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On stellar collapse and the black hole limit from a dynamical view

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ABSTRACT. — Dynamical considerations are applied to the motion of a test body within a star, during its gravitational collapse. The question investigated in this paper is whether the test body would have an inflection point in its motion of « free fall », changing to an outward motion, and if so, where this would occur. Using the Schwarzschild model of a star (constant density) from the view of the test body when it is near its inflection points, for times that are short compared with the total collapse time of the star, it is found that an inflection point occurs near $r_0/2$, where r_0 is the gravitational radius of the star. Thus it is concluded that in the cases of most stars, the contraction would change to an expansion long before the black hole limit would be reached. The expansion changes once again to a contraction at the second inflection point, r_0. The general motion is oscillatory. Rare cases where black holes may manifest themselves would be very dense stars that would oscillate in and out of the black hole state — revealing radiation emission during the normal star half-period, and no radiation emission in the black hole phase. This observation is noted to be the same as that of the « pulsar » data.

I

An interesting question for astrophysics, as well as a sensitive test of the validity of the theory of general relativity, is concerned with the process of gravitational contraction of a star. Assuming that the parameters of the star are such that it would not break apart along the way, because of non-gravitational effects, would the contraction continue irreversibly to the black hole limit? Or would the stellar matter reverse its path, changing from a gravitational contraction to a gravitational expansion, before the black hole limit would be reached?

If the Newtonian feature of the gravitational force, that it is attractive under all conditions, would also be true of the gravitational predictions of general relativity theory, then one might expect a continued contraction to the limit in which the corresponding curvature of space-time does indeed entail a family of geodesics that close on themselves in the domain of this stellar matter — the « black hole » limit. But the effective force field that acts on a test body, according to general relativity theory, is in terms of the *non-positive-definite* affine connection — that represents the geometrical features of the remainder of the star that acts on this test body. Thus, the collection of terms that make up this *net* force field could be attractive under some physical conditions (of relative separations, speeds and matter density), and repulsive under other conditions — even though the limiting Newtonian form of the force field in general relativity is predominantly attractive. The latter limit is a built-in feature of general relativity theory, that in the proper limit (of small mutual separations and matter density, and relative speeds of interacting matter), the formal structure of general relativity theory approaches that of Newton's theory of universal gravitation (an example of the use of the « correspondence principle »).

Thus it is possible that though the effective force field experienced by a test body within a star is attractive in the early stages of the stellar contraction, when the star's density is low enough, it could become repulsive later on in the contraction process when the stellar density becomes sufficiently great, though before the black hole limit of the star's contraction would be reached.

The problem of continued gravitational contraction was first studied by Oppenheimer and Snyder [*1*]. Since that time, the black hole limit of stellar collapse has been the focus of attention of a great deal of theoretical and experimental research [*2*]. The conclusions about black holes that these researchers have reached are based on what might be called a « kinematical approach ». That is, their conclusions are based on what might be said about the limit of the collapsed star *from the view of an outside observer* — assumed to be in a vacuum, looking down on the star. Thus, whatever conclusions have been reached about the black hole from the theoretical side (with the assumption that it is static or rotating, spherically symmetric, etc.) have been based on an analysis of *Einstein's vacuum equations:*

$$R_{\mu\nu} = 0 \tag{1}$$

In this paper I should like to present the initial stage of an approach to this problem from a dynamical, rather than a kinematical view. The general question asked in this investigation is: Precisely how would a constituent matter element of a star — a « test body » — *move*, when subjected to the gravitational field of the rest of the matter of the host star? The explicit question asked in this paper is: If the test body would indeed turn around at some stage of the stellar collapse, corresponding to the onset of the influence of a repulsive force in the (affine connection) force field, then precisely where would this happen, and would this be near the black hole limit of the collapse?

The field formalism that must be appealed to in answering these questions is the full set of Einstein's equations:

$$R_{\mu\nu} - (1/2)g_{\mu\nu}R = -(8\pi G/c^2)T_{\mu\nu} \qquad (2)$$

where $T_{\mu\nu}$ is the energy-momentum tensor field of the star, that in turn, is responsible for the motion of the test body within its domain.

The equation of motion of the test body, within the star, is the geodesic equation:

$$\ddot{x}^\rho = -\Gamma^\rho_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \qquad (3)$$

where

$$\Gamma^\rho_{\alpha\beta} = (1/2)g^{\rho\lambda}(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}) \qquad (4)$$

is the affine connection field of the Riemannian space-time, determined by the derivatives of the metric tensor solutions, $g_{\alpha\beta}$, of Einstein's field equations (2).

When the mutual interactions between the components of the interacting matter of the star become sufficiently great (and the curvature of space-time is correspondingly great), the Newtonian approximation for Einstein's predictions for the gravitational field breaks down. It is at this stage of the collapse where one might expect the repulsive force terms (on the right-hand side of eq. (3)) to become greater than the attractive force terms — that is, if the exact solutions of Einstein's equations (2) indicate that this would happen at all. In this case, the attractive force that draws a test body toward the center of the star (i. e. its « free fall ») would change to a repulsive force, causing it then to move away from the center of the star. [The build-up toward the inflection point in the motion of the test body in the collapsing star is analogous to the build-up of tension when inflating a balloon, in a field whose curvature becomes correspondingly great, until the tension might exceed the attractive two-body forces among the constituent elements of the matter field, leading to a rupture].

