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Cauchy data on a manifold

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INTRODUCTION

The Cauchy problem in General Relativity has been studied from the global in space point of view by Fischer and Marsden in [*5*], and by Hughes, Kato and Marsden [*8*] in the case that the space manifold is $\mathbb{R}^3$. These last authors also obtained results with weaker differentiability assumptions than in previous work ([*2*], [*3*], [*5*], [*7*]). In the present paper we study this problem for a general class of space manifolds. Our approach is based on a new explicit expression of the Hawking-Ellis reduced equations, and the introduction of modified Sobolev spaces. These spaces, which we call E-spaces, are a suitable framework for the study of field equations involving long range fields. Our method allows us not to make any hypothesis on the behaviour of the metric g at spatial infinity stronger than that of boundedness: only the derivatives (of order $0 < s \leqslant 3$) are supposed to be square integrable on the spacelike slices. Our proof of the existence and uniqueness theorems uses directly the energy estimates instead of relying on the theory of non-linear semigroups as in [*8*], without loosing generality concerning the differentiability requirements. The estimation in time of some bounds given here will be used in forthcoming papers to prove existence theorems with appropriate data at $t = -\infty$.

1. DEFINITIONS

Let M be a C^∞, n dimensional paracompact manifold, without boundary. In order to have a scale for tensor fields on M, and a volume element, we endow M with a C^∞ positive definite metric e. In the case where M is non compact we restrict (M, e) by the following hypothesis (1).

1. The radius of injectivity of the exponential mapping corresponding to e is, on M, bounded away from zero.
2. There exist a compact subset K of M outside of which e is flat (that is, has zero curvature).

Hypothesis 1 implies that M is a complete riemannian manifold. We recall the definition of *Sobolev spaces on (*M, *e)* and some of their properties.

Let u be a tensor field defined almost everywhere on M. We denote by $| u |$ its e norm, function defined almost everywhere on M. We say that $u \in L^p(M)$ if $| u |$ is p-integrable in the measure associated with e and set

$$\| u \|_{L^p(M)} = \int_M | u |^p d\mu(e)$$

We denote by $\overline{D}$ the operator of (generalized) (2) covariant differentiation in the metric e. The tensor field u is said to belong to $W_s^p(M)$ if its covariant derivatives of order $\leq s$ are p-integrable. We set

$$\| u \|_{W_s^p(M)} = \left\{ \int_M \sum_{0 \leq |i| \leq s} | \overline{D}^i u |^p d\mu(e) \right\}^{1/p}$$

The tensor fields of a given type which belong to $W_s^p(M)$ form a Banach space, which we will simply denote by $W_s^p(M)$ when no confusion can arise. If $u \in W_s^p(M)$ any of its contractions with the e-metric has the same property. Under the hypothesis made on (M, e) the following continuous embedding theorems are true (3), for an n-dimensional manifold M

$$\text{(1)} \qquad W_s^p(M) \subset L^q(M) \quad , \quad 1 < p \leq q \leq \frac{np}{n - sp} \quad , \quad s \leq \frac{n}{p}$$

$$\text{(2)} \qquad W_s^p(M) \subset C_b^0(M) \quad , \quad s > \frac{n}{p} \quad , \quad \| u \|_{C_b^0(M)} = \operatorname{Sup}_{x \in M} | u(x) |$$

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(1) These hypothesis are sufficient to imply the Sobolev imbedding theorems, and the density of $\mathcal{D} \equiv C_0^\infty$ in H_s. Hypothesis 2 is not necessary (boundedness conditions on the curvature and its derivatives would be sufficient, cf. [*1*]) but it will simplify the proof of the existence theorem later.

(2) *i. e.* in the sense of distributions, cf. Lichnerowicz [*10*].

(3) C_b^0 continuous and e-bounded tensor fields.

And the density theorem [4]:

(3) $C_0^\infty(M)$ is dense in $W_s^p(M)$

We list the simple continuous inclusion and multiplication properties (consequences of the preceding ones) that we will use in the sequel, for $n = 3$, $p = 2$ (with $H_s = W_s^2$)

(5) $H_2(M) \subset C_b^0(M) \quad , \quad H_1(M) \subset L^p(M) \quad , \quad 2 \leqslant p \leqslant 6.$

(4) $H_2(M) \times H_2(M) \to H_2(M) \quad , \quad H_1(M) \times H_1(M) \to L_2(M),$

by $(u, v) \to u \otimes v$

We consider now the product M × I, with I an interval of $\mathbb{R}$, which we endow with the positive definite metric $e \otimes 1$. We denote by D the generalized covariant derivative of a tensor field h (distribution on M × I) in the metric $e \otimes 1$ (note that $Du = (\partial u/\partial t, \bar{D}u)$.

We introduce the following spaces (we always mean by tensor fields, « tensor fields of some given type »).

DEFINITION 1. — $\tilde{E}_s$ is the space of tensor fields h on M × I such that: (*i*) the restriction h_t of h as well as the restriction $(D^\alpha h)_t$ of its derivatives of any order $|\alpha| \leqslant s$, to each $M_t = M \times \{ t \}$ is almost everywhere defined and square integrable in the metric e. We set

$$\| h \|_s^{M_t} = \left\{ \int_{M_t} \sum_{|\alpha| \leqslant s} | D^\alpha h(x, t) |^2 d\mu(e) \right\}^{1/2}$$

(*ii*) The mapping $I \to \mathbb{R}$ by $t \mapsto \| h \|_{s-1}^{M_t}$ is continuous and bounded, while $I \to \mathbb{R}$ by $t \mapsto \| h \|_s^{M_t}$ is measurable and essentially bounded.

