws, induced on Σ, when the latter is singular, which replace the induced connection (that cannot be defined, for the tangent space of M does not split into the tangent space to Σ and into its orthogonal one). Some of these maps, as div'' and $\operatorname{div}^\perp$, intervene in the physical conservation laws.

In physical applications we analyse the charged fluid, electromagnetic and gravitational field, as an example. We get compact formulas, that resemble Lichnerowicz's results. In particular we get the « shock conditions », the « conservation conditions » and an intrinsical definition of the shock energy tensor.

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(1) Lichnerowicz considers a $C^{(h,k)}$ manifold, to get the physical significant part of gravitational potentials. But, for our purposes it seems more simple to assume M C^∞ deferring to consideration on the Cauchy problem the statement about the physically significant part of $\varepsilon^k g$ (namely $\varepsilon^k g''$).

1. THE BASIC ASSUMPTIONS

Let M be a C^∞ manifold without boundary with dimension $n \geqslant 2$, connected, paracompact, oriented and endowed with a pseudo-Riemannian metric g at least of class C^0.

We are mainly concerned with the case in which $n = 4$ and g is Lorentz-type, for obvious physical reasons. But we don't need such a requirement, as our results are more general.

Moreover, let $j : \Sigma \to M$ be a C^∞ embedded orientable submanifold of M without boundary and with dimensions $n - 1$ ($\Sigma^\pm$ are the two orientations).

Σ will be the support of the shock waves. In General Relativity the physical fields satisfy equations which impose shock conditions for Σ. The most important among them is that Σ is « singular », i. e. the induced metric j^*g is degenerate. Thus we are led to make a study of the geometry of Σ which holds in the singular case too.

Let us introduce some notations :

$T_{(p,q)}M_{/\Sigma}$ is the subspace of tensors, p times contravariant and q times covariant of M, restricted to Σ (we say that such tensors are *on* Σ);

$T_{(p,0)}M''$ is the subspace of tensors, p times contravariant, of M, that are tangent to Σ;

$T_{(0,p)}M^\perp$ is the subspace of tensors, q times covariant, of M, generated by 1-forms orthogonal to Σ (by duality);

$T_{(p,0)}M^\perp$ is the subspace of tensors, p times contravariant, of M, generated by vectors orthogonal to Σ (by the metric).

If Σ is singular, $T_{(p,0)}\Sigma^\perp$ is the subspace, of tensors, p times contravariant, of Σ, generated by vectors orthogonal to Σ (by the metric).

The symbols « $_{/\Sigma}$ », « $''$ », « $^\perp$ » may be combined, with obvious meaning.

For simplicity, we write also T for $T_{(1,0)}$ and T* for $T_{(0,1)}$.

The spaces of sections, for each one of the preceding spaces, is denoted replacing « T » by « $\mathscr{T}$ ».

Furthermore, the spaces of antisymmetric tensors are denoted by Λ and those of their sections by Ω.

The class of differentiability of tensor fields is denoted by an upper suffix on $\mathscr{T}$ and Ω.

If necessary, the labels $\bar{t}$, $\underline{t}$ and $\hat{t}$ will denote the contravariant, covariant and mixed form (induced by the metric) of a tensor t.

2. THE ORIENTATION OF Σ

For the orientability of Σ there exists an « Orthogonal form » $0 \neq l \in \mathscr{T}^\infty_{(0,1)}M^\perp_{/\Sigma}$.

If Σ is singular, we have not the usual unitary normal, but l is defined up to a positive C^∞ function of Σ (2) and it is tangent to Σ.

2.1. Each orthogonal form l is « closed » in the following sense.

PROPOSITION. — Let $0 \neq l \in \mathscr{C}^\infty_{(0,1)} M^\perp_{/\Sigma}$ and let $x \in \Sigma$. Then there exists a neighbourhood $U \subset M$ of x and a C^∞ function

$$\phi : U \to R,$$

such that

$$l_{/U \cap \Sigma} = d\phi_{/U \cap \Sigma}.$$

Proof. — Let $\{ x^0, x^1, \ldots, x^{n-1} \}$ be an adapted chart on a neighbourhood of x.

Then, we have $l_{/U \cap \Sigma} = f\, dx^0$, with $f \gtrless 0$ of class C^∞.

Let $\overset{\circ}{f} : U \to R$ a C^∞ extension of f.

Then, $\phi \equiv \overset{\circ}{f} x^0$ is the required function $\doteq$

2.2. The orientability of Σ induces an important splitting of close enough neighbourhoods of Σ.

PROPOSITION. — There exist three C^∞ n dimensional submanifolds U, U^+, U^- of M, such that:

a) $U = U^+ \cup U^-$,

b) U is an open neighbourhood of Σ,

c) $U^\pm \cap \Sigma^\pm = \partial U^\pm$.

Furthermore, if V, V^+, V^- are three submanifolds of M, which satisfy the above properties, then, also $U \cap V$, $U \cap V^+$, $U \cap V^-$ satisfy the same properties.

Proof. — For each $x \in \Sigma$, there is a neighbourhood U_x of x and two C^∞ submanifolds U_x^+, U_x^-, which satisfy *a*) and *c*). Then

$$U \equiv \bigcup_{x \in \Sigma} U_x,$$

$$U^\pm \equiv \bigcup_{x \in \Sigma} U_x^\pm \text{ satisfy } a),\ b) \text{ and } c) \quad \doteq$$

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(2) We will see (6.3) that we can restrict the functions f to be constant along the integral lines of l.

3. TENSOR JUMPS

The actual purpose is to define the piece-wise differentiability and the jump of tensors, across Σ, which will be just the shock carrier.

3.1. DEFINITION. — *a*) Let $t \in \mathscr{T}_{(p,q)}\mathrm{M}$. Then, t is said to be of class $\mathrm{C}^{(r,\infty)}$, with $0 \leqslant r \leqslant \infty$, if t is of class C^r on M, C^∞ on $\mathrm{M} - \Sigma$ and if there exist two C^∞ tensors $\tilde{t}^{\pm}$ *on* $\mathrm{U}^{\pm}$ such that $t_{/\mathrm{U}^{\pm}-\mathrm{U}^{\pm}\cap\Sigma} = \tilde{t}^{\pm}_{/\mathrm{U}^{\pm}-\mathrm{U}^{\pm}\cap\Sigma}$.

The space of such tensors is denoted by $\mathscr{T}^{(r,\infty)}_{(p,q)}\mathrm{M}$.

b) Let $t \in \mathscr{T}_{(p,q)}\mathrm{M}$ or $t \in \mathscr{T}_{(p,q)}(\mathrm{M} - \Sigma)$. Then, t is said to be of class $\mathrm{C}^{(-1,\infty)}$ (« regularly discontinuous ») if it is C^∞ on $\mathrm{M} - \Sigma$ and there exist two C^∞ tensors $\tilde{t}^{\pm}$ *on* $\mathrm{U}^{\pm}$ such that $t_{/\mathrm{U}^{\pm}-\mathrm{U}\cap\Sigma} = \tilde{t}_{/\mathrm{U}^{\pm}-\mathrm{U}^{\pm}\cap\Sigma}$.

The space of such tensors is denoted by $\mathscr{T}^{(-1,\infty)}_{(p,q)}\mathrm{M}$, or by $\mathscr{T}^{(-1,\infty)}_{(p,q)}(\mathrm{M} - \Sigma)$ respectively $\doteqdot$

3.2. DEFINITION. — Let $t \in \mathscr{T}^{(-1,\infty)}_{(p,q)}\mathrm{M}$ or $t \in \mathscr{T}^{(-1,\infty)}_{(p,q)}(\mathrm{M} - \Sigma)$. Then, the ump of t is the tensor

$$[t] \in \mathscr{T}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma},$$

given by

$$[t] \equiv \tilde{t}^{+}_{/\Sigma} - \tilde{t}^{-}_{/\Sigma}, \quad \doteqdot$$

Note that if $t \in \mathscr{T}^{(-1,\infty)}_{(p,q)}\mathrm{M}$ and $[t] = 0$, then, there exists a unique $\tilde{t} \in \mathscr{T}^{(0,\infty)}_{(p,q)}\mathrm{M}$, such that

$$\tilde{t}_{/\mathrm{M}-\Sigma} = t_{/\mathrm{M}-\Sigma},$$

but, not necessarily

$$\tilde{t} = t.$$

On the other hand, if $t \in \mathscr{T}^{(0,\infty)}_{(p,q)}\mathrm{M}$, then $t_{/\Sigma} \in \mathscr{T}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma}$.

