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# Infinitesimal symplectic relations and generalized Hamiltonian dynamics

by

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SUMMARY. — Generalized Hamiltonian dynamics formulated originally by Dirac is reformulated in terms of symplectic geometry. A classification of constraints is given. Importance of first class constraints is stressed. Examples from particle dynamics illustrating various aspects of the generalization are given.

RÉSUMÉ. — On formule en termes de géométrie symplectique la généralization de la dynamique hamiltonienne donnée par Dirac. Après la classification des contraintes on souligne l'importance des contraintes de première classe. Plusieurs aspects de cette généralisation sont donnés par des exemples tirés de la dynamique d'une particule.

## 1. INTRODUCTION

In an attempt to provide a canonical framework for relativistic dynamics, Dirac [3] formulated a generalization of Hamiltonian dynamics associated with submanifolds of the phase space of a physical system. A classification of the Poisson algebra was given. Dirac's theory was formulated in the language of local coordinates. A coordinate independent discussion of dynamics associated with submanifolds of a symplectic manifold was given by Lichnerowicz [5]. Lichnerowicz also included the case of non-zero Hamiltonians.

We give a systematic discussion of the generalized Hamiltonian dynamics in terms of symplectic concepts such as lagrangian submanifolds and symplectic relations. Classification of constraints is given. Importance of first class constraints and integrability is stressed. Lagrangian description of dynamics is not included. Examples including both zero and non-zero Hamiltonians are given.

The program of symplectic interpretation of dynamics with constraints was initiated in a seminar conducted at the Warsaw Institute for Theoretical Physics by one of us (W. M. T.) in 1967-1968. Results remained unpublished except for a paper by Sniatycki [7] who as a participant in the seminar had access to the material.

## 2. SYMPLECTIC MANIFOLDS AND LAGRANGIAN SUBMANIFOLDS

DEFINITION 2.1. — Let P be a manifold, and $\omega$ a 2-form on P. The pair (P, $\omega$) is called a *symplectic manifold* if :

*i*) $\omega$ is non-degenerate;

*ii*) $\omega$ is closed: $d\omega = 0$.

A more explicit statement of condition *i*) is that if $\langle X \wedge Y, \omega \rangle = 0$ for each vector field Y, it follows that $X = 0$. Condition *i*) implies that the dimension of P is even: $\dim P = 2m$.

If local coordinates $(x^\alpha)$ are introduced in P and the local expression of $\omega$ is $\omega = \frac{1}{2}\omega_{\alpha\beta}dx^\alpha \wedge dx^\beta$ (1), then these two conditions can be expressed as:

*i'*) $\det \|\omega_{\alpha\beta}\| \neq 0$;

*ii'*) $\partial_{[\alpha}\omega_{\beta\gamma]} = 0$.

Coordinates $(q^i, p_j)$ of P such that $\omega = dq^i \wedge dp_i$ are called *canonical coordinates.* Local existence of canonical coordinates is guaranteed by Darboux's theorem.

DEFINITION 2.2. — Let (P, $\omega$) be a symplectic manifold. A submanifold $N \subset P$ is called a *lagrangian submanifold* of (P, $\omega$) if

*i*) $\omega | N = 0$;

*ii*) $\dim N = m = \frac{1}{2}\dim P$.

(1) We follow the tensor calculus conventions of reference [6]. In particular, the summation convention is used.

A submanifold $N \subset P$ is said to be *isotropic* if condition *i)* is satisfied. We show that the dimension of an isotropic submanifold can be at most equal to $m$. At any point $p \in P$, $\omega_p$ is a non-degenerate skew-symmetric form on $T_pP$. The vector space $T_pP$ together with the form $\omega_p$ give an example of what is called a *symplectic vector space.* For any subspace V of $T_pP$ we define the *symplectic polar*

$$V^{\S} = \{ v \in T_pP ; \langle v \wedge u, \omega \rangle = 0, \ \forall u \in V \} \tag{2.1}$$

The dimension of the symplectic polar is obviously equal to the codimension of V in $T_pP$. In particular, if $p$ is an element of an isotropic submanifold $N \subset P$, then the condition $\omega \mid N = 0$ implies $(T_pN)^{\S} \supset T_pN$. Let $n$ be the dimension of N. Since the dimension of $(T_pN)^{\S}$ is $2m - n$, we have $2m - n \geq n$, or $m \geq n$. Hence the dimension of N is at most $m$. It follows from the discussion above that a lagrangian submanifold is an isotropic submanifold of maximal dimension.

Let Q be a manifold and let $T^*Q$ be its cotangent bundle. We denote by $\pi_Q$ the bundle projection of $T^*Q$ onto Q. A 1-form $\vartheta_Q$ on $T^*Q$ is defined by:

$$\langle v, \vartheta_Q \rangle = \langle T\pi_Q(v), \tau_{T^*Q}(v) \rangle \qquad \text{for each} \qquad v \in TT^*Q .$$

The mapping $T\pi_Q$: $TT^*Q \to TQ$ is the tangent mapping of $\pi_Q$ and $\tau_{T^*Q}$: $TT^*Q \to T^*Q$ is the tangent bundle projection. The 1-form $\vartheta_Q$ is called the *canonical 1-form* on $T^*Q$ and the 2-form $\omega_Q = d\vartheta_Q$ is called the *canonical 2-form.*

Let $(q^i)$ be local coordinates of Q. At each point $q \in Q$, the differentials $dq^i$ of the coordinates form a basis of the cotangent space $T_q{}^*Q$. The components $p_j$ of a covector $p \in T_q{}^*Q$ together with the coordinates $q^i$ of $q$ form coordinates $(q^i, p_j)$ of the covector $p$. In this way we obtain coordinates $(q^i, p_j)$ of $T^*Q$ canonically associated with coordinates $q^i$. It is easily seen that the local expression of the canonical 1-form $\vartheta_Q$ is $\vartheta_Q = p_i dq^i$, and the local expression of the canonical 2-form $\omega_Q$ is $\omega_Q = dp_i \wedge dq^i$. It follows that $(T^*Q, \omega_Q)$ is a symplectic manifold and the coordinates $(q^i, p_j)$ are canonical coordinates.

If $\varphi$ is a 1-form on Q, then the pullback $\varphi^*\vartheta_Q$ of the canonical 1-form $\vartheta_Q$ by $\varphi$: $Q \to T^*Q$ is equal to $\varphi$. This property is proved by evaluating $\varphi^*\vartheta_Q$ on an arbitrary vector $v \in TQ$:

$$\begin{aligned}\langle v, \varphi^*\vartheta_Q \rangle = \langle T\varphi(v), \vartheta_Q \rangle &= \langle T\pi_Q(T\varphi(v)), \tau_{T^*Q}(T\varphi(v)) \rangle \\ &= \langle T(\pi_Q \circ \varphi)(v), \varphi(\tau_Q(v)) \rangle = \langle v, \varphi \rangle .\end{aligned}$$

From $\varphi^*\vartheta_Q = \varphi$, it follows that the image N of the mapping $\varphi$: $Q \to T^*Q$ is a lagrangian submanifold of $(T^*Q, \omega_Q)$ if and only if $\varphi$ is a closed 1-form. If $\varphi$ is exact, and $\varphi = dF$, then F is called a *generating function* of N. The image of a closed 1-form is a special case of a class of lagrangian submanifolds generated by functions defined on submanifolds of Q.

