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Section A :
*Physique théorique.*

# Iteration of endomorphisms on the real axis and representation of numbers

by

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ABSTRACT. — We study a class of endomorphisms of the set of real numbers $x$ of the form: $x \to \lambda f(x)$, $x \in [0, 2]$. The function $f$ is continuous, convex with a single maximum but otherwise arbitrary; $\lambda$ is a real parameter.

We focus our attention on periodic points: $x \in [0, 2]$ is periodic if there exists an integer $n$ such that the $n^{\text{th}}$ iterate of $x$ by $\lambda f$ coincides with $x$. Because of their special importance, we restrict ourselves to periods involving the maximum.

As shown by Metropolis *et al.*, for each mapping, one may represent in a non ambiguous way these periods by finite sequences of symbols R and L [the $i^{\text{th}}$ iterate of the maximum is represented by R (right) or L (left) depending on its position relatively to maximum] and these sequences have many universal properties. For example they can be ordered in a way which does not depend on details of the mapping.

In this paper, we prove two points:

*i*) the ordered set of all the symbolic sequences possesses a property of internal similarity: it is possible to find a monotonous application of the whole set into one of its subsets;

*ii*) we give a simple criterion for recognizing whether a sequence is allowed or not and to know in which order two given sequences appear.

For reasons of universality property, it will be sufficient to derive these results for the simplest case, namely the « linear » transform, i. e.

$$\begin{aligned} g(x) &= x, & 0 < x < 1 \\ &= 2 - x, & 1 < x < 2. \end{aligned}$$

We are led to define an expansion of real numbers analogous to $\beta$-expansion of Renyi. A few other properties of this peculiar case are briefly discussed.

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RÉSUMÉ. — On étudie une classe d'endomorphismes de l'ensemble des nombres réels $x$, de la forme

$$x \to \lambda f(x), \; x \in [0, 2].$$

La fonction $f$ est continue, convexe avec un seul maximum mais par ailleurs arbitraire; $\lambda$ est un paramètre réel.

Nous nous intéressons surtout aux points périodiques: $x \in [0,2]$ est périodique s'il existe un entier $n$ tel que le $n^{\text{ième}}$ itéré de $x$ par $\lambda f$ coïncide avec $x$. En raison de leur importance particulière, nous nous limitons aux périodes comprenant le maximum.

Comme l'ont montré Métropolis et ses collaborateurs, ces périodes peuvent être représentées de manière non ambiguë par des séquences finies de symboles R et L [le $i^{\text{ème}}$ itéré du sommet est représenté par R (« right ») ou L (« left ») suivant sa position relative par rapport au maximum] et ces séquences ont des propriétés universelles. Par exemple, on peut les ordonner d'une manière qui ne dépend pas de l'application.

Dans cet article, on clarifie 2 points :

*i*) l'ensemble ordonné de toutes ces séquences symboliques présente une propriété d'homothétie interne : on peut trouver une application monotone de tout l'ensemble dans l'une de ses parties,

*ii*) on donne un critère simple pour décider si une séquence est autorisée et pour savoir dans quel ordre relatif deux séquences apparaissent.

En raison de l'universalité, il suffit de montrer ces propriétés pour le cas le plus simple, l'application « linéaire »

$$\begin{aligned} g(x) &= x, & 0 < x < 1 \\ &= 2 - x, & 1 < x < 2. \end{aligned}$$

On est amené à définir un développement des nombres réels semblables au $\beta$-développement de Renyi. Quelques propriétés supplémentaires liées à ce cas particulier sont également discutées.

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## CONTENTS

# 1. INTRODUCTION

Let $f$ be a real valued function defined for $0 \leq x \leq 2$ which satisfies the following requirements

*i)* $f$ is continuous and differentiable in $[0,2]$

*ii)* $f(0) = f(2) = 0$ (1.1)

*iii)* $f$ has a unique maximum at $x = c$, $0 < c < 2$. $f(x) < f(c)$, if $x \neq c$. $f'(x) > 0$ if $x < c$ and $f'(x) < 0$ if $x > c$. (see fig. 1).

Let $\lambda$ be a real number $0 \leq \lambda \leq 2/f(c)$. Consider the transformation

$$T_\lambda(x) \equiv \lambda f(x) \tag{1.2}$$

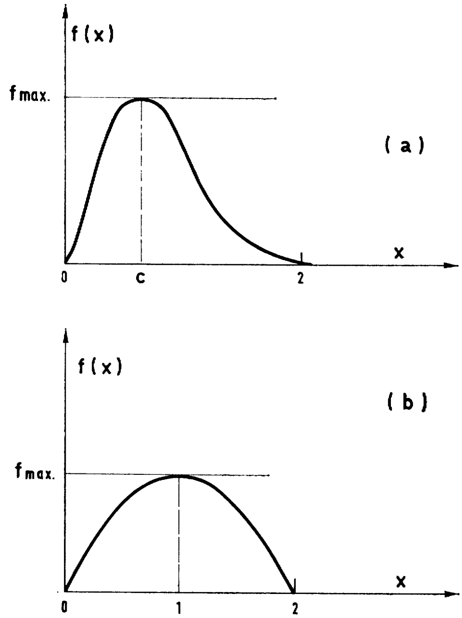


FIG. 1. — Plot of $f(x)$ as a function of $x$.
*a*) in the general case, $f(x)$ may not be convex,
*b*) for $f(x) = x(2 - x)$.

of the interval [0,2] into itself. The $l^{\text{th}}$ iterate of $T_\lambda$ denoted as $T_\lambda^{(l)}$, is defined by recursion, $T_\lambda^{(1)}(x) = T_\lambda(x)$, $T_\lambda^{(l)}(x) = T_\lambda[T_\lambda^{(l-1)}(x)]$, $l > 1$. The number $x$ belongs to a period of length $k$ iff $T_\lambda^{(k)}(x) = x$ while $T_\lambda^{(l)}(x) \neq x$, $l < k$, and, given $x$, this will happen only for particular values of $\lambda$. The periods containing $c$ and the corresponding values $\lambda$ are of special importance and we will consider only these in what follows. Such a period may be represented by a finite sequence of signs $\pm 1$ (or of letters R and L) as follows. Without loss of generality we start with $c$. If $T_\lambda(c) = c$, then $c$ belongs to a period of length 1 and we denote this period by a blank (or 0 or the letter *b*). If $T_\lambda(c) \neq c$ and $c$ belongs to a period of length $k > 1$, then $T_\lambda^{(k)}(c) - c = 0$, while $T_\lambda^{(l)}(c) - c \neq 0$ if $1 \leq l \leq k - 1$. The period (or the values of $\lambda$ for which this period exists) is represented by the $k - 1$ signs of the differences $c - T_\lambda^{(l)}(c)$ (or replace each $+ 1$ by an L and each $- 1$ by an R). Thus for example, if $T_\lambda(c) > c$, $T_\lambda^{(2)}(c) < c$ and $T_\lambda^{(3)}(c) = c$, then $c$ belongs to the period $(- 1, + 1)$ (or RL).

Note that if $T_\lambda(c) < c$, then $T_\lambda^{(l)}(c) = c$, $l \geq 1$ is impossible under the conditions on $f$. Thus every period starts with a $- 1$ (or the letter R). Also not every finite sequence of signs will represent a period.

The periods are completely ordered by the values of $\lambda$ corresponding to them. Thus if the periods $P = \sigma_1\sigma_2 \ldots \sigma_{k-1}$ and $Q = \tau_1\tau_2 \ldots \tau_{j-1}$, $\sigma_i = \pm 1$, $\tau_i = \pm 1$, arise for the values $\lambda_P$ and $\lambda_Q$ of $\lambda$, then P is said to be less (greater) than Q if $\lambda_P$ is less (greater) than $\lambda_Q$.

One may ask several questions:

1. Given a finite sequence of signs $\pm 1$ (or of letters R and L), how to decide whether it represents a period (i. e. whether a value of $\lambda$ exists for which this is the period)? When it does we say that the sequence is allowed.
2. Given two allowed sequences P and Q, which one is smaller?
3. Does the answer to the above questions depend on the details of $f$?

According to a theorem of Metropolis *et al.* [*1*] the answer to question (3) above is « no » for a large class of functions $f$, in particular for those satisfying conditions *i*)-*iii*) above. One may even relax some of them. We refer to this property as the universality property. In what follows we will try to answer the other two questions by choosing a particular $f$ for which all calculations can be carried to the end.

Metropolis, Stein and Stein [*1*] give an algorithm to get all the allowed sequences (U sequences in their paper, MSS sequences here).

The prescription is quite complicated and there is no simple relation between the length $k - 1$ of the sequence and the position on the real axis of the corresponding parameter $\lambda$. One of the aims of this article is to give a simple rule for ordering all sequences. As the order of occurrence of the MSS sequence is universal, it is sufficient to study it on a special transformation for which all necessary calculations may be done. We have considered the « broken linear » transformation

$$L_\lambda(x) = \lambda g(x) \quad 1 < \lambda < 2 \tag{1.3}$$

where $g(x)$ is the function

$$\begin{aligned} g(x) &= x & 0 < x < 1 \\ &= 2 - x & 1 < x < 2. \end{aligned} \tag{1.4}$$

Obviously, $g(x)$ is not differentiable at $x = 1$ and condition *i*) is violated. Nevertheless, the order of the periods is the same as for functions $f$ satisfying conditions (1.1) except that some of them (mostly « the harmonics ») are absent [*1*]. It is easy to see which ones and why and then to reconstruct the whole set of MSS sequences.

As shown in section 4, many properties of these sequences are connected with a representation of the real numbers, which we have called the $\lambda$-expansion by reference to the $\beta$-expansion of Renyi [*2*]-[*3*]. This $\lambda$-expansion is defined as follows. Given $\lambda$, $1 < \lambda < 2$, the $\lambda$-expansion of a number $x$ $1 < x < 2$ in the basis $\lambda$ is represented by the sequence $c_0, c_1, c_2, \ldots$ such that

*i*) $c_0 = 1$, $c_n = \pm 1$ or 0 for $n \geq 1$

*ii*) let $x_n = \sum_{k=0}^{n} \frac{c_k}{\lambda^k}$; then $x_n < x \Rightarrow c_{n+1} = +1$

$x_n > x \Rightarrow c_{n+1} = -1$

and $x_n = x \Rightarrow c_{n'} = 0$ for $n' > n$.

This is an expansion of the number $x$, as $| x - x_m | < 1/\lambda^m$. The digits used in a $\lambda$-expansion are $+1$ or $-1$, although the digits used in a $\beta$-expansion are 0 or 1 for $1 < \beta \leq 2$ (The digit 0 does not really exist in the $\lambda$-expansion, as it cannot be inserted between two non zero digits, it just marks the end of a $\lambda$-expansion when it is finite).

A particular case of $\lambda$-expansion is the so-called auto-expansion of the basis, that is the set $\{ c_n \}$ constructed by choosing for $\lambda$ and $x$ the same number. The « digits » of these auto-expansions are simply related with the symbolic sequences of Metropolis *et al.* This connection becomes clear when one considers the particular « broken linear » transform of eqs. (1.3)-(1.4).

More precisely, in sections 3 to 5, we derive:

*i*) a criterion indicating whether a given sequence of symbols R and L (in the sense of Metropolis *et al.*) does or does not correspond to a period (in other term, whether this sequence is « allowed » or not);

*ii*) the ordering criterion for the periods;

*iii*) the Sarkovskii theorem [*4-5-6-7*];

*iv*) some complementary properties of the $\lambda$-numbers related to the broken linear mapping $L_\lambda$.

In part 2, we show that the whole set of MSS sequences is similar to some parts of it. This property is called internal similarity and is proved just by using the algorithm of construction given by Metropolis *et al.* After recalling briefly the main results of ref. [*1*], we display the mapping which manifests the internal similarity property. The sequences given by the construction of Metropolis *et al.* is countable and fully ordered. Thus, one may define accumulation points for this set. Using the law of internal similarity we may show that the set of accumulation points has the power of the continuum.

## 2. INTERNAL SIMILARITY

In this section, we show that the family of MSS sequences is similar to some of its parts. This result does not depend on the particular expression of the function $f(x)$ of eqs. (1.1-1.2).

After recalling some notations and defining the ordering of the sequences, we build recursively all the MSS sequences (subsection 2.1).

It is then possible to define mappings of this set of sequences into itself which preserve the ordering relation. We exhibit a particular class of them which reduces the domain of values for the parameter $\lambda$ (subsection 2.2); we get thus the principle of internal similarity.

At the end of the section, we study other properties of this mapping considered as a composition law; one consequence of its existence is that the accumulation points of the parameter set associated to the MSS family of sequences has the power of the continuum.

### 2.1. The construction of Metropolis *et al.*

We intend to order the values of $\lambda$ for which MSS sequences exist i. e. values of $\lambda$ for which

$$T_\lambda^{(k)}(c) = c \qquad (2.1\,a)$$

with

$$T_\lambda^{(i)}(c) \neq c \quad \text{for} \quad 1 \leq i \leq k - 1. \qquad (2.1\,b)$$

The application $T_\lambda^{(i)}$ is defined recursively by

$$\begin{aligned} T_\lambda^{(o)}(x) &= x \\ T_\lambda^{(i)}(x) &= T_\lambda[T_\lambda^{(i-1)}(x)] \quad , \quad i \geq 1. \end{aligned} \qquad (2.2)$$

Following Metropolis *et al.* [*1*] one may associate to this period a sequence of $(k - 1)$ characters or symbols R (« right ») or L ( « left ») in the following manner:

$$P = \sigma_1\sigma_2 \ldots \sigma_{k-1} \qquad (2.3\,a)$$

where

$$\sigma_i = R \quad \text{if} \quad T_\lambda^{(i)}(c) > c \quad , \quad \sigma_i = L \quad \text{if} \quad T_\lambda^{(i)}(c) < c.$$

This may be written also

$$P = R^{\nu_1}L^{\mu_1}R^{\nu_2}L^{\mu_2} \ldots \qquad (2.3\,b)$$

with $\nu_i$, $\mu_i$ positive integers.

From now on, we shall denote by P or Q such sequences.

Several sequences may exist for a given $k$. For example, if $k = 5$, 3 sequences are found in the construction of Metropolis *et al.*:

$$RLR^2, \quad RL^2R, \quad RL^3.$$

These « allowed » sequences correspond to three values $\lambda_1$, $\lambda_2$, $\lambda_3$ of the parameter $\lambda$.

In their paper, Metropolis *et al.* [*1*] show that the $\lambda$ associated with the allowed sequences are ordered in a way which is independent of $f$ provided $f$ belongs to a large enough class of functions (roughly conditions 1.1). In the example above, one shows in this way that $\lambda_1 < \lambda_2 < \lambda_3$.

*Remarks.* — *a*) All sequences of R and L characters are not allowed. For example no allowed sequence begins with an L since in that case, we should have $T_\lambda(c) < c$, $T_\lambda^{(l)}(c) < T_\lambda^{(l-1)}(c)$ since $T_\lambda(x)$ is strictly increasing for $x < c$ and the equation $T_\lambda^k(c) = c$ is never satisfied.

*b*) The sequence P corresponding to $k = 1$ $[T_\lambda(c) = c]$ has length 0 and we shall denote it from now on by the symbol $b$ (= blank)

$$P = b \tag{2.4}$$

*Ordering relation on the sequences*

Being given P and P′ associated with $\lambda$ and $\lambda'$, we shall write

$$P < P' \quad \text{whenever} \quad \lambda < \lambda'. \tag{2.5}$$

This orders totally the MSS-sequences. Note that there is no simple connection between the length $k - 1$ (or cardinality) of a sequence and its order. For example

$$b < RLR^r < RL < RL^m \text{ for every } m, n.$$

2.1.1. HARMONIC AND ANTI-HARMONIC MAPPING

Again let P be an allowed sequence of $(k - 1)$ characters R or L.

*The harmonic of* P is defined by the mapping $H : P \to H(P)$ where

$$H(P) = P\sigma P \tag{2.6}$$

with $\sigma = L$ (resp. $\sigma = R$) if P contains an odd (resp. even) number of R symbols. For example,

when $P_1 = RL^2R$ and $P_2 = RLR^2$
one gets $H(P_1) = RL^2R^3L^2R$ and $H(P_2) = RLR^2LRLR^2$.

Metropolis *et al.* [*1*] prove that, if P is allowed, H(P) is allowed too (their theorem 1); we have $P < H(P)$ and the harmonics are adjacent i. e. no allowed sequence exists between P and H(P). When we iterate the process

$$P < H(P) < H^2(P) \ldots < H^j(P).$$

In a similar way, the *anti-harmonic mapping* $A : P \to A(P)$ is defined as

$$A(P) = P\tau P \tag{2.7}$$

with $\tau = R$ (resp. $\tau = L$) if P has an odd (resp. even) number of R symbols.

For the sequences $P_1$ and $P_2$ of the above example:

$$A(P_1) = RL^2RLRL^2R \quad \text{and} \quad A(P_2) = RLR^4LR^2.$$

In general, P being an allowed sequence, A(P) *is not* allowed and must be considered only as a mathematical tool.

Let $\lambda_1 < \lambda_2$ be two values of $\lambda$ corresponding to two allowed sequences $P_1$ and $P_2$. The theorems of Metropolis *et al.* give an iterative algorithm for constructing all the allowed sequences between $P_1$ and $P_2$, that is any sequence corresponding to a value of $\lambda$, say $\lambda^*$, such as $\lambda_1 < \lambda^* < \lambda_2$. For that purpose construct $H(P_1)$ and $A(P_2)$. If $H(P_1) = P^*\mu_1\mu_2 \ldots$ and $A(P_2) = P^*\nu_1\nu_2 \ldots$, $\mu_1 \neq \nu_1$, then $P^*$ is an allowed sequence lying between $P_1$ and $P_2$ and it has the smallest possible length. One may replace $P_1$ or $P_2$ by $P^*$ and start again.

*Example.* — Consider $P_1 = RLR^4$ and $P_2 = RLR^4LR$, then $P_2 > P_1$, $H(P_1) = RLR^4LRLR^4$, $A(P_2) = RLR^4LRLRLR^4LR$ and $P^* = RLR^4LRLR$.