4. INFINITESIMAL TENSOR JUMPS

The best way to calculate the derivatives jump of a tensor t involving only the differential structure of M, is to evaluate the jump of the Lie derivative $[\mathrm{L}_u t]$. Namely, we see that this jump is obtained by a tensor εt on Σ, which depends only on t.

In our treatment, we exclude the case when t is discontinuous across Σ, for we don't get reasonable results, the jumps of t and of the derivatives of t being inextricably bound.

4.1. LEMMA. — Let $t \in \mathscr{C}^{(r,\infty)}_{(p,q)}M$, $u \in \mathscr{C}^{(s,\infty)}_{(1,0)}M$, with $-1 \leqslant r, s \leqslant 0$. The Lie derivative $L_u t$, defined as a C^∞ tensor on $M - \Sigma$, belongs to $\mathscr{C}^{(-1,\infty)}_{(p,q)}(M - \Sigma)$.

Proof. — In fact, we have

$$(L_u t)^{\pm} = L_{\tilde{u}^{\pm}} \tilde{t}^{\pm},$$

where the Lie derivative $L_{\tilde{u}^{\pm}} \tilde{t}^{\pm}$ is defined on Σ (which is the boundary of $U^{\pm}$) by means of any local extension of $\tilde{u}^{\pm}$ and $\tilde{t}^{\pm}$ $\doteqdot$

4.2. We can now enunciate the fundamental theorem which gives a stronger version of Hadamard's formulas.

THEOREM. — Let $t \in \mathscr{C}^{(0,\infty)}_{(p,q)}M$. There exists a unique tensor

$$\varepsilon t \in \mathscr{C}^{\infty}_{(p,q+1)}M_{/\Sigma},$$

such that

$$[L_u t] = i_u \varepsilon t \equiv \varepsilon_u t,$$

for each $u \in \mathscr{C}^{\infty}_{(1,0)}M$.
t is $C^{(1,\infty)}$ if and only if $\varepsilon t = 0$.
Furthermore, we have

$$\varepsilon_u(t + s) = \varepsilon_u t + \varepsilon_u s,$$

$$\varepsilon_u(t \otimes s) = (\varepsilon_u t) \otimes s + t \otimes \varepsilon_u s,$$

$$\varepsilon_u C^i_j t = C^i_j \varepsilon_u t.$$

If $u \in \mathscr{C}^{\infty}_{(1,0)}M''$, then

$$\varepsilon_u t = 0.$$

Hence, for each $0 \neq l \in \mathscr{C}^{\infty}_{(0,1)}M^{\perp}_{/\Sigma}$ there exists a unique

$$t' \in \mathscr{C}^{\infty}_{(p,q)}M_{/\Sigma},$$

such that

$$\varepsilon t = l \otimes t'.$$

Finally, if $\{ x^0, x^1, \ldots, x^{n-1} \}$ is an adapted chart, then the local expression of t is

$$\varepsilon t = \left[\partial_0 t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}\right] dx^0 \otimes \partial x_{\alpha_1} \otimes \ldots \otimes \partial x_{\alpha_p} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}.$$

Proof. — Let $f \in \mathscr{C}^{\infty}_{(0,0)}M$. Then we have

$$\begin{aligned}
&[L_{fu} t] = f[L_u t], && \text{for} \quad p = 0 = q,\\
&[L_{fu} t] = f[L_u f] - [t . f] u = f[L_u t], && \text{for} \quad p = 1, q = 0\\
&[L_{fu} t] = f[L_u t] + \langle t, u \rangle [df] = f[L_u t], && \text{for} \quad p = 0, q = 1.
\end{aligned}$$

Hence the map $u \to [L_u t]$ is linear.

Furthermore, if $u \in \mathcal{T}^{\infty}_{(1,0)}M''$ we have

$$\widetilde{(L_u t)}^{\pm}(y) = L_{u/\Sigma}(\tilde{t}^{\pm})_{/\Sigma}(y) = L_{u/\Sigma}t_{/\Sigma}(y), \; \forall y \in \Sigma,$$

hence

$$[L_u t] = \widetilde{(L_u t)}^{+} - \widetilde{(L_u t)}^{-} = 0 \quad \doteqdot$$

4.3. DEFINITION. — Let $t \in \mathcal{T}^{(0,\infty)}_{(p,q)}M$. The INFINITESIMAL JUMP of t is the tensor $\varepsilon t \in \mathcal{T}^{\infty}_{(p,q+1)}M_{/\Sigma}$ $\doteqdot$

4.4. The calculation of the Lie derivative jump can be extended to the case when, both u and t are of class $C^{(0,\infty)}$.

Previously we introduce the following notation. « $\Diamond$ » denotes the bilinear map defined,

$$\forall h \in \mathcal{T}_{(1,1)}M, \quad r \equiv \bar{t} \otimes \underline{s} \equiv \bar{t}_1 \otimes \ldots \otimes \bar{t}_p \otimes \underline{s}^1 \otimes \ldots \otimes \underline{s}^q \in \mathcal{T}_{(p,q)}M,$$

by

$$h \Diamond r \equiv \sum_{1 \leqslant i \leqslant p} \bar{t}_1 \otimes \ldots \otimes \bar{t}_{i-1} \otimes h(\bar{t}_i) \otimes \bar{t}_{i+1} \otimes \ldots \otimes \bar{t}_p \otimes \underline{s}$$
$$- \sum_{1 \leqslant j \leqslant q} \bar{t} \otimes \underline{s}^1 \otimes \ldots \otimes \underline{s}^{j-1} \otimes h^*(\underline{s}^j) \otimes \underline{s}^{j+1} \otimes \ldots \otimes \underline{s}^q,$$

where h^* is the transpose of h.

THEOREM. — Let $u \in \mathcal{T}^{(0,\infty)}_{(1,0)}M$ and $t \in \mathcal{T}^{(0,\infty)}_{(p,q)}M$.

Then, we have

$$[L_u t] = \varepsilon_u t - \varepsilon u \Diamond t.$$

As, particular cases,

a) if $t \in \mathcal{T}^{(0,\infty)}_{(0,0)}M$, then $[L_u t] = \varepsilon_u t$;

b) if $t \in \mathcal{T}^{(0,\infty)}_{(1,0)}M$, then $[L_u t] = \varepsilon_u t - \varepsilon_t u$;

c) if $t \in \mathcal{T}^{(0,\infty)}_{(0,1)}M$, then $[L_u t] = \varepsilon_u t + \varepsilon u(t)$.

Proof. — If suffices to prove the last three cases.

For this purpose, let us notice that, if a certain map is linear on u and t, then it can be evaluated on the $C^{(1,\infty)}$ tensors u and t (in fact, the space of $C^{(0,\infty)}$ tensors is generated by the space of $C^{(1,\infty)}$ tensors, by means of $C^{(0,\infty)}$ functions).

Then, we see that

$$[L_u t] - \varepsilon_u t, \qquad t \in \mathcal{T}^{(0,\infty)}_{(0,0)}M,$$
$$[L_u t] - \varepsilon_u t + \varepsilon_t u, \qquad t \in \mathcal{T}^{(0,\infty)}_{(1,0)}M,$$
$$[L_u t] - e_u t - \varepsilon u(t), \qquad t \in \mathcal{T}^{(0,\infty)}_{(0,1)}M,$$

are linear respect to u and t, on the $C^{(0,\infty)}$ functions; moreover, these are zero, if u and t are of class $C^{(1,\infty)}$ $\doteq$

4.5. The jump of the exterior derivative is expressed by means of the infinitesimal jump.

PROPOSITION. — Let $t \in \Omega^{(0,\infty)}_{(0,q)}M$. Then it is

$$[d\,t] = \frac{A}{q\,!}\,\varepsilon t.$$

Hence, if we choose

$$0 \neq l \in \mathscr{T}^{\infty}_{(0,1)}M^{\perp}_{/\Sigma},$$

we have

$$[d\,t] = l \wedge t'.$$

Finally, if $\{ x^0, x^1, \ldots, x^{n-1} \}$ is an adapted chart, then the local expression of $[d\,t]$ is

$$[d\,t] = \sum_{1 \leqslant i_1 < \ldots < i_q \leqslant n-1} [\partial_0 t_{i_1 \ldots i_q}] dx^0 \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_q}.$$

Proof. — It follows from the definition of dt

$$(dt)(u_1, \ldots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} L_{u_i}(t(u_1, \ldots, u_{p+1})) + \sum_{i<j} (-1)^{i+j} t(L_{u_i}u_j, u_1, \ldots, \widehat{u_i}, \ldots, \widehat{u_j}, \ldots, u_{p+1}) \quad \doteq$$

4.6. We conclude this section introducing the « k-order infinitesimal jumps », for $C^{(k-1,\infty)}$ tensors, in the same way as the first one.