$\tilde{E}_s$ endowed with the norm

$$\| h \|_{\tilde{E}_t} = \operatorname*{Ess\,Sup}_{t \in I} \| h \|_s^{M_t}$$

is a Banach space.

DEFINITION 2. — E_s is the space of continuous and bounded tensor fields on M × I such that $Df \in \tilde{E}_{s-1}$. We endow E_s with the norm

$$\| f \|_{E_s} = \operatorname{Sup} (\| f \|_{C_b^0}, \| Df \|_{\tilde{E}_{s-1}})$$

E_s is a Banach space.

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[4] C_0^∞ infinitely differentiable tensor fields with compact support.

2. EMBEDDING THEOREMS AND MULTIPLICATION PROPERTIES

LEMMA 1. — If I is a bounded interval of $\mathbb{R}$, and $s > n/2 + 1$, E_s is identical to the space of tensor fields f on $M \times I$ with derivative $Df \in \tilde{E}_{s-1}$, which have a continuous and bounded restriction to some $t_0 \in I$.

Proof. — $Df \in \tilde{E}_{s-1}$ implies *a fortiori* that for each t the restriction $(Df)_t$ belongs to $H_{s-1}(M)$, thus $(Df)_t \in C_b^0(M)$ if $s > n/2 + 1$ and, moreover, the injection $H_{s-1}(M) \to C_b^0(M)$ is continuous; we have therefore

$$\| (Df)_t \|_{C_b^0(M)} \leqslant C(M, e) \| (Df)_t \|_{s-1}^{M_t} \quad , \quad s > \frac{n}{2} + 1,$$

the mapping $t \mapsto \| (Df)_t \|_{C_b^0(M)}$ is measurable and bounded, that is

$$Df \in L^\infty(I, C_b^0(M))$$

If $f_{t_0} \in C_b^0(M)$ we thus have also $f \in C_b^0(M \times I)$ for any bounded interval I, since f is given by

$$f_t = f_{t_0} + \int_{t_0}^{t} \frac{\partial f}{\partial t} dt$$

thus

$$\|f\|_{C_b^0(M\times I)} \leqslant \|f_{t_0}\|_{C_b^0(M)} + l(I)C(M, e) \| Df \|_{\tilde{E}_{s-1}}$$

Note that we also have for all $t \in I$:

$$\|f_t - f_{t_0}\|_{L^2(M)} \leqslant l(I) \| Df \|_{\tilde{E}_0}$$

The following lemma is immediate:

LEMMA 2. — If $s > \frac{n}{2} + 1$, E_s is an algebra.

The lemma that we shall use in the sequel is the following:

LEMMA 3. — If $s > \frac{n}{2} + 1$ we have the continuous multiplication property

$$E_s \times \tilde{E}_{s-1} \times \tilde{E}_{s-1} \to \tilde{E}_{s-1},$$

by $(u, v, w) \mapsto u \otimes v \otimes w$

PROOF. — If $u \in E_s$, $v \in \tilde{E}_{s-1}$, $w \in \tilde{E}_{s-1}$, $s > \frac{n}{2} + 1$ then $v_t \in H_{s-1}(M) \subset C_b^0(M)$, $w_t \in H_{s-1}(M) \subset C_b^0(M)$ while $\left(\frac{\partial^m v}{\partial t^m}\right)_t \in H_{s-m-1}(M)$, $\left(\frac{\partial^m w}{\partial t^m}\right)_t \in H_{s-m-1}(M)$, if $m \leqslant s - 1$, thus, if $s > \frac{n}{2} + 1$, $(v \otimes w)_t \in H_{s-1}(M) \subset C_b^0(M)$, and $u \otimes v \otimes w \in C_b^0(M)$ the proof relies now on the multiplication properties of the spaces H_σ recalled above in the particular case $s = 3$ and the Leibniz formula for $D^\alpha(u \otimes v \otimes w)$.

3. THE REDUCED EINSTEIN EQUATIONS

M is now 3-dimensional.

We use the « brackground » metric $e \times 1$ on $M \times I$, to write globally the Einstein equations on $M \times I$, for the contravariant form g of a lorentzian metric, as a hyperbolic system, under the covariant gauge condition (5)

$$F \equiv g.(\Gamma(g) - \Gamma(e \times 1)) = 0$$
$$(\text{i. e. } F^\lambda = g^{\alpha\beta}(\Gamma^\lambda_{\alpha\beta}(g) - \Gamma^\lambda_{\alpha\beta}(e \times 1))$$

$(\Gamma(g) - \Gamma(e \times 1)$ is the 3-tensor difference of the connexions of g and e). The equivalence of this hyperbolic system, for initial data satisfying the constraints, with the original Einstein equations will be proved along standard lines. It is called e-reduced Einstein equations.

THEOREM. — The e-reduced Einstein equations (in empty space) read:

$$\frac{1}{2} g.D^2 g + P(g)(Dg, Dg) + g.g \text{ Riem}(e \times 1) = 0$$

that is, in coordinates

$$\frac{1}{2} g^{\alpha\beta} D_\alpha D_\beta g^{\lambda\mu} + P^{\lambda\mu,\rho\sigma}_{\alpha\beta,\gamma\delta} D_\rho g^{\alpha\beta} D_\sigma g^{\gamma\delta} + \frac{1}{2} g^{\alpha\beta} g^{\nu(\mu} \overline{R}^{\lambda)}{}_{\alpha,\nu\beta} = 0$$

$P^{\lambda\mu,\rho\sigma}_{\alpha\beta,\gamma\delta}$ is a polynomial in $g^{\nu\varepsilon}$ and $g_{\nu\varepsilon}$, that is a rational function in $g^{\nu\varepsilon}$ with denominator a power of det $g^{\nu\varepsilon}$, while $\overline{R}_{\alpha\nu,\beta}{}^\lambda$ is the Riemann tensor of the metric $e \times 1$.