PROPOSITION 2.1. — *Let* $C \subset Q$ *be a submanifold and let* $F : C \to \mathbb{R}$ *be a differentiable function. The set*

$$N = \{ p \in T^*Q ; \ q = \pi_Q(p) \in C, \ \ \forall u \in T_qC, \ \langle u, p \rangle = \langle u, dF \rangle \} \quad (2.2)$$

*is a lagrangian submanifold of* $(T^*Q, \omega_Q)$.

*Proof.* — Using canonical coordinates of $T^*Q$, one can easily show that N is a submanifold of $T^*Q$ of dimension $m = \dim Q$. Let $z$ be any vector in $TN \subset TT^*Q$. Then

$$\langle z, \vartheta_Q \rangle = \langle T\pi_Q(z), \tau_{T^*Q}(z) \rangle = \langle T\pi_Q(z), dF \rangle = \langle z, d\bar{F} \rangle ,$$

where $\bar{F} = F \circ (\pi_Q \mid N)$. It follows that $\vartheta_Q \mid N = d\bar{F}$ and consequently $\omega_Q \mid N = 0$. Since N is isotropic and $\dim N = \frac{1}{2} \dim T^*Q$, it follows that N is lagrangian, q. e. d.

## 3. THE STRUCTURE OF THE TANGENT BUNDLE OF A SYMPLECTIC MANIFOLD

Let $(P, \omega)$ be a symplectic manifold. Then the tangent bundle TP is isomorphic to the cotangent bundle $T^*P$. The isomorphism is established by the vector bundle morphism $\beta$

$$\beta : TP \to T^*P : u \to u \lrcorner \omega . \quad (3.1)$$

Objects in TP corresponding to $\pi_P$, $\vartheta_P$ and $\omega_P$ are $\tau_P = \pi_P \circ \beta$, $\chi = \beta^*\vartheta_P$ and $\dot{\omega} = d\chi = \beta^*\omega_P$, respectively.

We note that $(TP, \dot{\omega})$ is a symplectic manifold. Since TP is isomorphic to the cotangent bundle $T^*P$, it is possible to generate Lagrangian submanifolds of $(TP, \dot{\omega})$ from functions defined on submanifolds of P. Applications of such submanifolds to particle dynamics will be studied in subsequent sections.

## 4. INFINITESIMAL RELATIONS

Let $(P, \omega)$ be a symplectic manifold representing the *phase space* of a mechanical system [2]. Let us consider the set of all differentiable curves in P. Possible histories form a subset $\mathscr{D}$ of the set of all curves. The set $\mathscr{D}$ itself, or any law which characterizes $\mathscr{D}$ completely will be referred to as

[2] In the present section we make no use of the symplectic structure $\omega$.

the *dynamics* of the system. It usually happens that the set $\mathcal{D}$ is the set of all solutions of an ordinary differential equation of first order.

DEFINITION 4.1. — A submanifold D′ of TP is called a *first order differential equation.* A differentiable mapping $\gamma : I \to P$ is called an *integral curve* of D′ if the vector $\dot{\gamma}(t)$ tangent to $\gamma$ at $t$ belongs to D′, for each $t \in I$. We denoted by I an open neighbourhood of $0 \in \mathbb{R}$.

When a differential equation D′ is used to describe the dynamics of a mechanical system, it is important that each element of D′ is tangent to an integral curve.

DEFINITION 4.2. — A differential equation $D' \subset TP$ is said to be *integrable* if for each element $u \in D'$ there is an integral curve $\gamma$ of D′ such that $\dot{\gamma}(0) = u$. An integrable differential equation will be called an *infinitesimal relation.*

*Examples.* — *a*) Let $X : P \to TP$ be a vector field: then the image $D' = \mathrm{Im}\,(X)$ is a differential equation. It follows from the geometric version of Cauchy's theorem that D′ is integrable. A vector field is frequently called an infinitesimal transformation since it generates a (local) one-parameter group of transformations. An integral curve of D′ is called an integral curve of the vector field X.

*b*) Let D′ be a distribution on P (3) : D′ is again an example of a differential equation. Integrability of distributions is the subject of Frobenius theory. This theory deals with the problem of existence of submanifolds of P whose tangent vectors belong to the distribution. Such submanifolds are called integral submanifolds. A distribution is said to be completely integrable if for each point there is an integral submanifold of dimension equal to the dimension of the distribution. Integrability in the sense of Definition 4.2 always holds for a distribution. A completely integrable distribution leads to a foliation of P by maximal connected integral manifolds; this foliation can be interpreted as defining an equivalence relation in P. Consequently, an integrable distribution can be regarded as an infinitesimal equivalence relation.

*c*) Let (P, B, $\mu$) be a differential fibration [2], and let $X : B \to TB$ be a vector field. We define a differential equation

$$D' = \{ u \in TP ;\ T\mu(u) = X(\mu(\tau_P(u))) \} .$$

Let $u \in T_pP$ belong to D′ and let $\gamma : I \to B$ be an integral curve of X such that $\gamma(0) = \mu(p)$. It is easily seen that any lift of $\gamma$ to P is an integral curve of D′; in particular a lift can be chosen to be tangent to $u$. It follows that D′ is an infinitesimal relation.

(3) A distribution on P is a subbundle of TP, also known as a *field of* p-*directions.*

Let $\varphi_t : \mathrm{B} \to \mathrm{B}$ be the one-parameter group of transformations generated by X. For each $t$ we have a submanifold $\mathrm{D}_t \subset \mathrm{P} \times \mathrm{P}$ defined by

$$\mathrm{D}_t = \{ (p', p) \in \mathrm{P} \times \mathrm{P} \,;\; \mu(p') = \mu(\varphi_t(p)) \} .$$

This submanifold is not the graph of a mapping but can still be considered a relation in P. The one-parameter family $\{ \mathrm{D}_t \}$ of relations is in a sense generated by D′. This justifies the use of the term « infinitesimal relation » for D′.

## 5. HAMILTONIAN DYNAMICS

Dynamics of nonrelativistic particle systems is usually described by a hamiltonian vector field on a symplectic manifold $(\mathrm{P}, \omega)$ representing the phase space of the system. We show that a hamiltonian vector field provides a special example of a lagrangian submanifold of $(\mathrm{TP}, \dot{\omega})$.

DEFINITION 5.1. — A vector field $\mathrm{X} : \mathrm{P} \to \mathrm{TP}$ is said to be *locally hamiltonian* if the form $\mathrm{X} \lrcorner\, \omega$ is closed: $d(\mathrm{X} \lrcorner\, \omega) = 0$. The field X is said to be *globally hamiltonian* if $\mathrm{X} \lrcorner\, \omega$ is exact. A function $\mathrm{H} : \mathrm{P} \to \mathbb{R}$ is called a *Hamiltonian* for the field X if $\mathrm{X} \lrcorner\, \omega = -\,d\mathrm{H}$ [*1*].