### 2.1.2. CONSTRUCTION OF ALL PERIODS

The above prescription makes possible the construction of any allowed sequence between two given allowed sequences $P_1$ and $P_2$ corresponding to $\lambda_1$, $\lambda_2$ with $\lambda_1 < \lambda_2$.

If $\lambda < c/f(c)$, $T_\lambda^{(l)}c = c$ has no solution. The first period appears for $\lambda = c/f(c)$ and corresponds to a sequence of length zero,

$$P_1 = b.$$

When $i$ increases the set of sequences $RL^i$ increases (i. e. $RL^i > RL^j$ if $i > j$) it is easy to see that the greatest allowed value for $\lambda$, i. e. $\lambda = 2$ corresponds to the « limit » sequence when $i \to \infty$, which is written $RL^\infty$ for obvious reasons (*).

(*) The notion of infinite sequence is intuitive in the case of $RL^\infty$. More generally, as the set of all allowed sequences is countable, there are certainly accumulation points of the corresponding set of the parameter $\lambda$. These accumulation points correspond to infinite sequences that are defined as follows. One first considers an equivalence relation in the monotonous sets of sequences. Let $\{ P_i \}$ and $\{ P'_j \}$ be two increasing sets [$\{ S_i \}$ is increasing iff $S_i > S_k \Leftrightarrow i > k$], then $\{ P_i \} \sim \{ P'_j \}$ iff any Q that is larger than any $P_i$ is also larger than any $P'_j$ and conversely. If $\{ P_i \}$ is increasing and $\{ P'_j \}$ decreasing, then $\{ P_i \} \sim \{ P'_j \}$ iff, given a sequence Q larger than any $P_i$, a sequence $P'_j$ exists that is smaller than Q although any $P'_j$ is greater than any $P_i$. The quotient set of the monotonous sequences by this equivalence relation is by definition the set of the infinite sequences. They are totally ordered in an obvious way, finite allowed sequences being a subset of them. Furthermore, this set of infinite sequences is closed under the formation of accumulation points by increasing and decreasing sets. A simple example of such an infinite sequence is sequence $RL^\infty$ above. Similarly, the set $\{ RLR^{2l}, l \in \mathbb{N} \}$ is increasing although $\{ RLR^{2l+1}, l \in \mathbb{N} \}$ is decreasing (fig. 2). It is not difficult to show that, in the

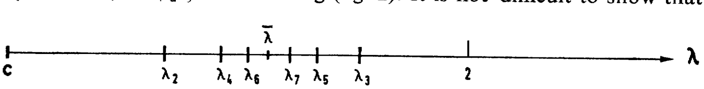


FIG. 2. — Values of the parameters defining the sequences R and $RLR^n$ ($n = 0,4$); $\bar{\lambda}$ is the accumulation point.

All sequences appear when $\lambda$ increases from $cf(c)$ to 2 between the limitings equences

$$P_1 = b, \qquad P_2 = RL^\infty.$$

Using the method of Metropolis *et al.* which extends at once to the case of infinite sequences, one gets step by step all the allowed sequences which are between $b$ and $RL^\infty$. The first steps of the construction are shown on table I. As $P_1$ and $P_2$ do not depend on the exact form of the defining function $f(x)$, the ordering is universal.

TABLE I. — *Construction of all the first sequences by the MSS method.*

| | | | |
|---|---|---|---|
| *a*) $k' = 1$ | $P_1 = b$ | *e*) $k = 1$ | $b$ |
| | $P_2 = RL^\infty$ | $k = 2$ | R |
| | | $k = 4$ | RLR |
| *b*) $k = 1$ | $b$ | $k = 5$ | $RLR^2$ |
| $k = 2$ | R | $k = 3$ | RL |
| | $RL^\infty$ | $k = 5$ | $RL^2R$ |
| | | $k = 4$ | $RL^2$ |
| *c*) $k = 1$ | $b$ | $k = 5$ | $RL^3$ |
| $k = 2$ | R | | $RL^\infty$ |
| $k = 3$ | RL | | |
| | $RL^\infty$ | *f*) $k = 1$ | $b$ |
| | | $k = 2$ | R |
| *d*) $k = 1$ | $b$ | $k = 4$ | RLR |
| $k = 2$ | R | $k = 6$ | $RLR^3$ |
| $k = 4$ | RLR | $k = 5$ | $RLR^2$ |
| $k = 3$ | RL | $k = 3$ | RL |
| $k = 4$ | $RL^2$ | $k = 6$ | $RL^2RL$ |
| | $RL^\infty$ | $k = 5$ | $RL^2R$ |
| | | $k = 6$ | $RL^2R^2$ |
| | | $k = 4$ | $RL^2$ |
| | | $k = 6$ | $RL^3R$ |
| | | $k = 5$ | $RL^3$ |
| | | $k = 6$ | $RL^4$ |
| | | | $RL^\infty$ |

Before ending this subsection let us point out some auxiliary results which are of some interest and will be recovered in a simple manner thereafter

above sense these two sets define a single accumulation point that is the infinite sequence $RLR^\infty$. It must be noticed that it is yet not proved—although presumably true as judged from numerical calculations—that the upper bound of the increasing set $\{ \lambda_{2l} \}$ and the lower bound of the decreasing set $\{ \lambda_{2l+1} \}$ are the same. This common bound, if it exists should be the value of $\lambda$ corresponding to the infinite sequence $RLR^\infty$. For the broken linear transform to be studied below, any infinite sequence defined as above, does actually correspond to a single value of the parameter $\lambda$.

*i*) R is the smallest sequence (after $b$) as no sequence can be built between $b$ and R. So $R < RL^n$ for every integer $n$.

*ii*) if $p < n$, $R < RL^p < RL^n$.

Between $RL^{n-1}$, and $RL^n$, all the sequences begin necessarily by the pattern $RL^nR^\alpha \dots$ Conversely $RL^{n-1} < Q < RL^n$ whenever $Q = RL^nR^\alpha \dots$, $\alpha \neq 0$.

## 2.2. The internal similarity

One can thus reconstruct the whole set of allowed MSS sequences starting with $P_1 = b$ and $P_2 = RL^N$, N arbitrarily large but finite. We will see in this section that this set is similar (in a sense to be made more precise) to some of its own parts; we will refer to this property as the law of internal similarity.

Starting with $P'_1$ and $P'_2$, $P_1 < P'_1 < P'_2 < P_2$, one can, by the algorithm of Metropolis *et al.* described above, construct all sequences $P'$ such that $P'_1 < P' < P'_2$. With a convenient choice of $P'_1$ and $P'_2$ we will find a monotonous bijection between the sequences P, $P_1 < P < P_2$ and the sequences $P'$, $P'_1 < P' < P'_2$.

### 2.2.1. The * composition law

Let $P = \sigma_1\sigma_2 \dots \sigma_{p-1}$ with $\sigma_i$ either R or L, be a sequence, allowed or not, of $p - 1$ symbols. Similarly let Q be a sequence of $n - 1$ symbols R or L. Define $Q * P_1 \equiv Q * b = Q$ for $p = 1$, while for $p > 1$

$$Q * P = Q\sigma_1Q\sigma_2Q \dots Q\sigma_{p-1}Q \tag{2.8 a}$$

if the number of R symbols in Q is even, and

$$Q * P = Q\tau_1Q\tau_2Q \dots Q\tau_{p-1}Q, \qquad \tau_i \neq \sigma_i \tag{2.8 b}$$

otherwise.

For example, $R * RL^n = RLR^{2n+1}$, while

$$RLR * RL^n = RLR^3(LR)^{2n+1}.$$

Some properties of the $*$ mapping may be noted.

*i*) For two sequences $Q_1$ and $Q_2$ of symbols R or L, $Q_1 * Q_2 \neq Q_2 * Q_1$ in general. For example, $R * RL = RLR^3$, while $RL * R = RL^2RL$.

*ii*) The associative law holds, i. e.

$$Q_1 * (Q_2 * Q_3) = (Q_1 * Q_2) * Q_3.$$

This can be verified directly by first observing that the number of R symbols in $Q_1$ and $Q_2$ has the same parity as that number in $Q_1 * Q_2$.

*iii*) If H(P) is the harmonic and A(P) the anti-harmonic of P, then,

$$\mathrm{H(P)} = \mathrm{P} * \mathrm{R}, \qquad \mathrm{A(P)} = \mathrm{P} * \mathrm{L}. \tag{2.9}$$

So that for any Q,

$$\mathrm{Q} * \mathrm{H(P)} = \mathrm{H(Q} * \mathrm{P)}, \qquad \mathrm{Q} * \mathrm{A(P)} = \mathrm{A(Q} * \mathrm{P)}. \tag{2.10}$$

*iv*) If P and Q are allowed sequences, then so is P * Q. The period of P * Q is the product of the periods of P and Q.

*v*) If P, $Q_1$ and $Q_2$ are allowed sequences and $Q_1 < Q_2$, then $P * Q_1 < P * Q_2$.

A direct proof of this point is rather tedious. It becomes obvious by using the criterion of classification and ordering of MSS sequences given in section 4.

*vi*) Let the allowed sequences P, $Q_1$, $Q_2$, $Q'_1$, $Q'_2$ satisfy

$$P * Q_1 = Q'_1 < Q'_2 = P * Q_2.$$

Then corresponding to any allowed $Q'$ with $Q'_1 < Q' < Q'_2$ one can always find an allowed sequence Q such that $Q' = P * Q$.

We show on Table II the first allowed sequences between $Q'_1$ and $Q'_2$ when $P = b$, R and RL respectively.

TABLE II. — *MSS method applied together to sequences* Q *of lower period* $k(\leq 4)$ *and to sequences between* $Q'_1 = R$, $Q'_2 = R * RL^\infty$ *(multiplication of periods by 2) and sequences between* $Q''_1 = RL$, $Q''_2 = RL * RL^\infty$.

| Q | | R * Q | | RL * Q | |
|---|---|---|---|---|---|
| *a*) $k = 1$ | $P_1 = b$ | $k = 2$ | $Q'_1 = R$ | $k = 3$ | $Q''_1 = RL$ |
| | $P_2 = RL^\infty$ | | $Q'_2 = RLR^\infty$ | | $Q''_2 = RL^2(RLR)^\infty$ |
| *b*) $k = 1$ | $b$ | $k = 2$ | R | $k = 3$ | RL |
| $k = 2$ | R | $k = 4$ | RLR | $k = 6$ | $RL^2RL$ |
| | $RL^\infty$ | | $RLR^\infty$ | | $RL^2(RLR)^\infty$ |
| *c*) $k = 1$ | $b$ | $k = 2$ | R | $k = 3$ | RL |
| $k = 2$ | R | $k = 4$ | RLR | $k = 6$ | $RL^2RL$ |
| $k = 3$ | RL | $k = 6$ | $RLR^3$ | $k = 9$ | $RL^2RLR^2L$ |
| | $RL^\infty$ | | $RLR^\infty$ | | $RL^2(RLR)^\infty$ |
| *d*) $k = 1$ | $b$ | $k = 2$ | R | $k = 3$ | RL |
| $k = 2$ | R | $k = 4$ | RLR | $k = 6$ | $RL^2RL$ |
| $k = 4$ | RLR | $k = 8$ | $RLR^3LR$ | $k = 12$ | $RL^2RLR^2L^2RL$ |
| $k = 3$ | RL | $k = 6$ | $RLR^3$ | $k = 9$ | $RL^2RLR^2L$ |
| $k = 4$ | $RL^2$ | $k = 8$ | $RLR^5$ | $k = 12$ | $RL^2RLR^2LR^2L$ |
| | $RL^\infty$ | | $RLR^\infty$ | | $RL^2(RLR)^\infty$ |

*vii*) For any allowed sequence Q, one has

$$b < Q < Q * Q < Q * Q * Q < \dots$$

and

$$RL^n > Q * RL^n > Q * Q * RL^n > Q * Q * Q * RL^n > \dots$$

*viii*) Let $P_1 = b$, $P_2 = RL^n$ and P and Q any allowed sequences, $P_1 < P$, $Q < P_2$. Set $P'_i = Q * P_i$, $i = 1,2$; $P' = Q * P$. Then to every P corresponds a P'. Conversely for any P' with $Q * P_1 = P'_1 < P' < P'_2 = Q * P_2$ corresponds a P with $P_1 < P < P_2$, $P' = Q * P$.

This is the law of internal similarity alluded to at the beginning of the section.

*Remark 1.* — Given any sequence Q, can it be written as $Q_1 * Q_2$ with $Q_1 \neq b$, $Q_2 \neq b$? As the number of symbols in $Q_1 * Q_2$ is $(q_1 + 1)(q_2 + 1) - 1$ where $q_1$ and $q_2$ are the numbers of symbols in $Q_1$ and $Q_2$ respectively, one sees that if the number $q$ of symbols in Q is such that $q + 1$ is a prime integer, then one cannot factorize Q as $Q_1 * Q_2$. Even if $q + 1$ were not a prime integer, the factorization of Q may not exist. For example, the (allowed) sequences $RL^2R^2$, $RL^3R$ and $RL^4$ cannot be so factorized. Such sequences are called primary sequences.

For a given $q$, the number of allowed sequences of length $q$ is known [*1*] and is roughly $2^q/(q + 1)$. If $q + 1$ is prime, all of these are primary. If $q + 1$ is composite, the number of factorizable allowed sequences becomes rapidly negligible for large $q$. For $q = 14$, there are 1,091 allowed sequences ($2^{14}/15 \sim 1{,}092.3$), out of which 6 are non-primary.

*Remark 2.* — The mapping $P \to Q * P$, Q fixed, considered above defines the law of internal similarity, because of property (*iii*) above. The other mapping $P \to P * Q$ does not define any law of internal similarity.

### 2.2.2. Accumulation points

Let Q be an allowed sequence, so that

$$Q^{*j} = \underbrace{Q * \dots * Q}_{j\text{-times}}$$

is allowed. Let $\lambda_j$ and $\mu_j$ be the values of the parameter $\lambda$ for the sequences $Q^{*j} * b$ and $Q^{*j} * RL^n$ respectively. Then according to (*v*) and (*vii*) above $\lambda_{j-1} < \lambda_j < \mu_j < \mu_{j-1}$ for any $j$. The $\lambda_j$'s and $\mu_j$'s tend to definite limits $\lambda_\infty$ and $\mu_\infty$ with $\lambda_\infty \leq \mu_\infty$. It is conjectured that $\lambda_\infty = \mu_\infty$. If so, we may allow the infinite sequence $Q^{*\infty}$, since $Q^{*\infty} * P$ for any allowed P will be independent of P (fig. 3).

This infinite sequence may be also considered as an accumulation point, and we shall prove that the set of the accumulation points has the power of the continuum.

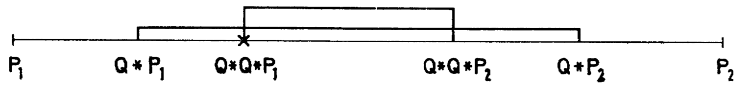


FIG. 3. — Nested intervals corresponding to sequences $Q^{*j} * P_1$ and $Q^{*j} * P_2$ ($P_1 = b$, $P_2 = RL^\infty$) when $j$ increases.

*Proof.* — Let $Q_1$ and $Q_2$ be two allowed finite sequences and suppose there exists a finite sequence P′ such that $Q_1 * P < P' < Q_2$ whatever P is. This is the case, for instance, with $Q_1 = R$ and $Q_2 = RL$. We look to the sequences

$$Q_1^{*\alpha_1} * Q_2^{*\beta_1} * Q_1^{*\alpha_2} * Q_2^{*\beta_2} * \dots *Q_1^{*\alpha_i} * Q_2^{*\beta_i} * P, \tag{2.11}$$

$\alpha_k, \beta_k \in N_+$, $1 \leq k \leq i$.

The property of internal similarity shows that any sequence of the form (2.11) is included between the sequence corresponding to $P = b$ and to $P = RL^\infty$ respectively, and any sequence located between these two bounds is of the form (2.11). Let us call this subset an *interval* in the set of the allowed sequences.

We shall consider the structure of these intervals as

$$N = \sum_{j=1}^{i} \alpha_j + \sum_{m=1}^{i} \beta_m \quad \text{grows, with} \quad Q_1 = R \quad \text{and} \quad Q_2 = RL.$$

If $N = 1$, either $\alpha_1 = 1$, $\beta_1 = 0$ or $\alpha_1 = 0$ and $\beta_1 = 1$. The intervals generated by these two different choices are $[Q_1 * b, Q_1 * RL^\infty[$ and $[Q_2 * b, Q_2 * RL^\infty[$. These intervals are disconnected due to our particular choice of $Q_1$ and $Q_2$ (a limite sequence P′ exists such as $Q_1 * RL^\infty < P' < Q_2 * b$).

Let $N = 2$, then four intervals are found, corresponding to four different choices for $\{ \alpha_i \}$ and $\{ \beta_i \}$:

$$[Q_1^{*2} * b, Q_1^{*2} * RL^\infty[, [Q_1 * Q_2 * b, Q_1 * Q_2 * RL^\infty[,$$
$$[Q_2 * Q_1 * b, Q_2 * Q_1 * RL^\infty[ \text{ and } [Q_2^{*2} * b, Q_2^{*2} * RL^\infty[.$$

As $Q_2 > Q_1 * P$ whatever P is, and as the mapping $Q^*$ is strictly monotonous (if $P > S$ and $P \neq S$, then $Q * P > Q * S$ and $Q * P \neq Q * S$), the above four intervals are disconnected, the first two being included into $[Q_1 * b, Q_1 * RL^\infty[$ and the two others into $[Q_2 * b, Q_2 * RL^\infty[$. Iterating this construction, one finds at the order N, $2^N$ disconnected intervals, these intervals being included by pair into the intervals of the previous order. This construction is similar to the one of the triadic Cantor set, and generates a set of accumulation point with the power of the continuum. As the whole set of the *accumulation* points is a subset of the values of the real parameter $\lambda$, the whole set of the accumulation points has the power of the continuum (fig. 4).

