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Scale symmetry and virial theorem

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ABSTRACT. — Scale symmetry (or dilatation invariance) is discussed in terms of Noether's Theorem, expressed in terms of a symmetry group action on phase space endowed with a symplectic structure. The conventional conceptual approach expressing invariance of some Hamiltonian under scale transformations is re-expressed in alternate form by infinitesimal automorphisms of the given symplectic structure. That is, the vector field representing scale transformations leaves the symplectic structure invariant. In this model, the conserved quantity or constant of motion related to scale symmetry is the virial. It is shown that the conventional virial theorem can be derived within this framework.

I. INTRODUCTION

Symmetry principles and their associated conservation laws in high energy physics are of two types: «.Space-time »-symmetries and « internal » symmetries. For internal symmetries, properties such as charge, baryon number, etc., are conserved whereas for space-time symmetries, kinematic properties such as energy and momentum are conserved. A further classification refers to exact and approximate symmetries. So far, only internal approximate symmetries have been successfully exploited in high energy physics. As regards approximate space-time symmetries, scale or dilatation invariance is of this last type, since this symmetry is exact if and only if strongly interacting particles are massless. That is, exact dilatation symmetry implies the mass spectrum is either continuous or all masses are zero. So scale symmetry must be broken for strong interactions. This situation refers to a quantum field-theoretical approach where conserved quantities are generators of symmetry transformations in Hilbert-space satisfying certain commutation relations.

In this paper the nature of dilatation symmetry is studied within an intrinsic manifold-theoretical framework. Such a frame seems to provide complementary features to a quantum field-theoretical description of symmetries. It is shown that the virial theorem is a consequence of scale invariance.

II. SCALE SYMMETRY IN TERMS OF A DIFFERENTIAL-TOPOLOGICAL VERSION OF NOETHER'S THEOREM

Let (M^4, ds^2), $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = g_{\mu\nu}dx^\mu dx^\nu$, be the four-dimensional space-time manifold and consider C(4), the 15-parameter conformal group of automorphisms of M^4. Its infinitesimal representation is given by the 15 infinitesimal generators

$$\Lambda_{\mu\nu} = g_{\nu\lambda}x^\lambda \frac{\partial}{\partial x^\mu} - g_{\mu\lambda}x^\lambda \frac{\partial}{\partial x^\nu} \text{(Lorentz-generators)} \quad A_\mu = x^2 \frac{\partial}{\partial x^\mu} - 2g_{\mu\nu}x^\nu x^\rho \frac{\partial}{\partial x^\rho}$$

$$P_\mu = \frac{\partial}{\partial x^\mu} \qquad \mu, \nu, \lambda = 0, 1, 2, 3$$

and

$$X_D = x^\mu \frac{\partial}{\partial x^\mu} \tag{1}$$

the latter being the dilatation-generator. Note that scale-transformations in M^4 are defined by

$$x^{\mu'} = \lambda x^\mu \qquad \lambda \in \mathbb{R} \tag{2}$$

$$ds'^2 = \lambda^2 ds^2 \tag{3}$$

Let T^*M^4 be the cotangent bundle of M^4. A symplectic structure Ω on T^*M^4 is a distinguished closed 2-form ($d\Omega = 0$) of maximal rank $2n$ defined everywhere on T^*M^4 ([*1*], [*2*], p. 534). This nondegenerate bilinear form has by Darboux's theorem the local expression [*1*]

$$\Omega = \Sigma dp_\mu \wedge dx^\mu \tag{4}$$

It may be regarded as a linear isomorphism $\Omega(x) : T_x \to T_x^*$ for each $x = (p_\mu, x^\mu) \in T^*M^4$ and hence as a bundle isomorphism (a diffeomorphism from one bundle to another which maps isomorphically fibres into fibres):

$$\Omega : TM^4 \to T^*M^4 \quad , \quad \Omega(X) = i(X)\Omega \tag{5}$$

(Refer to formula (50), p. 541, Ref. [*2*]). Here TM^4 is the tangent bundle of M^4, $X : M^4 \to TM^4$ a C^∞ vector field and $i(X)$ the interior product operator [*1*], [*2*].