Since $\mathscr{L}_{\mathrm{X}}\omega = d(\mathrm{X} \lrcorner\, \omega) + \mathrm{X} \lrcorner\, d\omega$ and $\omega$ is closed, a vector field $\mathrm{X} : \mathrm{P} \to \mathrm{TP}$ is locally hamiltonian if and only if $\mathscr{L}_{\mathrm{X}}\omega = 0$. It follows that, if X is locally hamiltonian, it generates a one-parameter group of symplectic transformations. For this reason, locally hamiltonian vector fields are called *infinitesimal symplectic transformations.*

PROPOSITION 5.1. — *A vector field* $\mathrm{X} : \mathrm{P} \to \mathrm{TP}$ *is locally hamiltonian if and only if the image* D′ *of* X *is a lagrangian submanifold of* $(\mathrm{TP}, \dot{\omega})$.

*Proof.* — It is obvious that D′ is isotropic if and only if $\mathrm{X}^*\dot{\omega} = d(\mathrm{X}^*\chi) = 0$. We show that $\mathrm{X}^*\chi$ is closed if and only if X is locally hamiltonian:

$$\mathrm{X}^*\chi = \mathrm{X}^*\beta^*\vartheta_{\mathrm{P}} = (\beta \circ \mathrm{X})^*\vartheta_{\mathrm{P}} = \beta \circ \mathrm{X} = \mathrm{X} \lrcorner\, \omega .$$

Since $\dim \mathrm{D}' = \dim \mathrm{P} = \frac{1}{2} \dim \mathrm{TP}$, D′ is lagrangian if and only if X is locally hamiltonian, q. e. d.

Relativistic generalizations of hamiltonian dynamics require passing from infinitesimal transformations to infinitesimal relations. Proposition 5.1 suggests a generalization of the concept of a locally hamiltonian vector field. We consider infinitesimal relations in P which are lagrangian submanifolds of $(\mathrm{TP}, \dot{\omega})$.

DEFINITION 5.2. — A differential equation $\mathrm{D}' \subset \mathrm{TP}$ is called an *infinitesimal symplectic relation* if it is integrable and is a lagrangian submanifold of $(\mathrm{TP}, \dot{\omega})$.

## 6. GENERALIZED HAMILTONIAN DYNAMICS

A generalization of Hamiltonian dynamics was proposed by Dirac [3]. The characteristic feature of this generalization is the appearance of a constraint submanifold on which the Hamiltonian is defined. We give a construction of differential equations appearing in Dirac's generalized dynamics independent of local coordinates.

PROPOSITION 6.1. — *Let* $K \subset P$ *be a submanifold and* $H : K \to \mathbb{R}$ *a differentiable function. The set*

$$D' = \{ w \in TP \,;\; p = \tau_P(w) \in K,\ \forall u \in T_pK,\ \langle w \wedge u, \omega \rangle = -\langle u, dH \rangle \} \quad (6.1)$$

*is a lagrangian submanifold of* $(TP, \dot{\omega})$.

*Proof.* — Let

$$N = \{ r \in T^*P \,;\; p = \pi_P(r) \in K,\ \forall u \in T_pK,\ \langle u, r \rangle = - \langle u, dH \rangle \}$$

be the lagrangian submanifold of $(T^*P, \omega_P)$ generated by H in the sense of Proposition 2.1. The inverse image $\beta^{-1}(N)$ of this submanifold by the diffeomorphism $\beta : TP \to T^*P$ defined in Section 3 is a lagrangian submanifold of $(TP, \dot{\omega})$. It follows from the definition of $\beta$ that $\pi_P(\beta(w)) = \tau_P(w)$ and $\langle u, \beta(w) \rangle = \langle u, w \lrcorner \omega \rangle = \langle w \wedge u, \omega \rangle$. We see that $w \in D'$ if and only if $\beta(w) \in N$. Hence, $\beta^{-1}(N) = D'$ and $D'$ is a lagrangian submanifold of $(TP, \dot{\omega})$, q. e. d.

DEFINITION 6.1. — The lagrangian submanifold $D'$ introduced in Proposition 6.1 is said to be *generated* by the function H, and H is called a *Hamiltonian* of $D'$. The submanifold K is called the *constraint submanifold* and $D'$ is called a *generalized Hamiltonian system.*

If $D'$ is to describe the dynamics of a mechanical system, it must be integrable, that is it must be an infinitesimal relation. Integrability of $D'$ depends on certain properties of the constraint submanifold K. We give a classification of constraints in the next section and return to the problem of integrability in Section 8.

## 7. CLASSIFICATION OF CONSTRAINTS

At each point $p$ of a submanifold $K \subset P$ we consider spaces $T_pK \subset T_pP$ and $(T_pK)^{\S}$. In terms of these two spaces we have the following definitions [11] :

*a*) K is said to be *isotropic* at $p$ if $T_pK \subset (T_pK)^{\S}$.

*b*) K is said to be *coisotropic* at $p$ if $T_pK \supset (T_pK)^{\S}$.

A coisotropic submanifold $K \subset P$ is also called a *first class constraint submanifold*.

*c*) K is said to be *lagrangian* at $p$ if it is isotropic and coisotropic: $T_pK = (T_pK)^\S$.

*d*) K is said to be *symplectic* at $p$ if $T_pK \cap (T_pK)^\S = \{0\}$.

These definitions are related to the classification of submanifolds of P, introduced by Dirac [3]. We derive Dirac's classification of constraints by coordinate independent methods.

Let $N_p$ be the intersection of $T_pK$ and $(T_pK)^\S$, and let $l$ and $k$ denote the dimension of $N_p$ and the codimension of K, respectively.

DEFINITION 7.1. — The pair of numbers $(l, k - l)$ is called the *class* of the submanifold K at $p$.

We note that $k - l$ is even. The 2-form $\omega_p$ restricted to $(T_pK)^\S$ is in general degenerate. However, the 2-form induced on the quotient space $(T_pK)^\S/N_p$ is non-degenerate. Since $(T_pK)^\S/N_p$ is a symplectic vector space, and $k-l$ is its dimension, it follows that $k - l$ is even. Further, we note that

$$l \leq k, \quad l \leq 2m - k,$$

and consequently $l \leq m$.

Since $k - l$ is the rank of $\omega \mid K$ at $p$, the class of K at $p$ can also be defined as the pair $(k - r, r)$, where $k$ is the codimension of K and $r$ is the rank of $\omega \mid K$ at $p$. The integer $r$ is even being the rank of a 2-form.

In terms of the class of a submanifold K we have the following criteria:

*a*) K is isotropic at $p$ if and only if $l = 2m - k$.

*b*) K is coisotropic at $p$ if and only if $l = k$.

*c*) K is langrangian at $p$ if and only if $l = k = m$.

*d*) K is symplectic at $p$ if and only if $l = 0$.