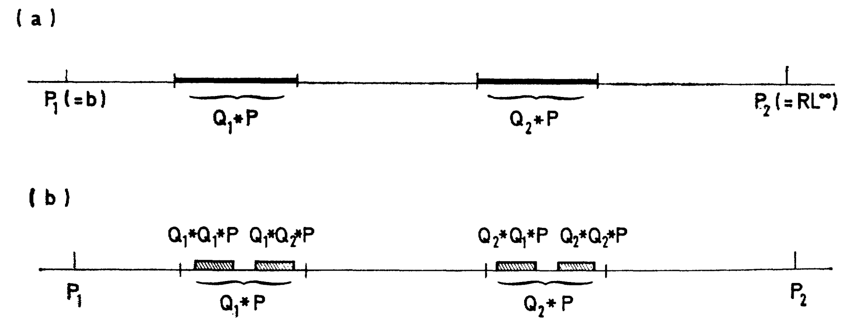


FIG. 4. — Construction of a Cantor ensemble of accumulation points by $Q_1^{*\alpha_1} * Q_2^{*\beta_1} * \ldots * P$, $N = \Sigma\alpha_i + \Sigma\beta_i$.

*a*) $N = 1$ *b*) $N = 2$.

## 3. $\lambda$-EXPANSIONS

In subsections 3 to 5, we shall study the « broken linear » mapping

$$L_\lambda(x) = \lambda x \qquad 0 \leq x \leq 1 \qquad (3.1)$$
$$= 2\lambda - \lambda x \qquad 1 \leq x \leq 2$$

which satisfies assumptions *ii*) and *iii*) of (1.1) in the introduction not *i*) as it is not differentiable at $x = 1$. Doing this, we are able to make explicit calculations and to find the values of the parameter $\lambda$ corresponding to the various periods and their order of occurrence which must be the same as for more general mappings (1.1-1.2) because of the universality property. A difficulty will remain, as, in example (3.1) several periods may collapse at the same value $\lambda$. This ambiguity may be removed either by using results of section 2 or by studying more general endomorphisms [8].

Practically, we replace the MSS sequences of symbols R and L by some related finite sequences of $+1$ and $-1$, that we denote $\{ a_i \}$. We shall see that (eq. 4.8)

$$a_0 = 1$$
$$a_i = \alpha_1 \ldots \alpha_i$$

with

$$\alpha_i = +1 \text{ whenever } T_\lambda^{(i)}(\lambda) < 1 \text{ (or the } i+1^{\text{th}} \text{ symbol is L)}$$
$$= -1 \text{ whenever } T_\lambda^{(i)}(\lambda) > 1 \text{ (or the } i+1^{\text{th}} \text{ symbol is R)}$$
$$= 0 \text{ if } T_\lambda^{(i)}(\lambda) = 0\text{, end of the period.}$$

All sequences $\{ a_i \}$ with $+1$ and $-1$ are not allowed and we get a criterion to select them (theorem 2 or eq. 4.13). Then, we are led to consider polynomials which are simply related to these sequences. The *largest real* root of these polynomials corresponds to the value of the parameter in eq. (3.1) such that 1 belongs to the period (Theorem 3). The order of occurrence of the periods is then given by the natural ordering relation on these sequences.

To prove all the above statements, we have introduced the $\lambda$-expansions of numbers in analogy with the $\beta$-expansion of Renyi [2] and Parry [3]. Definitions and theorems related to $\lambda$-expansions are given in section 3; the application to the MSS sequences will appear in section 4 only. In section 5 we give some complementary results.

### 3.1. Definition of the $\lambda$-expansion

Given $\lambda$, $1 < \lambda < 2$, to each real $x$, $1 < x < 2$ we associate *its $\lambda$-expansion in the $\lambda$ basis* which is defined at each order $n$ by the two numbers $x_n$, $c_n$ *depending both on $x$ and $\lambda$* such that

*i*) $$c_n = \pm 1,0 \qquad x_n = \sum_{k=0}^{n} \frac{c_k}{\lambda^k} \tag{3.2}$$

*ii*) $$\begin{array}{ll} \text{if } x_{n-1} < x & c_n = +1 \\ \text{if } x_{n-1} > x & c_n = -1 \\ \text{if } x_{n-1} = x & c_n = 0. \end{array} \tag{3.3}$$

We take as initial conditions $x_0 = c_0 = 1$ (and necessarily $c_1 = 1$). If $c_n = 0$, then $c_n' = 0$ for every $n' > n$ and it is easy to show by induction that, for every $m$,

$$| x - x_m | < \frac{1}{\lambda^m} \text{ strictly, unless } c_{m+1} = 0.$$

Our standard notation for a $\lambda$-expansion will be $\{ c_i \}$, as usual for denumerable sets. The expansion $\{ c_i \}$ is unique, the sequence $x_n$ converges to $x$ and two distinct numbers $x$ and $y$ have distinct $\lambda$-expansions. The quantity

$$r_n(x \mid \lambda) = \sum_{k=1}^{\infty} \frac{c_{n+k}}{\lambda^k}$$

which depends both on $x$ and $\lambda$ may be thought as a remainder and

$$-1 < r_n(x \mid \lambda) < +1 \text{ strictly, unless } c_{n+2} = 0.$$

Notice that the sum $\sum_{k=1}^{\infty} \frac{c_{n+k}}{\lambda^k}$ has then the same sign as $c_{n+1}$. i. e. the rest has the same sign as the first neglected term.

Conversely, for a given $\lambda$, a sequence $c_0, c_1, c_2 \ldots$ of $\pm 1$ (eventually with 0 after a finite order) is the $\lambda$-expansion of its sum $x = \sum c_i/\lambda^i$ if and only if

$$\forall n \quad \left| \sum_{k=1}^{\infty} \frac{c_{n+k}}{\lambda^k} \right| < 1 \tag{3.4}$$

The $\lambda$-expansion looks very much like the $\beta$-expansion of Renyi [*2*] and Parry [*3*] for real numbers. The main difference is that in the $\beta$-expansion of $x$

$$x = \sum_{i=0}^{\infty} \frac{c_i}{\beta^i},$$

$c_i$ is a non negative integer $0 \leq c_i \leq [\beta]$, $[\beta]$ being the integer part of the basis, although in $\lambda$-expansions $c_i = \pm 1$. Theorems are not very different from Renyi's but slight complications appear in the proofs as functions of $\lambda^{-1}$ as $\sum_{i=0}^{\infty} c_i/\lambda^i$ are, *a priori*, no more monotonous in the case of the $\lambda$-expansion.

It seems arbitrary to study only the $\lambda$-expansion of numbers $x$, $1 \leq x < 2$. It is sufficient for our purpose, nevertheless the definitions (3.2-3) and results are still valid where setting for initial conditions $x_0 = c_0 = [x]$ where $[x]$ is the integer part of $x$ and $x \geq 1$. The extension to $\lambda > 2$ is less obvious as this implies the use of more than two digits.

*Remark 1.* — The expansion is defined by an ordered sequence of $+1$ and $-1$ (and perhaps of zeros from a certain order) which depends both on the basis $\lambda$ and the number $x$. For example, in basis $\lambda_1 = \frac{1+\sqrt{5}}{2}$ (« golden number »), $x = \sqrt{5} - 1$ is written

$$\sqrt{5} - 1 = 1 + \frac{1}{\lambda_1} - \frac{1}{\lambda_1^2}$$

although in basis $\lambda_2 = \sqrt{2}$, its $\lambda$-expansion is infinite, not periodic and reads

$$\sqrt{5} - 1 = 1 + \frac{1}{\lambda_2} - \frac{1}{\lambda_2^2} + \frac{1}{\lambda_2^3} - \frac{1}{\lambda_2^4} - \frac{1}{\lambda_2^5} + \frac{1}{\lambda_2^6} - \frac{1}{\lambda_2^7} + \frac{1}{\lambda_2^8} \cdots$$

*Remark 2.* — Due to (3.4) all sequences of $+1$ and $-1$ are not allowed. For example if $\lambda = \lambda_1 = \frac{1+\sqrt{5}}{2}$, a sequence beginning with

$c_0 = c_1 = c_2 = 1$ *must* go on with $c_3 = -1$ otherwise the first order rest $\sum_{i=1}^{\infty} c_i/\lambda^i$ were greater than 1.

*Remark 3.* — Extending definitions (3.2-3.3) when $\lambda = 2$ and setting then $c_i = 1 + 2\theta_i$ ($\theta_i = 0$ or 1) we get a $\lambda$-expansion which looks very much like the usual binary expansion of real numbers. All sequences $(c_0, c_1, c_2, \ldots)$ are allowed except if all the $c_i$'s are the same from a certain order N. If $c_N = -1$ and $c_i = +1$ for $i > N$, the sequence $(\ldots -1, 1, 1, \ldots)$ is surely not a $\lambda$-expansion and it must be replaced by the finite sequence $(c_0, c_1, \ldots, c_{N-1}, 0, \ldots, 0, \ldots)$. With this restriction, to be compared to the convention $0.9999 \ldots = 1.00 \ldots$, *all* expansions are allowed.

*Remark 4.* — The basis itself has its own $\lambda$-expansion or *auto-expansion.* We shall keep for this particular expansion the notation $(a_0, a_1, a_2, \ldots) = \{ a_i \}$. We shall study later on its properties. Let us give some examples

$$\lambda = \frac{1+\sqrt{5}}{2} \quad a_0 = a_1 = 1 \quad a_n = 0 \quad n > 1 \quad S_0 = (1, 1, 0, 0, \ldots) \tag{3.5 a}$$

$$\lambda = \sqrt{2} \qquad a_0 = 1 \quad a_1 = 1 \quad a_2 = -1 \qquad \tilde{S} = (1, 1, -1, 1, -1, \ldots)$$

$$a_{2n+1} = 1 \quad a_{2n+2} = -1 \quad \tilde{S} = 1(1, -1)^{\infty} \tag{3.5 b}$$

$$\lambda = \sqrt[4]{2} \quad \{ a_i \} = (1, 1, -1, 1, -1, -1, 1, 1, -1, \ldots)$$

$$= 1, 1, -1, (1, -1, -1, 1)^{\infty}$$

*Ordering relation on sequences*

Let $\mathcal{S}$ be the set of the infinite and finite sequences made of $+1$ and $-1$. A finite sequence of length L is written, by convention as an infinite set $\{ a_i \}$ where $a_i = \pm 1$ if $0 \leq i \leq L - 1$ and $a_i = 0$ if $i \geq L$. A number $\lambda$ being given, $1 < \lambda < 2$, some elements of $\mathcal{S}$ do not correspond to the $\lambda$-expansion of a number, and this subset of the allowed sequences depends itself on $\lambda$.

We shall use the total ordering relation in $\mathcal{S}$ defined as follows

$$\{ b_i \} < \{ c_i \} \qquad \{ b_i \}, \{ c_i \} \in \mathcal{S} \tag{3.6}$$

if $b_m < c_m$, $m$ being the smallest integer for which $b_m \neq c_m$. This definition is valid even when one of the coefficients is zero.

### 3.2. Characterization of the $\lambda$-expansions

In this subsection the sequence $\{ a_i \}$ is the given auto-expansion of the basis $\lambda$; $\{ c_i \}$ is some sequence in $\mathcal{S}$ and we look for the conditions $\{ c_i \}$ must satisfy to be the $\lambda$-expansion of its sum.

PROPOSITION 1. — A necessary condition for $\{ c_i \}$ to be the $\lambda$-expansion of the sum $\sum c_i/\lambda^i$ is that the inequalities

$$\begin{aligned} (c_n, c_{n+1}, \ldots) &< (a_0, a_1, a_2, \ldots) \\ -(c_n, c_{n+1}, \ldots) &< (a_0, a_1, a_2, \ldots) \end{aligned} \tag{3.7}$$

are *strictly* satisfied for every $n \geq 1$.

We denote by $-(c_n, c_{n+1}, \ldots)$ the sequence

$$-(c_n, c_{n+1}, \ldots) \equiv (-c_n, -c_{n+1}, -c_{n+2}, \ldots) \tag{3.8}$$

*Proof.* — From (3.4)

$$\left| \sum_{i=n}^{\infty} \frac{c_i}{\lambda^i} \right| < \frac{1}{\lambda^{n-1}} \qquad \text{for} \qquad n \geq 1$$

or

$$\left| \sum_{i=n}^{\infty} \frac{c_i}{\lambda^i} \right| < \frac{1}{\lambda^n} \lambda = \frac{1}{\lambda^n} \sum_{i=0}^{\infty} \frac{a_i}{\lambda^i}$$

or

$$\left| \sum_{k=0}^{\infty} \frac{c_{n+k}}{\lambda^k} \right| < \sum_{k=0}^{\infty} \frac{a_k}{\lambda^k} \qquad n \geq 1 \tag{3.9}$$

To show now the inequalities (3.7), we assume first $c_n = +1$, then the second inequality (3.7) holds. We turn now to the first one. Let $n' > 0$ be the first integer such that $c_{n+n'} \neq a_{n'}$. To show that $c_{n+n'} < a_{n'}$, we assume the contrary, i. e. $c_{n+n'} > a_{n'}$, and prove it is impossible. By construction of the $\lambda$-expansion,

$$\sum_{i=0}^{\infty} \frac{c_{n+n'+i}}{\lambda^i} \text{ has the sign of } c_{n+n'}, \text{ i. e. } \sum_{i=0}^{\infty} \frac{c_{n+n'+i}}{\lambda^i} > 0$$

and is zero only if $c_{n+n'} = 0$; furthermore $\sum_{i=0}^{\infty} \frac{a_{n'+i}}{\kappa^i}$ has the sign of $a_{n'}$, i. e. $\sum_{i=0}^{\infty} \frac{a_{n'+i}}{\lambda^i} < 0$ and is zero only if $a_{n'} = 0$; this is in contradiction with (3.9). ■

THEOREM 1. — *i*) If the auto-expansion $\{ a_i \}$ of $\lambda$ is infinite ($a_i \neq 0\ \forall i$) conditions (3.7) are necessary and sufficient for $\{ c_i \}$ to be the $\lambda$-expansion of $\sum_i c_i/\lambda^i$.

*ii*) If the auto-expansion $\{ a_i \}$ of $\lambda$ is finite of length $L + 1$

$$[\{ a_i \} = (a_0, a_1, \ldots, a_L, 0 \ldots), \quad a_L \neq 0]$$

conditions (3.7) are necessary and sufficient except if there exists an integer $q$ such that from this order $q$, $\{ c_i \}$ reproduces infinitely—eventually to a minus sign—the finite pattern $-1, a_0, a_1, \ldots, a_L$ (i. e. $c_q = \mp 1$, $c_{q+1} = \pm a_0$, $c_{q+2} = \pm a_1 \ldots$). The $\lambda$-expansion of the sum $\sum_i c_i/\lambda^i$ reduces then to the finite sequence $(c_0, c_1, \ldots, c_{q-1}, 0, 0, \ldots)$.

A proof is given in appendix A.

To give an example, consider the case where the auto-expansion of $\lambda$ is

$$\lambda = 1 + \frac{1}{\lambda}$$

so that $1/\lambda^k - 1/\lambda^{k+1} - 1/\lambda^{k+2} = 0$ for any integer $k$, and let $x = \sum_{q=0}^{L} c_q/\lambda^q$ be the finite $\lambda$-expansion of a number $x$ on the basis $\lambda$. One may obviously add to $\sum_{q=0}^{L} c_q/\lambda^q$ a series of the form

$$\sum_{k=0}^{\infty} \varepsilon_k \frac{1}{\lambda^{L+1}} \left( \frac{1}{\lambda^{3k}} - \frac{1}{\lambda^{3k+1}} - \frac{1}{\lambda^{3k+2}} \right)$$

with $\varepsilon_k = \pm 1$, without changing the value of the sum.

Applying result of theorem 1 to the auto-expansion $\{ a_i \}$ of $\lambda$, we get

THEOREM 2. — A necessary condition for $\{ a_i \}$ to be an auto-expansion is that $\forall n \geq 1$ the *strict* inequalities

$$(a_n, a_{n+1}, \ldots) < (a_0, a_1, a_2, \ldots)$$

and

$$-(a_n, a_{n+1}, \ldots) < (a_0, a_1, a_2, \ldots) \tag{3.10}$$

hold.

The sufficient condition defining an auto-expansion is rather cumbersome to formulate, owing to the possibility that, after a certain order, patterns yielding a null contribution to the sum are again and again repeated. If this is not this case, the inequalities (3.10) are sufficient. If this is the case, i. e. if $a_i \neq 0 \; \forall i$, and if L exists such as $\lambda = \sum_{q=0}^{L} \frac{a_q}{\lambda^q}$ where $\{ a_q, 0 \leq q \leq L \}$

satisfies (3.10) and if $\sum_{q=L+1}^{\infty} \frac{a_q}{\lambda^q} = 0$, then the auto-expansion of $\lambda$ is $\sum_{q=0}^{L_0} \frac{a_q}{\lambda^q}$, where $L_0$ is the smallest L satisfying the above requirements.

*Remark 1.* — In this last case the equation defining $\lambda$ can be factorized as

$$\lambda - \sum_{q=0}^{\infty} \frac{a_q}{\lambda^q} = \left(\lambda - \sum_{q=0}^{L_0} \frac{a_q}{\lambda^q}\right)\left(1 - \sum_{j=0}^{\infty} \frac{\varepsilon_j}{\lambda^{(j+1)(L_0+2)}}\right) \tag{3.11}$$

where $\varepsilon_j = \pm 1$, unless $j_0$ exists such as $\varepsilon_j = 0$ for $j \geq j_0$ and $\varepsilon_j = \pm 1$ for $j < j_0$. As $\{ a_q \}$ satisfies (3.10):

$$\pm (\varepsilon_n, \varepsilon_{n+1}, \dots) < (\varepsilon_0, \varepsilon_1, \dots) \ \forall n \geq 1.$$

*Remark 2.* — The ordered sequence $\{ c_i \}$ is made of $(+1)$ and $(-1)$. Let $M_1$ be the number of consecutive $+1$ from the left at the beginning

$$c_0 = c_1 = \dots = c_{M_1 - 1} = +1, \qquad c_{M_1} = -1.$$

Let $M_2$ be the number of consecutive $-1$ just following

$$c_{M_1} = \dots = c_{M_1+M_2-1} = -1 \qquad c_{M_1+M_2} = +1$$

and more generally $M_3, M_4, \dots$ the number of consecutive $+1$ or $-1$. The sequence $S = \{ a_i \}$ for the basis (or auto-expansion) can be similarly, represented by successive integers $L_1, L_2, L_3, \dots$ ($L_i \in N_+$).