PROPOSITION. — Let $t \in \mathscr{T}^{(k-1,\infty)}_{(p,q)}M$. There exists a unique tensor

$$\varepsilon^k\, t \in \mathscr{T}^{\infty}_{(p,q+k)}M_{/\Sigma}.$$

such that $[L_{u_k} \ldots L_{u_1} t] = i_{u_k} \ldots i_{u_1} \varepsilon^k t \equiv \varepsilon^k_{u_1,\ldots,u_k} t$, for each $u_1, \ldots, u_k \in \mathscr{T}^{\infty}_{(1,0)}M$. t is $C^{(k,\infty)}$ if and only if $\varepsilon^k t = 0$.

The tensor $\varepsilon^k t$ is symmetrical in the first k indices.

Furthermore, we have

$$\varepsilon^k_{u_1,\ldots,u_k}(t + s) = \varepsilon^k_{u_1,\ldots,u_k} t + \varepsilon^k_{u_1,\ldots,u_k} s,$$
$$\varepsilon^k_{u_1,\ldots,u_k}(t \otimes s) = \left(\varepsilon^k_{u_1,\ldots,u_k} t\right) \otimes s + t \otimes \varepsilon^k_{u_1,\ldots,u_k} s,$$
$$\varepsilon^k_{u_1,\ldots,u_k} C^i_j t = C^i_j \varepsilon^k_{u_1,\ldots,u_k} t.$$

If $u_i \in \mathscr{T}^{\infty}_{(1,0)}M''$, $1 \leqslant i \leqslant k$, then $\varepsilon_{u_1,\ldots,u_k} t = 0$.

Hence, for each $0 \neq l \in \mathscr{T}^{\infty}_{(0,1)}\mathrm{M}^{\perp}_{/\Sigma}$, there exists a unique

$$t^k \in \mathscr{T}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma},$$

such that

$$\varepsilon^k t = \underset{1}{l} \otimes \ldots \otimes \underset{k}{l} \otimes t^{(k)}$$

Finally, if $\{ x^0, x^1, \ldots, x^{n-1} \}$ is an adepted chart, then the local expression of $\varepsilon^k t$ is

$$\varepsilon^k t = [\underset{1}{\partial_0} \ldots \underset{k}{\partial_0} t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}] \underset{1}{dx^0} \otimes \ldots \otimes \underset{k}{dx^0} \otimes \partial x_{\alpha_1} \otimes \ldots \otimes \partial x_{\alpha_p} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q} \doteq$$

The proposition 4.4 can be extendend to the k order, in a suitable way.

5. THE RIEMANNIAN CONNECTION JUMP

Henceforth, we suppose that the assumed metric g is at least of class $C^{(0,\infty)}$. If g is of class $C^{(k-1,\infty)}$, $1 \leqslant k$, we get the following jumps

$$[\underline{g}] = 0 \qquad \varepsilon^i \underline{g} = 0, \qquad 1 \leqslant i \leqslant k-1, \qquad \text{and} \qquad \varepsilon^k \underline{g}.$$

The physically significant « part » of $\varepsilon^k \underline{g}$ is

$$\varepsilon^k g^{||} \equiv \underset{1}{l} \otimes \ldots \otimes \underset{k}{l} \otimes j^* \underline{g}^k \in \mathscr{T}^{\infty}_{(0,k)}\mathrm{M}^{\perp}_{/\Sigma} \otimes \mathscr{T}^{\infty}_{(0,2)}\Sigma$$

Notice that $\varepsilon^k \underline{g}$ is different from $\varepsilon^k \bar{g}$. More precisely, we have

$$\varepsilon^k \underline{g} = - \mathrm{C}^{12}_{13} \underline{g} \otimes \underline{g} \otimes \varepsilon^k \bar{g} = - \underset{1}{l} \otimes \ldots \otimes \underset{k}{l} \otimes \mathrm{C}^{12}_{13} \underline{g} \otimes \underline{g} \otimes \bar{g}^k.$$

Let g be of class $C^{(0,\infty)}$. Then, the Riemannian connection ∇ is defined on $\mathrm{M} - \Sigma$. If $t \in \mathscr{T}^{(r,\infty)}_{(p,q)}\mathrm{M}$, with $-1 \leqslant r \leqslant \infty$, then $\nabla t \in \mathscr{T}^{(-1,\infty)}_{(p,q+1)}(\mathrm{M} - \Sigma)$. In each chart, the Christoffel symbols are of class $C^{(-1,\infty)}$.

5.1. We can express the jump of the connection by a tensor. For physical reasons we are concerned with the cases $k = 1,2$.

THEOREM. — Let g be of class $C^{(k-1,\infty)}$, with $k = 1,2$.

The map $\qquad [\Gamma]^k : \mathscr{T}^{(0,\infty)}_{(k,0)}\mathrm{M} \times \mathscr{T}^{(k-1,\infty)}_{(1,0)}\mathrm{M} \rightarrow \mathscr{T}^{\infty}_{(1,0)}\mathrm{M}_{/\Sigma}$

given by

$$[\Gamma]^k : (u, t) \rightarrow i_u[\nabla^k t] - \varepsilon^k_u t, \text{ (where } \nabla^2 \equiv \nabla\nabla) \text{ is } k+1 \text{ linear.}$$

Then, we can identify $[\Gamma]^k$ with a tensor

$$[\Gamma]^k \in \mathscr{T}^{\infty}_{(1,k+1)}\mathrm{M}_{/\Sigma}.$$

$[\Gamma]^k$ is symmetrical in the last two covariant indices.

Furthermore, $[\Gamma]^k$ is expressed by $\varepsilon^k \underline{g}$, as

$$[\Gamma]^1(u, t) = \frac{1}{2}(\mathrm{C}^{12}_{12} + \mathrm{C}^{12}_{21} - \mathrm{C}^{12}_{23})u \otimes t \otimes \varepsilon^1 \underline{g},$$

$$[\Gamma]^2(u, t) = \frac{1}{2}(\mathrm{C}^{123}_{123} + \mathrm{C}^{123}_{132} - \mathrm{C}^{123}_{134})u \otimes t \otimes \varepsilon^2 \underline{g}.$$

Hence, for each $0 \neq l \in \mathscr{T}^{\infty}_{(0,1)}\mathrm{M}^{\perp}_{/\Sigma}$, we have

$$[\Gamma]^1 = \frac{1}{2}(\underline{l} \otimes \widehat{g}^1 + \widehat{g}^1 \otimes \underline{l} - \underline{g}^1 \otimes \bar{l}),$$

$$[\Gamma]^2 = \frac{1}{2}(\underline{l} \otimes \underline{l} \otimes \widehat{g}^2 + \underline{l} \otimes \widehat{g}^2 \otimes \underline{l} - \underline{l} \otimes \underline{g}^2 \otimes \bar{l}).$$

Proof. — It suffices to prove that $[\Gamma]^k$ is $k + 1$ linear with respect to $\mathrm{C}^{(k-1,\infty)}$ functions and then to evaluate $[\Gamma]^k$ on tensors of class C^{∞}, taking into account the Riemannian expression of ∇ (see [*11*], p. 127).

Namely, we get

$$[\Gamma]^1(u, ft) \equiv [\nabla_u(ft)] - \varepsilon_u(ft) = f[\nabla_u t] + [u.f]t - f\varepsilon_u t - [u.f]t = f[\Gamma]^1(u, t)$$

and, $\forall a, b, c \in \mathscr{T}^{\infty}_{(1,0)}\mathrm{M}$,

$$[\Gamma]^1(a \otimes b, c) = \frac{1}{2}([\mathrm{L}_a g](b, c) + [\mathrm{L}_b g](a, c) - [\mathrm{L}_c g](a, b))\,;$$

$$[\Gamma]^2(a \otimes b, ft) = i_{a\otimes b}[\nabla\nabla(ft)] - e^2_{a\otimes b}(ft) = f[\Gamma]^2(a \otimes b, t)$$

and, $\forall a, b, c, d \in \mathscr{T}^{\infty}_{(1,0)}\mathrm{M}$,

$$\begin{aligned}[\Gamma^2](a \otimes b \otimes c, d) &= [\nabla_a \nabla_b c]\cdot d - [\nabla_{\nabla_a b} c]\cdot d = [\nabla_a \nabla_b c]\cdot d \\ &= \frac{1}{2}[\mathrm{L}_a(\nabla_b c\cdot d) + \mathrm{L}_{\nabla_b} c(a\cdot d) + (\mathrm{L}_a \nabla_b c)\cdot d - (\mathrm{L}_d g)(a,\ \nabla_b c)] \\ &= [\mathrm{L}_a(\nabla_b c\cdot d)] \\ &= \frac{1}{2}([\mathrm{L}_a \mathrm{L}_b g](c, d) + [\mathrm{L}_a \mathrm{L}_c g](b, d) - [\mathrm{L}_a \mathrm{L}_d g](b, c)) \quad \doteq\end{aligned}$$

The expression of $[\Gamma]^k$ gives the following results, if $g \in \mathscr{T}^{(k,\infty)}_{(0,2)}\mathrm{M}$.