After the bulk of the matter of the star would pass its inflection point, with the star exploding outward from its center, its matter density would continuously rarefy until the predominantly repulsive force of its matter constituents changes to a predominantly attractive force. The attraction would then cause the outward explosion of the star to slow down and then turn it around once again into an implosion. Thus, the dynamics of the star implies an oscillatory motion, though damped, since radiation energy would be lost in each cycle, until the star would die.

The heuristic for tacitly assuming that the actual dynamics of the star is oscillatory is that only this type of motion is relativistically covariant, in a strict sense; that is, so long as the specification of any particular time (say, the beginning of a particular cycle) is a function of the reference frame of the moving test body within the star from which one wishes to measure a particular duration. For if there would be a single objective contraction (or expansion) of the star, this would imply that a special (universal) time measure could be associated with all relatively moving frames of reference within the star — any test body would move with a time measure that entails the same universal time of contraction (or expansion) from a unique beginning of the star to its death (in time) — or else to its limiting black hole state.

II

To investigate the specific question: *where*, during the gravitational contraction of a star, would a test body within it reach an inflection point, changing its inward motion to an outward motion? a particular stellar model will have to be used. For this purpose, it will be assumed in this paper that near the inflection points in its motion, for times that are short compared with the total collapse time of the star, the test body would react to an environment that has constant mass density and is spherically symmetric, up to the « gravitational radius » r_0 of the star. According to the definition of r_0, it is further asserted that for $r > r_0$, the vacuum equations (1) would apply, yielding the exterior Schwarzschild solution there. Thus, for these short times in the test body's motion, the model of the star utilized is that of Schwarzschild [*2 a*, p. 468] — applied to the test body's trajectory near its inflection points.

The aim of the present investigation is to use this model in order to estimate one particular detail of the test body's motion — the spatial location within the star where the inflection points would indeed occur.

The well known *Schwarzschild interior solution* is as follows:

$$g_{00} = [(3/2)(1 - (r_0/R_0)^2)^{1/2} - (1/2)(1 - (r/R_0)^2)^{1/2}]^2 ,$$
$$g_{11} = - [1 - (r/R_0)^2]^{-1} = 1/g^{11}$$
$$g_{22} = - r^2, \quad g_{33} = - r^2 \sin^2\theta, \quad g_{\mu \neq \nu} = 0 \tag{5}$$

where $r \leq r_0$, $R_0^2 = 3c^2/8\pi G\rho$ is determined as a constant of the integration, ρ is the assumed constant mass density of the star, and r_0 is the star's gravitational radius — that is, its radius when the contraction begins, in any particular cycle.

According to the diagonal form of the solution (5), it follows that the pertinent affine connection terms are:

$$\Gamma^1_{\alpha\beta} = (1/2)g^{11}(\partial_\alpha g_{1\beta} + \partial_\beta g_{1\alpha} - \partial_1 g_{\alpha\beta})$$

The « geometrical force field » (per mass of the test body) in the equation of motion (3) is then:

$$- \Gamma^1_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = -(1/2)g^{11}[\partial_1 g_{11}(\dot{r})^2 - \partial_1 g_{00}(\dot{x}^0)^2] \tag{6}$$

From the oscillatory dynamics of the star, the motion of the test body predicated by this force field has two inflection points. The first is the place where the body's motion changes from a « free fall » toward the center of the star, to an outward motion along r (I will call this inflection point $r = \mathrm{R}$). The second inflection point is where the body's outward motion changes once again to the « free fall » (ar $r = r_0$). At these two inflection points, the body's velocity is zero ($\dot{r} = 0$). Further, in the neighborhood of the test body, at these times, it is a good approximation to take $\dot{x}^0 = c$. Thus, the geometrical force field (per mass of the test body) takes the following form at the inflection points:

$$- \Gamma^1_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = (c^2/2)g^{11}\partial_1 g_{00} \qquad \text{at } r = \mathrm{R}, r_0 \tag{7}$$

Considering neighboring cycles in the star's dynamics, and assuming that it is exactly cyclical, the magnitude of the acceleration of the test body, $\ddot{r}$, must be equal at each of the inflection points. Thus, the location of the inflection point, $r = \mathrm{R}$, for the inward motion, may be determined from the « force » relation implied by eqs. (3) and (7):

$$[g^{11}\partial_1 g_{00}]_{r=\mathrm{R}} = [g^{11}\partial_1 g_{00}]_{r=r_0}$$

With the interior Schwarzschild solution (5), this relation then gives the following equation in the unknown R:

$$3\left[\frac{1 - (r_0/\mathrm{R}_0)^2}{1 - (\mathrm{R}/\mathrm{R}_0)^2}\right]^{1/2} - 1 = 2(r_0/\mathrm{R})\,\frac{1 - (r_0/\mathrm{R}_0)^2}{1 - (\mathrm{R}/\mathrm{R}_0)^2} \tag{8}$$