The condition (3.7) for $\{ c_i \}$ may be written

$$M_1 \leq L_1 + 1$$
$$M_i \leq L_1 \quad \forall i > 1$$

If $M_i = L_1$ then $M_{i+1} \geq L_2$

If $M_i = L_1$ and $M_{i+1} = L_2$ then $M_{i+2} \leq L_3$

If $M_i = L_1$

$M_{i+1} = L_2$

and $M_{i+2} = L_3$ then $M_{i+3} \geq L_4$, etc.

If we are specially interested in the auto-expansion, condition (3.10) reads

$L_i \leq L_1 \quad \forall i > 1$

If $L_i = L_1$ then $L_{i+1} \geq L_2$ (3.12)

If $L_i = L_1$ and $L_{i+1} = L_2$ then $L_{i+2} \leq L_3$, etc.

If the sequence S is finite, and $L_p$ is the last non-zero L then $L_p < L_1$. If $L_2 = L_1$ then the auto-expansion is infinite and $L_i = L_1 \ \forall i$. The corresponding $\lambda$-number is the (unique) root of

$$\lambda^{L_1} - 2\lambda^{L_1 - 1} + 1 = 0 \text{ that is larger than 1.}$$

Actually this expansion is not a $\lambda$-expansion as the pattern $-\underbrace{1\,1\,\ldots\,1}_{L_1-1}$ is indefinitely reproduced and has a null contribution, the auto-expansion being defined in this case by the finite sequence $(\underbrace{1, 1, 1, \ldots, 1}_{L_1 - 1 \text{ times}})$.

### 3.3. Characteristic equations. Study of the roots

In the present subsection and in most of the following we shall restrict ourselves to auto-expansions. A necessary and (nearly) sufficient condition for a sequence $\{ a_i \} = S$ to be the auto-expansion of one of its roots is that inequalities (3.10) hold (in short this sequence is « allowed »). The complementary condition of non factorization is not really essential and we shall not deal with it presently.

The problem now is to know which, if any, of the roots of the characteristic equation $\varphi_S(x) = 0$ where

$$\varphi_S(x) = x - \sum_{i=0}^{\infty} \frac{a_i}{x^i} \tag{3.13}$$

has $\{ a_i \}$ for auto-expansion. We shall prove two results.

THEOREM 3. — The root with the auto-expansion $\{ a_i \}$ is unique, it is the largest real root $x(S)$ of $\varphi_S(x) = 0$ ($x(S) < 2$ as $| a_i | = 1$).

THEOREM 4. — Let two real numbers $\lambda$, $\mu$ defined by their auto-expansion, $S = \{ a_i \}$ $T = \{ b_j \}$; $1 < \lambda, \mu < 2$. Then

$$\lambda < \mu \Rightarrow \{ a_i \} < \{ b_i \}$$

and conversely

$$\{ a_i \} < \{ b_i \} \Rightarrow \lambda \leq \mu,$$

the equality may occur only if the characteristic equations related to S and T have as a common factor a polynomial in $\lambda^{-1}$ with coefficients $\pm 1$.

The proof is tedious and depends on two lemmas which are given in Appendix B. We indicate here the main steps. We use the following notations and definitions; let us call *initial length* of $S = \{ a_i \}$ the integer $L_1$ ($\geq 2$) which is the number of consecutive coefficients $+ 1$ at the head of the sequence: $a_i = + 1$, $0 \leq i < L_1$, $a_{L_1} = - 1$, furthermore $x_1$ (resp. $x_1'$) is the only real root greater than 1 of

$$x^{L_1+1} - 2x^{L_1} + 1 = 0 \qquad (\text{resp. of } x^{L_1} - 2x^{L_1-1} + 1 = 0)$$

corresponding to the sequences of length $L_1$ (resp. $L_1 - 1$) with only $+ 1$ coefficients; we have $1 < x_1' < x_1 < 2$. In what follows we shall often consider the particular sequence $\tilde{S} = 1(1, -1)^{\infty}$, which is the auto-expansion of $\sqrt{2}$ and has been already considered in (3.5 *b*).

PROPOSITION 1. — Let S be an allowed sequence such as $S < \tilde{S}$, then

$$x^2\varphi_S(x) = (x - 1)\varphi_T(x^2) \tag{3.14}$$

where $T = \{ b_j \}$ is an allowed sequence.

*Proof.* — Condition (3.10) implies that $S < \tilde{S}$ is of the form

$$S = 1(1, -1)^{n_1}(-1, 1)^{n_2}(1, -1)^{n_3} \dots$$

without the possibility for the final pattern to be 1 or $-1$ alone. There are the supplementary conditions

$$1 \leq n_i \leq n_1$$
$$n_2 \neq n_1$$

(unless $n_i = n_1 \; \forall i$ and the sequence is infinite but forbidden) and

if $n_i = n_1$ then $n_{i+1} \geq n_2$
if $n_i = n_1$ and $n_{i+1} = n_2$ then $n_{i+2} \leq n_3$
if $n_i = n_1, n_{i+1} = n_2$ and $n_{i+2} = n_3$ then $n_{i+3} \geq n_4 \dots$

then it is easy to show that T is the sequence with

$n_1$ coefficients $+ 1$ first, then
$n_2$ coefficients $- 1$
$n_3$ coefficients $+ 1 \dots$

and the conditions on the $n_i$'s are the one we had written in (3.12) for the $L_i$'s. ■

Moreover, there exists an allowed sequence U and a minimal integer $l$ such that $S < \tilde{S}$ factorize as

$$x^{2+2^2+\dots+2^l}\varphi_S(x) = (x - 1)(x^2 - 1) \dots (x^{2^{l-1}})\varphi_U(x^{2^l})$$

with $U > \tilde{S}$.

Then, all the theorems we shall set for sequences $S > \tilde{S}$ (especially prop. 4) can be extended without difficulty to sequences $S < \tilde{S}$.

PROPOSITION 2. — If S is the auto-expansion of $x_0[\varphi_S(x_0) = 0]$ then $x_1' < x_0 < x_1$.

*Proof.* — The characteristic equation reads for $x = x_0$ as

$$x_0 = \left(1 + \dots + \frac{1}{x_0^{L_1 - 1}}\right) - \frac{1}{x_0^{L_1}} \pm \dots$$

If S is the auto-expansion of $x_0$, the remainder obtained by cutting the expansion after the order $x_0^{-(L_1-1)}$ is negative and

$$x_0 < 1 + \dots + \frac{1}{x_0^{L_1-1}} \qquad \text{or} \qquad x_0 < x_1 .$$

Similarly, from

$$x_0 = 1 + \dots + \frac{1}{x_0^{L_1-2}} + \frac{1}{x_0^{L_1-1}} \dots$$

we have

$$x_0 > 1 + \dots + \frac{1}{x_0^{L_1-2}} \qquad \text{or} \qquad x_0 > x_1' \quad \blacksquare$$

PROPOSITION 3. — (It results from the lemma in Appendix B).

If S and S′ are two allowed sequences with the same initial length $L_1$ then

*i)* $\varphi_S(x)$ and $\varphi_{S'}(x)$ have real roots.

*ii)* $x_1' < x(S)$, $x(S') < x_1$ where $x(S)$ and $x(S')$ are the greatest real roots of $\varphi_S(x)$ and $\varphi_{S'}(x)$ resp.

*iii)* $S < S' \Rightarrow x(S) < x(S')$.

PROPOSITION 4. — (Again a consequence of Appendix B and proposition 1).

Let S be an allowed sequence:

*i)* if $S > \tilde{S}$, then the characteristic equation $\varphi_S(x) = 0$ has one and only one root larger than $\sup(\sqrt{2}, x_1')$ and this is $x(S)$ ;

*ii)* if $S < \tilde{S}$, then there exists an integer $l$ and a sequence $T > \tilde{S}$ such that the characteristic equation $\varphi_S(x) = 0$ has one and only one root larger than $[\sup \sqrt{2}, x_1']^{2^{-l}}$. This root is $x(S)$ and $x(S) = [x(T)]^{1/2^l}$.

PROPOSITION 5. — (From propositions 2-3-4).

If S is allowed and does not factorize, then the characteristic equation has real roots and the largest one is the only one having S as auto-expansion.

If S begins from the left by a finite allowed sequence T such as $x(S)$ is a root of $\varphi_T(x) = 0$, the auto-expansion of $x(S)$ is given by the sequence T with the smallest length.

Theorems 3 and 4 result from proposition 5.

### 3.4. $\lambda$-simple numbers

Every real number $\lambda$, $1 < \lambda < 2$ is the largest root of a characteristic equation $\lambda = \sum_{i=0}^{\infty} a_i/\lambda^i$ where $a_i = \pm 1, 0$, $\{ a_i \}$ being an allowed sequence such as the characteristic equation cannot be factorized in form (3.11).

When the sequence $\{ a_i \}$ is finite of length L + 1 (i. e. there exists an integer L such that $a_i \neq 0, 0 < i \leq L, a_{L+k} = 0, k \geq 1$), then the largest root of

$$\lambda = \sum_{i=0}^{L} \frac{a_i}{\lambda^i}$$

is said to be a $\lambda$-simple number (this denomination is, of course, very reminiscent of the one of $\beta$-simple numbers used by Parry [3]) $\lambda$-simple numbers form of countable set on the interval [1, 2] as the finite sequences $(a_0, \ldots, a_L, 0, 0, \ldots)$ are countable.

THEOREM 5. — The $\lambda$-simple numbers are dense on the interval [1, 2].

*Proof.* — Let $x$ be some real number $1 \leq x \leq 2$ and $S = \{ a_i \}$ is its auto-expansion. If $x$ is not $\lambda$-simple, the $a_i$'s are never zero and the sequence is the succession of

$$\begin{aligned} &L_1 \text{ signs } + 1 \\ &L_2 \text{ signs } - 1 \\ &L_3 \text{ signs } + 1 \ \ldots \end{aligned}$$

with conditions (3.12) on the $L_i$'s; S may be written as

$$S \equiv L_1 L_2 L_2 \ldots L_{2i+1} \ldots$$

Either there exists infinitely many indices $i_k$ $(i_1 < i_2 < i_3 \ldots)$ such that $L_{2i_k+1} < L_1$ or there exists infinitely many indices $j_l$ such that $L_{2j_l} < L_1$ since otherwise the sequence will be forbidden.

Any finite sequence $S_k \equiv L_1 L_2 L_3 \ldots L_{2i_k+1}$ which satisfies (3.12) defines a $\lambda$-simple number $x_k$. As

$$S_k > S_{k+1} > \ldots > S \text{ we have } x < \ldots < \bar{x}_{k+1} < x_k.$$

Let $\bar{x}$ be the lower bound of the set $\{ x_k \}$, it is larger than $x$, and has the same auto-expansion as $x$, then $\bar{x} = x$.

## 4. BROKEN LINEAR TRANSFORMATIONS AND MSS SEQUENCES

We now come back to the broken linear transformation $L_\lambda(x)$ defined by (3.1):

$$\begin{aligned} L_\lambda(x) &= \lambda x & 0 \leq x \leq 1 \\ &= 2\lambda - \lambda x & 1 \leq x \leq 2 \end{aligned}$$

or in a more concise way

$$L_\lambda(x) = \lambda[(1 - \beta) + \beta x] \tag{4.1}$$

where

$$\begin{array}{llll} \beta = +1 & \text{when} & x < 1 & (\text{or } L_\lambda \text{ is increasing}) \\ \beta = -1 & \text{when} & x > 1 & (\text{or } L_\lambda \text{ is decreasing}). \end{array} \qquad (4.2)$$

From Metropolis *et al.* [*1*], we know that the order of occurrence of its allowed sequences is the same as for the more general mappings $T_\lambda(x)$ of eqs. (1.1-1.2), though the conditions (1.1) are not all satisfied in that case. For that mapping many calculations may be done explicitly and the condition that an ordered sequence of R and L is found in the construction of Metropolis *et al.* is only a rewriting of the condition that a sequence $\{ a_i \}$ is allowed, in the sense of the inequalities (3.10). The price to be paid for this simplification of the mapping is the loss of some periods (6 among 1,091 for period 15). These periods correspond to a well defined class of non primary sequences (see sec. 2.2) and to the corresponding class of factorizable characteristic functions of section 3. However the criterion for recognizing and ordering the MSS sequences, derived from the study of the broken linear mapping $L_\lambda(x)$ can be extended at once to all the MSS sequences.

In this section, we first introduce the $\lambda$-sequences then calculate the iterates of the maximum in the broken linear case and show that the equation for $\lambda$ defined by the MSS sequences is analogous to the characteristic equations of section 3. We investigate the case of non-primary sequences. The theorems we derived in section 3 for auto-expansions are « translated » in terms of mappings and MSS sequences for any kind of transformations $T_\lambda$. Finally, we recover the Šarkovskii theorem [*4*]-[*6*].

### 4.1. Iterates of the maximum

The $(l + 1)^{\text{th}}$ iterate of $x$ reads

$$L_\lambda^{(l+1)}(x) = \lambda^{l+1}\beta_0 \ldots \beta_l(x - 1) + \lambda + (\lambda - 1)\sum_{i=1}^{l}\beta_l \ldots \beta_{l-i+1}\lambda^i$$

where

$$\beta_i = +1 \quad \text{when} \quad L_\lambda^{(i)}(x) < 1$$

and

$$\beta_i = -1 \quad \text{when} \quad L_\lambda^{(i)}(x) > 1,$$

$\beta_i$ depends both on $x$ and $\lambda$ and

$$L_\lambda^{(l+1)}(x) = L_\lambda[L_\lambda^{(l)}(x)],\ L_\lambda^{(1)}(x) = L_\lambda(x).$$

If $x = \lambda$, this can be written

$$\frac{L_\lambda^{(l+1)}(\lambda) - 1}{\lambda - 1} = 1 + \sum_{i=1}^{l+1}\alpha_l \ldots \alpha_{l-i+1}\lambda^i$$

$$= \alpha_0 \ldots \alpha_l \mathscr{P}_l(\lambda \mid \lambda) \qquad (4.3)$$

$\alpha_0 = -1$ and $\alpha_i = +1$ when the $(i+1)^{\text{th}}$ symbol of MSS sequence is L (or $L_\lambda^{(i)}(\lambda) < 1$)

$$\alpha_i = -1 \qquad \text{when} \qquad L_\lambda^{(i)}(\lambda) > 1 \tag{4.4}$$

and

$$\mathscr{P}_l(\lambda \mid x) = x^{l+1} + \sum_{i=0}^{l} x^{l-i}\alpha_0 \ldots \alpha_i \tag{4.5}$$

is a polynomial whose coefficients ($= \pm 1$) depend explicitly on $\lambda$ through the sequence $\{ \alpha_i \}$.

If there exists an L such that $L_\lambda^{(L)}(\lambda) = 1$, then we must put $\alpha_L = 0$. In this case (which corresponds to a finite period), as $L_\lambda^{(L+1)}(\lambda) = \lambda$, the next iterates are deduced at once from $L_\lambda^{(j)}(\lambda)$ with $1 \leq j \leq L + 1$.

Consider, for instance, the case where $\lambda$ is the largest root of $\lambda = 1 + \frac{1}{\lambda}$. As $\lambda > 1$, $L_\lambda^{(1)}(\lambda) = 2\lambda - \lambda^2$ and $\alpha_1 = +1$ as $0 < L_\lambda^{(1)}(\lambda) < 1$, $L_\lambda^{(2)}(\lambda) = 2\lambda^2 - \lambda^3 = 1$ then $\alpha_2 = 0$. The polynomials $\mathscr{P}_l(\lambda \mid x)$ take the form

$$\mathscr{P}_0(\lambda \mid x) = x - 1$$
$$\mathscr{P}_1(\lambda \mid x) = x^2 - x - 1.$$

For every $\lambda > 1$, $\alpha_0 = -1$, $\alpha_1 = +1$ and the $\alpha_i$ are correlated. For example, if $\alpha_2 = -1$, then necessarily $\alpha_3 = -1$ (but $\alpha_4$ may be $+1$ or $-1$).

The correspondence with the symbols of Metropolis *et al.* is obvious: $\alpha_i = +1$ (resp. $\alpha_i = -1$) corresponds to the case where the $(i+1)^{\text{th}}$ term of the MSS sequence is L (resp. R). The notation is also roughly the same as that used by Milnor and Thurston [9].

If $\lambda$ is periodic of period $k$,

$$L_\lambda^{(k-1)}(\lambda) = 1$$
$$L_\lambda^{(l)}(\lambda) \neq 1 \qquad l < k - 1$$

and

$$\mathscr{P}_{k-2}(\lambda \mid \lambda) = 0 \tag{4.6}$$

or, written differently

$$\lambda = 1 + \sum_{i=1}^{k-2} \frac{\alpha_1 \ldots \alpha_i}{\lambda^i} = \sum_{i=0}^{k-2} \frac{a_i}{\lambda^i} \tag{4.7}$$

where

$$a_0 = 1$$
$$a_i = \alpha_1 \ldots \alpha_i \quad , \quad i \geq 1 \tag{4.8}$$

and $a_i = \pm 1$, $a_1 = 1$. If $a_2 = -1$, then $a_3 = +1 \ldots$ Notice that if $\lambda$ is periodic with period $k$, $\{ a_i \}$ is of length $k - 1$.

If $\lambda$ is not periodic $\dfrac{L_\lambda^{(l+1)}(\lambda) - 1}{(\lambda - 1)\lambda^l} \underset{l \to \infty}{\to} 0$ as $L_\lambda^{(l+1)}(\lambda)$ is bounded and we get for $\lambda$ an equation with an infinite number of coefficients

$$\lambda = \sum_{i=0}^{\infty} \frac{a_i}{\lambda^i} \tag{4.9}$$

the $a_i$'s being defined as in (4.8) and depending on $\lambda$.