5.2. Corollary. — The following conditions are equivalent.

a) $[\nabla^k t] = 0, \qquad \forall t \in \mathscr{T}^{(k,\infty)}_{(0,1)}\mathrm{M}$.
b) $\varepsilon^k g = 0 \quad \doteq$

5.3. Corollary. — The following conditions are equivalent.

a) $[\nabla^k t] = 0 \qquad \forall t \in \mathscr{T}^{(k,\infty)}_{(1,0)}\mathrm{M}''$.
b) $\varepsilon^k g'' = 0 \quad \doteq$

5.4. We can calculate $[\operatorname{div} t] \equiv - C_1^1[\nabla t]$ and $[\nabla \operatorname{div} t] = C_2^1[\nabla\nabla t]$.

COROLLARY.

We have, $\forall t \in \mathscr{T}^{(k-1,\infty)}_{(1,0)}M$:

$$C_k^1[\nabla^k t] = (C_k^1[\Gamma]^k)(t) + C_k^1\varepsilon^k t$$

$$= \begin{cases} \frac{1}{2}\langle l, t\rangle C_1^1\widehat{g}^1 + \langle l, t^1\rangle & \text{if} \quad k = 1 \\ l \otimes \left(\frac{1}{2}\langle l, t\rangle C_1^1\widehat{g}^2 + \langle l, t^2\rangle\right) & \text{if} \quad k = 2 \end{cases} \quad \doteqdot$$

5.5. In the study of shock waves we find a condition on $\varepsilon^k g$, which we want to characterize in an interesting way.

COROLLARY. — The following conditions (« harmonicity condition ») are equivalent.

$a)$ $\left(\frac{1}{2} C_{k+1,k+2} - C_{k,k+1}\right)\varepsilon^k g = 0 = \frac{1}{2} l C_1^1\widehat{g}^k - \underline{g}^k(\bar{l})$

$b)$ $[\Delta t] = 0, \qquad \forall t \in \mathscr{T}^{(k+1,\infty)}_{(0,k)}M.$

Proof. — We utilise the previous corollary taking into account that

$$(\overline{dt})^1 = - \bar{g}^1(dt), \qquad \text{for} \quad k = 1$$
$$(\bar{t})^2 = - \bar{g}^2(\underline{t}), \qquad \text{for} \quad k = 2 \quad \doteqdot$$

5.6. COROLLARY. — Let $k = 2$.

We have $\quad [\nabla_u \nabla t] = [L_u \nabla t], \qquad \forall u,\ t \in \mathscr{T}^{(1,\infty)}_{(1,0)}M.$

Hence $\quad \varepsilon^1 \nabla t = [\Gamma]^2(t) + \varepsilon^2 t, \qquad \forall t \in \mathscr{T}^{(1,\infty)}_{(1,0)}M.$

Proof. — It $v \in \mathscr{T}^{(1,\infty)}_{(1,0)}M$, we have

$$i_v[\nabla_u \nabla t] = [\nabla_u \nabla_v t] - [\nabla_{\nabla_u v} t] = [\nabla_{\nabla_v t} u] + [L_u \nabla_v t] = [L_u \nabla_v t] \quad \doteqdot$$

From the expression of $[\Gamma]^2$ we get the jumps of the Riemannian tensor $\underline{R}$, the Ricci tensor $\underline{r}$ and the scalar curvature r_0.

5.7. COROLLARY. — Let $k = 2$.

We have

$$[\underline{R}] = \frac{1}{2} A_{12}A_{34}S_{23}\varepsilon^2\underline{g} = \frac{1}{2}(l \otimes l) \wedge \underline{g}^2$$

$$[r] = C_{13}[R] = \frac{1}{2}(l^2\underline{g}^2 + C_1^1\widehat{g}^2 l \otimes l - l \vee \underline{g}^2(\bar{l}))$$

$$[r_0] = C_{11}[r] l^2 C_1^1\widehat{g}^2 - \underline{g}^2(l, l)$$

Proof. — $[\underline{R}] = A_{12}[\Gamma]^2 = \frac{1}{2}(A_{12}\varepsilon^2 g + A_{12}A_{34}S_{23}\varepsilon^2 g)$ ∸

5.8. COROLLARY. — Let $k = 2$.
The following conditions are equivalent.

a) $[R] = 0$.
b) $\varepsilon^2 g'' = 0$ ∸

5.9. We can extend theorem 5.1 in several ways. For example we will use the following result in the physical applications.

PROPOSITION. — Let $k = 2$. Let $t \in \mathscr{C}^{\infty}_{(1,0)}M$, $u, v \in \mathscr{C}^{(0,\infty)}_{(1,0)}M$. Then we have

$$[\nabla_u \nabla_v t] = [\Gamma]^2(u, v, t) + \nabla_{\varepsilon_{uv}} t.$$

Proof. — $[\nabla_u \nabla_v t] = i_v[\nabla_u \nabla t] + \nabla_{[\nabla_u v]} t = [\Gamma]^2(u, v, t) + \nabla_{\varepsilon_{uv}} t$ ∸

6. CONNECTIONS INDUCED ON Σ

It is well known that if Σ is not singular (i. e. the induced metric j^*g on Σ is not degenerate or $l^2 \neq 0$), then the connection ∇ of M can be decomposed into the tangent connection respect to Σ and the second fundamental form of Σ.

But our main interest is, for physical reasons, towards the singular case. In such a case, we have not a tangent projection and a unitary normal to Σ and the vectors orthogonal to Σ belong to its tangent space. On the other hand, in the singular case we find other interesting properties of the connection. In this section we assume g at least of class $C^{(0,\infty)}$ and Σ singular.

6.1. PROPOSITION. — The two maps

$$\tilde{\nabla}^{\pm} : \mathscr{C}^{\infty}_{(1,0)}\Sigma \times \mathscr{C}^{\infty}_{(p,0)}\Sigma \to \mathscr{C}^{\infty}_{(p,0)}M_{/\Sigma}$$

given by

$$\tilde{\nabla}^{\pm} : (u, t) \to (\nabla_u \tilde{t})^{\pm}_{/\Sigma},$$

where $\tilde{t}$ is an extension of t, are well defined (independent of the choice of the extension) and are derivation laws.

Proof. — We contend that, if $p = 1$ and $\tilde{t}_{/\Sigma} = 0$, then $(\nabla_u \tilde{t})^{\pm}_{/\Sigma} = 0$. In fact, $\forall v \in \mathscr{C}^{(0,\infty)}_{(1,0)}M_{/\Sigma}$, it is

$$2v\cdot(\nabla_u \tilde{t})^{\pm}_{/\Sigma} = \tilde{u}.(\tilde{v}\cdot\tilde{t})_{/\Sigma} + \tilde{t}.(\tilde{v}\cdot\tilde{u})_{/\Sigma} + (L_{\tilde{u}}\tilde{t})\cdot v - (L_v g)^{\pm}(u, t),$$

where each term is zero ∸

6.2. PROPOSITION. — The two maps

$$\nabla^{\perp\pm} : \mathscr{C}^{\infty}_{(1,0)}\Sigma^{\perp} \times \mathscr{C}^{\infty}_{(p,q)}\Sigma \to \mathscr{C}^{\infty}_{(p,q)}\Sigma,$$

given by

$$\nabla^{\perp\pm} : (l, t) \to \tilde{\nabla}^{\pm}_{l}\tilde{t}$$

for contravariant tensors and given by duality for covariant tensors, are derivation laws.

Proof. — It suffices to prove that, for controvariant vectors we have

$$l\cdot\nabla_l t = l\cdot\nabla_t l - l\cdot \mathrm{L}_l t = \frac{1}{2}\nabla_t l^2 = 0 \quad ∸$$

This proposition can be generalized in such a way as to concern the « jump type tensors » by introducing the new derivations ∇'' and $\nabla^{\perp}$.