Proof. — This expression can be deduced from the classical formulas in local coordinates as follows:

$$R^{\lambda\mu} \equiv R^{\lambda\mu}_{(h)} + L^{\lambda\mu}$$

with

$$R^{\lambda\mu}_{(h)} = \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g^{\lambda\mu}}{\partial x^\alpha \partial x^\beta} + H^{\lambda\mu}$$

(5) This condition, introduced by Hawking and Ellis [7] is equivalent to the introduction of harmonic coordinates used previously (cf. [2]), when e is the euclidean metric.

where

$$H^{\lambda\mu} = P^{\lambda\mu,\rho\sigma}_{\alpha\beta,\gamma\delta}\partial_\rho g^{\alpha\beta}\partial_\sigma g^{\gamma\delta}$$

with $P^{\lambda\mu,\rho\sigma}_{\alpha\beta,\gamma\delta}$ a polynomial in $g_{\nu\tau}$, $g^{\nu\tau}$, and

$$L^{\lambda\mu} \equiv \frac{1}{2}\left(g^{\mu\alpha}\frac{\partial\Gamma^\lambda}{\partial x^\alpha} + g^{\lambda\alpha}\frac{\partial\Gamma^\mu}{\partial x^\alpha}\right)$$

with

$$\Gamma^\alpha = g^{\lambda\mu}\Gamma^\alpha_{\lambda\mu}$$

We now set:

$$F^\alpha \equiv \Gamma^\alpha - g^{\lambda\mu}\overline{\Gamma}^\alpha_{\lambda\mu}$$

and define the « e-reduced » Ricci tensor $R^{\lambda\mu}_{(e)}$ by:

$$R^{\lambda\mu}_{(e)} = R^{\lambda\mu} - \frac{1}{2}(g^{\mu\alpha}D_\alpha F^\lambda + g^{\lambda\alpha}D_\alpha F^\mu)$$

The tensor $R^{\lambda\mu}_{(e)}$ can be written:

$$R^{\lambda\mu}_{(e)} = \frac{1}{2}g^{\alpha\beta}D_\alpha D_\beta g^{\lambda\mu} + f^{\lambda\mu}$$

where $f^{\lambda\mu}$ does not depend on the second derivatives of g.

To compute the explicit form of $f^{\lambda\mu}$ we choose coordinates such that, at a point x, the Christoffel symbols of e vanish: $\overline{\Gamma}^\alpha_{\lambda\mu} = 0$ (this is no restriction). At such a point $\partial_\rho g^{\alpha\beta} = D_\rho g^{\alpha\beta}$ and $D_\alpha D_\beta g^{\mu\nu} = \partial^2_{\alpha\beta}g^{\mu\nu} + g^{\lambda(\mu}\partial_\alpha\overline{\Gamma}^{\nu)}_{\beta\lambda}$, while $\partial_\alpha\Gamma^\mu = \partial_\alpha F^\mu + g^{\rho\sigma}\delta_\alpha\overline{\Gamma}^\mu_{\rho\sigma}$. In such coordinates, at the point x:

$$\begin{aligned} R^{\mu\nu} = {} & \frac{1}{2}g^{\alpha\beta}D_\alpha D_\beta g^{\mu\nu} - \frac{1}{2}g^{\alpha\beta}g^{\lambda(\mu}\partial_\alpha\overline{\Gamma}^{\nu)}_{\beta\lambda} \\ & + P^{\mu\nu,\rho\sigma}_{\alpha\beta,\gamma\delta}(g^{\lambda\tau}, g_{\lambda\tau})D_\rho g^{\alpha\beta}D_\sigma g^{\gamma\delta} \\ & + \frac{1}{2}g^{\alpha(\mu}\nabla_\alpha F^{\nu)} + \frac{1}{2}g^{\alpha(\mu}g^{\rho\sigma}\partial_\alpha\overline{\Gamma}^{\nu)}_{\rho\sigma} \end{aligned}$$

We remark that, at the point x where $\overline{\Gamma}^\lambda_{\alpha\beta} = 0$:

$$g^{\alpha\beta}g^{\lambda\mu}\partial_\alpha\overline{\Gamma}^\nu_{\beta\lambda} + g^{\alpha\mu}g^{\rho\sigma}\partial_\alpha\overline{\Gamma}^\nu_{\rho\sigma} = g^{\lambda\mu}g^{\alpha\beta}\overline{R}^\nu{}_{\alpha,\lambda\beta}$$

(with $\overline{R}_{\alpha\lambda,\beta}{}^\nu$ the curvature tensor of e). Thus we have, identically now

$$f^{\mu\nu} = P^{\mu\nu,\rho\sigma}_{\alpha\beta,\gamma\delta}(g^{\lambda\tau}, g_{\mu\tau})D_\rho g^{\alpha\beta}D_\sigma g^{\gamma\delta} + \frac{1}{2}g^{\alpha\beta}g^{\lambda(\mu}\overline{R}^\nu{}_{\alpha,\lambda\beta}$$

4. DEFINITION OF A NON LINEAR MAPPING $E_s \to E_s$

We associate with the reduced Einstein equations the linear system obtained by replacing in all terms except the second derivatives, g by a given tensor field γ. That is:

$$\frac{1}{2}\gamma^{\lambda\mu}D_\lambda D_\mu g^{\alpha\beta} + f^{\alpha\beta}(\gamma^{\lambda\mu}, \gamma_{\rho\sigma}, D_\rho\gamma^{\lambda\mu}) = 0 \tag{4-1}$$

LEMMA. — If $\gamma \in E_s(M \times I)$ is a non degenerate metric and if $s \geqslant 3$, the tensor $f \equiv (f^{\alpha\beta})$ is in $\tilde{E}_{s-1}$.