LEMMA 1. — In terms of local coordinates (p_μ, x^μ) the map (5) and its inverse Ω^{-1} are given by

$$\Omega\left(a_\mu \frac{\partial}{\partial p_\mu} + b_\mu \frac{\partial}{\partial x^\mu}\right) = - b_\mu dp_\mu + a_\mu dx^\mu \tag{6}$$

and

$$\Omega^{-1}(c_\mu dp_\mu + e_\mu dx^\mu) = e_\mu \frac{\partial}{\partial p_\mu} - c_\mu \frac{\partial}{\partial x^\mu} \tag{7}$$

Proof. — The proof follows from the definition of the interior product $i(X)$ [1], [2]. In fact, let $\Omega = \Sigma dp_\mu \wedge dx^\mu$, then

$$\begin{aligned} i(X)\Omega &= i(X)(dp_\mu \wedge dx^\mu) = i(X)dp_\mu \wedge dx^\mu + (-1)^1 dp_\mu \wedge i(X)dx^\mu \\ &= X(p_\mu)dx^\mu - X(x^\mu)dp_\mu \\ &= a_\mu dx^\mu - b_\mu dp_\mu . \end{aligned}$$

This proves (6). Here we used the following general rules

$$i(X)\Omega(Y) = \Omega(X, Y) \tag{8}$$

$$i(X)(\omega \wedge \eta) = i(X)\omega \wedge \eta + (-1)^{\deg \omega}\omega \wedge i(X)\eta \tag{9}$$

$$i(X)df = X(f) \tag{10}$$

where ω and η are any differentiable differential forms of degree p and q, respectively, f a C^∞-map, X and Y are C^∞ vector fields. Formula (7) is obtained along similar lines. ■

COROLLARY 2. — The lift of the infinitesimal dilatation vector field (1) on M^4 is the vector field

$$X_{\bar{D}} = x^\mu \frac{\partial}{\partial x^\mu} - p_\mu \frac{\partial}{\partial p_\mu} \tag{11}$$

on T^*M^4, if $\bar{D} = x^\mu p_\mu$.

Proof. — Let the observable

$$\bar{D} : T^*M^4 \to \mathbb{R} \; ; \; (p_\mu, x^\mu) \mapsto \bar{D}(p_\mu, x^\mu) = x^\mu p_\mu \tag{12}$$

Then

$$d\bar{D} = \sum \frac{\partial \bar{D}}{\partial p_\mu} dp_\mu + \frac{\partial \bar{D}}{\partial x^\mu} dx^\mu \tag{13}$$

so one obtains by formula (7)

$$\Omega^{-1}(d\bar{D}) = X_{\bar{D}} \quad ■ \tag{14}$$

Next define the Lie derivative of a differential form ω with respect to the vector field $X_{\bar{D}}$ by

$$(L_{\bar{D}}\omega)(x) = \lim_{t \to 0} \frac{\omega_t(x) - \omega(x)}{t} \qquad t \in \mathbb{R} \tag{15}$$

Then one proves the following

PROPOSITION 3. — The infinitesimal transformation X_D (eq. (11)) is an infinitesimal automorphism of the symplectic structure $\Omega = \Sigma dp_\mu \wedge dx^\mu$, that is

$$L_{\bar{D}}(\Omega) = 0 \tag{16}$$

Proof. — Set

$$X_{D_1} = x^\mu \frac{\partial}{\partial x^\mu} \tag{17 a}$$

and

$$X_{D_2} = p_\mu \frac{\partial}{\partial p_\mu} \tag{17 b}$$

then, on account of the general rules

$$L_{X_1+X_2} = L_{X_1} + L_{X_2} \tag{18}$$

and

$$L_X = i(X)d + di(X) \tag{19}$$

we have

$$L_{\bar{D}}(\Omega) : = L_{D_1-D_2}(\Omega) = L_{D_1}(\Omega) - L_{D_2}(\Omega)$$

and

$$L_{D_1}(\Omega) = i(D_1)d\Omega + d(i(D_1)\Omega = d(i(D_1)\Omega)$$

Now

$$i(D_1)\Omega = i\left(x^\mu \frac{\partial}{\partial x^\mu}\right)\Omega = i\left(x^\mu \frac{\partial}{\partial x^\mu}\right)(dp_\mu \wedge dx^\mu) = - x^\mu dx_\mu .$$