6.3. PROPOSITION. — The two maps

$$\nabla''^{\pm} : \mathscr{C}^{\infty}_{(1,0)}\Sigma \times \big(\mathscr{C}^{\infty}_{(r,0)}\Sigma^{\perp} \otimes \mathscr{C}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma}\big) \to \mathscr{C}^{\infty}_{(r,0)}\Sigma \otimes \mathscr{C}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma},$$

given by

$$\nabla''^{\pm} : (u, l \otimes \ldots \otimes l \otimes t) \to \nabla_u(\tilde{l} \otimes \ldots \otimes \tilde{l} \otimes t)^{\pm},$$

are derivation laws.

Proof. — It suffices to prove the statement for $p = 0 = q$, $r = 1$. In fact, we have

$$l\cdot\tilde{\nabla}^{\pm}_{u} b = \tilde{\nabla}^{\pm}_{u}(l\cdot b) - (\tilde{\nabla}^{\pm}_{u} l)\cdot b = 0 \quad ∸$$

6.4. PROPOSITION. — The two maps

$$\nabla^{\perp\pm} : \mathscr{C}^{\infty}_{(1,0)}\Sigma^{\perp} \times \big(\mathscr{C}^{\infty}_{(r,0)}\Sigma^{\perp} \otimes \mathscr{C}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma}\big) \to \mathscr{C}^{\infty}_{(r,0)}\Sigma^{\perp} \otimes \mathscr{C}^{\infty}_{(p,q)}\mathrm{M}_{/\Sigma}$$

given by

$$\nabla^{\perp\pm} : (l, l \otimes \ldots \otimes l \otimes t) \to \nabla_l(\tilde{l} \otimes \ldots \otimes \tilde{l} \otimes t)^{\pm},$$

are derivation laws.

Proof. — It suffices to prove the statement for $p = 0 = q$, $r = 1$. In fact, we have $\forall u \in \mathscr{C}^{\infty}_{(1,0)}\Sigma$,

$$u\cdot\tilde{\nabla}^{\pm}_{l} l = \tilde{\nabla}^{\pm}_{l}(u\cdot l) - (\tilde{\nabla}^{\pm}_{l} u)\cdot l = 0 \quad ∸$$

Wence, $\nabla^{\perp\pm}$ may be viewed as a connection on each integral manifold L of $\bar{l}$, writing

$$\nabla^{\mathrm{L}\pm} : \mathscr{C}_{(1,0)}\mathrm{L} \times \mathscr{C}_{(p,0)}\mathrm{L} \to \mathscr{C}_{(p,0)}\mathrm{L},$$

i. e. L is a geodesical submanifold of M, with respect to ∇.

Then $\nabla^{L\pm}$ induces two affine structures on each L (at least locally) and we can normalise l, up to a positive constant along each L (in such a way that $\tilde{\nabla}_l^{\pm} l = 0$). Such an l is said to be a « normal ».

6.5. Proposition 6.3 suggests the definition of an interesting differential operator.

COROLLARY. — The two maps

$$\operatorname{div}''^{\pm} \equiv -\, C_1^1 \nabla''^{\pm} : \mathscr{T}^{\infty}_{(1,0)}\Sigma^{\perp} \otimes \mathscr{T}^{\infty}_{(p,q)}M_{/\Sigma} \to \mathscr{T}^{\infty}_{(p,q)}M_{/\Sigma}$$

are well defined and are given (for any normal l) by

$$\operatorname{div}^{\pm} : l \otimes t \to (\operatorname{div}''^{\pm} l)t - (\nabla_l t) \quad \dot{-}$$

6.6. Proposition 6.4 suggests the definition of an interesting differential operator.

COROLLARY. — The two maps

$$\operatorname{div}^{\perp\pm} \equiv C_1^1 \nabla^{\perp\pm} : \mathscr{T}^{\infty}_{(1,0)}\Sigma^{\perp} \otimes \mathscr{T}^{\infty}_{(p,q)}M_{/\Sigma} \to \mathscr{T}^{\infty}_{(p,q)}M_{/\Sigma}$$

are well defined and are given (for any normal l) by

$$\operatorname{div}^{\perp\pm} : l \otimes t \to -\, (\nabla_l t)^{\pm} \quad \dot{-}$$

6.7. PROPOSITION. — The map

$$[\tilde{\Gamma}] : \mathscr{T}^{\infty}_{(1,0)}\Sigma \times \mathscr{T}^{\infty}_{(1,0)}\Sigma \to \mathscr{T}^{\infty}_{(1,0)}\Sigma,$$

given by

$$[\tilde{\Gamma}] : (u, t) \to [\nabla_u \tilde{t}],$$

is the restriction of $[\Gamma]^1$ and we get

$$[\tilde{\Gamma}] = -\frac{1}{2}\bar{\varepsilon} g'' = -\frac{1}{2}\bar{l} \otimes j^* g' \quad \dot{=}$$

Moreover, a sufficient condition to get

$$\nabla''^{+} = \nabla''^{-} \qquad \text{and} \qquad \nabla^{\perp +} = \nabla^{\perp -}$$

is that the harmonicity condition holds $\dot{-}$

6.8. We have introduced only those induced derivation laws of tensors on Σ we need for applications. Further ones can be interesting.

As an example we mention two of them.

a) The map $\quad \underset{\sim}{\nabla} : \mathscr{T}^{\infty}_{(1,0)}\Sigma \times \mathscr{T}^{\infty}_{(p,0)}\Sigma \to \mathscr{T}^{\infty}_{(0,p)}\Sigma,$

given by

$$\underset{\sim}{\nabla} : (u, t) \to j^*(\underline{\tilde{\nabla}_u^+ t}) = j^*(\underline{\tilde{\nabla}_u^- t})$$

is a derivation law.

Moreover, $\underset{\sim}{\nabla}$ is the Riemannian connection induced on Σ by j^*g, as we have, $\forall u, v, t \in \mathscr{T}^{\infty}_{(1,0)}\Sigma$,

$$2 \langle \underset{\sim}{\nabla}_u t, v \rangle = \mathrm{L}_u(j^*g)(t, v) + \mathrm{L}_t(j^*g)(u, v) + (j^*g)(\mathrm{L}_u t, v) - (\mathrm{L}_v j^*g)(u, t) \quad \dot{-}$$

b) The bilinear map

$$\overset{0}{\Omega} : \mathscr{T}^{\infty}_{(1,0)}\Sigma \times \mathscr{T}^{\infty}_{(1,0)}\Sigma \to \Omega^{\infty}_{(0,n-1)}\Sigma,$$

given by

$$\overset{0}{\Omega} : (u, v) \to j^* * (\tilde{\nabla}_u^+ v) = j^* * (\tilde{\nabla}_u^- v),$$

and the linear map

$$\overset{0}{\Gamma} : \mathscr{T}^{\infty}_{(0,1)}\mathrm{M} \to \mathscr{T}^{\infty}_{(0,2)}\Sigma,$$

given by

$$\overset{0}{\Gamma} : l \to j^*(\nabla^+ l) = j^*(\nabla^- l),$$

resemble the second fundamental form relative to the non singular case $\dot{-}$

7. FURTHER USEFUL FORMULAS

In this section we assume that g is of class $C^{(1,\infty)}$, Σ is singular and the harmonicity condition ([3]) holds.

We have not calculated $[\nabla\nabla t]$ for $C^{(0,\infty)}$ tensors and $[\nabla t]$ for $C^{(-1,\infty)}$ tensors. But we can calculate $[\mathrm{div}\, \nabla t]$ for $C^{(0,\infty)}$ tensors and $j^*_{12}[\mathrm{div}\, \underline{\mathrm{R}}]$. Such results will be fundamental for physical applications.

7.1. Lemma. — Let

$$j^* : \mathscr{T}^{(0,\infty)}_{(0,1)}\mathrm{M} \to \mathscr{T}^{\infty}_{(0,1)}\Sigma$$

and

$$i^* : \mathscr{T}^{(0,\infty)}_{(0,1)}\mathrm{M} \to (\mathscr{T}^{\infty}_{(1,0)}\Sigma^{\perp})^*$$

be the canonical projections.

([3]) This condition is physically interesting, but it is not necessary for the following theorems. However it gives simplified formulas.

Let $t \in \mathscr{T}^{(0,\infty)}_{(1,1)}\mathrm{M}$. Then the following conditions are equivalent.