Proof. — It is a consequence of the expression of $f^{\alpha\beta}$, of the lemmas 2 and 3 § 2 and the hypothesis that $\overline{R}^{\nu}{}_{\alpha,\lambda\beta}$ vanishes outside a compact set [6].

DEFINITION. — We say that M_t is uniformly space-like for γ if there exist strictly positive numbers a_0, A_0, a_1, A_1 such that on M_t.

$$A_0 > \gamma_t^{\perp\perp} > a_0 \tag{4-2 a}$$

$$A_1 e(X, X) > -\gamma_t^{||}(X, X) > a_1 e(X, X) \qquad \text{for all vectors X.} \tag{4-2 b}$$

Where $\gamma_t^{\perp\perp}$ and $\gamma_t^{||}$ denote the perp-perp and parallel projections of γ_t on M (notations of Kuchar), i. e. γ^{00} and γ^{ij}. We note $a = (a_0, a_1)$, $A = (A_0, A_1)$.

DEFINITION. — We say that γ is regularly hyperbolic on $M \times I$ if M_t is uniformly space-like for $t \in I$ and if the numbers a_0, A_0, a_1, A_1 do not depend on t.

We shall prove existence and uniqueness for the solution of the Cauchy problem for linear equations of the type 4-1, in the space E_s, by using a refinement of the Leray-Gårding-Dionne energy estimates [7] and written here on $M \times I$.

5. FUNDAMENTAL ENERGY ESTIMATE

Let $u = (u_{\alpha_1 \dots \alpha_p}{}^{\beta_1 \dots \beta_q})$ be a tensor field on $M \times I$. We consider the linear equation:

$$\gamma . D^2 u + \alpha . Du + \beta . u = v \tag{5-1}$$

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(6) Such a strong hypothesis is not required at this stage, but will simplify proofs later.

(7) In the second order case see also earlier work of Sobolev [11].

that is, in coordinates

$$\gamma^{\lambda\mu}D_\lambda D_\mu u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p} + \alpha^{\rho\beta_1\ldots\beta_q,\mu_1\ldots\mu_p}_{\alpha_1\ldots\alpha_p,\lambda_1\ldots\lambda_q}D_\rho u^{\lambda_1\ldots\lambda_q}_{\mu_1\ldots\mu_p} + \beta^{\beta_1\ldots\beta_q,\mu_1\ldots\mu_p}_{\alpha_1\ldots\alpha_p,\lambda_1\ldots q_p}u^{\lambda_1\ldots\lambda_q}_{\mu_1\ldots\mu_p} = v^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}$$

(thus α, β, and v are given tensor fields on $M \times I$, with the indicated variance).

LEMMA 5.1. — *Hypothesis:* (1) $\gamma \in C^0_b(M \times I)$, $D\gamma \in L^\infty(M \times I)$, and γ is regularly hyperbolic (inequalities (4-2))

(2) $\alpha \in L^\infty(M \times I)$

(3 *a*) $\beta \in L^\infty(M \times I)$, (3 *b*) $\beta \in \tilde{E}_0(M \times I)$

(4) $v \in \tilde{E}_0$

(5 *a*) $u \in \tilde{E}_2(M \times I)$, (5 *b*) $u \in E_2(M \times I)$

Conclusions:

Under the hypothesis (1), (2), (3 *a*), (4), (5 *a*) (case I) or (1), (2), (3 *b*), (4) (5 *b*) (case II), a tensor field u satisfying 5-1 satisfies the fundamental energy estimates written below (5-3 *a*, 5-3 *b*).

Proof. — In both cases we have identically (almost everywhere for each t)

(5-2)
$$\partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}\gamma^{\lambda\mu}D_\lambda D_\mu u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$$
$$\equiv \frac{1}{2}\partial_t\left\{\gamma^{00}\partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}\partial_t u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \gamma^{ij}D_i u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}D_j u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\right\}$$
$$- \frac{1}{2}\partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}\partial_t u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_t\gamma^{00} + \frac{1}{2}D_i u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}D_j u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_t\gamma^{ij}$$
$$- \partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}\partial_t u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_u}D_i\gamma^{0i} - \partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}D_j u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}D_i\gamma^{ij}$$
$$+ D_i\left\{\gamma^{0i}\partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}\partial_t u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \gamma^{ij}\partial_t u^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p}D_j u^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\right\}$$

Thus since $C^\infty_0(M)$ is dense in $H_1(M)$, and $(Du)_t \in H_1(M)$, by integration:

$$\int_{M_t}\{\gamma^{00}|\partial_t u|^2 - \gamma^{ij}D_i u.D_j u\}\,d\mu(e) = \int_{M_0}\{\gamma^{00}|\partial_t u|^2 - \gamma^{ij}D_i u.D_j u\}\,d\mu(e)$$
$$+ \int_0^t\int_{M_\tau}\{\partial_t\gamma^{00}|\partial_t u|^2 - \partial_t\gamma^{ij}D_i u.D_j u + 2D_i\gamma^{0i}|\partial_t u|^2 + 2D_i\gamma^{ij}D_j u.\partial_t u\}\,d\mu(e)d\tau$$
$$-2\int_0^t\int_{M_\tau}\{\partial_t u.(\alpha.Du) + \partial_t u.(\beta.u)\}\,d\mu(e)d\tau + 2\int_0^t\int_{M_\tau}\{\partial_t u.v\}\,d\mu(e)d\tau$$

We note that, in both cases

$$\| u_t - u_0 \|^2_{L^2(M)} \leqslant t\int_0^t (\| Du \|^{M_\tau}_0)^2 d\tau$$

([8]) Recall that indices are raised and lowered in the metric e.