Eqs. (4.7)-(4.9) show that we have substituted to the MSS or $\{ \alpha_i \}$ sequence a new sequence $\{ a_i \}$ (of length $k - 1$ if $\lambda$ is periodic of period $k$), made of $+1$ or $-1$ coefficients and biunivoquely related to it. It will be convenient to give it a name, we choose to call it the *$\lambda$-sequence.*

Notice that eq. (4.9) has a meaning only in the broken linear case (4.1) though the construction of $\{ \alpha_i \}$ and $\{ a_i \}$ requires the knowledge of the MSS sequence only. The $\lambda$-sequences $\{ a_i \}$ include then all $\lambda$-sequences corresponding to periods of $L_\lambda$ and some complementary sequences which do not correspond to periods of $L_\lambda$ [*1*].

One can write Eq. (4.3) also as

$$\left[\lambda - \sum_{i=1}^{\infty} \frac{a_i}{\lambda^i}\right] = \frac{L_\lambda^{(l+1)}(\lambda) - 1}{\lambda - 1} \alpha_0 \ldots \alpha_l. \tag{4.10}$$

As

$$\left| \frac{L_\lambda^{(l+1)}(\lambda) - 1}{\lambda - 1} \right| < 1$$

the remainder of the series $\displaystyle\sum_{i=0}^{l} \frac{a_i}{\lambda^i}$ when cutting after order $l$ is less than $1/\lambda^l$ and is never zero for $l + 1 < k$. *Then the finite or infinite set $\{ a_i \}$ is the auto-expansion of $\lambda$* and from section 3 we know that the coefficients $\{ a_i \}$ satisfy inequalities (3.10). Moreover the characteristic equation cannot factorize in the form (3.11), $\lambda$ being the root of the characteristic equation made from any finite allowed subsequence $(a_0, \ldots, a_L)$. We shall shorten these conditions by saying that $\{ a_i \}$ is not $\lambda$-factorizable.

Conversely, being given a sequence $\{ a_i \}$ satisfying (3.10), if its greatest real root $\mu$ is not the root of a finite allowed subsequence $(a_0, \ldots, a_L, 0, \ldots, 0)$ $[L + 2 < k]$, $\{ a_i \}$ is the auto-expansion of $\mu$ and the iterates $L_\mu^{(l+1)}(\mu)$ are given by (4.10).

From section 3, we know quite a few things about these $\lambda$-sequences, especially how to recognize and order them. Before translating in terms of mappings and periods the results for auto-expansions of last section, we want to clear the problem of the forgotten periods.

### 4.2. Non primary and « forgotten » sequences

In the broken linear case, only part of the whole set of allowed MSS sequences corresponds to periods. Let us call « forgotten » sequences, those which exist in the MSS set, but are not found in the broken linear case. These forgotten sequences may be characterized in the following two (equivalent) ways:

*i*) They are non primary (a sequence P is non primary if two sequences $Q_1$ and $Q_2$ exist, $Q_1$ and $Q_2 \neq b$, such as $P = Q_1 * Q_2$) and P is not of the form $P = R * Q$, Q being itself primary (this last condition may be equally stated as $P > \tilde{S}$).

*ii*) The polynomial built up from the $\{ a_i \}$ characterizing the forgotten sequences satisfies (3.10) and is $\lambda$-factorizable, its largest root being larger than $\sqrt{2}$.

We have already seen (see section 3, eqs. 3.11) that a $\lambda$-sequence $\{ a_i \}$ which is $\lambda$-factorizable defines a finite subsequence $(a_0, a_1, \ldots, a_L, 0 \ldots)$ which is primary (and corresponds to a MSS sequence $P_m$) and another quotient sequence $\{ \varepsilon_i \}$. Both $(a_0, a_1, \ldots, a_L)$ and $\{ \varepsilon_i \}$ satisfy (3.10). If $\{ \varepsilon_i \}$ is not $\lambda$-factorizable, it corresponds to a MSS sequence Q and the whole sequence $\{ a_i \}$ corresponds to the MSS sequence.

$$P = P_m * Q$$

where the symbol $*$ has been defined in subsection (2.2.1).

It remains to find out how the $*$ composition law, that generates non primary sequences, acts on the polynomial built up from these sequences. This representation of the $*$ mapping is derived straigtforwardly from its definition and from the definition of the set $\{ a_i \}$ and $\{ \varepsilon_i \}$ as given in (4.4) and (4.8). The result is given in the following

THEOREM. — Let P, Q be two MSS sequences of period $n$ and $m$ respectively, $\varphi_P$, $\varphi_Q$ and $\varphi_{P*Q}$ be the characteristic functions associated with P, Q and $P * Q$ respectively, then

$$x^n \varphi_{P*Q}(x) = \varphi_P(x) \varphi_Q(x^n) \qquad (4.11)$$

*Remark 1.* — We recover from (4.11) that the composition law $*$ is not commutative and that it multiplies the period of the MSS sequences.

*Remark 2.* — If $P = R(n = 2)$, then

$$x^2 \varphi_{R*Q}(x) = (x - 1) \varphi_Q(x^2) \qquad (4.12)$$

the $\lambda$-simple number associated with $R * Q$ is the square root of the one associated with Q, $\varphi_{R*Q}(x)$ is then factorizable, but not $\lambda$-factorizable and the sequence $R * Q$ exists in the broken linear case (although it would not exist if P was different from R).

*Remark 3.* — If $Q = R(m = 2)$,

$$x^n \varphi_{P*R}(x) = (x^n - 1)\varphi_P(x),$$

which shows that the $\lambda$-simple number associated with $P * R$ is the same as the one associated with P. As already seen, $P * R$ is the harmonic of P (in the sense of Metropolis *et al.*). For the broken linear transform all the harmonics of a given sequence (that is the sequences $P * R$, $P * R^2$, $P * R^3, \ldots$) correspond to the same value of $\lambda$ although if $T_\lambda$ is a $C^1$ mapping, the values of $\lambda$ corresponding to a sequence and to its harmonic are, in general, distinct [*1*] and no sequence P′ exists such as $P < P' < P * R$.

*Remark 4.* — If $U < \tilde{S}$, then U may be factorized as

$$U = R^{*l} * Q \qquad \text{with} \qquad Q > \tilde{\tilde{S}},$$

and

$$x^{2+4+\ldots+2^l} \varphi_U(x) = (x - 1)(x^2 - 1) \ldots (x^{2^{l-1}} - 1)\varphi_Q(x^{2^l}).$$

Furthermore Q, if not primary, may be factorized as

$$Q = Q_1 * Q_2 \ldots * Q_\mu,$$

where $Q_1 \neq R$ and $\{ Q_j, 1 \leq j \leq \mu \}$ is a set of primary sequences. The greatest real root of $\varphi_U(x)$ is the $\lambda$-number associated with $Q_1$.

REMARK 5. — Let $Q_1$ and $Q_2$ be two MSS sequences and P be any primary MSS sequence, then the same $\lambda$-simple number is associated with $P * Q_1$ and $P * Q_2$ and the periods associated with $P * Q_1$ and $P * Q_2$ do not exist in the broken linear case. Nevertheless if $Q_1 > Q_2$, then $P * Q_1 > P * Q_2$.

### 4.3. Collected results

Using theorems of section 3 and remarks of subsection 4.2 we get the following results.

#### 4.3.1. CHARACTERIZATION OF THE MSS SEQUENCES

*i*) For a general mapping $T_\lambda$, a necessary and sufficient condition for a sequence P to be a MSS sequence is that the related $\lambda$-sequence $\{ a_i \}$ satisfies conditions (3.10) i. e.

$$\forall n \geq 1 \quad \pm (a_n, a_{n+1}, \ldots) < (a_0, a_1, \ldots) \tag{4.13}$$

(the order relation being that of 3.1).

Non primary sequences (except those of the form P = R * Q) correspond biunivoquely to $\lambda$-factorizable characteristic equations for $\lambda$-sequences.

*ii*) For the broken-linear mapping $L_\lambda$, conditions (4.13) are necessary. They are sufficient if the characteristic equation is not $\lambda$-factorizable.

If it $\lambda$-factorizes, the corresponding MSS sequence is forgotten, as it defines the same parameter $\lambda$ as the minimal characteristic equation.

4.3.2. ORDERING

Let P and Q be two MSS sequences with associated $\lambda$-sequences S and T.

The inequality for MSS sequences

$$P < Q$$

implies the inequality for $\lambda$-sequences

$$S < T$$

and conversely $S < T \Rightarrow P < Q$, though the characteristic equation of P and Q of S and T may determine the same $\lambda$-simple number.

In the broken-linear case (4.1), finite periods appear for values of the parameter $\lambda$ which are the largest real roots of the corresponding characteristic equations.

In the general case, the parameter $\lambda$ of the mapping $T_\lambda$ is related in a very complicated manner to the solutions $\psi(\lambda)$ of the related characteristic equation. However, we know [*1*] and [*9*] that $\psi$ is a (non strictly) monotonous function of $\lambda$ i. e.

$$\lambda < \mu \Rightarrow \psi(\lambda) \leq \psi(\mu)$$

4.3.3. PERIOD 8 SEQUENCES (table III)

As an example, let us consider all the $\{ a_i \}$ sequences corresponding to a period 8. The sequences are in decreasing order; there are 7 numbers $a_i$ and $a_0 = a_1 = 1$. We write all the sequences compatible with (3.10) or (4.13). There are 16 (numbered from 1 to 16) and we verify that they appear in the order of Metropolis *et al.* [*1*].

Two characteristic polynomials are not primary and correspond to product of MSS sequences:

sequence n$^0$ 8 (polynomial $(x^3 - x^2 - x - 1)(x^4 - 1)$) is the harmonics $RL^2 * R$,

sequence n$^0$ 16 (polynomial $(x - 1)(x^2 - 1)(x^4 - 1)$) is the third harmonics R * R * R.

They correspond, in the broken linear case, to a period 4 and to the fixed point respectively. Sequence n$^0$ 15 (polynomial $(x - 1)(x^6 - x^4 - x^2 - 1)$) is the product $R * RL^2$ and defines a $\lambda$-number which is the square root of the $\lambda$-number related to sequence $RL^2$.

TABLE III. — *Table of all the allowed sequences of period $k = 8$ by decreasing order. The 3 representations $(a_i)$, $\{ \alpha_i \}$, MSS are given.*

| N° | $a_i$ $(i = 0, \ldots, 6)$ | | | | | | | $\alpha_i$ $(i = 1, \ldots, 6)$ | | | | | | MSS sequences |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $RL^6$ |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | −1 | 1 | 1 | 1 | 1 | 1 | −1 | $RL^5R$ |
| 3 | 1 | 1 | 1 | 1 | 1 | −1 | 1 | 1 | 1 | 1 | 1 | −1 | −1 | $RL^4R^2$ |
| 4 | 1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | 1 | $RL^4RL$ |
| 5 | 1 | 1 | 1 | 1 | −1 | 1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | $RL^3R^2L$ |
| 6 | 1 | 1 | 1 | 1 | −1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 | −1 | $RL^3R^3$ |
| 7 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | 1 | −1 | $RL^3RLR$ |
| 8 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | −1 | 1 | 1 | $RL^3RL^2$ |
| 9 | 1 | 1 | 1 | −1 | 1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 | $RL^2R^2LR$ |
| 10 | 1 | 1 | 1 | −1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 | $RL^2R^4$ |
| 11 | 1 | 1 | 1 | −1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | −1 | 1 | $RL^2R^3L$ |
| 12 | 1 | 1 | 1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | 1 | −1 | 1 | $RL^2RLRL$ |
| 13 | 1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 | 1 | −1 | −1 | $RL^2RLR^2$ |
| 14 | 1 | 1 | −1 | 1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 | −1 | $RLR^2LR^2$ |
| 15 | 1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | −1 | −1 | −1 | −1 | $RLR^5$ |
| 16 | 1 | 1 | −1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | −1 | 1 | −1 | $RLR^3LR$ |

### 4.4. Internal similarity and Sarkovskii theorem

#### 4.4.1. INTERNAL SIMILARITY

Again $\tilde{S} = 1(1, -1)^\infty$ is the auto-expansion of $\sqrt{2}$, which is associated to the MSS sequence $RLR^{2n+1} = R * RL^n$, $n \to \infty$.

Now, let P be some MSS sequence. If $P < \tilde{S}$, there exists a smallest integer $l$ such that $P = R^{*l} * P'$ with $P' > \tilde{S}$ and

$$R^{*(l+1)} * RL^n < P < R^{*l} * RL^n.$$

Moreover, if $Q' > \tilde{S}$ and $Q = R^{*l} * Q'$, then

$$P' < Q' \Leftrightarrow P < Q.$$

We recover in this particular case, the transformation $R^{*l} *$ (section 2.2.1) which preserves the ordering and maps the whole set of MSS sequences into one of its parts.

#### 4.4.2. SARKOVSKII THEOREM [*4-6*]

The period of any sequence smaller than $\tilde{S}$ is even. Any odd period corresponds to a MSS sequence greater than $\tilde{S}$. Let $S_0$ be the only sequence of period 3 : $S_0 = (1, 1, 0, \ldots)$ (or RL) and consider sequences S, $\tilde{S} < S < S_0$. Because of inequalities (3.10) or (4.13), they begin with the pattern

$$S_n = 1(1, -1)^n 1. \tag{4.14}$$

They may stop there or go on as

$$1(1, -1)^n 1\ 1 \ldots \tag{4.15}$$

The $\lambda$-sequence $S_n$ of (4.14) is related to an odd period $q = 2n + 3$ and it corresponds to the MSS sequence $RLR^{2n}$. By comparison with (4.15), we see that the smallest sequence of period $q = 2n + 3$ is necessarily $S_n$. The same result has been already obtained by Stefan [5], Cosnard [6] and Li and Yorke [7] (fig. 5). We have

$$n < m \Rightarrow S_n > S_m.$$

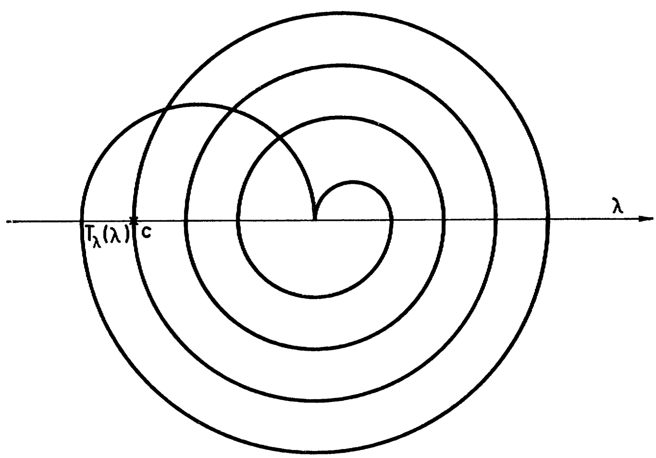


FIG. 5. — Order of the iterates of the maximum, in the broken linear case for sequence $RLR^{2n}(\equiv S_n)$, $n = 3$ (from ref. [5]).

Following Stefan [5] we shall write that

$$k \vdash k'$$

if period $k$ appears *after* period $k'$.

As
$$S_n < \ldots < S_2 < S_1 < S_0$$
we have

$$3 \vdash 5 \vdash 7 \vdash 9 \ldots$$

and as
$$b < R < R * R < \ldots < R^{*l}$$
$$\ldots \vdash 2^l \ldots \vdash 2^3 \vdash 2^2 \vdash 2 \vdash 1.$$

Any other even period $k = 2^l m$ ($l > 1$ integer, $m$ odd integer) appears first between these two cases and all integers are ordered in this way. The first period $k = 2^l m$ appears for $R^{*l} * S_m$ and as

$$l < l' \Rightarrow R^{*l'} * S_m < R^{*l} * S_{m'} \text{ for every } m, m'$$

we get the ordering of Sarkovskii

$$\begin{aligned} &3 \vdash 5 \vdash 7 \vdash 9 \ldots \vdash 2 \times 3 \vdash 2 \times 5 \vdash 2 \times 7 \ldots \vdash \\ &\ldots \vdash 2^l \times 3 \vdash 2^l \times 5 \vdash 2^l \times 7 \ldots \vdash \\ &\ldots \vdash 2^{l'} \ldots \vdash 2^2 \vdash 2 \vdash 1 \end{aligned} \tag{4.16}$$

Notice that the way of writing the periods in (4.16) reflects quite well internal similarity.

*Remark.* — Our theorem is proved for the ordering of the MSS sequences. Sarkovskii's result tells something slightly different. It says that if for a given $\lambda$

$$k \vdash k'$$

then the existence of a period $k$ implies the existence of a period $k'$. (A period $k$ exists, in the sense of the Sarkovskii theorem, if a number $x$ exists such as $T_\lambda^k(x) = x$ with $T_\lambda^j(x) \neq x \ \forall j < k$, but the maximum is not necessarily a point of the period, so that different periods may « coexist » for a given $\lambda$). Actually for the broken linear mapping $L_\lambda$ periods, when they exist are necessarily MSS periods for the value of $\lambda$ for which they appear. For $\mu > \lambda$, they still exist and the two theorems are equivalent.

Furthermore the Sarkovskii theorem is valid for any continuous mapping, without any restriction about the number of its critical points. Thus its application to functions with a *single critical* point, as in the present case, does not cover its full range of validity. However it must be emphasized that the Sarkovskii theorem is proved to be the « best possible » by means of examples of mappings with a *single critical* point. Accordingly one may believe that stronger statements might be proved for transform with more than one critical point.

## 5. OTHER RESULTS

In this section, we have collected some complementary theorems for the broken linear transform $L_\lambda$. We shall use mainly results of sections 3 and 4 without trying to extend them to more general mapping $T_\lambda$.