Let K be isotropic at $p$. Then $T_pK \subset (T_pK)^\S$, or equivalently $T_pK = N_p$. The dimensions of $T_pK$ and $N_p$ are $2m - k$ and $l$ respectively. It follows that $2m - k = l$. Conversely, if $2m - k = l$, then $T_pK = N_p$ since $N_p$ by its definition is contained in $T_pK$. It follows that $T_pK \subset (T_pK)^\S$. This proves criterion *a*). The remaining criteria are equally easy to prove.

We establish a relation between our classification of constraints, Dirac's original classification and the interpretation given by Lichnerowicz.

Let $\mathscr{X}$ denote the Lie algebra of vector fields on P and let $\mathscr{H}$ be the subalgebra of (globally) hamiltonian fields. We define subalgebras $\bar{\mathscr{K}} \subset \mathscr{X}$ and $\mathscr{K} \subset \mathscr{H}$:

$$\bar{\mathscr{K}} = \{ X \in \mathscr{X} ; X(K) \subset TK \}, \tag{7.1}$$

$$\mathscr{K} = \{ X \in \mathscr{H} ; X(K) \subset TK \}. \tag{7.2}$$

We also introduce subsets $\mathscr{K}^\S \subset \mathscr{H}$ and $\mathscr{N} \subset \mathscr{H}$:

$$\mathscr{K}^\S = \left\{ X \in \mathscr{H} ; X(K) \subset (TK)^\S = \bigcup_{p \in K} (T_pK)^\S \right\}, \tag{7.3}$$

$$\mathscr{N} = \left\{ X \in \mathscr{H} ; X(K) \subset N = \bigcup_{p \in K} N_p \right\} = \mathscr{K} \cap \mathscr{K}^\S. \tag{7.4}$$

PROPOSITION 7.1. — *The set $\mathcal{N}$ is an ideal of $\mathcal{K}$.*

*Proof.* — Let X, Y and Z be elements of $\mathcal{N}$, $\mathcal{K}$ and $\bar{\mathcal{K}}$ respectively. Then

$$\langle [Y, X] \wedge Z, \omega \rangle | K = [\mathcal{L}_Y \langle X \wedge Z, \omega \rangle - \langle X \wedge [Y, Z], \omega \rangle - \langle X \wedge Z, d(Y \lrcorner \omega) \rangle + \langle X \wedge Y \wedge Z, d\omega \rangle] | K = 0 .$$

It follows that $[X, Y] \in \mathcal{N}$. Hence $\mathcal{N}$ is an ideal of $\mathcal{K}$, q. e. d.

Dirac's classification of constraints is based on the Poisson bracket structure, which is equivalent to the symplectic structure [*4*] [*10*]. Let $f$ and $g$ be functions on P. The *Poisson bracket* of $f$ and $g$ is the function

$$\{ f, g \} = \langle \beta^{-1} \circ dt, dg \rangle . \tag{7.5}$$

Introducing a bivector field G on P such that

$$G \llcorner \mu = \beta^{-1} \circ \mu \tag{7.6}$$

for each 1-form $\mu$, we write

$$\{f, g\} = \langle G, df \wedge dg \rangle . \tag{7.7}$$

The family $\mathcal{P}$ of functions on P together with the Poisson bracket $\{ \ , \ \}$ form a Lie algebra called the *Poisson algebra.* Following Lichnerowicz [*5*] we define subsets $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{A}$ of $\mathcal{P}$:

$$\mathcal{C} = \{ f \in \mathcal{P} ; f | K = \text{const} \} , \tag{7.8}$$

$$\mathcal{B} = \{ f \in \mathcal{P} ; \{ f, g \} | K = 0 \quad \text{for each} \quad g \in \mathcal{C} \} , \tag{7.9}$$

$$\mathcal{A} = \mathcal{C} \cap \mathcal{B} . \tag{7.10}$$

Elements of $\mathcal{C}$ are called *constraints,* elements of $\mathcal{B}$ are *first class functions* and elements of $\mathcal{A}$ are called *first class constraints.*

PROPOSITION 7.2. — *A function $f$ belongs to $\mathcal{C}$ if and only if* $X = G \llcorner df \in \mathcal{K}^{\S}$.

*Proof.* — Let $Y \in \bar{\mathcal{K}}$. We have $\langle Y, df \rangle = \langle X \wedge Y, \omega \rangle$. If $X \in \mathcal{K}^{\S}$ then $\langle Y, df \rangle | K = \langle X \wedge Y, \omega \rangle | K = 0$. It follows that $f \in \mathcal{C}$. If $f \in \mathcal{C}$ then $\langle X \wedge Y, \omega \rangle | K = \langle Y, df \rangle | K = 0$. Hence $X \in \mathcal{K}^{\S}$, q. e. d.

PROPOSITION 7.2. — *A function $f$ belongs to $\mathcal{B}$ if and only if* $X = G \llcorner df \in \mathcal{K}$.

*Proof.* — Let $g \in \mathcal{C}$. We have $\{ f, g \} = \langle G, df \wedge dg \rangle = \langle X, dg \rangle$. If $X \in \mathcal{K}$ then $\langle X, dg \rangle | K = 0$. It follows that $\{ f, g \} | K = 0$. Consequently $f \in \mathcal{B}$. If $f \in \mathcal{B}$ then $\{ f, g \} | K = 0$. Hence $\langle x, dg \rangle | K = 0$. It follows that $X \in \mathcal{K}$, q. e. d.

The following conclusions can be drawn from the two propositions above:

*i*) $f \in \mathcal{A}$ if and only if $X = G \llcorner df \in \mathcal{N}$,

*ii*) $\mathcal{B}$ is a subalgebra of the Poisson algebra $\mathcal{P}$,

*iii*) $\mathcal{A}$ is an ideal of $\mathcal{B}$.

Throughout the remainder of this section it is assumed that the class of K is constant. An immediate consequence of this assumption is that $N = \bigcup_{p \in K} N_p$ is a distribution on K. We note that N is exactly the *characteristic distribution* of $\omega \mid K$:

$$N = \{ u \in TK ; \; u \lrcorner (\omega \mid K) = 0 \} . \tag{7.11}$$

PROPOSITION 7.4. — *The distribution* N *is completely integrable.*

*Proof.* — Let $X : K \to TK$ and $Y : K \to TK$ be vector fields such that $\mathrm{im}\,(X) \subset N$ and $\mathrm{im}\,(Y) \subset N$. Then $X \lrcorner (\omega \mid K) = 0$ and $Y \lrcorner (\omega \mid K) = 0$. The identity

$$[X, Y] \lrcorner \rho = Y \lrcorner d(X \lrcorner \rho) - X \lrcorner d(Y \lrcorner \rho) + d((X \wedge Y) \lrcorner \rho) + (X \wedge Y) \lrcorner d\rho$$

holds for any differential form $\rho$ on K. Setting $\rho = \omega \mid K$ we obtain

$$[X, Y] \lrcorner (\omega \mid K) = 0 .$$

Hence $\mathrm{im}\,([X, Y]) \subset N$. It follows from Frobenius' theorem that N is completely integrable, q. e. d.

The foliation of K by maximal integral manifolds of N is called the characteristic foliation of $\omega \mid K$.