Thus in case I

$$\left(\| u \|_0^{M_t}\right)^2 \leqslant 2\left(\| u \|_0^{M_0}\right)^2 + 2t\int_0^t \left(\| Du \|_0^{M_\tau}\right)^2 d\tau$$

which leads to the familiar inequality:

$$(5\text{-}3\,a)\quad \left(\| u \|_1^{M_t}\right)^2 \leqslant C_0\left(\| u \|_1^{M_0}\right)^2 + \int_0^t C_1(\tau)\left(\| u \|_1^{M_\tau}\right)^2 d\tau + \int_0^t C_2(\tau) \| u \|_1^{M_\tau} d\tau$$

with $C_1(\tau) = C_1\left(\| D\gamma \|_{C_b^0}^{M_0} + \| \alpha \|_{C_b^0}^{M_\tau} + \| \beta \|_{C_b^0}^{M_\tau}\right) + 2t$

$$C_2(\tau) = C_2 \| v \|_0^{M_\tau}$$

C_0, C_1, C_2 are constants depending only on M, e, a, A.
The last integral can be bounded by:

$$\int_0^t C_2(\tau) \| u \|_1^{M_\tau} d\tau < \frac{1}{2}\int_0^t (C_2(\tau))^2 d\tau + \frac{1}{2}\int_0^t \left(\| u \|_1^{M_\tau}\right)^2 d\tau$$

In case II we write the fundamental energy inequality:

$$(5\text{-}3\,b)\quad \left(\| Du \|_0^{M_t}\right)^2 \leqslant C_0\left(\| Du \|_0^{M_0}\right)^2 + \int_0^t C_1(\tau)\left\{\left(\| Du \|_0^{M_\tau}\right)^2 + \left(\| u \|_{C_b^0}^{M_\tau}\right)^2\right\} d\tau + \int_0^t C_2(\tau) d\tau$$

wich $C_1(\tau) = C_1\left(\| \alpha \|_{C_b^0}^{M_\tau} + \| D\gamma \|_{C_b^0}^{M_\tau} + \| \beta \|_0^{M_\tau} + 1\right)$

and

$$C_2(\tau) = C_2\left(\| v \|_0^{M_\tau}\right)^2$$

C_0, C_1, C_2 depending only on M, e, a, A.
From (5-1) we deduce

UNIQUENESS THEOREM. — Under the hypothesis (1), (2), (3 a), (4) the linear equation (5-1) has at most one solution $u \in E_2$, with prescribed Cauchy data on M_0.

Proof. — If u and w are two such solutions their difference $u - w$ is in $\tilde{E}_2$ (it has Cauchy data zero); the energy estimate 5-3 a gives the conclusion.

6. SECOND ENERGY ESTIMATE

If u is a tensor field satisfying 5-1 its covariant derivative $u' = Du$ satisfies (if the considered products are defined) the equation:

$$(6\text{-}1)\qquad \gamma . D^2 u' + \alpha' . Du' + \beta' . u' = v'$$

with

$$(6\text{-}2)\qquad \begin{aligned} \alpha' &= \delta \times \alpha + \mathrm{D}\gamma \\ \beta' &= \mathrm{D}\alpha + \beta + \gamma.\Sigma\mathrm{D}\ \mathrm{Riem}\,(e \times 1) \\ v' &= \mathrm{D}v - \mathrm{D}\beta.u + \gamma.\Sigma\mathrm{D}\ \mathrm{Riem}\,(e \times 1).u \end{aligned}$$

(in coordinates $u' = \left(\mathrm{D}_\sigma u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}\right)$, and 6-1 reads :

$$\begin{aligned} &\gamma^{\lambda\mu}\mathrm{D}_\lambda\mathrm{D}_\mu\mathrm{D}_\sigma u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} + \alpha^{\rho\beta_1\dots\beta_q,\mu_1\dots\mu_p}{}_{\alpha_1\dots\alpha_p,\lambda_1\dots\lambda_q}\mathrm{D}_\sigma\mathrm{D}_\rho u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} \\ &\quad + \mathrm{D}_\sigma\gamma^{\lambda\mu}\mathrm{D}_\lambda\mathrm{D}_\mu u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} + \mathrm{D}_\sigma\alpha^{\rho\beta_1\dots\beta_q,\mu_1\dots\mu_p}{}_{\alpha_1\dots\alpha_p,\lambda_1\dots\lambda_q}\mathrm{D}_\rho u^{\lambda_1\dots\lambda_q}_{\mu_1\dots\mu_p} \\ &\quad + \beta^{\beta_1\dots\beta_q,\mu_1\dots\mu_p}{}_{\alpha_1\dots\alpha_p,\lambda_1\dots\lambda_q}\mathrm{D}_\sigma u^{\lambda_1\dots\lambda_q}_{\mu_1\dots\mu_p} + \mathrm{D}_\sigma\beta^{\beta_1\dots\beta_q,\mu_1\dots\mu_p}{}_{\alpha_1\dots\alpha_p,\lambda_1\dots\lambda_q}u^{\lambda_1\dots\lambda_q}_{\mu_1\dots\mu_p} \\ &\quad + \gamma^{\lambda\mu}\Big(2\sum_i \overline{\mathrm{R}}^{\alpha_i}{}_{\rho,\sigma\mu}\mathrm{D}_\lambda u^{\alpha_1\dots\rho\dots\alpha_p}_{\beta_1\dots\beta_q} - \overline{\mathrm{R}}^{\rho}{}_{\mu,\sigma\lambda}\mathrm{D}_\rho u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} \\ &\quad + \gamma^{\lambda\mu}\Big(2\sum_i \overline{\mathrm{R}}_{\beta_i}{}^{\rho}{}_{,\sigma\mu}\mathrm{D}_\lambda u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\rho\dots\beta_q}\Big) \\ &= \mathrm{D}_\sigma v^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} - \gamma^{\lambda\mu}\Big(\sum_i \mathrm{D}_\lambda\mathrm{R}^{\alpha_i}{}_{\rho,\sigma\mu}u^{\alpha_1\dots\rho\dots\alpha_p}_{\beta_1\dots\beta_q} - \sum_i \mathrm{D}_\lambda\mathrm{R}_{\beta_i}{}^{\rho}{}_{,\sigma\mu}u^{\alpha_1\dots\alpha_p}_{\beta_1\dots\rho\dots\beta_q}\Big) \end{aligned}$$