$a)$ $(i_{\bar{x}} i_{l} t)_{/\Sigma} = 0, \qquad \forall \bar{x} \in \mathscr{T}^{\infty}_{(1,0)}\mathrm{M}''$.

$a')$ $(j^* t)_{/\Sigma} \in \mathscr{T}^{\infty}_{(1,0)}\Sigma \otimes \mathscr{T}^{\infty}_{(0,1)}\Sigma$.

$b\cdot$ $(i_{\underline{x}} i_{l} t^{+})_{/\Sigma} = 0, \qquad \forall \bar{x} \in \mathscr{T}^{\infty}_{(1,0)}\mathrm{M}''$.

$b')$ $(i^* t^{+})_{/\Sigma} \in (\mathscr{T}^{\infty}_{(1,0)}\Sigma^{\perp})^* \otimes \mathscr{T}^{\infty}_{(1,0)}\Sigma^{\perp}$ $\dot{-}$

7.2. LEMMA. — Let $t \in \mathscr{T}^{(0,\infty)}_{(1,1)}\mathrm{M}$. Let the previous conditions hold. Then we have

$$\mathrm{C}^1_1 t = \mathrm{C}^1_1 j^* t + \mathrm{C}^1_1 i^* t^{+}.$$

Proof. — We can easily prove this algebraic formula by any adapted basis $\dot{-}$

7.3. THEOREM. — Let $t \in \mathscr{T}^{(0,\infty)}_{(p,q)}\mathrm{M}$. Then we have

$$[\operatorname{div} \nabla t] = (\operatorname{div}'' + \operatorname{div}^{\perp})\varepsilon t.$$

Hence, if l is a normal, we get

$$[\operatorname{div} \nabla t] = (\operatorname{div}'' l)t' - 2\nabla_l t'.$$

Proof. — Let $x \in \mathscr{T}^{\infty}_{(1,0)}\Sigma$. Then the formula

$$i_x i_l [\nabla\nabla t] = [\nabla_l \nabla_x t] - [\nabla_{\nabla_l x} t] = 0 - \varepsilon_{\nabla_l x} t = 0$$

shows that condition 7.2 a holds.

Then we can write

$$\mathrm{C}^1_1[\overline{\nabla}\nabla t] = \mathrm{C}^1_1 j^*_1[\nabla\nabla t] + \mathrm{C}^1_1 i^*_1[\nabla\overline{\nabla} t].$$

Moreover we get:

$$a) \qquad \begin{aligned} j^*_1[\overline{\nabla}\nabla t] &= j^*_1[\nabla\overline{\nabla} t] - j^*_1[\mathrm{R}] \diamond t \\ &= \nabla''[\overline{\nabla} t] - j^*_1[\mathrm{R}] \diamond t \\ &= \nabla''\varepsilon t - j^*_1[\mathrm{R}] \diamond t, \end{aligned}$$

$$b) \qquad \begin{aligned} i^*_1[\nabla\overline{\nabla} t] &= \nabla^{\perp}[\overline{\nabla} t] \\ &= \nabla^{\perp}\varepsilon t. \end{aligned}$$

The statement follows taking into account that, by the harmonicity condition, we get

$$\mathrm{C}^1_1 j^*_1[\mathrm{R}] = \frac{1}{2}\mathrm{C}^1_1(\mathrm{A}_{34}\mathrm{S}_{23}((j^*_1 g^2) \otimes l \otimes l)) = \frac{1}{2}\mathrm{A}(g^2(l) \otimes l) = 0 \quad \dot{-}$$

7.4. THEOREM. — We have

$$j_{12}^*[\operatorname{div} R] = j_{12}^*(\operatorname{div}'' + \operatorname{div}^\perp)[\Gamma]^2 = j_{12}^* S_{13}(\operatorname{div}'' + \operatorname{div}^\perp)\varepsilon^2 g.$$

Hence if l is a normal, we get

$$j_{12}^*[\operatorname{div} R] = \left(\left(-\frac{1}{2}(\operatorname{div}'' l) + \nabla_l\right) j^* g^2\right) \otimes l.$$

Proof. — Let $x \in \mathscr{C}^\infty_{(1,0)}\Sigma$. Then the formula

$$i_x i_l j_{34}^*[\nabla R] = j_{12}^* i_x i_l [\nabla R] = j_{12}^* i_x \nabla_l [R] = j_{12}^* i_x (l \otimes l \wedge \nabla_l g^2) = 0$$

shows that condition 7.1 *a* holds.

Then we can write

$$j_{12}^*[\operatorname{div} R] = -\,C_{12} j_{34}^*[\nabla R] = -\,C_1^1 j_{34}^* i_1^*[\nabla R] - C_{12} j_{34}^* j_2^*[\nabla R].$$

Moreover we get:

a) $$j_{34}^* i_1^*[\nabla R] = j_{34}^* \nabla^\perp [\Gamma]^2 ;$$

b) $$j_{34}^* j_2^*[\nabla R] = j_{34}^* S_{12}(\nabla''[\Gamma]^2 - S_{13}\nabla''[\Gamma]^2),$$

in fact, $\forall x, y, z \in \mathscr{C}^\infty_{(1,0)}M''$, $\forall u \in \mathscr{C}^\infty_{(1,0)}M$, we have

$$\begin{aligned}
[\nabla R](u, x, y, z) &= [\nabla_u \nabla_x \nabla_y z] - [\nabla_u \nabla_y \nabla_x z] - [\nabla_u \nabla_{L_{xy}} z] \\
&\quad - [R](\nabla_u x, y, z) - [R](x, \nabla_u y, z) - [R](x, y, \nabla_u z) \\
&= \nabla_x([\Gamma]^2(u, y, z)) + [\Gamma]^2(L_u x, y, z) + [R](u, x, \nabla_y z) \\
&\quad - \nabla_y([\Gamma]^2(u, x, z)) - [\Gamma]^2(L_u y, x, z) - [R](u, y, \nabla_x z) \\
&\quad - [\Gamma]^2(L_x u, y, z) - [\Gamma]^2(\nabla_u x, y, z) + [\Gamma]^2(\nabla_u y, x, z) \\
&= (\nabla_x[\Gamma]^2)(u, y, z) - (\nabla_y[\Gamma]^2)(u, x, z).
\end{aligned}$$

The statement follows taking into account that, by the harmonicity condition, we get

$$C_{12}S_{12}S_{13}\nabla''[\Gamma]^2 = C_{12}S_{13}\nabla''[\Gamma]^2 = C_{23}\nabla''[\Gamma]^2 = \nabla'' C_{12}[\Gamma]^2 = 0 \quad \dot{-}$$

This theorem can be viewed as a particular case of the previous one, if we take into account that the Riemannian tensor R is locally the covariant derivative of a $C^{(0,\infty)}$ tensor.

8. ELECTROMAGNETIC AND GRAVITATIONAL SHOCK WAVES

We apply now our theory to a physical case, namely to the relativistic electromagnetic and gravitational shock waves.

Henceforth, M represents the space-time manifold and Σ represents the support of the shock waves.

8.1. DEFINITION. — A « self-interacting system constituted by a gravitational field an electromagnetic field and an incoherent charged fluid » with a shock of 1° kind is a 6-plet

$$(M, \Sigma, g, F, C, \mu, \rho)$$

where

M is a C^∞ manifold without boundary, with dimension 4, connected, paracompact, oriented and time oriented with respect to g;

Σ is a C^∞ embedded submanifold of M, without boundary, with dimension 3, oriented;

g is a $C^{(1,\infty)}$ Lorentz metric;

F is a $C^{(0,\infty)}$ 2-form;

C is a family $\{ D_p \}_{p\in P}$ of embedded, connected, time like, maximal submanifolds, such that $D \equiv \bigcup_{p\in P} D_p$ is open and there exists locally a $C^{(1,\infty)}$ chart adapted to the family;

μ is a positive $C^{(0,\infty)}$ function of M which is zero on $M - D$, ρ is the function $\rho = K\mu$, with $K \in R - \{ 0 \}$

Such that

$$r - \frac{1}{2} r_0 g = \tau \tag{1}$$

$$dF = 0 \tag{2}$$

$$\operatorname{div} F = - J, \tag{3}$$

where

$$\tau \equiv \frac{1}{4} F^2 g + C_{13} F \otimes F - \mu v \otimes v$$

$$J \equiv \rho v$$

v is the unique vector field tangent to the family C, normalized and future oriented $\doteq$

It is known that from (1) (2) and (3), by means of Bianchi identity, we get further equations

$$\operatorname{div} (\mu v) = 0 \tag{4}$$

$$\mu \nabla_v v = i_J F \tag{5}$$

8.2. THEOREM. — Let $(M, \Sigma, g, F, C, \mu, \rho)$ be a self interacting system as in the previous definition.

Moreover, we assume

$$\varepsilon^2 g''(x) \neq 0, \qquad (6)\ (^4)$$

$$\varepsilon^1 F(x) \neq 0, \qquad \forall x \in \Sigma. \qquad (7)$$

(4) Such assumption is suggested by considerations on the Cauchy problem (see [1] and [10]) in order to get an effective shock.