In fact

$$i(D_1)(dp_\mu \wedge dx^\mu) = i(D_1)dp_\mu \wedge dx^\mu + (-1)^1 dp_\mu \wedge i(D_1)dx^\mu$$

$$i(D_1)dp_\mu = X_{D_1}(p_\mu) = 0 \quad \text{and} \quad i(D_1)dx^\mu = X_{D_1}(x^\mu) = x^\mu \frac{\partial x^\mu}{\partial x^\mu} = x^\mu$$

by formulae (10) and (17 *a*). Therefore

$$di(D_1)\Omega = - dx^\mu \wedge dp_\mu \tag{20 a}$$

$$L_{D_1}\Omega = \Omega \tag{21 a}$$

Likewise, one finds on account of (10) and (17 *b*):

$$i(D_2)dp_\mu = p_\mu, \qquad i(D_2)dx^\mu = 0, \qquad i(D_2)\Omega = p_\mu dx^\mu$$

$$\Rightarrow di(D_2)\Omega = dp_\mu dx^\mu \tag{20 b}$$

$$\Rightarrow L_{D_2}\Omega = \Omega \tag{21 b}$$

The conjunction of (20 *a*) and (20 *b*), or (21 *a*) and (21 *b*), respectively, yields the assertion (16), thus proving the proposition. ■

By virtue of the Corollary 2 and Proposition 3 one infers.

PROPOSITION 4. — An observable $f : T^*M^4 \to \mathbb{R}$ is a constant of motion associated with scale symmetry $X_{\bar{D}} = x^\mu \frac{\partial}{\partial x^\mu} - p_\mu \frac{\partial}{\partial p_\mu}$ iff f represents the four-dimensional virial

$$\bar{D} = x^\mu p_\mu = f \tag{22}$$

Proof. — By Corollary 2 ; $\bar{D} = x^\mu p_\mu$ implies $X_{\bar{D}} = x^\mu \frac{\partial}{\partial x^\mu} - p_\mu \frac{\partial}{\partial p_\mu}$. Conversely, let $X_{\bar{D}}$ be given by (11), then, by (5) and (6):

$$\Omega(X_{\bar{D}}) = i(\bar{D})\Omega = - p_\mu dx^\mu - x^\mu dp_\mu = - df = - d\bar{D}$$

and hence $f = x^\mu p_p = \bar{D}$. ■

The observable f is constant on U_t-orbits (where $\{ U_t \mid - \infty < t < + \infty \}$ is a dynamical 1-parameter group) i. e.

$$X_{\bar{D}}(f) = \frac{d}{dt}[f(U_t(p_\mu, x^\mu)]_{t=0} = \lim_{t \to 0} \frac{U_t^* f - f}{t} = 0 \tag{23}$$

In fact:

$$X_{\bar{D}}(f) = \left(x^\mu \frac{\partial}{\partial x^\mu} - p_\mu \frac{\partial}{\partial p_\mu}\right)(f) = \left(x^\mu \frac{\partial}{\partial x^\mu} - p_\mu \frac{\partial}{\partial p_\mu}\right)(x^\mu p_\mu) = x^\mu p_\mu - p_\mu x^\mu = 0.$$

Remark 5. — The constant of motion $f = x^\mu p_\mu$ associated with a dynamical system is also obtained as follows:

$$i(\bar{D})\Omega = i(\bar{D})(dp_\mu \wedge dx^\mu) = X_{\bar{D}}(p_\mu)dx^\mu - X_{\bar{D}}(x^\mu)dp_\mu$$