In a suitable neighbourhood of each point of K it is possible to find $k$ independent constraints $\psi^a$; $a = 1, \ldots, k$ such that $\psi^a \mid K = 0$. Simple algebraic considerations show that functions $G^{ab} = \{ \psi^a, \psi^b \} \mid K$ form a matrix of constant rank $k - l$. Let functions $C^\alpha_a$; $\alpha = 1, \ldots, l$ be $l$ linearly independent solutions of equations $(C^\alpha_a \mid K)G^{ab} = 0$. Then $(C^\alpha_a \psi^a) \mid K = 0$ and $\{ C^\alpha_a \psi^a, \psi^b \} \mid K = (C^\alpha_a \mid K)G^{ab} = 0$. It follows that functions $\varphi^\alpha = C^\alpha_a \psi^a$ are first class constraints. Let $(\varphi^\alpha, \varphi^A)$; $\alpha = 1, \ldots, l$; $A = 1, \ldots, k - l$ are independent constraints then functions $\{ \varphi^A, \varphi^B \} \mid K$ form a nonsingular matrix. Dirac calls functions $\varphi^A$ *second class constraints.*

The following proposition relates our classification of constraints to Dirac's classification.

PROPOSITION 7.5. — *The class of a submanifold* $K \subset P$ *is* $(l, k - l)$ *if and only if in a neighbourhood of each point of* K *there is a system of independent constraints of which* $l$ *are first class and* $k - l$ *are second class.*

One part of the proposition is proved above. The other part is easy to prove.

## 8. INTEGRABILITY

We return to the problem of integrability of generalized Hamiltonian systems introduced in Section 6.

THEOREM 8.1. — *The generalized Hamiltonian system* D′ *generated by* *a Hamiltonian* H *defined on a constraint submanifold* K *is integrable if and only if* K *is a first class constraint manifold and* H *is constant on leaves of the characteristic foliation of* $\omega \mid \mathrm{K}$.

*Proof.* — We assume first that D′ is integrable. If $\gamma : \mathrm{I} \to \mathrm{P}$ is an integral curve of D′, then we have $\dot{\gamma}(t) \in \mathrm{D}'$ and $\tau_{\mathrm{P}}(\dot{\gamma}(t)) = \gamma(t)$ for each $t \in \mathrm{I}$. It follows from $\tau_{\mathrm{P}}(\mathrm{D}') \subset \mathrm{K}$ that $\gamma(t) \in \mathrm{K}$ for each $t \in \mathrm{I}$. Since D′ is integrable, each vector $w \in \mathrm{D}'$ is tangent to an integral curve and consequently belongs to TK. We conclude that $\mathrm{D}' \subset \mathrm{TK}$. At each point $p \in \mathrm{K}$, we have the set $\mathrm{D}'_p = \{ u \in \mathrm{T}_p\mathrm{P} ; \forall v \in \mathrm{T}_p\mathrm{K}, \langle v \wedge u, \omega \rangle = - \langle u, d\mathrm{H} \rangle \}$. Comparing this set with the subspace $(\mathrm{T}_p\mathrm{K})^\S = \{ w \in \mathrm{T}_p\mathrm{P} ; \forall v \in \mathrm{T}_p\mathrm{K}, \langle v \wedge u, \omega \rangle = 0 \}$, we see that $(\mathrm{T}_p\mathrm{K})^\S = \mathrm{D}'_p - w_p$, where $w_p$ is any fixed element of $\mathrm{D}'_p$. Since $\mathrm{D}'_p \subset \mathrm{T}_p\mathrm{K}$, it follows that $(\mathrm{T}_p\mathrm{K})^\S \subset \mathrm{T}_p\mathrm{K}$. We conclude that K is a first class constraint. For a first class constraint the characteristic distribution is $(\mathrm{TK})^\S = \bigcup_{p \in \mathrm{K}} (\mathrm{T}_p\mathrm{K})^\S$. Let $u$ be an element of $(\mathrm{T}_p\mathrm{K})^\S$ and let $w$ be any element of $\mathrm{D}'_p$. Then $w \in \mathrm{T}_p\mathrm{K}$ and $\langle u, d\mathrm{H} \rangle = - \langle w \wedge u, \omega \rangle$. Since $w \in \mathrm{T}_p\mathrm{K}$ and $u \in (\mathrm{T}_p\mathrm{K})^\S$, it follows that $\langle w \wedge u, \omega \rangle = 0$. Hence, $\langle u, d\mathrm{H} \rangle = 0$ and consequently H is constant on the characteristic foliation of $\omega \mid \mathrm{K}$.

Now we assume that K is a first class constraint. Let $w$ be an element of $\mathrm{D}_{p'}$ and let S be a local section of the characteristic foliation of $\omega \mid \mathrm{K}$ which passes through $p$ and is tangent to $w$. We show that S is a symplectic submanifold of P; this means that at each point $p' \in \mathrm{S}$ we have $\mathrm{T}_{p'}\mathrm{S} \cap (\mathrm{T}_{p'}\mathrm{S})^\S = \{ 0 \}$. Since S is a section of the characteristic foliation, we have $\mathrm{T}_{p'}\mathrm{S} \cap (\mathrm{T}_p\mathrm{K})^\S = \{ 0 \}$ and $\mathrm{T}_{p'}\mathrm{S} + (\mathrm{T}_{p'}\mathrm{K})^\S = \mathrm{T}_{p'}\mathrm{K}$. It follows that

$$
\begin{aligned}
\mathrm{T}_{p'}\mathrm{S} \cap (\mathrm{T}_{p'}\mathrm{S})^\S &= \{ u \in \mathrm{T}_{p'}\mathrm{S} ; \forall v \in \mathrm{T}_{p'}\mathrm{S}, \langle u \wedge v, \omega \rangle = 0 \} \\
&\subset \mathrm{T}_{p'}\mathrm{S} \cap (\mathrm{T}_{p'}\mathrm{K})^\S = \{ u \in \mathrm{T}_{p'}\mathrm{S} ; \forall v \in \mathrm{T}_{p'}\mathrm{K}, \langle u \wedge v, \omega \rangle = 0 \} = \{ 0 \}.
\end{aligned}
$$

The Hamiltonian H restricted to S generates a Hamiltonian vector field $\mathrm{X} : \mathrm{S} \to \mathrm{TS} \subset \mathrm{TP}$ such that $\mathrm{X} \lrcorner (\omega \mid \mathrm{S}) = - d(\mathrm{H} \mid \mathrm{S})$, let $\gamma : \mathrm{I} \to \mathrm{S} \subset \mathrm{P}$ be an integral curve of X. Then $\dot{\gamma}(t) = \mathrm{X}(\gamma(t))$ and for each vector $u \in \mathrm{T}_{\gamma(t)}\mathrm{K}$ we have $\langle \mathrm{X}(\gamma(t)) \wedge u, \omega \rangle = \langle u, \mathrm{X} \lrcorner \omega \rangle = - \langle u, d\mathrm{H} \rangle$. It follows that $\gamma$ is an integral curve of D′. If in particular $\gamma(0) = p$, then $\dot{\gamma}(0) = \mathrm{X}(p)$. Since $\mathrm{X}(p) \in \mathrm{D}'_p$ and $w$ is the unique element of $\mathrm{D}'_p$ which is tangent to S, we conclude that $\dot{\gamma}(0) = w$. Hence D′ is integrable, q. e. d.