### 5.1. Number of critical points of $L_\lambda^{(n)}(x)$

We denote by

$N(n)$ the number of extrema (or of critical points) of $L_\lambda^{(n)}$: $1 \leq N(n) \leq 2^n$,

$b(n)$ the number of values of $x$, $0 < x \leq 1$ such that $L_\lambda^{(n)}(x) = 1$; we have the relation

$$N(n+1) = N(n) + 2b(n) \tag{5.1}$$

and $b(n)$ can be written in an integral form as

$$b(n) = \lambda^n \int_0^1 \delta[L_\lambda^{(n)}(x) - 1]dx$$
$$= \lambda^{n-1} \int_0^\lambda \delta[L_\lambda^{(n-1)}(x) - 1]dx$$

where $\delta$ is the Dirac « function ». We need the intermediate functions

$$c(n) \quad = \lambda^n \int_1^\lambda \delta[L_\lambda^{(n)}(x) - 1]dx$$

and

$$d(k, n) = \lambda^n \int_{T_\lambda^k(\lambda)}^1 \delta[L_\lambda^{(n)}(x) - 1]dx \tag{5.2}$$

$$[d(0, n) = -c(n)].$$

We have

$$b(n) = b(n-1) + c(n-1) \tag{5.3}$$

and

$$d(l, n) = \alpha_l[c(n-1) + d(l+1, n-1)] \tag{5.4}$$

where $\{ \alpha_l = \pm 1 \}$ is the set defining the $\lambda$-expansion as in eq. (4.4). For every $l$

$$c(n) = c(n-1) + \alpha_1 c(n-2) + \alpha_1\alpha_2 c(n-3) \ldots$$
$$+ \alpha_1 \ldots \alpha_l[c(n-l-1) + d(l+1, n-1-l)]$$

or

$$c(n) = \sum_{i=0}^{l} a_i c(n-i-1) + a_l d(l+1, n-1-l). \tag{5.5}$$

From (5.2) and (5.5) for $l = L_1$ and $l = L_1 - 1$

$$c(n) < c(n-1) + \ldots + c(n-L_1)$$
$$c(n) > c(n-1) + \ldots + c(n-L_1+1)$$

where $L_1 \geq 2$ is again the number of $+1$ at the head of the $\{ a_i \}$ sequence $[a_0 = a_1 = \ldots = a_{L_1-1} = 1,\ a_{L_1} = -1]$. The growth of $c(n)$ (and of $b(n)$ and $N(n)$ through eqs. (5.1), (5.3)) is like

$$x_1' < \frac{\ln c(n)}{n} < x_1 \tag{5.6}$$

where $x_1$ and $x_1'$ are the real solutions of $x^{L_1+1} - 2x^{L_1} + 1 = 0$ and $x^{L_1} - 2x^{L_1-1} + 1 = 0$ respectively as in section 4. We have two cases:

CASE *i*) : *$\lambda$ is $\lambda$-simple of period $k = l + 2$*

Then $d(l + 1, n - 1 - l) = 0$ and $c(n) \sim \mu^n P_k(n)$ where $P_k$ is some polynomial of degree less than $k$ and $\mu$ is one of the roots of the characteristic equation for $\lambda$

$$\mu^{k-1} + \alpha_0\mu^{k-2} + \ldots + \alpha_0 \ldots \alpha_{k-2} \equiv \mu^{k-1} - \sum_{i=0}^{k-2} a_i\mu^{k-2-i} = 0.$$

We must reject all the imaginary and negative roots as then $c(n)$ is surely not always increasing with $n$. From (5.6)

$$\mu = \lambda. \tag{5.7}$$

This leads to the conjecture that $\lambda$ is not only the largest *real* root of the characteristic equation, but that it is also the root with the largest modulus.

CASE *ii*) : *$\lambda$ is not $\lambda$-simple*

Then $d(l + 1, n - 1 - l) \neq 0$ for every $n$, $l$ and $c(n) = \sum_{i=0}^{n} a_i c(n - 1 - i)$.

There again $\dfrac{\ln c(n)}{n} \sim \mu$ where $\mu$ is a real root of the characteristic equation

$$\mu = \sum_{i=0}^{\infty} \frac{a_i}{\mu^i}$$

and from (5.6) again $\mu = \lambda$.

It can be shown by using its definition as presented by Adler *et al.* [*11*] that, in the broken linear case, $\ln \lambda$ is the topological entropy. More generally, for every mapping $T_\lambda(x)$ with only one maximum

$$\lim_{n\to\infty} \frac{\ln N(n)}{n}$$

exists and is again the topological entropy. But it is then related to the parameter $\lambda$ in a more complicated manner which depends on the way the mapping $T_\lambda$ is related to some broken linear mapping $L_{\psi(\lambda)}$ [*9*].

## 5.2. Stefan's [*12*] matrices

The topological entropy roughly measures the amount of information we have on the mapping $L_\lambda$ (or $T_\lambda$). One way of seeing it, is the following.

For $\lambda$, $\lambda$-simple of period $k$, let us divide the interval $[L_\lambda(\lambda), \lambda]$ into $(k-1)$ adjacent intervals $I_i$ $(i = 1, \ldots, k-1)$ ending at the iterates of $\lambda$.

In other words let $\{ I_i \} = \{ ]L_\lambda^{(m)}(\lambda), L_\lambda^{(p)}(\lambda)[ \}$, such that no integer $m'$ exists with $L_\lambda^{(m)}(\lambda) < L_\lambda^{(m')}(\lambda) < L_\lambda^{(p)}(\lambda)$, so that

$$L_\lambda(I_i) = \sum_j r_{ij} I_j \qquad i, j = 1, \ldots, k-1$$

with

$$\left. \begin{aligned} r_{ij} &= 0 \quad \text{whenever} \quad L_\lambda(I_i) \cap I_j = \emptyset \\ &= 1 \quad \text{if} \quad L_\lambda(I_i) \supset I_j \end{aligned} \right\} \tag{5.8}$$

then, the information on the transformation is contained in the matrix

$$(r_{ij})_{i,j=1,\ldots,k-1} \tag{5.9}$$

with only 0 and + 1 matrix elements.

Let $X_i$ be the *length* ($\geq 0$) of interval $I_i$. Through application $L_\lambda$, this length becomes $\lambda X_i$ and is the sum of the lengths of the intervals $I_j$ for which $r_{ij} \neq 0$:

$$\lambda X_i = \sum_j r_{ij} X_j. \tag{5.10}$$

Thus $\lambda$ is a root of the secular equation

$$\det (r_{ij} - \lambda \delta_{ij}) = 0 \tag{5.11}$$

where

$$\begin{aligned} \delta_{ij} &= +1 \quad \text{if} \quad i = j \\ &= 0 \text{ otherwise} \end{aligned}$$

is the Kronecker symbol.

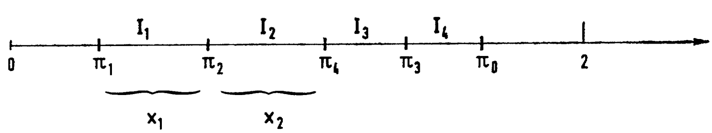


FIG. 6. — Example of repartition of the real axis of the iterates of $\lambda$ for a period $k = 5$; $\pi_i = L_\lambda^{(i)}(\lambda)$.

As an example, let us take the sequence $RL^2R$ of period 5, which corresponds to the ordered succession of transforms of $c$ (fig. 6)

$$\pi_1 = T_\lambda(\lambda), \quad \pi_2 = T_\lambda^2(\lambda), \quad \pi_4 = c, \quad \pi_3 = T_\lambda^3(\lambda), \quad \pi_0 = \lambda$$

and to the 4 × 4 matrix

$$(r_{ij}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

of secular equation $\lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1 = 0$. This is exactly the characteristic equation of $\lambda$ for the sequence $RL^2R$ as defined in section 4.

This is not a coincidence but a general result. The order of the iterates of $\lambda$ on the real axis may be represented by a permutation $\sigma$ of integers $1, 2, 3, \ldots, k$ (if $\lambda$ is periodic of period $k$); the $i^{\text{th}}$ point on the line corresponds to some iterate $L_\lambda^{(l)}(\lambda)$ of $\lambda$ and we have $\sigma(l) = i$.

We have $\sigma(1) = 1$, $\sigma(k) = k$ always and if the last iterate $L_\lambda^{k-1}(\lambda)$ [$= 1$] is the $(p + 1)^{\text{th}}$ point on the line $\sigma(k - 1) = p + 1$. In the above example $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 4$, $\sigma(4) = 3$, $\sigma(5) = 5$.

Let $X_i$ be again the length of the $i^{\text{th}}$ segment, i. e. the length between the $i^{\text{th}}$ and $(i + 1)^{\text{th}}$ point on the axis [take care that this order of the points on the line is not connected in a simple way to their transformation law, for instance the $(i + 1)^{\text{th}}$ points in general is not the iterate of the $i^{\text{th}}$ point], and $Y_i$ denotes the distance of the $i^{\text{th}}$ point to 1. We have

$$\begin{aligned} Y_1 &= X_1 + \ldots + X_p \\ Y_2 &= X_2 + \ldots + X_p \\ \ldots\ Y_p &= X_p \\ Y_{p+1} &= 0 \\ Y_{p+2} &= X_{p+1} \\ Y_{p+3} &= X_{p+1} + X_{p+2} \\ \ldots\ Y_{k-1} &= X_{p+1} + X_{p+2} + \ldots + X_{k-1} (= \lambda - 1). \end{aligned}$$

From the definition of the broken linear mapping,

$$L_\lambda^{(l+1)}(\lambda) - 1 = (\lambda - 1) + \lambda\alpha_l[L_\lambda^{(l)}(\lambda) - 1]$$

and

$$- \alpha_{l+1} Y_{\sigma(l+1)} = Y_{k-1} - \lambda Y_{\sigma(l)}.$$

Setting now $Z_l = Y_{\sigma(l)}$, we get the homogeneous system of $k - 1$ equations

$$\begin{aligned} - \alpha_1 Z_1 &= Z_{k-1} - \lambda Z_{k-1} && (l = k - 1) \\ - \alpha_{l+1} Z_{l+1} &= Z_{k-1} - \lambda Z_l && 1 \leq l \leq k - 3 \\ 0 &= Z_{k-1} - \lambda Z_{k-2} && (l = k - 2) \end{aligned}$$

which has only the trivial solution $Z_i = 0$ unless

$$\det \begin{vmatrix} 1-\lambda & \alpha_1 & 0 & 0 \\ 1 & -\lambda & \alpha_2 & \\ \vdots & & \ddots & \ddots \\ \vdots & & & \ddots & \alpha_{k-2} \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

We recover the characteristic equation for $\lambda$ (eq. 4.7) and, as the transformations $X_i \to Y_i \to Z_i$ are invertible, the systems for the $X_i$'s and $Z_i$'s have the same secular equation.

These results are in agreement with ref. [*9*].

### 5.3. Invariant measures

For every $\lambda$, we are looking for a measure $\nu_\lambda$ absolutely continuous with respect to the Lebesgue measure and invariant under $L_\lambda(x)$ on the stable interval $I = (2\lambda - \lambda^2, \lambda)$. It is unique (Lasota-Yorke [*13*]) and there exists a measurable function $h_\lambda(x)$ (Radon-Nikodym theorem) such that

$$\nu_\lambda(E) = \int_E h_\lambda(x)dx \tag{5.12}$$

for every Lebesgue measurable set E.

If $\nu_\lambda$ is invariant under $L_\lambda(x)$, then $\nu_\lambda(E) = \nu_\lambda[L_\lambda^{-1}(E)]$ for all Lebesgue sets of I, i. e.

$$\lambda h_\lambda(x) = \sum_{L_\lambda(y)=x} h_\lambda(y) \tag{5.13}$$

and

$$h_\lambda(x) = 0 \quad \text{if} \quad x < 2\lambda - \lambda^2 \quad \text{or} \quad x > \lambda$$

or

$$\begin{aligned} \lambda h_\lambda(x) &= h_\lambda\left(\frac{x}{\lambda}\right) + h_\lambda\left(2 - \frac{x}{\lambda}\right) \qquad 2\lambda - \lambda^2 < x < \lambda \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.14}$$

Let now $\varphi_\lambda(x)$ be the step function

$$\begin{aligned} \varphi_\lambda(x) &= \sum_{\lambda > x > L_\lambda^{(n+1)}(\lambda)} \frac{a_n}{\lambda^{n+1}} \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{5.15}$$

where the $\{ a_i \}$ are that of the auto-expansion of $\lambda$ and the summation may be finite if $\lambda$ is $\lambda$-simple; $\varphi_\lambda(x)$ is defined except for the countable set of the iterates of $\lambda$ and (5.14) holds almost everywhere for $x < \lambda$.

*Proof.* — For $x < \lambda$

$$\varphi_\lambda\left(\frac{x}{\lambda}\right) = \sum_{x > \lambda L_\lambda^{(n+1)}(\lambda)} \frac{a_n}{\lambda^{n+1}} = \sum_{\substack{L_\lambda^{(n+1)}(\lambda) < 1 \\ x > L_\lambda^{kn+2)}(\lambda)}} \frac{a_n}{\lambda^{n+1}} = \lambda \sum_{\substack{L_\lambda^{(n+1)}(\lambda) < 1 \\ x > L_\lambda^{(n+2)}(\lambda)}} \frac{a_{n+1}}{\lambda^{n+2}}$$

where we have replaced $a_n$ by $a_{n+1} = \alpha_{n+1}a_n (= a_n$, as the summation runs over all $n$ such that $L_\lambda^{(n+1)}(\lambda) < 1)$

$$\varphi_\lambda\left(2 - \frac{x}{\lambda}\right) = \sum_{x < 2\lambda - \lambda L_\lambda^{(n+1)}(\lambda)} \frac{a_n}{\lambda^{n+1}} = \sum_{L_\lambda^{(n+1)}(\lambda) < 1} \frac{a_n}{\lambda^{n+1}} + \sum_{\substack{L_\lambda^{(n+1)}(\lambda) > 1 \\ x < L_\lambda^{(n+2)}(\lambda)}} \frac{a_n}{\lambda^{n+1}}$$

$$= \sum_{L_\lambda^{(n+1)}(\lambda) < 1} \frac{a_n}{\lambda^{n+1}} - \lambda \sum_{\substack{L_\lambda^{(n+1)}(\lambda) > 1 \\ x < L_\lambda^{(n+2)}(\lambda)}} \frac{a_{n+1}}{\lambda^{n+2}} (a_{n+1} = \alpha_{n+1}a_n = - a_n)$$

$$= \sum_{L_\lambda^{(n+1)}(\lambda) < 1} \frac{a_n}{\lambda^{n+1}} - \lambda \sum_{L_\lambda^{(n+1)}(\lambda) > 1} \frac{a_{n+1}}{\lambda^{n+2}} + \lambda \sum_{\substack{L_\lambda^{(n+1)}(\lambda) > 1 \\ x > L_\lambda^{(n+2)}(\lambda)}} \frac{a_{n+1}}{\lambda^{n+2}}$$

Now

$$\varphi_\lambda\left(\frac{x}{\lambda}\right) + \varphi_\lambda\left(2 - \frac{x}{\lambda}\right) = \sum_n \frac{a_n}{\lambda^{n+1}} + \lambda \sum_{x > L_\lambda^{(n+2)}(\lambda)} \frac{a_{n+1}}{\lambda^{n+2}}$$

$$= \left[\sum_n \frac{a_n}{\lambda^{n+1}} - a_0\right] + \lambda \sum_{x > L_\lambda^{(n+1)}(\lambda)} \frac{a_n}{\lambda^{n+1}}.$$

The term into brackets is the characteristic equation of $\lambda$ and gives 0. ■

Furthermore, $\varphi_\lambda(x)$ is a continuous function of $\lambda$ at $x$ fixed. When $\lambda$ is not $\lambda$-simple, $\lambda$ tends to $\lambda'$ is equivalent to say that the auto-expansions of $\lambda$ and $\lambda'$ are the same up to an order $n_0$, arbitrarily large. Then, one may readily find two MSS sequences corresponding to two $\lambda$-simple numbers say $\lambda_0$ and $\lambda'_0$ which are as close as one wants of $\lambda$ and $\lambda'$ and such as

$$\lambda_0 < \lambda,\ \lambda' < \lambda'_0.$$

These two finite MSS sequences begin with the same pattern as the one defined by the auto-expansion of $\lambda$, up to an order $n_0$, arbitrarily large and for any $l$ smaller than or equal to $n_0 + 1$:

$$L_\lambda^{(l)}(\lambda),\ L_{\lambda'}^{(l)}(\lambda') \in [L_{\lambda_0}^{(l)}(\lambda_0),\ L_{\lambda'_0}^{(l)}(\lambda'_0)]$$

and

$$|\varphi_\lambda(x) - \varphi_{\lambda'}(x)| \sim \left| \sum_{\substack{n > n_0 \\ x > L_\lambda^{(n+1)}(\lambda)}} \frac{a_n}{\lambda^n} - \sum_{\substack{n > n_0 \\ x > L_{\lambda'}^{(n+1)}(\lambda')}} \frac{a'_n}{\lambda'^n} \right| \to 0 \quad \text{if} \quad x \neq L_\lambda^{(l+1)}(\lambda)$$

When $\lambda$ is $\lambda$-simple, the auto-expansion of $\lambda'$ must reproduce to a minus sign the pattern $(-1, a_0, a_1, \ldots, a_{k-2})$ and the proof of the continuity of $\varphi_\lambda(x)$ is similar.

Similarly, $\lambda$ being fixed, $\varphi_\lambda(x)$ is a continuous function of $x$ for $x \neq L_\lambda^{(n)}(\lambda)$ as $x$ and $x'$ $[|x - x'| \to 0]$ are in different intervals $L_\lambda^{(l)}(\lambda)\ L_\lambda^{(l')}(\lambda)$ for some large value of $l$ or of $l'$.

It remains to prove that $\varphi_\lambda(x)$ is positive. This can be done in two steps: one proves it first when $\lambda$ is $\lambda$-simple, then as $\varphi_\lambda(x)$ is continuous with respect to $\lambda$ at $x$ fixed and, as the $\lambda$-simple numbers are dense in $]1, 2[$, then $\varphi_\lambda(x)$ is positive (or eventually zero) for any $\lambda$ between 1 and 2.