Now

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial p_\mu} dp_\mu \quad \text{hence} \quad \frac{\partial f}{\partial x^\mu} = X_{\bar{D}}(p_\mu) = x^\mu \frac{\partial p_\mu}{\partial x^\mu} - p_\mu \frac{\partial p_\mu}{\partial p_\mu} = - p_\mu$$

and

$$\frac{\partial f}{\partial p_\mu} = - X_{\bar{D}}(x^\mu) = - x^\mu \frac{\partial x^\mu}{\partial x^\mu} + p_\mu \frac{\partial x^\mu}{\partial p_\mu} = - x^\mu.$$

$$\Rightarrow df = - p_\mu dx^\mu - x^\mu dp_\mu \Rightarrow f = x^\mu . p_\mu.$$

Remark 6. — On account of Lemma 1, formula (7) one may re-formulate the result of proposition 4 in terms of the Poisson bracket:

$$(\Omega^{-1} d\bar{D})(f) = \{ \bar{D}, f \} = \frac{\partial \bar{D}}{\partial p_\mu}\frac{\partial f}{\partial x^\mu} - \frac{\partial \bar{D}}{\partial x^\mu}\frac{\partial f}{\partial p_\mu} = X_{\bar{D}}(f) = 0.$$

The constant of motion $f = x^\mu p_\mu$ which was obtained from propositions 3 and 4 relates to Nöther's theorem whose intrinsic formulation may be given within a differential topological approach as follows:

PROPOSITION 7. — If G is any n-parameter Lie group which is a dynamical symmetry group of a canonical system (T^*M^4, Ω), such that $\beta_1(T^*M^4) = 0$ (*) (β_1: first Betti number), then with each parameter of G is associated an integral of the system.

Proof. — Let G act differentiably on T^*M^4, i. e. $\varphi : G \to \text{Diff}\,(T^*M^4)$ (diffeomorphisms of T^*M^4); $g \mapsto \varphi(g)$ is a group homomorphism and

(*) The number of linearly independent cohomology classes of order p is given by the p-th Betti number $\beta_p(M)$ of a manifold M.

$\emptyset: G \times T^*M^4 \to T^*M^4; (g, x) \mapsto \varphi(g)x$ is a C^∞-map. Then G is said to be a dynamical group of symmetries, iff it leaves the 2-form Ω invariant, i. e. iff $\varphi^*(g)\Omega = \Omega$ for $g \in G$. The transformation group of all diffeomorphisms φ of the phase space T^*M^4 that preserve Ω is

$$\text{Diff}(T^*M^4, \Omega) := \{ \varphi \in \text{Diff}(T^*M^4) \mid \varphi^*(g)\Omega = \Omega \}$$

and its « Lie algebra »

$$\text{diff}(T^*M^4, \Omega) = \{ X \in \mathscr{X}(T^*M^4) \mid L_X(\Omega) = 0 \}$$

Let now $Y \in d\varphi(g)\mathfrak{G} \subset \text{diff}(T^*M, \Omega)$ be an infinitesimal transformation which is a locally Hamiltonian vector field:

$$L_Y(\Omega) = \lim_{t \to 0} 1/t(U_t^*\Omega - \Omega) = 0 \qquad \text{(cf. formula (15))}$$

Thus, by formula (19) one has.

$i(Y)d\Omega + d(i(Y)\Omega) = 0$. By the assumption $B_1(T^*M^4) = 0$ there exists an $f \in C^\infty(T^*M^4, \mathbb{R})$, such that $i(Y)\Omega = df$. Hence:

$$i(Y)\Omega(X) = \Omega(Y, X) = -\Omega(X, Y) = -i(X)\Omega(Y) = 0$$

thus, $df(X) = X(f) = 0$. ■

Remark 8. — If $\{ X_1, \ldots, X_r \}$ is a basis of generators for the Lie algebra $\mathfrak{G}(G)$ of the symmetry group G, the X_i have images $Y_i = d\varphi(g)X_i$, i. e. the action $\varphi : G \to \text{Diff}(T^*M^4)$ determines a basis $\{ Y_1, \ldots, Y_r \}$ for $d\varphi(g)\mathfrak{G} \subset \text{diff}(T^*M, \Omega)$. This basis uniquely characterizes the action φ. The one-forms $i(Y_i)\Omega$ may locally be integrated as differentials $\{ df_1, \ldots, df_r \}$ of functions $f_i \in C^\infty(T^*M^4, \mathbb{R})$ which form a system of local first integrals, since the $\{ df_1, \ldots, df_r \}$ are linearly independent.