In view of application to non linear equations we now prove:

LEMMA. — *Hypothesis:* the coefficients of equation (5-1) are such that:

(1) $\gamma \in \mathrm{C}^0_b(\mathrm{M} \times \mathrm{I})$, $\mathrm{D}\gamma \in \mathrm{L}^\infty(\mathrm{M} \times \mathrm{I})$ and γ is regularly hyperbolic.

(2) $\alpha \in \mathrm{E}_2(\mathrm{M} \times \mathrm{I})$

(3) $\beta \in \tilde{\mathrm{E}}_1(\mathrm{M} \times \mathrm{I})$

(4) $v \in \tilde{\mathrm{E}}_1(\mathrm{M} \times \mathrm{I})$

Conclusion: every tensor field $u \in \mathrm{E}_3(\mathrm{M} \times \mathrm{I})$ satisfying (5-1) satisfies the second energy estimate:

$$\left(\| \mathrm{D}u \|^{\mathrm{M}_t}_1\right)^2 \leqslant \mathrm{C}'_0\left(\| \mathrm{D}u \|^{\mathrm{M}_0}_1\right)^2 + \int_0^t \left\{ \mathrm{C}'_1(\tau)\left(\| \mathrm{D}u \|^{\mathrm{M}_\tau}_1\right)^2 + \mathrm{C}'_2(\tau)\left(\| u \|^{\mathrm{M}_\tau}_{\mathrm{C}^0_b}\right)^2 \right\} d\tau + \int_0^t \mathrm{C}'_3(\tau)d\tau$$

where C'_0 is a constant, $\mathrm{C}'_1(\tau)$, $\mathrm{C}'_2(\tau)$ and $\mathrm{C}'_3(\tau)$ measurable functions on I, bounded by numbers depending only on the norms of γ, α, β, v in the indicated spaces. Note that $\mathrm{C}'_2(\tau) = 0$ if $\beta = 0$ and Riem $(e) = 0$.

REMARK. — If $u \in \mathrm{E}_2$, $\| u \|^{\mathrm{M}_\tau}_{\mathrm{C}^0_b} \leqslant \mathrm{C} \, \| u \|^{\mathrm{M}_\tau}_2$.

Proof. — $\mathrm{D}u = u' \in \tilde{\mathrm{E}}_2(\mathrm{M} \infty \mathrm{I})$ satisfies (6-1). We apply to it the equality (5-2). But we now estimate

$$\left| \int_{\mathrm{M}_\tau} 2\partial_t u' . \beta' . u' d\mu(e) \right| \leqslant \left(\| \mathrm{D}u \|^{\mathrm{M}_\tau}_1\right)^2 + \left(\| \beta' . \mathrm{D}u \|^{\mathrm{M}_\tau}_0\right)^2$$

We denote by C constants depending only on M and e, C_e constants depending in M, e, a, A which vanish if Riem $(e) = 0$.

We have (cf. (5), § 1) :

$$\| \beta' . \mathrm{D}u \|_0^{\mathrm{M}_\tau} \leqslant \mathrm{C} \| \beta' \|_1^{\mathrm{M}_\tau} \| \mathrm{D}u \|_1^{\mathrm{M}_\tau}$$

while (cf. 6-2)

$$(\| \beta' \|_1^{\mathrm{M}_\tau})^2 \leqslant 2(\| \mathrm{D}\alpha \|_1^{\mathrm{M}_\tau})^2 + 2(\| \beta \|_1^{\mathrm{M}_\tau})^2 + \mathrm{C}_e$$

we also have

$$(\| v' \|_0^{\mathrm{M}_\tau})^2 \leqslant 2(\| v \|_1^{\mathrm{M}_\tau})^2 + \mathrm{C}(\| \beta \|_1^{\mathrm{M}_\tau})^2(\| u \|_{\mathrm{C}_b^0}^{\mathrm{M}_\tau})^2 + \mathrm{C}_e(\| u \|_{\mathrm{C}_b^0}^{\mathrm{M}_t})^2$$

7. MAIN ENERGY ESTIMATE

Analogous estimates can be obtained for the higher derivatives, with appropriate estimates on the coefficients. In the case $n = 3$, the following will be sufficient.

LEMMA. — *Hypothesis 7-1*

(1) $\gamma \in \mathrm{E}_3(\mathrm{M} \times \mathrm{I})$ and is regularly hyperbolic.