The proof that $\varphi_\lambda(x)$ is positive when $\lambda$ is $\lambda$-simple goes as follows: Let $(x_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(k)})$ be the finite ordered set of the iterates of $\lambda$ ($\sigma$ is the above defined permutation of the integers from 1 to $k$). From (5.15), $\varphi_\lambda(x)$ is zero when $x$ is less than $x_{\sigma^{-1}(1)}$ or larger than $x_{\sigma^{-1}(k)}$. Furthermore $\varphi_\lambda(x)$ takes a constant value in any interval $]x_{\sigma^{-1}(j)}, x_{\sigma^{-1}(j+1)}[$ $1 \leq j < k - 1$, and it is easy to compute from (5.15) the value of $\varphi_\lambda$ in the two end intervals: $\varphi_\lambda(x) = \frac{1}{\lambda}$ if $x_{\sigma^{-1}(1)} < x < x_{\sigma^{-1}(2)}$ and $\varphi_\lambda(x) = 1$ if $x_{\sigma^{-1}(k-1)} < x < x_{\sigma^{-1}(k)}$. From this knowledge of $\varphi_\lambda$ in the two end intervals it is possible to show recursively from (5.14) that $\varphi_\lambda$ is everywhere positive.

### 5.4. Periodic points

If $\lambda$ is a $\lambda$-simple number, it corresponds to a MSS sequence Q of period $k$. This means that a set of numbers $x_j$, $1 \leq j \leq k$ exists such as $x_1 = 1$, $x_2 = L_\lambda(x_1), \ldots, x_k = L_\lambda^{(k-1)}(x_1)$, $x_1 = L_\lambda^{(k)}(x_1)$, $x_i \neq x_j$ if $i \neq j$.

When $\lambda$ changes continuously, we can ask whether this period $k$ still exists, i. e. if a set of numbers $\{ y_j, 1 \leq j \leq k \}$ exists such as $y_j = L_\lambda^{(j)}(y_1)$, $\ldots, y_1 = L_\lambda^{(k)}(y_1)$ and $y_i \neq y_j$ if $i \neq j$. These conditions are very similar to the one defining the set $\{ x_j \}$ associated with the $\lambda$-simple number, except that one does not require anymore that 1 belongs to the period, as this implies that $\lambda$ takes a particular value, solving the equation $1 = L_\lambda^{(k)}(1)$.

For $\lambda' < \lambda$, in the broken linear case, the period $\{ y_i \}$ does not exist; but for $C^1$ transformation $T_\lambda(x)$, the set $\{ y_j \}$ exists.

For $\lambda' > \lambda$ in any case, this period $k$ exists and $y_1, y_2, \ldots, y_k$ are functions of $\lambda'$.

*Example.* — The MSS sequence corresponding to period 3 is RL. The $\lambda$-simple number for this sequence is $\lambda = \frac{\sqrt{5}+1}{2}$.

If $\lambda' < \lambda$, we cannot find $x_1$, $x_2$, $x_3$ all distinct verifying $L_{\lambda'}^{(3)}(x_1) = x_1$.
If $\lambda' = \lambda$ we have a solution $x_1 = 1$, $x_2 = \lambda$, $x_3 = 2\lambda - \lambda^2 = \lambda - 1$.

If $\lambda' > \lambda$ we have

$$y_1 = \frac{2\lambda'^2}{\lambda'^3 + 1}, \qquad y_2 = \frac{2\lambda'^3}{\lambda'^3 + 1}, \qquad y_3 = \frac{2\lambda'}{\lambda'^3 + 1},$$

and $y_1 \to 1$, $y_2 \to \lambda$, $y_3 \to 2\lambda - \lambda^2$ as $\lambda' \to \lambda$.

To show the connection existing between the $\lambda$-expansion of $y_1$ and the sequence Q it is convenient to rewrite the broken linear mapping as

$$\widehat{L}_\lambda(x) = 1 + \beta\lambda x \tag{5.16}$$

where $\beta = -\, sgx$. This mapping maps into itself the interval $[-1, +1]$.

Following the same procedure as in sections 3 and 4 one may write any $x$ between $-1$ and $+1$ as

$$x = -\frac{\beta_0}{\lambda}\left[1 + \frac{\beta_1}{\lambda} + \ldots + \frac{\beta_1 \ldots \beta_{k-1}}{\lambda^{k-1}} - \frac{\beta_1 \ldots \beta_{k-1}}{\lambda^{k-1}}\widehat{L}_\lambda^{(k)}(x)\right]$$

since $|\widehat{L}_\lambda^k(x)| \leq 1$, $r_k = -\beta_k\widehat{L}_\lambda^{(k)}(x)$ may be thought as a remainder: $0 \leq r_k \leq 1$.

Like in section 4

$$\begin{aligned} \beta_i &= +1 \quad \text{if} \quad \widehat{L}_\lambda^{(i)}(x) < 0 \quad (x \text{ less than the maximum}) \\ &= -1 \quad \text{if} \quad \widehat{L}_\lambda^{(i)}(x) > 0 \quad (x \text{ larger than the maximum}). \end{aligned}$$

If the maximum is not an iterate of $x$, $r_k \neq 0, 1$ for every $k$ and

$$x = -\frac{\beta_0}{\lambda}\left[1 + \sum_{i=1}^{\infty}\frac{\beta_1 \ldots \beta_i}{\lambda^i}\right] \tag{5.17}$$

is the $\lambda$-expansion of $x$; the $j^{\text{th}}$ iterate of $x$ corresponds to the shifted $\beta$-sequence

$$\widehat{L}_\lambda^{(j)}(x) = -\frac{\beta_j}{\lambda}\left[1 + \sum_{i=1}^{\infty}\frac{\beta_{j+1} \ldots \beta_{j+i}}{\lambda^i}\right] \tag{5.18}$$

and for this mapping, the relation between $\lambda$-expansion and iterates is striking. (In $L_\lambda(x)$, we should consider the $\lambda$-expansion of $\frac{x-1}{\lambda-1}$ to get a similar result).

This is true for every $x$, mainly for the $y_j$ which belongs to a period but have not the maximum in their sequence of iterates; the $\lambda$-expansion is then periodic.

The $\lambda$-expansion of 1 reads

$$1 = \frac{1}{\lambda}\left[1 + \sum_{i=1}^{\infty} \frac{\alpha_1 \ldots \alpha_i}{\lambda^i}\right]$$

where the $\alpha_i = - sg\widehat{L}_\lambda^{(i)}(1)$ and we recover the characteristic equation of $\lambda$.

Setting
$$\begin{aligned} b_n &= - \beta_0\beta_1 \ldots \beta_n \\ a_n &= - \alpha_0\alpha_1 \ldots \alpha_n \\ &= \alpha_1 \ldots \alpha_n \end{aligned}$$

we get again the conditions for $\{ b_i \}$ to be a $\lambda$-expansion, i. e.

$$\pm (b_n, b_{n+1} \ldots) < (a_0, a_1, \ldots) \qquad \text{for} \qquad \text{every } n \geq 1.$$

*Remark.* — If the maximum is an iterate of $x$, the remainder is zero for some finite $j$, the expansion is no more infinite and (5.18) no more holds. However, in view of keeping this remarkable property, we may adopt the convention that the infinite expansion (5.17) holds by completing it with the $\lambda$-expansion of 0 (or 1), i. e. the characteristic equation for $\lambda$; $\beta_j$ is no more zero but chosen arbitrarily as $\pm 1$ and

$$x = - \frac{\beta_0}{\lambda}\left[1 + \sum_{i=1}^{j-1} \frac{\beta_1 \ldots \beta_i}{\lambda^i}\right] + \frac{1}{\lambda^j}\left[1 + \sum_{i=1}^{\infty} \frac{\beta_{j+1} \ldots \beta_{j+i}}{\lambda^i}\right].$$

If $\lambda$ is $\lambda$-simple, of period $k$, one may choose freely $\beta_j, \beta_{j+k}, \beta_{j+2k}, \beta_{j+3k}, \ldots$ For example the characteristic equation for $\lambda$ reads:

$$0 = \left[1 + \sum_{i=1}^{k-1} \frac{\beta_1 \ldots \beta_i}{\lambda^i}\right]\left[1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\lambda^{kn}}\right],$$

where $\varepsilon_n = \pm 1$ is arbitrary. This problem has already been studied in Part 4.

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# APPENDIX A

**Proof of the sufficient condition** (eqs. 3.10 and 3.7).

We first show the

**Lemma**, if for $n \geq 1$

$$(b_n, b_{n+1}, \ldots) < (a_0, a_1, \ldots)$$

and

$$-(b_n, b_{n+1}, \ldots) < (a_0, a_1, \ldots), \text{ then}$$

*i*) the inequalities between sequences

$$(b_n, b_{n+1}, \ldots) < (a_m, a_{m+1}, \ldots) \; [\text{resp. } -(b_n, b_{n+1}, \ldots) < (a_m, a_{m+1}, \ldots)]$$

imply the inequalities between the sums of series

$$\sum_{i=0}^{\infty} \frac{b_{n+i}}{\lambda^i} \leq \sum_{i=0}^{\infty} \frac{a_{m+i}}{\lambda^i} \left[\text{resp. } -\sum_{i=0}^{\infty} \frac{b_{n+i}}{\lambda^i} \leq \sum_{i=0}^{\infty} \frac{a_{m+i}}{\lambda^i}\right] \tag{A.1}$$

*ii*)

$$b_n \sum_{i=0}^{\infty} \frac{b_{n+i}}{\lambda^i} \geqslant 0. \tag{A.2}$$

*** We prove *i*) by showing by recurrence that

$(b_n, b_{n+1}, \ldots, b_{n+q}) \leq (a_m, \ldots, a_{m+q})$ [resp. $-(b_n, b_{n+1}, \ldots, b_{n+q}) \leq (a_m, \ldots, a_{m+q})$]

implies

$$\sum_{i=0}^{q} \frac{b_{n+i}}{\lambda^i} \leq \sum_{i=0}^{q} \frac{a_{m+i}}{\lambda^i} + \frac{2q}{\lambda^q} \tag{A.3}$$

$$\left[\text{resp. } -\sum_{i=0}^{q} \frac{b_{n+i}}{\lambda^i} \leq \sum_{i=0}^{q} \frac{a_{m+i}}{\lambda^i} + \frac{2q}{\lambda^q}\right]$$

it is true for $q = 0$. We assume it is true for $q = s - 1$ and prove it for $q = s$ and for $(b_n, \ldots, b_{n+s})$ (same proof for $-(b_n, \ldots, b_{n+s})$).

* *either* $b_n = a_m$ and (A.3) is a consequence of the recurrence hypothesis

* *either* $b_n < a_m$ ; let $\Delta = \sum_{i=0}^{s} \frac{b_{n+i}}{\lambda^i} - \sum_{i=0}^{s} \frac{a_{m+i}}{\lambda^i}$

*If* $b_n = 0$, *then* $a_m = 1$ *and*

$$\Delta = -\lambda r_{m-1}(\lambda \mid \lambda) + a_{m+s+1} \frac{r_{m+s}(\lambda \mid \lambda)}{\lambda^s} < \frac{1}{\lambda^s}$$

where again $r_{m-1}(\lambda \mid \lambda)$ is the rest of the auto-expansion of $\lambda$.

*If* $b_n = -1$, *either* $a_m = 0$ *and*

$$\Delta = -1 + \frac{1}{\lambda} \sum_{i=0}^{s-1} \frac{b_{n+1+i}}{\lambda^i}$$

as $(b_{n+1}, \ldots, b_{n+r}) \leq (a_0, \ldots, a_{r-1})$, we get from the recurrence hypothesis

$$\sum_{i=0}^{s-1} \frac{b_{n+1+i}}{\lambda^i} \leq \sum_{i=0}^{s-1} \frac{a_i}{\lambda^i} + \frac{2(s-1)}{\lambda^{s-1}} = \lambda - a_s \frac{r_{s-1}(\lambda \mid \lambda)}{\lambda^{s-1}} + \frac{2(s-1)}{\lambda^{s-1}}$$

and

$$\Delta < \frac{2(s-1)}{\lambda^s} - a_s \frac{r_{s-1}(\lambda \mid \lambda)}{\lambda^s} < \frac{2s}{\lambda^s}$$

*either* $a_m = +1$ and let $n' > 0$ be the first integer such that $b_{n+n'} \neq a_{m+n'}$

$$\Delta = -2 + \frac{1}{\lambda^{n'}} \left[ \sum_{i=0}^{s-n'} \frac{b_{n+n'+i}}{\lambda^i} - \sum_{i=0}^{s-n'} \frac{a_{m+n'+i}}{\lambda^i} \right]$$

if $b_{n+n'} < a_{m+n'}$ it is a consequence of the recurrence hypothesis,
if $b_{n+n'} > a_{m+n'}$ it is straigtforward if $b_{n+n'} = 0$, or $a_{m+n'} = 0$,

if $a_{m+n'} \neq 0$ $$\sum_{i=0}^{s-n'} \frac{a_{m+n'+i}}{\lambda^i} = \lambda a_{m+n'} r_{m+n'-1}(\lambda \mid \lambda) - a_{m+s+1} \frac{r_{m+s}(\lambda \mid \lambda)}{\lambda^{s-n'}}$$

by definition of the $\lambda$-expansions.

On the other hand $(b_{n+n'}, \ldots, b_{n+s}) \leq (a_0, \ldots, a_{n'-s})$
whence

$$\sum_{i=0}^{s-n'} \frac{b_{n+n'+i}}{\lambda^i} \leq \sum_{i=0}^{s-n'} \frac{a_i}{\lambda^i} + \frac{2(s-n')}{\lambda^{s-n'}},$$

and

$$\Delta < \frac{2(s-n') - a_{s-n'+1} r_{s-n'}(\lambda \mid \lambda) - a_{m+s+1} r_{m+s}(\lambda \mid \lambda)}{\lambda^s} \frac{2s}{\lambda^s}.$$

The proof of inequality (A.2) is similar. We first prove that $\forall q$

$$b_n \sum_{i=0}^{q} \frac{b_{n+i}}{\lambda^i} \geq -\frac{2q}{\lambda^q}. \tag{A.4}$$

This is obvious if $b_n = b_{n+1}$; if $b_n$ and $b_{n+1}$ have different signs we use the minoration (A.3) and the identity

$$b_n \sum_{i=0}^{q} \frac{b_{n+i}}{\lambda^i} = 1 - \frac{1}{\lambda} \sum_{i=0}^{q-1} \left( -\frac{b_{n+i+1}}{\lambda^i} \right),$$

as

$$\sum_{i=0}^{q-1} \left( -\frac{b_{n+i+1}}{\lambda^i} \right) \leq \sum_{i=0}^{q-1} \frac{a_i}{\lambda^i} + \frac{2(q-1)}{\lambda^{q-1}}, \qquad \text{(from eq. (A.3))},$$

then

$$b_n \sum_{i=0}^{q} \frac{b_{n+i}}{\lambda^i} \geq 1 - \frac{1}{\lambda} \left[ \sum_{i=0}^{q-1} \frac{a_i}{\lambda^i} \right] - \frac{2(q-1)}{\lambda^q} > -\frac{2q}{\lambda^q}. \quad \blacksquare$$

*Proof of theorem 1.* — The inequalities (A.1) and (A.2) are consequences of inequalities (A.3) and (A.4) when $q \to \infty$.

Then we are assured of

$$0 \leq \left| \sum_{i=n}^{\infty} \frac{b_i}{\lambda^i} \right| \leq \frac{1}{\lambda^{n-1}} \tag{A.5}$$

and we must now drop the equalities to achieve the proof of the theorem 1.

One may assume for instance that, $b_n = +1$. Let $n' \geq 0$ be the first integer so that $b_{n+n'} \neq a_{n'}$; necessarily $b_{n+n'} < a_{n'}$. We distinguish two cases:

*The auto expansion of the basis is infinite* ($a_i \neq 0\ \forall i$) then

$$\sum_{i=0}^{\infty} \frac{a_{n'+i}}{\lambda^i} = \lambda r_{n'-1}(\lambda \mid \lambda) > 0 \text{ strictly}$$

$$\sum_{i=0}^{\infty} \frac{b_{n+n'+1}}{\lambda^i} \leq 0 \qquad \text{(eq. (A.2))}$$

whence the strict inequality $\left| \sum_{i=n}^{\infty} \frac{b_i}{\lambda^i} \right| < \frac{1}{\lambda^{n-1}}$.

We prove now that $\left| \sum_{i=n}^{\infty} \frac{b_i}{\lambda^i} \right| \neq 0$ if $b_n \neq 0$; From (A.2) it is necessary that $b_{n+1} = -1$ and then

$$\lambda = - \sum_{i=0}^{\infty} \frac{b_{n+1+i}}{\lambda^i} = \sum_{i=0}^{\infty} \frac{a_i}{\lambda^i}$$

this is equivalent to $-(b_{n+1}, b_{n+2}, \ldots) = (a_0, a_1, a_2, \ldots)$ which is impossible.

*The auto-expansion of the basis is finite of length $k - 1$*

$$\lambda = \sum_{i=0}^{k-2} \frac{a_i}{\lambda^i}.$$

If $a_{n'} \neq 0$ the proof of the right hand side inequality of (A.5) is the same as in the infinite case.

If $a_{n'} = 0$ it amounts to look when the equality $\sum_{i=0}^{\infty} \frac{b_{n+n'+i}}{\lambda^i} = 0$ holds and this equality may be included in the discussion of the *l. h. s.* inequality of (A.5).

Let $q$ be the first integer so that $\sum_{i=q}^{\infty} \frac{b_i}{\lambda^i} = 0$; for example $b_q = -1$, necessarily (eq. (A.2)) $b_{q+1} = +1 = a_0$ and

$$\sum_{i=0}^{\infty} \frac{b_{q+i+1}}{\lambda^i} = \sum_{i=0}^{k-2} \frac{a_i}{\lambda^i}$$

then

$$b_{q+i+1} = a_i \qquad 0 \leq i \leq k - 2 \quad \text{(proof by recurrence on } i\text{)}$$

i. e.

$$\begin{aligned} b_q &= -1 \\ b_{q+1} &= a_0 \\ b_{q+i+1} &= a_i \\ b_{q+k-1} &= a_{k-2} \end{aligned}$$

and $\sum_{i=0}^{\infty} \frac{b_{q+k+i}}{\lambda^i} = 0$. We are let with the same problem $k$ steps further and more generally, the expansion of $x$ can be written for $n \geq 0$

$$\left.\begin{aligned} b_{q+nk} &= -\varepsilon_n \\ b_{q+nk+1} &= \varepsilon_n a_0 \\ \ldots\ b_{q+nk+i+1} &= \varepsilon_n a_i \end{aligned}\right\} \tag{A.6}$$

$\varepsilon_n = \pm 1, 0, \varepsilon_n$ remaining the same for $k$ consecutive indices (if $\varepsilon_n = 0$ for $n = n_0$, then $\varepsilon_n = 0$ for $n > n_0$).