Let K be a constraint submanifold of constant class but not necessarily purely first class. Integrability does not hold for the lagrangian submanifold D′ generated by a Hamiltonian H constant on the leaves of characteristic foliation although $\mathrm{D}' \cap \mathrm{TK}$ can be shown to be integrable. The infinitesimal relation $\mathrm{D}' \cap \mathrm{TK}$ is an isotropic submanifold but not a lagrangian submanifold when second class constraints are present.

We do not consider second class constraints important for dynamics. In our opinion it is unreasonable to expect dynamics with second class constraints to be the classical limit of quantum dynamics. No natural examples of second class constraints in particle mechanics are known. Artificial examples can be easily given: interpreting the phase space of a system as the configuration space leads to purely second class constraints. It is our opinion that second class constraints arise only when the phase space of a system is incorrectly interpreted.

## 9. EXAMPLES

*a*) Nonrelativistic dynamics of a charged particle. Gauge dependent formulation.

We consider the motion of a particle of mass $m$ and charge $e$ in an external static electromagnetic field. The configuration space of the particle is a riemannian manifold M of dimension 3. The riemannian structure of M is represented by *metric tensors*

$$g : \mathrm{M} \to \mathrm{T^*M} \otimes \mathrm{T^*M} \qquad \text{and} \qquad \tilde{g} : \mathrm{M} \to \mathrm{TM} \otimes \mathrm{TM}\,.$$

The electromagnetic field is represented by the *scalar potential* $\mathrm{V} : \mathrm{M} \to \mathbb{R}$, the *vector potential* $\mathrm{A} : \mathrm{M} \to \mathrm{T^*M}$, the *electric field* $\mathrm{E} = -d\mathrm{V}$ and the *magnetic induction* $\mathrm{B} = d\mathrm{A}$.

The phase space of the particle is the cotangent bundle $\mathrm{P} = \mathrm{T^*M}$, there are no constraints and a Hamiltonian $\mathrm{H} : \mathrm{T^*M} \to \mathbb{R}$ is defined by

$$\mathrm{H}(p) = \frac{1}{2m} \langle\, g, (p - e\mathrm{A}(x)) \otimes (p - e\mathrm{A}(x)) \,\rangle + e\mathrm{V}(x), \tag{9.1}$$

where $x = \pi_{\mathrm{M}}(p)$. The infinitesimal symplectic relation D′ generated by the Hamiltonian is the image of the hamiltonian vector field $\mathrm{X} = -\mathrm{G} \llcorner d\mathrm{H}$. We decompose this field into horizontal and vertical components using the riemannian connection in T*M.

If $(x^i)$; $i = 1, 2, 3$ is a coordinate system of M and $(x^i, p_j)$; $i, j = 1, 2, 3$ is the induced coordinate system of P such that $\vartheta_{\mathrm{M}} = p_i dx^i$ then $\omega_{\mathrm{M}} = dp_i \wedge dx^i$. This is true in particular when $(x^i)$ is a geodesic normal coordinate system at any point $x \in \mathrm{M}$. It follows easily that

$$\langle\, u \wedge v, \omega_{\mathrm{M}} \,\rangle = \langle\, \mathrm{hor}\,(v),\ \mathrm{ver}\,(u) \,\rangle - \langle\, \mathrm{hor}\,(u),\ \mathrm{ver}\,(v) \,\rangle\,, \tag{9.2}$$

where $u$ and $v$ are vectors in $\mathrm{T}_p\mathrm{P}$ at some point $p \in \mathrm{P}$, hor $(u)$ and hor $(v)$ are vectors in $\mathrm{T}_x\mathrm{M}$, $x = \pi_{\mathrm{M}}(p)$ obtained by projecting to TM the horizontal components of $u$ and $v$, ver $(u)$ and ver $(v)$ are covectors in $\mathrm{T}^*_x\mathrm{M}$ obtained by identifying vertical components of $u$ and $v$ with elements of T*M.

Let $u \in \mathrm{T}_p\mathrm{P}$ be a vertical vector. From $\mathrm{X} \lrcorner\, \omega_{\mathrm{M}} = -d\mathrm{H}$ we obtain

$$\langle\, \mathrm{hor}\,(\mathrm{X}),\ \mathrm{ver}\,(u) \,\rangle = \langle\, u, d\mathrm{H} \,\rangle = \frac{1}{m} \langle\, \tilde{g}, \mathrm{ver}\,(u) \otimes (p - e\mathrm{A}(\mathrm{X})) \,\rangle\,. \tag{9.3}$$

If $v \in T_pP$ is a horizontal vector then

$$\langle \text{hor}(v), \text{ver}(X) \rangle = -\langle v, dH \rangle$$
$$= \frac{e}{m}\langle \tilde{g}, \nabla_{\text{hor}(v)} A \otimes (p - eA(X)) \rangle - e\langle \text{hor}(v), dV \rangle, \quad (9.4)$$

where $\nabla_{\text{hor}(v)}A$ is the covariant derivative of A in the direction of hor $(v)$. Let $(x^i)$; $i = 1, 2, 3$ be a coordinate system of M and $(x^i, p_j)$; $i, j = 1, 2, 3$ the induced coordinate system of P. The content of equations (9.3) and (9.4) is transcribed into differential equations

$$\frac{dx^i}{dt} = \frac{1}{m} g^{ij}(p_j - eA_j) \quad (9.5)$$

and

$$\frac{Dp_j}{dt} = \frac{e}{m} g^{kl}\nabla_j A_l(p_k - eA_k) - e\partial_j V \quad (9.6)$$

equivalent to the familiar second order system:

$$p_i = mg_{ij}\frac{dx^j}{dt} + eA_i, \quad (9.7)$$

$$mg_{ij}\frac{D}{dt}\frac{dx^j}{dt} = e(\partial_i A_j - \partial_j A_i)\frac{dx^j}{dt} - e\partial_i V = eB_{ij}\frac{dx^j}{dt} + eE_i. \quad (9.8)$$

We denoted by $g_{ij}$ and $g^{kl}$ components of tensors $g$ and $\tilde{g}$ respectively, $\frac{D}{dt}$ is the absolute (covariant) derivative and $t$ is interpreted as time.

*b*) Relativistic dynamics of a charged particle. Gauge dependent formulation.

We consider the motion of a particle of mass $m$ and charge $e$ in an external electromagnetic field in space-time. The configuration space of the particle is a pseudoriemannian manifold M of dimension 4 representing space-time. The pseudoriemannian structure is represented by *metric tensors* $g : M \to T^*M \otimes T^*M$ and $\tilde{g} : M \to TM \otimes TM$. The electromagnetic field is represented by the *potential* $A : M \to T^*M$ and the *field* $F = -dA$.