(2) $\alpha \in \mathrm{E}_3(\mathrm{M} \times \mathrm{I})$

(3) $\beta \in \tilde{\mathrm{E}}_2(\mathrm{M} \times \mathrm{I})$

(4) $v \in \tilde{\mathrm{E}}_2(\mathrm{M} \times \mathrm{I})$

Conclusion: every tensor field $u \in \mathrm{E}_4(\mathrm{M} \times \mathrm{I})$, satisfying 5-1 satisfies the energy estimate:

$$(\| \mathrm{D}u \|_2^{\mathrm{M}_t})^2 \leqslant \mathrm{C}_0 \left\{ (\| \mathrm{D}u \|_2^{\mathrm{M}_0})^2 + \int_0^t \mathrm{C}_1(\tau)(\| \mathrm{D}u \|_2^{\mathrm{M}_\tau})^2 d\tau + \int_0^t \mathrm{C}_2(\tau)(\| u \|_{\mathrm{C}_b^0}^{\mathrm{M}_\tau})^2 d\tau + \int_0^t \mathrm{C}_3(\tau) d\tau \right\}$$

with

$$\mathrm{C}_1(\tau) \leqslant \mathrm{C}_1((\| \mathrm{D}\gamma \|_2^{\mathrm{M}_t})^2 + (\| \alpha \|_{\mathrm{C}_b^0}^{\mathrm{M}_\tau})^2 + (\| \mathrm{D}\alpha \|_2^{\mathrm{M}_\tau})^2 + 1 + (\| \beta \|_2^{\mathrm{M}_\tau})^2 + e_1)$$

$$\mathrm{C}_2(\tau) \leqslant \mathrm{C}_2(\| \beta \|_2^{\mathrm{M}_\tau})^2 + e_2$$

$$\mathrm{C}_3(\tau) \leqslant \mathrm{C}_3((\| v \|_2^{\mathrm{M}_\tau})^2)$$

where C_1, C_2, C_3 depend only on M and C_0 depends on M, e, a, A and e_1, e_2 vanish if Riem $(e \times 1) = 0$.

Proof. — Obtained as previously, by derivating 6-1.

We now express $\| \mathrm{D}u \|_2^{\mathrm{M}_0}$ with the values for $t = 0$, u_0 and $(\partial_t u)_0$, through the equations (5-1) and their derivative; we set:

$$u_0 = \varphi, \qquad (\partial_t u)_0 = \psi$$

$$(\text{i. e. } u^{\alpha_1 \dots \alpha_p} = \varphi^{\alpha_1 \dots \alpha_p}, \qquad (\partial_t u^{\alpha_1 \dots \alpha_p})_0 = \psi^{\alpha_1 \dots \alpha_p})$$

with

$$\varphi \in C_b^0(M),\ \bar{D}\varphi \in H_2(M),\ \psi \in H_2(M). \tag{7-2}$$

$(Du)_0$ and $(\bar{D}Du)_0$ are known, and $(\partial_{tt}^2 u)_0$ is computed from 5-1. Analogously the t derivative of 5-1 permits the determination of $(D^3 u)_0$ in terms of φ, ψ. We have

$$(\| Du \|_2^{M_0})^2 \leqslant l \left\{ (\| \bar{D}\varphi \|_{H_2(M)})^2 + (\| \varphi \|_{C_b^0(M)})^2 + (\| \psi \|_{H_2(M)})^2 + (\| v \|_1^{M_0})^2 \right\}$$

where l is a number depending on M, e, a, A, $\| D\gamma \|_2^{M_0}$, $\| \alpha \|_{C_b^0}^{M_0}$, $\| D\alpha \|_2^{M_0}$, $\| \beta \|_2^{M_0}$.

We note that

$$\| u \|_{C_b^0(M_\tau)} \leqslant \| \varphi \|_{C_b^0(M)} + C \int_0^\tau \| Du \|_2^{M_\tau} d\tau$$

thus

$$\int_0^t (\| u \|_{C_b^0(M_\tau)})^2 d\tau \leqslant 2t(\| \varphi \|_{C_b^0(M)})^2 + C^2 t^2 \int_0^t (\| Du \|_2^{M_\tau})^2 d\tau$$

Finally the inequality may be written

$$(\| Du \|_2^{M_t})^2 \leqslant k_0(t) + k_1(t) \int_0^t (\| Du \|_2^{M_t})^2 d\tau \tag{7-5}$$

where $k_0(t)$ and $k_1(t)$ are positive numbers, continuously increasing with t depending only on the norms of the coefficients and the norms of the Cauchy datas.

If $\varphi = \psi = 0$ and $v = 0$, then $k_0(t) = 0$. The same result is true when $\beta = 0$, $v = 0$ and only $\bar{D}\varphi = \psi = 0$.

The inequality 7-5 implies that $\| Du \|_2^{M_t}$ is bounded for every $t \in I$ (finite), by the solution of the corresponding equality, namely

$$(\| Du \|_2^{M_t})^2 \leqslant k_0(t) + k_2(t)$$

with

$$k_2(t) = k_1(t) \exp K_1(t) \int_0^t k_0(\tau) \exp(-K_1(\tau)) d\tau$$

$$K_1(t) = \int_0^t k_1(\tau) d\tau.$$

8. EXISTENCE FOR THE LINEAR SYSTEM

THEOREM. — The linear equation (5-1) with coefficients satisfying hypothesis 7-1, has one and only one solution $u \in E_3(M \times I)$, I arbitrary bounded interval of R including zero, u taking on M_0 the Cauchy data φ, ψ satisfying 7-2.

Proof. — Consider first a system of the indicated type, but moreover with C^∞ coefficients, and C^∞ Cauchy data φ, ψ.

It is a globally hyperbolic system on $M \times I$ and it results from the Leray theory (1952) that it has a C^∞ solution u on $M \times I$.

If moreover $\bar{D}\varphi$, ψ have a compact support K on M_0 and if β and v vanish outside $K \times I$, it results from the support properties of the solution that Du vanishes (9) outside $K' \times I$, with K' a compact subset of M.