Then we get the following results.

a) Σ is singular, i. e.

a') $$l^2 = 0.$$

b) $$\left(C_{13} - \frac{1}{2}C_{34}\right)\varepsilon^2 g = 0,$$

i. e.

b') $$g^2(l) = \frac{1}{2}\operatorname{tr}\widehat{g^2}l.$$

c) $$\begin{cases} j^*_{23}\varepsilon F = 0 \\ C_{23}\varepsilon F = 0 \end{cases}$$

i. e.

c') $$F' = l \wedge F^0,$$

c'') $$F^0(l) = 0$$

(where F^0 is defined up to a multiple of l, while j^*F^0 is uniquely given).

d) $$\varepsilon v = 0$$

e) $$\varepsilon \mu = 0$$

f) $$j^*_{32}(\operatorname{div}'' + \operatorname{div}^{\perp})\varepsilon^2 g = j^*_{23}\varepsilon\tau$$

i. e.

f') $$(\operatorname{div}'' l - 2\nabla_l)j^*g^2 = -F(l, F^0)j^*g + j^*(F^0 \vee F(l)) \quad \text{for} \quad \nabla_l l = 0$$

g) $$(\operatorname{div}'' + \operatorname{div}^{\perp})\varepsilon F = A_{12}C^{12}_{23}(\varepsilon^2 g \otimes F)$$

i. e.

g') $$(\operatorname{div}'' l - 2\nabla_l)j^*F^0 = -(j^*g^2)(\bar{F}(l)) \quad \text{for} \quad \nabla_l l = 0$$

h) If $u \in \mathscr{C}^\infty_{(1,0)}M$ is such that $u^2 = 1$, $L_u v = 0$ and $u \cdot v = 0$

(we can find such an u, at least locally) then we get the geodesic derivation formula

$$[\nabla_v \nabla_v u] = K i_v \varepsilon_u F + [R](v; u, v).$$

If (e_0, e_1, e_2, e_3) is an orthonormal local basis, such that $v = e_0$, $l = \lambda(e_0 + e_1)$, e_2 and e_3 are the eigenvectors of the restriction of g^2 to the plane orthogonal to e_0 and e_1 with eigenvalues γ_2 and γ_3, then we get (choicing $\lambda = 1$)

$$\begin{aligned}
[\nabla_v \nabla_v u].e_0 &= 0 \\
[\nabla_v \nabla_v u].e_1 &= 0 \\
[\nabla_v \nabla_v u].e_2 &= -Ku^1F^0_2 + \frac{1}{2}\gamma u^2 \\
[\nabla_v \nabla_v u].e_3 &= -Ku^1F^0_3 - \frac{1}{2}\gamma u^3,
\end{aligned}$$

where $\gamma \equiv \gamma_2 = -\gamma_3$.

Proof. — Let us prove *a*) and *b*):

(1) gives

$$0 = [r] = \frac{1}{2}(l^2 g^2 + c_1^1 \widehat{g}^2 l \otimes l \vee g^2(l)),$$

which is equivalent to

$$l^2 \neq 0 \qquad \text{and} \qquad j^* g^2 = 0$$

or

$$l^2 = 0 \qquad \text{and} \qquad g^2(l) = \frac{1}{2} \operatorname{tr} \widehat{g}^2 l.$$

But the first condition is excluded by (6).

Let us prove *c*):

(2) gives $$l \wedge F' = 0,$$

i. e. $$F' = l \wedge F^0;$$

moreover (7) gives $$j^* F^0 \neq 0 \qquad \forall x \in \Sigma.$$

(3) gives $$l \cdot F^0 = 0$$

(and $$l^2 = 0).$$

Let us prove *d*):

(5) gives $$0 = K[i_v F] = [\nabla_v v] = l(v) v',$$

hence $$v' = 0,$$

v being time-like and l null.

Let us prove *e*):

(4) gives $$0 = [\mu \operatorname{div} v] - [v . \mu] = - l(v)\mu$$

hence $$\mu' = 0.$$

Let us prove *f*):

Taking into account the Bianchi identity, theorem 7.4 gives

$$\begin{aligned} j_{23}^*(\operatorname{div}'' + \operatorname{div}^\perp)\varepsilon^2 g &= j_{23}^* S_{13}[\operatorname{div} R] = j_{23}^* A_{13}[\nabla r] \\ &= j_{23}^* A_{13}\left[\nabla\left(\tau - \frac{1}{2}\operatorname{tr}\tau\, g\right)\right] \\ &= j_{23}^* \varepsilon^1\left(\tau - \frac{1}{2}\operatorname{tr}\tau\, g\right) \\ &= -\frac{1}{2}(C_{23}^{12}(\varepsilon^1 F \otimes \overline{F})) \otimes j^* g \\ &\quad + C_{24}\varepsilon^1 F \otimes F + C_{12}C_{14} F \otimes \varepsilon^1 F \\ &= l \otimes (- F' \cdot F j^* g + S_{12} j^*(i_l F \otimes F^0)) \end{aligned}$$

hence

$$(\operatorname{div}'' l) j^* g^2 - 2\nabla_l j^* g^2 = - F(l, F^0) j^*(F^0 \vee F(l)).$$

Let us prove *g*):

Theorem 7.3 gives

$$\begin{aligned}(\mathrm{div}'' + \mathrm{div}^{\perp})\varepsilon\mathrm{F} &= [\mathrm{div}\ \nabla\mathrm{F}] \\ &= [\mathrm{div}\ d\mathrm{F}] - [\mathrm{A}_{12}\mathrm{C}_{13}\nabla\nabla\mathrm{F}] \\ &= -\ [\mathrm{A}_{12}\nabla\mathrm{C}_{12}\nabla\mathrm{F}] - [\mathrm{A}_{12}\mathrm{C}_{13}[\widehat{\mathrm{R}} \langle\rangle \underline{\mathrm{F}})] \\ &= -\ \mathrm{A}_{12}\varepsilon\mathrm{J} - \mathrm{A}_{12}\mathrm{C}_{13}(\widehat{[\mathrm{R}]} \langle\rangle \underline{\mathrm{F}}) \\ &= -\ 2\mathrm{C}_{13}^{12}(\underline{[\mathrm{R}]} \otimes \overline{\mathrm{F}})\end{aligned}$$

hence

$$(\mathrm{div}''\, l)j^*\mathrm{F}^0 - 2\nabla_l j^*\mathrm{F}^0 = -\ (j^*g^2)(\overline{\mathrm{F}}(l)).$$

Let us prove *h*):

(5) gives

$$\begin{aligned}[\nabla_v\nabla_v u] &= [\nabla_v\nabla_u v] + [\nabla_v \mathrm{L}_v u] \\ &= [\nabla_u\nabla_v v] + [\mathrm{R}](v, u, v) + \varepsilon_v(\mathrm{L}_v u) \\ &= \mathrm{K}[\nabla_u i_v \mathrm{F}] + [\mathrm{R}](v, u, v) \\ &= \mathrm{K} i_v \varepsilon_u \mathrm{F} + [\mathrm{R}](v, u_\perp, v),\end{aligned}$$

where $u^{\perp}$ is the component of u orthogonal to v and l.

Moreover

$$\begin{aligned}&i_l[\nabla_v\nabla_v u] = 0, \qquad \text{for} \qquad i_l\mathrm{F}' = 0 = i_l[\mathrm{R}]; \\ &i_v[\nabla_v\nabla_v u] = 0, \qquad \text{for} \qquad i_v i_v \mathrm{F}' = 0 = i_v i_v[\mathrm{R}]; \\ &[\mathrm{R}](v, u\,;\, v, e_i) = \frac{1}{2} g^2(u, e_i), \qquad i = 2, 3.\end{aligned}$$

Furthermore $\gamma_2 = -\ \gamma_3$ follows form *b*) ∸

Let us remark that the formulas *b*), *c*), *f*) and *g*) are compatibility conditions on initial data and they involve only $j^*\mathrm{F}^0$ and j^*g^2.

Moreover the equations *f*) and *g*) result into ordinary differential equations along the null geodesic generated by l.