The phase space is the cotangent bundle $P = T^*M$. A first class constraint submanifold $K \subset P$ of codimension 1 is defined by

$$K = \{ p \in P ; \langle \tilde{g}, (p - eA(x)) \otimes (p - eA(x)) \rangle = m^2, \ x = \pi_M(p) \} \quad (9.9)$$

and the Hamiltonian is zero. The infinitesimal symplectic relation D′ generated by the Hamiltonian is the characteristic distribution N of $\omega_M | K$. Phase space trajectories of the particle are the integral manifolds of D′.

It is convenient for technical reasons to introduce the hamiltonian vector field $\bar{X} = - G \llcorner d\bar{H}$, where $\bar{H}$ is the function

$$\bar{H} : P \to \mathbb{R} : p \to \langle \tilde{g}, (p - eA(x)) \otimes (p - eA(x)) \rangle^{\frac{1}{2}} \quad (9.10)$$

differentiable everywhere except when $p = eA(x)$. Since H is a first class constraint the field $X = \bar{X} \mid K$ belongs to the ideal N. Integral curves of X are parametrized phase space trajectories of the particle. From

$$\bar{X} \lrcorner\, \omega_M = - d\bar{H} \tag{9.11}$$

we obtain

$$\begin{aligned} \langle \operatorname{hor}(\bar{X}), \operatorname{ver}(u) \rangle &= \langle u, d\bar{H} \rangle \\ &= (\bar{H}(p))^{-1} \langle \tilde{g}, \operatorname{ver}(u) \otimes (p - eA(x)) \rangle \end{aligned} \tag{9.12}$$

for each vertical vector $u \in T_pP$ and

$$\begin{aligned} \langle \operatorname{hor}(v), \operatorname{ver}(\bar{X}) \rangle &= - \langle v, d\bar{H} \rangle \\ &= e(\bar{H}(p))^{-1} \langle g, \nabla_{\operatorname{hor}(v)} A \otimes (p - eA(x)) \rangle \end{aligned} \tag{9.13}$$

for each horizontal vector $v \in T_pP$.

The field X satisfies

$$\langle \operatorname{hor}(X), \operatorname{ver}(u) \rangle = \frac{1}{m} \langle \tilde{g}, \operatorname{ver}(u) \otimes (p - eA(x)) \rangle \tag{9.14}$$

for each vertical vector $u \in T_pP$, and

$$\langle \operatorname{hor}(v), \operatorname{ver}(X) \rangle = \frac{e}{m} \langle \tilde{g}, \nabla_{\operatorname{hor}(v)} A \otimes (p - eA(x)) \rangle \tag{9.15}$$

for each horizontal vector $v \in T_pP$. From (9.14) we see that the horizontal component of $X(p)$ projected to TM is a unit vector in $T_xM$. It follows that integral curves of X are trajectories parametrized by proper time.

We write differential equations for parametrized trajectories in terms of coordinates $(x^\varkappa)$; $\varkappa = 0, 1, 2, 3$ of M and the induced coordinates $(x^\varkappa, p_\lambda)$; $\varkappa, \lambda = 1, 2, 3$ of P:

$$\frac{dx^\varkappa}{ds} = \frac{1}{m} g^{\varkappa\lambda}(p_\lambda - eA_\lambda), \tag{9.16}$$

$$\frac{Dp_\lambda}{ds} = \frac{e}{m} g^{\mu\nu} \nabla_\lambda A_\mu (p_\nu - eA_\nu). \tag{9.17}$$

These first order equations are equivalent to the second order system:

$$p_\varkappa = m g_{\varkappa\lambda} \frac{dx^\lambda}{ds} + eA_\varkappa \tag{9.18}$$

$$m g_{\varkappa\lambda} \frac{D}{ds} \frac{dx^\lambda}{ds} = e(\partial_\varkappa A_\lambda - \partial_\lambda A_\varkappa) \frac{dx^\lambda}{ds} = - eF_{\varkappa\lambda} \frac{dx^\lambda}{ds}. \tag{9.19}$$

*c*) Nonrelativistic dynamics of a charged particle. Gauge independent formulation.

Let M be the riemannian manifold of Example *a*) and let S be a *principal* *fibre bundle* [2] with *base* M and *projection* $\xi : S \to M$. The structure group is the group $\mathbb{R}$ of real numbers with respect to addition. The action of the structure group on S defines a one-parameter group $\gamma_\tau$, $\tau \in \mathbb{R}$ of differentiable transformations of S. The infinitesional generator of this group is a vector field $Z : S \to TS$ called the *fundamental vector field.* There is a *connection form* $\alpha$ on S and a *curvature form* $\beta = d\alpha$. The connection form $\alpha$ is a 1-form such that $\langle Z, \alpha \rangle = 1$ and $\mathscr{L}_Z\alpha = 0$. Each vector $w \in TS$ has a unique decomposition $w = w^v + w^h$ into the *vertical component* $w^v$ in the direction of Z and the *horizontal component* $w^h$ satisfying $\langle w^h, \alpha \rangle = 0$. The curvature form $\beta$ satisfies conditions $Z \lrcorner \beta = 0$ and $\mathscr{L}_Z\beta = 0$ necessary and sufficient for the existence of a 2-form B on M such that $\beta = \xi^*B$. The form B represents the magnetic induction and $\alpha$ is the *gauge invariant potential.* The electric field is represented by the 1-form $E = -dV$, where $V : M \to \mathbb{R}$ is the scalar potential.

We consider the motion of a particle of mass $m$ and charge $e$. The configuration space is S and the phase space is $R = T^*S$. The true configuration space of a classical particle is M. The principal fibre bundle S is used only as a device for introducing gauge independent quantities. No direct interpretation of S can be given except in quantum mechanics and the Hamilton-Jacobi theory [9]. Each element $r \in T_s^*S$ has a unique decomposition $r = r^v + r^h$ into the *vertical component* $r^v = r - \langle Z, r \rangle \alpha(s)$ such that $\langle u, r^v \rangle = 0$ for each vertical vector $u \in T_sS$, and the *horizontal component* $r^h = \langle Z, r \rangle \alpha(s)$ such that $\langle w, v^h \rangle = 0$ for each horizontal vector $w \in T_sS$. The number $q = \langle Z, r \rangle$ representing the horizontal component $r^h$ of $r$ is interpreted as the charge of the particle. We introduce the mapping $\varkappa : R \to T^*M$ defined by $\pi_M \circ \varkappa = \xi \circ \pi_S$ and $\langle T\xi(w), \varkappa(r) \rangle = \langle w, r^v \rangle$ for each vector $w \in T_sS$, $s = \pi_S(r)$. The covector $p = \varkappa(r)$ representing the vertical component of $r$ is interpreted as the *gauge invariant canonical momentum* of the particle.

A constraint submanifold $K \subset R$ is defined by

$$K = \{ r \in R ;\ \langle Z, r \rangle = e \} \tag{9.20}$$

and a Hamiltonian $H : K \to \mathbb{R}$ is defined by

$$H(r) = \frac{1}{2m} \langle \tilde{g}, \varkappa(r) \otimes \varkappa(r) \rangle + eV(x),\ x = \xi(\pi_S(r)). \tag{9.21}$$

The physical meaning of the constraint is clear: the charge $q$ of the particle is equal to $e$. The Hamiltonian represents the energy of the particle.