Consider now the system 5-1 with coefficients and Cauchy data satisfying 7-1 and 7-2, and a sequence of coefficients and data, γ_n, α_n, β_n, v_n and φ_n, ψ_n, all C^∞, with β_n, v_n of support $K \times I$, $\bar{D}\varphi_n$, ψ_n of support K, a compact set, tending respectively to γ, α, β, v and φ, ψ in the appropriate norms (cf. lemma (3)). We deduce from the energy inequality that the corresponding C^∞ solutions u_n have a uniformly bounded $E_3(M \times I)$ norm. We consider now the linear equation satisfied by a difference $u_n - u_m$:

$$\gamma_n . D^2(u_n - u_m) + \alpha_n . D(u_n - u_m) + \beta_n . (u_n - u_m) = (\gamma_m - \gamma_n)D^2 u_m + (\alpha_m - \alpha_n) . Du_m + (\beta_n - \beta_m) . u_m + v_n - v_m$$

The right hand side is in $\tilde{E}_1$, while $u_n - u_m \in \tilde{E}_3$. The second energy estimate (cf. Remark) shows that this difference tends to zero in $E_2(M \times I)$ norm, therefore u_n tends to a tensor field $u \in E_2(M \times I)$, generalized solution of the given Cauchy problem. The fact that this u is indeed in $E_3(M \times I)$ is proved as follows (10): since $(Du_n)_t$ is uniformly bounded in the $H_2(M)$ norm, there is a subsequence $(Du_{n_i})_t$ which converges weakly to some $f_t \in H_2(M)$. But since $(Du_n)_t$ converges (strongly) in $H_1(M)$ to $(Du)_t$, we have $f_t = (Du)_t$, thus $(Du)_t \in H_2(M)$, and is uniformly bounded in $H_2(M)$ norm for $t \in I$. It results from known theorems (11) that $t \mapsto \| (Du)_t \|_{H_2(M_t)}$ is measurable, thus finally, $u \in E_3$.

9. SOLUTION OF THE REDUCED EINSTEIN EQUATIONS

If γ is regularly hyperbolic on $M \times I$ and $\gamma \in E_3(M \times I)$ the equations 4-1 satisfy the hypothesis 7-1, with $\alpha = \beta = 0$, $v = -f(\gamma, D\gamma) \in \tilde{E}_2(M \times I)$, $g = u$. We can apply the previous results and associate with the Cauchy data (φ, ψ) and lorentzian metric γ a tensor $g \in E_3(M \times I)$, unique solution of 4-1 with Cauchy data (φ, ψ), $\bar{D}\varphi \in H_2(M)$, $\psi \in H_2(M)$, $\varphi \in C^0_b(M)$.

(9) It is in order to have this property that we have supposed Riem (e) zero outside a compact subset of M.

(10) Cf. Dionne (1962) and Hawking-Ellis (1973) in the case of H_4^{loc}.

(11) Cf. Bourbaki XIII, ch. 4, § 5.

If we suppose moreover that φ is a lorentzian tensor satisfying at each point of M the inequalities 4-2 (i. e. M uniformly space-like for φ), the same property will be true of g when the interval I is small enough, i. e. $l(\mathrm{I}) < \varepsilon$. The solution of the Cauchy problem (φ, ψ) for 4-1 defines therefore, if $l(\mathrm{I}) < \varepsilon$, a (continuous) mapping $\gamma \mapsto g$ from an open set Ω of $\mathrm{E}_3(\mathrm{M} \times \mathrm{I})$ into itself, where Ω will be defined by inequalities of the type

$$\| \gamma \|_{\mathrm{E}_3(\mathrm{M} \times \mathrm{I})} < k$$

$$\gamma^{\perp\perp} > a \quad , \quad -\gamma^{||}(\mathrm{X}, \mathrm{X}) > ae(\mathrm{X}, \mathrm{X})$$

Moreover the difference $u = g_1 - g_2$ of the images of two points γ_1, γ_2, satisfies the linear equation

$$(9\text{-}1) \qquad \frac{1}{2}\gamma_1 . \mathrm{D}^2 u = -f(\gamma_1, \mathrm{D}\gamma_1) + f(\gamma_2, \mathrm{D}\gamma_2) + \frac{1}{2}(\gamma_2 - \gamma_1)\mathrm{D}^2 g_2$$

with Cauchy data zero.

The second energy estimates applied to 9-1 shows that the mapping $\gamma \mapsto g = \mathrm{F}(\gamma)$ is contracting in the E_2 norm if $l(\mathrm{I}) < \varepsilon$, thus it admits a unique fixed point $\bar{g} \in \mathrm{E}_2$. We obtain the result that $\bar{g} \in \mathrm{E}_3$ as for the linear equation, § 8, from the fact that the iterates $g_n = \mathrm{F}^n(\gamma)$ are uniformly bounded in E_3 norm, thus $(\mathrm{D}g_n)_t$ is uniformly (in n and t) bounded in $\mathrm{H}_2(\mathrm{M})$ norm, and therefore the sequence is weakly compact.

The $\bar{g}$ thus obtained is solution of the given Cauchy problem.

In conclusion we have proved.

THEOREM. — Let (M, e) be a 3-manifold satisfying the hypothesis of § 1. Let (φ, ψ) be Cauchy data for Einstein equations; $g_0 = \varphi$, $(\partial_t g)_0 = \psi$, with $\varphi \in \mathrm{C}_b^0(\mathrm{M})$, $\overline{\mathrm{D}\varphi} \in \mathrm{H}_2(\mathrm{M})$, $\psi \in \mathrm{H}_2(\mathrm{M})$, M regularly space-like for φ, then the reduced Einstein equations have one and only one solution g on some 4-manifold $\mathrm{M} \times \mathrm{I}$, $g \in \mathrm{E}_3(\mathrm{M} \times \mathrm{I})$, taking on M_0 these Cauchy data.

Standard methods show that if the Cauchy data satisfy the constraints and the initial gauge condition $\mathrm{F}_0 = 0$ the solution obtained is a solution of the full Einstein equations.

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