III. THE VIRIAL THEOREM DERIVED FROM THE CONSTANT OF MOTION $\bar{D}$

By proposition 4 the constant of motion

$$\bar{D} = x^\mu p_\mu = E.t - x^k p_k \tag{24}$$

is constant on U_t-orbits. Hence

$$\frac{d\bar{D}}{dt} = 0 = E - \frac{d}{dt}\Big(\sum_k p_k x^k\Big) \Rightarrow E = \frac{d}{dt}\Big(\sum p_k x^k\Big) = \sum_k p_k v_k + \sum x_k \dot{p}_k$$

$$E = 2T + \sum x_k \dot{p}_k = 2T - \sum x_k \frac{\partial V}{\partial x^k}$$

on account of $\dot{p}_k = -\dfrac{\partial V}{\partial x^k}$. Therefore

$$\overline{2T} = \overline{\sum_k x_k \frac{\partial V}{\partial x^k}} \tag{25}$$

For a motion in a bounded region of $\mathbb{R}^3$ the quantity $\sum x_k p_k$ is bounded and the time average $\overline{\dfrac{d}{dt}\left(\sum_k x_k p_k\right)}$ vanishes.

PROPOSITION 9 (virial theorem). — Let (T^*M^4, Ω), $\Omega = dp_\mu dx^\mu$, be a canonical system and C(4) the group of conformal automorphisms of M^4. Let $\overline{D} = x^\mu p_\mu$ be the constant of motion associated with dilatations (2). Define an energy function T on TM^4 by

$$T = m/2 \langle \, , \rangle : TM^4 \to \mathbb{R}; \quad T(X) = m/2 \langle X, X \rangle, \quad X \in TM^4 \tag{26}$$

Then $2T = \overline{\sum x_k \dfrac{\partial V}{\partial x^\mu}}$ holds where V is a homogeneous $C^\infty(\mathbb{R}^3, \mathbb{R})$ function of degree k.

The assumption of homogeneity for

$$V : \mathbb{R}^3 \to \mathbb{R}, \qquad V(tx) = t^k V(x), \qquad k \in \mathbb{Z}^+, \qquad t \in \mathbb{R} - \{0\}$$

is required so that Euler's relation

$$\sum_i x_i \frac{\partial V}{\partial x_i} = kV(x)$$

may be applied. This yields for instance in the case of the virial theorem for the inverse square law

$$V(r) = \frac{C}{r} \quad , \quad C = \text{const.} \tag{26}$$

i. e. Coulomb interaction (homogeneous function of degree $k = -1$), the familiar result

$$2\overline{T} = -\overline{V}. \tag{27}$$

Note that, conventionally, eq. (26) is obtained in terms of a direct argument along the following lines: Consider the eq. of motion

$$\vec{F} = \frac{C}{r^2}\frac{\vec{r}}{r} = m\dot{\vec{v}} \tag{28}$$

then

$$\vec{r}\vec{F} = m\vec{r}\dot{\vec{v}} = \frac{C}{r} = V(r) \qquad \text{(eq. (26))}$$

so

$$m\frac{d}{dt}(\vec{r}\,\vec{v}) = m\dot{\vec{r}}\,\vec{v} + m\vec{r}\,\dot{\vec{v}} = m\vec{v}\,\vec{v} + m\vec{r}\,\dot{\vec{v}}$$

$$m\frac{d}{dt}(\vec{r}\,\vec{v}) = 2\mathrm{T} + \mathrm{V}. \tag{29}$$

For any attractive potential, $\mathrm{C} < 0$, bound states occur, i. e. a particle remains indefinitely within a volume around the force centre. The time average of $d(\vec{r}\,\vec{v})/dt$ in a bound state must be zero if one averages over many cycles of the motion. Thus the last equation gives for the time average again eq. (27).

REFERENCES

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