8.3. If $j^*(\mathrm{F}(l)) \neq 0$, then an electromagnetic shock induces effectively a gravitational shock and *vice versa*. More precisely we get the following result.

PROPOSITION. — If $j^*\mathrm{F}^0 \neq 0$, then the following conditions are equivalent:

a) $(\mathrm{div}''\, l - 2\nabla_l)j^*g^2 = 0$

b) $j^*\mathrm{F}(\bar{l}) = 0$.

If $j^*g^2 \neq 0$, then the following conditions are equivalent:

c) $(\mathrm{div}''\, l - 2\nabla_l)j^*\mathrm{F}^0 = 0$

b) $j^*\mathrm{F}(l) = 0$.

Proof. — *a*) ⇒ *b*) j^*g and $j^*(\mathrm{F}^0 \vee \mathrm{F}(l))$ are linearly independent, for the first one is not decomposable while the second one is.

Hence $j^*(F(\bar{l})) = 0$.

$c) \Rightarrow b)$ let $\bar{y} \equiv \bar{F}(l) \in \mathcal{C}^{\infty}_{(1,0)}\Sigma$. Let $j^*(g^2(\bar{y})) = 0$ and $j^*y = 0$.

Then choicing a basis $\{ e_0, e_1, e_2, e_3 \}$ such that

$$e_1 \equiv l \quad , \quad e_2 = y \quad , \quad e_3 \in \mathcal{C}^{\infty}_{(1,0)}\Sigma \cap \{ e_2 \}^{\perp} \quad , \quad e_0 \in \{ e_2, e_3 \}^{\perp}$$

and

$$e_3^2 = 1 \quad , \quad e_0^2 = -1, \qquad \text{we find} \qquad g^2_{22} = g^2_{33} = 0$$

Hence, taking into account 8.2 b, we get

$$j^*g^2 = 0$$

$b) \Rightarrow a)$ and $b) \Rightarrow c)$ are trivial $\dotplus$

Let us remark that $j^*(\underline{F}(l)) = 0$ means that each observer sees the electric and the magnetic field parallel to the observed direction of $\bar{l}$.

8.4. Let us remark that if V is a one dimensional vector space then $\underset{p}{\otimes}$ V is one dimensional and, if $p = 2q$, it has a natural orientation.

LEMMA. — The tensors

$$W_e \equiv C_{36}(\varepsilon^1 F \otimes \varepsilon^1 F) = (F^0)^2 l \otimes l \otimes l \otimes l \in \mathcal{C}^{\infty}_{(0,4)}M_{/\Sigma}$$

$$W_g \equiv \left(\frac{1}{2} C_{36}C_{48} - C_{23}C_{67}\right)\varepsilon^2 g \otimes \varepsilon^2 g$$

$$= \frac{1}{2}\left((g^2)^2 - \frac{1}{2}(\text{tr } g^2)^2\right) l \otimes l \otimes l \otimes l \in \mathcal{C}^{\infty}_{(0,4)}M_{/\Sigma},$$

depend only on (j^*F^0) and (j^*g^2) and are positive.

Proof. — $a)$ $(F^0)^2$ depends only on j^*F^0 and it is positive. In fact, taking in to account (c''), we get

$$\begin{matrix} j^*G^0 = j^*F^0 \\ F^0 \cdot l = 0 \end{matrix} \Rightarrow (G^0)^2 = (F^0 + \alpha l)^2 = (F^0)^2 > 0$$

$b)$ $(g^2)^2 - \frac{1}{2}(\text{tr } g^2)^2$ depends only on j^*g^2 and it is positive. In fact, taking into account $(b)'$ and using a basis $\{ e_0, e_1, e_2, e_3 \}$ such that

$$e_1 \equiv l \quad , \quad e_0 \cdot e_1 = 1 \quad , \quad e_0^2 = -1 \quad , \quad e_2^2 = e_3^2 = 1 \quad , \quad \{ e_2, e_3 \} = \{ e_0, e_1 \}^{\perp},$$

e_1 and e_2 are the eigenvectors of g^2 restricted to their plane, we find

$$(g^2)^2 - \frac{1}{2}(\text{tr } g^2)^2 = 2\gamma^2, \qquad \text{where} \qquad g^2_{22} = -g^2_{33} = \gamma \quad \dotplus$$

Note that $(g^2)^2 - \frac{1}{2}(\operatorname{tr} g^2)^2 = b^2$, where $b \in \mathscr{C}^{\infty}_{(0,2)}\mathrm{M}_{/\Sigma}$ is any tensor such that $j^*b = j^*g$ and $b(l) = 0$. Hence we get that $b \in \mathscr{C}^{\infty}_{(2,0)}\Sigma$ and $\operatorname{tr} \widehat{b} = 0$.

8.5. The preceding result suggests to assume as a measure of the shock the following tensors.

DEFINITION. — The « energy of the electromagnetic shock » is the tensor

$$\mathrm{W}_e \equiv \mathrm{C}_{36}(\varepsilon^1\mathrm{F} \otimes \varepsilon^1\mathrm{F})$$

and the « energy of the gravitational shock » is the tensor

$$\mathrm{W}_g \equiv \left(\frac{1}{2}\mathrm{C}_{36}\mathrm{C}_{48} - \mathrm{C}_{23}\mathrm{C}_{67}\right)\varepsilon^2 g \otimes \varepsilon^2 g.$$

The « energy of the gravitational electromagnetic shock » is the tensor

$$\mathrm{W} \equiv \mathrm{W}_e + \mathrm{W}_g \quad \dot{-}$$

8.6. For the energy tensor $\mathrm{W} \equiv \mathrm{W}_e + \mathrm{W}_g$ we find the following conservation law.

PROPOSITION. — $\operatorname{div}'' \mathrm{W} = 0$

Proof.

$$\begin{aligned}
\operatorname{div}'' \mathrm{W} &= (\operatorname{div}'' l - \nabla_l)\left((\mathrm{F}^0)^2 + \frac{1}{2}(g^2)^2 - \frac{1}{4}(\operatorname{tr} g^2)^2\right) l \otimes l \otimes l \\
&= (\langle \mathrm{F}^0, (\operatorname{div}'' l - 2\nabla_l)j^*\mathrm{F}_0 \rangle + \frac{1}{2}\langle \bar{b}, (\operatorname{div}'' l - 2\nabla_l)j^*g^2 \rangle) l \otimes l \otimes l \\
&= -j^*g^2(\bar{\mathrm{F}}(l), \bar{\mathrm{F}}^0) + \frac{1}{2}\langle j^*(\mathrm{F}^0 \vee \mathrm{F}(l)), \bar{b} \rangle - \frac{1}{2}\mathrm{F}(l, \bar{\mathrm{F}}^0)\langle \bar{b}, j^*g \rangle \\
&= -j^*g^2(\bar{\mathrm{F}}(l), \bar{\mathrm{F}}^0) + \frac{1}{2}\langle \mathrm{F}^0 \vee \mathrm{F}(l), \bar{b} \rangle \\
&= -j^*g^2(\bar{\mathrm{F}}(l), \bar{\mathrm{F}}^0) + j^*g^2(\bar{\mathrm{F}}^0, \bar{\mathrm{F}}(l)) = 0 \quad \dot{-}
\end{aligned}$$

Here we have developed the interacting gravitational electromagnetic field as an example. In an analogous way one can easily study the three fields separately.

LIST OF SYMBOLS

$\mathrm{C}^{i_1 \dots i_p}_{j_1 \dots j_p}$ is the contraction of the contravariant index $i_1 \dots i_p$ with the covariant index $j_1 \dots j_p$.

$C_{i_1 \dots i_p, j_1 \dots j_p}$ is the contraction of the index $i_1 \dots i_p$ with the index $j_1 \dots j_p$, identifying covariant and contravariant indices by means of the metric.

$S \equiv \sum_{\sigma \in S(q)} \sigma$ is the symmetrization operator.

$A \equiv \sum_{\sigma \in S(q)} \varepsilon(\sigma)\sigma$ is the antisymmetrization operator ($\sigma \equiv$ permutation, $S(q) \equiv$ symmetric group of order q).

$\vee \equiv S \circ \otimes$ is the symmetrized tensor product.

$\wedge \equiv A \circ \otimes$ is the antisymmetrized tensor product.

$*$ is the Hodge contraction with the unitary volume form.

t^* is the transpose of the tensor t, by means of the duality.

t^+ is the adjoint of the tensor t, by means of the metric.

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(Manuscrit reçu le 6 novembre 1978).