Sequences with this peculiar periodicity must be rejected though the necessary condition holds.

This situation may be compared with the ambiguity where one replaces somewhat arbitrarily

$$1,999 \ldots \text{ by } 2$$

in the decimal expansions.

# APPENDIX B

A SEQUENCE S SATISFYING CONDITION (3.10) AND SUCH THAT $S > \tilde{S}$ HAS ONE ROOT AND ONLY ONE FOR $x > \sup(\sqrt{2}, x_1')$.

As in the bulk of the paper $\tilde{S}$ is the sequence of the auto-expansion of $\sqrt{2}$, $x_1$ and $x_1'$ are the roots $1 < x_1' < x_1$ of $x^{L_1+1} - 2x^{L_1} + 1 = 0$ and $x^{L_1} - 2x^{L_1-1} + 1 = 0$ respectively.

We shall show successively that

*i*) the greatest real root of S, $x(S)$, exists and for $S > \tilde{S}$, $x(S) > \sup(\sqrt{2}, x_1')$

*ii*) there is no other real root for $x > \sup(\sqrt{2}, x_1')$.

To prove the two propositions, we shall substitute to the notation

$$S = \{ a_0, a_1, \ldots, \} \text{ the notation}$$
$$S \equiv L_1L_2 \ldots L_{p+1},$$

where the $L_i$'s are integers:

$$1 \leq L_i \leq L_1, \quad L_1 \geq 2, \quad L_2 < L_1 \tag{B.1}$$

and denote the number of consecutive + 1 or − 1 in the $\{ a_i \}$ sequence: we have $L_1$ coefficients + 1, then $L_2$ coefficients − 1, $L_3$ coefficients + 1, ... or said differently

$$a_0 = a_1 = \ldots = a_{L_1-1} = +1$$
$$a_{L_1} = a_{L_1+1} = \ldots = a_{L_1+L_2-1} = -1$$

and more generally if

$$N_i = \sum_{j=1}^{i} L_j, \quad a_{N_i} = a_{N_i+1} = \ldots = a_{N_i+L_{i+1}-1} = (-)^{i-1}.$$

Actually, inequalities (3.10) insure some more restrictive conditions than conditions (B.1) for example $L_{p+1} < L_1$, and if $L_i = L_1$ then $L_{i+1} \geq L_2$ ... but it will be sufficient to assume conditions (B.1). We deal then with a larger class of sequences than that generated with condition (3.10). With these notations $x_1$ and $x_1'$ are the greatest real roots of the sequences $S_1 \equiv L_1$, $S_1' \equiv (L_1 - 1)$.

In the following we use the auxiliary sequences

$$T \equiv L_1 \ldots L_{p-1}L_p$$
$$U \equiv L_1 \ldots L_{p-1}$$

which satisfy (B.1) and we denote by L the sum $L = L_2 + \ldots + L_p$ and by $\psi_S(x)$ the function

$$\psi_S(x) = x^{L_1-1}(x-1)\varphi_S(x) \tag{B.2}$$

*Lemma 1.* — *i*) $\psi_S(x_1) > 0$, $\psi_S(x_1') < 0$

*ii*) if $L_1 = 2$ and $S > \tilde{S}$ we have the inequality: $\psi_S(\sqrt{2}) < 0$.

*Corollary.* — $x(S)$ exists and

$$x(S) > \sup\left(\sqrt{2}, x'_{-}\right) \begin{cases} = x'_1, & \text{if} \quad L_1 \geq 3 \\ = \sqrt{2} & \text{if} \quad L_1 = 2 \end{cases}$$

*i*) We prove the first part of the lemma by induction; *it is true for p = 1*; from

$$\psi_S(x) = x^{L_1+1} - 2x^{L_1} + 2 - \frac{1}{x^{L_2}}$$

we get

$$\psi_S(x_1) = 1 - \frac{1}{x_1^{L_2}} > 0$$

$$\psi_S(x'_1) = 2 - x'_1 - \frac{1}{x_1'^{L_2}} = - \frac{x_1'^{L_2+1} - 2x_1'^{L_2} + 1}{x_1'^{L_2}}$$

which is negative as $L_2 < L_1$.

*We assume it is true for p = l − 1 and prove it for p = l*, we have the identity

$$\psi_S(x) = \psi_T(x) + \frac{(-)^p}{x^{L_2+\ldots+L_p}}\left(1 - \frac{1}{x^{L_{p+1}}}\right) \tag{B.3}$$

$$\psi_S(x) = \psi_U(x) + (-)^{p-1}\left[\frac{1}{x^{L_2+\ldots+L_{p-1}}} - \frac{2}{x^{L_2+\ldots+L_p}} + \frac{1}{x^{L_2+\ldots+L_p+L_{p+1}}}\right]. \tag{B.4}$$

From the recurrence hypothesis

$$\psi_T(x_1),\ \psi_U(x_1) > 0$$
$$\psi_T(x'_1),\ \psi_U(x'_1) < 0$$

a) *p odd*

from (B.3) $\qquad \psi_S(x'_1) < \psi_T(x'_1) < 0$

from (B.4) $\qquad \psi_S(x) > \psi_U(x) + \frac{1}{x^L} - \frac{2}{x^{L+1}} + \frac{1}{x^{L+L_1+1}}$

as $\psi_S(x)$ is maximum for $L_p = 1$ and $L_{p+1} = L_1$.

The last inequality then gives $\psi_S(x_1) > \psi_U(x_1) > 0$.

b) *p even*

from (B.3) $\qquad \psi_S(x_1) > \psi_T(x_1) > 0$

from (B.4) $\qquad \psi_S(x) < \psi_U(x) - \frac{1}{x^L} + \frac{2}{x^{L+L_1}} - \frac{1}{x^{L+L_1+1}}$

as $\psi_S(x)$ is minimum for $L_p = L_1$, $L_{p+1} = 1$.

Whence

$$\psi_S(x'_1) < \psi_U(x'_1) - \frac{\left[x_1'^{L_1+1} - 2x'_1 + 1\right]}{x_1'^{L+L_1+1}} \leq \psi_U(x'_1) < 0$$

*ii*) if $L_1 = 2$ and $S > \tilde{S}$

either $S \equiv 1(1-1)^{n_1}\,1$ or $S \equiv 1(1-1)^{n_1}1\ 1-1$ and the proof is direct either

$$S \equiv 1(1-1)^{n_1}1\ 1-1\ \ldots,$$

then

$$\mathrm{T} \equiv 1(1 \ -1)^{n_1}1 \ 1 - 1 \ \ldots \qquad \text{and} \qquad \mathrm{T} > \tilde{\mathrm{S}}$$
$$\mathrm{U} \equiv 1(1 \ -1)^{n_1}1 \ 1 \ \ldots\ldots \qquad \text{and} \qquad \mathrm{U} > \tilde{\mathrm{S}}$$

then $\psi_{\mathrm{T}}(\sqrt{2}) < 0$, $\psi_{\mathrm{U}}(\sqrt{2}) < 0$ by induction.

Using (B.3) and (B.4) we get the announced result.

*Lemma 2.* — For $p \geq 1$, $\psi_{\mathrm{S}}(x)$ is increasing for $x > \sup(\sqrt{2}, x_1')$.

*Corollary.* — $x(\mathrm{S})$ is the unique real root greater than $x_1'$ and

$$x_1' < x(\mathrm{S}) < x_1.$$

Though more complicated the proof is similar to that of Lemma 1. We indicate briefly the first steps of the proof and develop only the last part which is more delicate.

*i)* For $p = 1$

$$\psi_{\mathrm{S}}'(x) > 1.2 \qquad \text{if} \qquad \mathrm{L}_1 \geq 3 \qquad \text{and} \qquad x > x_1'$$
$$\psi_{\mathrm{S}}'(x) > 0.843 \qquad \text{if} \qquad \mathrm{L}_1 = 2 \qquad \text{and} \qquad x > \sqrt{2}$$

$$\left[\text{we study the variation of } \psi_{\mathrm{S}}'(x) = (\mathrm{L}_1 + 1)x^{\mathrm{L}_1} - 2\mathrm{L}_1 x^{\mathrm{L}_1 - 1} + \frac{\mathrm{L}_2}{x^{\mathrm{L}_2+1}}\right]$$

*ii)* For $p = 2$, $\psi_{\mathrm{S}}'(x) > 0$ for $x > \sup(\sqrt{2}, x_1')$.

$$\left[\text{From } \psi_{\mathrm{S}}'(x) = (\mathrm{L}_1 + 1)x^{\mathrm{L}_1} - 2\mathrm{L}_1 x^{\mathrm{L}_1 - 1} + \frac{2\mathrm{L}_2}{x^{\mathrm{L}_2+1}} - \frac{(\mathrm{L}_2 + \mathrm{L}_3)}{x^{\mathrm{L}_2+\mathrm{L}_3+1}}\right]$$

*iii)* If $p$ is even ($p = 4, 6, 8, \ldots$). $\psi_{\mathrm{S}}'(x) > 0$ whenever $\psi_{\mathrm{T}}'(x) > 0$. We have

$$\psi_{\mathrm{S}}'(x) = \psi_{\mathrm{T}}'(x) + \left[\frac{(\mathrm{L}_2 + \ldots + \mathrm{L}_p)}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{p+1}}} - \frac{(\mathrm{L}_2 + \ldots + \mathrm{L}_p + \mathrm{L}_{p+1})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_p+\mathrm{L}_{p+1}+1}}\right]$$

as $\mathrm{L}_2 + \ldots + \mathrm{L}_p \geq (p - 1) \geq 3$, the term in brackets is positive.

Then it is enough to prove the Lemma for $p$ odd: $p = 2k + 1$.

*iv)* $p = 2k + 1$, $k \geqslant 1$. Let $\mathrm{S}_m \equiv \mathrm{L}_1\mathrm{L}_2$. We already know that

$$\psi_{\mathrm{S}_m}'(x) > 1.2 \qquad \text{for} \qquad x > x_1', \qquad \text{and} \qquad \mathrm{L}_1 \geq 3$$
$$> 0.843 \qquad \text{for} \qquad x > \sqrt{2} \qquad \text{and} \qquad \mathrm{L}_1 = 2 \text{ (part } i).$$

We have

$$\psi_{\mathrm{S}}'(x) = \psi_{\mathrm{S}_1}'(x) + \left[\frac{(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k}+1}} - \frac{2(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k+1})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k+1}+1}} + \frac{(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k+2})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k+2}+1}}\right] \tag{B.5}$$

which is minimum for $\mathrm{L}_{2k+2} = \mathrm{L}_1$.

*iv.*1) $\mathrm{L}_1 \geq 3$ we neglect the last term in the brackets

$$\psi_{\mathrm{S}}'(x) > \psi_{\mathrm{S}_1}'(x) + \frac{(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k}+1}} - \frac{2(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k+1})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k+1}+1}}$$
$$> \psi_{\mathrm{S}_1}'(x) + \frac{(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k})}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k}+1}} - \frac{2(\mathrm{L}_2 + \ldots + \mathrm{L}_{2k} + 1)}{x^{\mathrm{L}_2+\ldots+\mathrm{L}_{2k}+2}}$$

as the minimum is for $\mathrm{L}_{2k+1} = 1$.

Now let $L = L_2 + \dots + L_{2k} \geq (2k - 1)$. When $x > x'_1$

$$\frac{L}{x^{L+1}} - \frac{2(L+1)}{x^{L+2}}$$

is minimal if L is minimum, i. e. $L = 2k - 1$ and we have the minoration

$$\psi'_S(x) > \psi'_{S_1}(x) + \frac{(2k-1)}{x^{2k}} - \frac{4k}{x^{2k+1}} \qquad \text{if} \qquad x > x'_1.$$

Then doing the same for $\psi'_{S_1}(x)$ we get

$$\begin{aligned} \psi'_S(x) &> \psi'_{S_m}(x) + \left[\frac{2k-1}{x^{2k}} - \frac{4k}{x^{2k+1}}\right] + \dots + \left[\frac{1}{x^2} - \frac{4}{x^3}\right] \\ &> \psi'_{S_m}(x) + \sum_{k=1}^{\infty}\left(\frac{2k-1}{x^{2k}} - \frac{4k}{x^{2k+1}}\right) \\ &= \psi'_{S_m}(x) + \frac{x^2 - 4x + 1}{(x^2-1)^2} \\ &> \inf \psi'_{S_m}(x) + \inf \frac{x^2 - 4x + 1}{(x^2-1)^2} > 1.2 + \frac{9 - 5\sqrt{5}}{2} > 0 \end{aligned}$$

*iv*.2) $L_1 = 2$ and $L_2 = 1$, $L_3 = 1,2$.

We keep the last term in (B.5); the minimum is for $L_{2k+2} = 2$; let again $L = L_2 + \dots + L_{2k}, 2k - 1 \leq L < 2(2k-1)$, for $L_{2k+2} = 2$, the minimum in $L_{2k+1}$ is got

* either for $L_{2k+1} = 2$ when $L = 1$, i. e. S is the sequence $S_M \equiv 2\,122$ and we prove directly in that case that $\psi'_{S_M}(x) > 0.882$ for $x > \sqrt{2}$

* either for $L_{2k+1} = 1$ when $L > 1$. Then

$$\psi_S(x) > \psi'_{S_1}(x) + \frac{L}{x^{L+4}}(x-1)(x^2 - x - 1) + \frac{3 - 2x^2}{x^{L+4}}$$

For $x > x_1\left(= \frac{1+\sqrt{5}}{2}\right)$ we drop the middle term which is positive and replace L by its smallest value $L = 2k - 1$. Then,

$$\begin{aligned} \psi'_S(x) &> \psi'_{S_1}(x) + \frac{3 - 2x^2}{x^{2k+3}} > \psi'_{S_M}(x) + \frac{(3-2x^2)}{x^5}\sum_{k=0}^{\infty}\frac{1}{x^{2k}} \\ &> \inf \psi'_{S_M}(x) + \inf \frac{(3-2x^2)}{x^3(x^2-1)} > 0 \end{aligned}$$

For $\sqrt{2} < x < x_1$ we keep the whole expression and give a minoration for $L = 2k - 1$ again. The proof is similar to the previous one. ■

*Lemma 3.* — Let

$$\begin{aligned} S &\equiv L_1 \dots L_p L_{p+1} \\ S &\equiv L_1 \dots L_p L'_{p+1} \qquad 1 \leq L'_{p+1} \leq L_{p+1} < L_1 \\ & \qquad\qquad\qquad\qquad 1 \leq L_2 < L_1 \end{aligned}$$

and

$$\begin{aligned} T &\equiv L_1 \dots L_{p-1} L_p \\ V &\equiv L_1 \dots L_{p-1}(L_p - 1) \end{aligned}$$

if $p$ is odd (resp. even)

$$V < S < S' < T \Rightarrow x(V) < x(S) < x(S') < x(T)$$
$$(\text{resp. } V > S > S' > T \Rightarrow x(V) > x(S) > x(S') > x(T))$$

*Corollary.* — If S and S′ are sequences satisfying condition (3.10) then

$$S < S' \Rightarrow x(S) < x(S').$$

*** If $p$ is odd we have the inequalities

$$\varphi_V(x) > \varphi_S(x) > \varphi_{S'}(x) > \varphi_T(x)$$

(the inequalities are reversed when $p$ is even).

The first inequality comes from the identity

$$\varphi_S(x) = \varphi_V(x) - \frac{1}{x^\nu}\left[1 - \frac{1}{x} \cdots - \frac{1}{x^{L_{p+1}}}\right]$$

the term into brackets being positive for $L_{p+1} < L_1$.

From Lemma, 1 $x(V)$, $x(S)$, $x(S')$, $x(T)$ exist.

We have $\varphi_S(x)\,|_{x=x(V)} < 0$ whence $x(S) > x(V)$

$$\varphi_{S'}(x)\,|_{x=x(S)} < 0 \quad \text{whence} \quad x(S') > x(S)$$
$$\varphi_T(x)\,|_{x=x(S')} < 0 \quad \text{whence} \quad x(T) > x(S')$$

i. e. $V < S < S' < T \Rightarrow x(V) < x(S) < x(S') < x(T)$. ■

## REFERENCES


[*1*] N. METROPOLIS, M. L. STEIN and P. R. STEIN, *J. of Combinatorial theory*, t. A **15**, 1973, p. 25.
See also J. GUCKENHEIMER, *Inventiones Math.*, t. **39**, 1977, p. 165.

[*2*] A. RENYI, *Acta Math. Acad. Sci. Hung.*, t. **8**, 1957, p. 477.

[*3*] W. PARRY, *Acta Math. Acad. Sci. Hung,*, t. **11**, 1960, p. 401.

[*4*] A. N. ŠARKOVSKII, *Ukrainian Math. J.*, t. **16**, n° 1, 1964, p. 61.

[*5*] P. ŠTEFAN, *Comm. Math. Phys.*, t. **54**, 1977, p. 237.

[*6*] M. Y. COSNARD, A. EBERHARD, *Sur les cycles d'une application continue de la variable réelle*, Séminaire analyse numérique n° 274 Lab. Math. Appl., 1977. Université Scientifique et Mathématique de Grenoble, U. S. M.G.

[*7*] T. Y. LI and J. A. YORKE, *A. M. M.*, t. **82**, 10, 1975, p. 985.

[*8*] B. DERRIDA, Y. POMEAU, *in preparation.*

[*9*] J. MILNOR, W. THURSTON, « The kneading matrix ». *Preprint*, IHES (1977).

[*10*] B. DERRIDA, A. GERVOIS, Y. POMEAU, *in preparation.*

[*11*] R. L. ADLER, A. G. KONHEIM and M. H. MC ANDREW, *Trans. Amer. Math. Soc.*, t. **114**, 1965, p. 309.

[*12*] P. ŠTEFAN, *Private communication.*

[*13*] A. LASOTA, J. A. YORKE, *Trans. Amer. Math. Soc.*, t. **186**, 1973, p. 481.


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