Using the change of variables:

$$x = \left[\frac{1 - (r_0/\mathrm{R}_0)^2}{1 - (\mathrm{R}/\mathrm{R}_0)^2}\right]^{1/2} = \left[\frac{1 - \lambda}{1 - u\lambda}\right]^{1/2} \tag{9}$$

where $\lambda = (r_0/\mathrm{R}_0)^2$ and $u = (\mathrm{R}/r_0)^2$, eq. (8) then takes the form:

$$(x^2 - 1)(3x - 1)^2 = \lambda(4x^6 - 9x^2 + 6x - 1) \tag{10}$$

Because of the curvature of space, the weighting factor $\sqrt{-g} = \sqrt{-\det g_{\alpha\beta}}$, determined from the solution (5), must be inserted into the integral that determines the volume of a star, in calculating its mass density [*2 a*, p. 474]. It is then readily found that the density has the form

$$\rho = (\mathrm{M}/\mathrm{V}_0)[1 - 0.3 r_s/r_0 + 0(r_s^2/r^2)]$$

where r_s is the Schwarzschild radius, $2\mathrm{GM}/c^2$. Since $r_s/r_0 \ll 1$ (it is the order of 10^{-6} for the sun), the remainder in this series may be safely neglected. It then follows that

$$\lambda \equiv (r_0/R_0)^2 = (r_s/r_0)(1 - 0.3r_s/r_0) \simeq r_s/r_0 \ll 1$$

Thus, a perturbation method is appropriate in order to determine the solutions of the polynomial algebraic equation (10).

In the zeroth approximation, $\lambda = 0$, and the solutions of (10) reduce to the roots, $x = \pm 1, 1/3$. Next, to determine the first order correction to the root, $x = -1$, we alter it to the root, $x = -1 + \delta$, where δ is the same order of magnitude as λ. In this case, eq. (10) takes the following form, to order λ:

$$-32\delta = -12\lambda$$

giving $\delta = 3\lambda/8$. Substituting this value of δ back into the first order solution, $x = -(1 - 3\lambda/8)$ and comparing this with the definition of x (eq. (9)), we have to first order in λ:

$$x^2 = 1 - 3\lambda/4 = (1 - \lambda)/(1 - u\lambda)$$

or

$$1 - u\lambda = (1 - \lambda)/(1 - 3\lambda/4) = (1 - \lambda)(1 + 3\lambda/4) = 1 - \lambda/4$$

The required first order solution is then $u = (R/r_0)^2 = 1/4$, yielding

$$R = r_0/2$$

as the inflection point where the inward fall of a test body, within the star, would change its direction, then moving outward along the radial direction.

The other roots of the polynomial equation (10), corrected to first order, $x = (1 + \delta)$ and $x = (1/3 + \delta)$, both yield values of R that are greater than r_0. Since the interior Schwarzschild solution (5) is only valid for $r \leq r_0$, these latter roots do not match the boundary conditions of the problem, and thus must be rejected, as unphysical. The solution, $R = r_0/2$, is then unique for this problem.

III

The conclusion is then reached from the present analysis that the limit of stellar contraction is generally far from the black hole radius. The star, as a whole, should continue to oscillate between contraction and expansion losing energy in each cycle due to emitted radiation, until the star would die. But the salient point of this analysis is that, within the approximations and assumptions used, it seems unlikely that most stars would ever reach the black hole state of stellar collapse, primarily because of the predominance of an effective repulsive force after a sufficiently small radius of the star (i. e. sufficiently high matter density) would be reached.

Nevertheless, there could be two exceptions, occurring in rare cases. First, the contraction of an unusually small and extremely dense star, that is not initially in the black hole state, into the black hole state, cannot be ruled out. However, the dynamics implied by this analysis would require this star to expand out of the black hole state again, and then back into it, in oscillatory fashion. In these cases, what the astronomer should look for is a periodic appearance and disappearance of radiation from a (quite hot) visible star — *similar to the observation of a pulsar.* Indeed, it may be speculated that at least a portion of the stellar objects classified as « pulsars » may be explained in terms of the dynamics of pulsating stars, expanding and contracting out of and into the black hole state, periodically.

The second rare exception to the conclusion of the present paper might be cases of black holes that are born that way, and never come out of the black hole state. But according to the present analysis, such a star would still be in continual oscillation, though never leaving the black hole density. A possible observable consequence of such a star's existence is that it should scatter light (and matter) in an apparently anomalous fashion, whenever this light (or matter) would propagate past the outer regions of the highly curved space-time whose centers are the black holes — i. e. sufficiently far from the center not to be « trapped » into the region. Thus, starlight should be seen to scatter anomalously from each « black hole scattering center », with periodic changes in « bending » — if indeed such black holes do exist.

The final proof about whether or not the black hole limit can exist, according to general relativity theory, depends on the demonstration of a corresponding *stable solution* of the full form (2) of Einstein's field equations, incorporating the dynamics of the material elements of the star that take part in its formation. Studies on this problem are currently in progress.

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