The action of the structure group on S generates a one-parameter group $\gamma_\tau^*$, $\tau \in \mathbb{R}$ of symplectic transformations of R. It is easily seen that the first class constraint submanifold K is invariant under the group action. It follows that orbits belonging to K form the characteristic foliation of $\omega_S \mid K$. It also follows that the *reduced symplectic manifold* exists [11]. The action of the structure group preserves the connection form $\alpha$ and the decompositions into vertical and horizontal components defined in terms of $\alpha$. We have $(\gamma_\tau^*(r))^v = \gamma_\tau^*(r^v)$ and also $\xi \circ \gamma_\tau = \xi$. It follows that

$$\begin{aligned}\langle T\xi(w), \varkappa(\gamma_\tau^*(r)) \rangle &= \langle w, (\gamma_\tau^*(r))^v \rangle \\ &= \langle w, \gamma_\tau^*(r^v) \rangle \\ &= \langle T\gamma_\tau(w), r^v \rangle \\ &= \langle T\xi(T\gamma_\tau(w)), \varkappa(r) \rangle \\ &= \langle T\xi(w), \varkappa(r) \rangle . \end{aligned} \tag{9.22}$$

Hence $\varkappa(\gamma_\tau^*(r)) = \varkappa(r)$. The mapping $\varkappa$ restricted to K is a submersion onto $P = T^*M$ with one-dimensional fibres. It follows that the fibres of $\varkappa \mid K$ are the orbits of the structure group. Consequently the reduced phase space is diffeomorphic (although not symplectomorphic) to P. In order to find the correct reduced symplectic structure of P we calculate $(\varkappa \mid K)^*\vartheta_M$. Let $u \in TR$ be a vector tangent to K. Then

$$\begin{aligned}\langle u, \varkappa^*\vartheta_M \rangle &= \langle T\varkappa(u), \vartheta_M \rangle \\ &= \langle T\pi_M(T\varkappa(u)), \tau_P(T\varkappa(u)) \rangle \\ &= \langle T(\pi_M \circ \varkappa)(u), \varkappa(\tau_R(u)) \rangle \\ &= \langle T(\xi \circ \pi_S)(u), \varkappa(\tau_R(u)) \rangle \\ &= \langle T\xi(T\pi_S(u)), \varkappa(\tau_R(u)) \rangle \\ &= \langle T\pi_S(u), (\tau_R(u))^v \rangle \\ &= \langle T\pi_S(u), \tau_R(u) \rangle - e\langle T\tau_S(u), \alpha \rangle \\ &= \langle u, \vartheta_S - e\pi_S^*\alpha \rangle .\end{aligned}$$

Hence

$$(\varkappa \mid K)^*\vartheta_M = (\varkappa^*\vartheta_M) \mid K = (\vartheta_S - e\pi_S^*\alpha) \mid K . \tag{9.23}$$

By differentiating we obtain

$$\begin{aligned}\omega_S \mid K &= (\varkappa^*\omega_M + e\pi_S^*\beta) \mid K \\ &= (\varkappa^*\omega_M + e\pi_S^*\xi^*B) \mid K \\ &= (\varkappa^*\omega_M + e(\xi \circ \pi_S)^*B) \mid K \\ &= (\varkappa^*\omega_M + e(\pi_M \circ \varkappa)^*B) \mid K \\ &= (\varkappa^*\omega_M + e\varkappa^*\pi_M^*B) \mid K \\ &= (\varkappa \mid K)^*(\omega_M + e\pi_M^*B) .\end{aligned}$$

The result is that the reduced symplectic manifold is $(P, \hat{\omega})$, where $\hat{\omega} = \omega_M + e\pi_M^*B$.

The Hamiltonian H is constant on leaves of the characteristic foliation of $\omega_S \mid K$. Consequently there is the *reduced Hamiltonian*

$$\hat{H} : P \to \mathbb{R} : p \to \frac{1}{2m}\langle \tilde{g}, p \otimes p \rangle + eV(x), \; x = \pi_M(p) \tag{9.24}$$

such that $H = \hat{H} \circ (\varkappa \mid K)$. Trajectories of the particle in the reduced phase space are integral curves of the hamiltonian vector field $\hat{X}$ generated by $\hat{H}$.

Corresponding to (9.2) we have the formula

$$\langle u \wedge v, \hat{\omega} \rangle = \langle \mathrm{hor}(v), \mathrm{ver}(u) \rangle - \langle \mathrm{hor}(u), \mathrm{ver}(v) \rangle + e \langle \mathrm{hor}(u) \wedge \mathrm{hor}(v), \mathrm{B} \rangle . \quad (9.25)$$

From $\hat{\mathrm{X}} \lrcorner \hat{\omega} = - d\hat{\mathrm{H}}$ we obtain

$$\langle \mathrm{hor}(\hat{\mathrm{X}}), \mathrm{ver}(u) \rangle = \langle u, d\hat{\mathrm{H}} \rangle = \frac{1}{m} \langle \tilde{g}, \mathrm{ver}(u) \otimes p \rangle \quad (9.26)$$

for each vertical vector $u \in \mathrm{T}_p\mathrm{P}$ and

$$\langle \mathrm{hor}(v), \mathrm{ver}(\hat{\mathrm{X}}) \rangle + e \langle \mathrm{hor}(\hat{\mathrm{X}}) \wedge \mathrm{hor}(v), \mathrm{B} \rangle = - \langle v, d\hat{\mathrm{H}} \rangle = - e \langle \mathrm{hor}(v), d\mathrm{V} \rangle \quad (9.27)$$

for each horizontal vector $v \in \mathrm{T}_p\mathrm{P}$. In terms of coordinates $(x^i, p_j)$; $i, j = 1, 2, 3$ of P we have the differential equations

$$\frac{dx^i}{dt} = \frac{1}{m} g^{ij} p_j \quad (9.28)$$

and

$$\frac{\mathrm{D}p_j}{dt} = \frac{e}{m} g^{kl} \mathrm{B}_{jl} p_k - e\partial_j \mathrm{V} \quad (9.29)$$

equivalent to

$$p_i = m g_{ij} \frac{dx^j}{dt} \quad (9.30)$$

and

$$m g_{ij} \frac{\mathrm{D}}{dt} \frac{dx^j}{dt} = e\mathrm{B}_{ij} \frac{dx^j}{dt} + e\mathrm{E}_i . \quad (9.31)$$

These equations contain no gauge dependent quantities.

The above example is an elegant illustration of the version of generalized dynamics formulated by Lichnerowicz [5]. The Hamiltonian is not zero and leads to a hamiltonian vector field in the reduced phase space.

*d*) Relativistic dynamics of a charged particle. Gauge independent formulation (cf. [8] [9]).

Let M be the pseudoriemannian manifold of Example *b*) and let S be a principal fibre bundle with base M, projection $\xi : \mathrm{S} \to \mathrm{M}$ and structure group $\mathbb{R}$. There is a connection form $\alpha$ and a curvature form $\varphi = - d\alpha$. The 2-form F on M such that $\varphi = \xi^*\mathrm{F}$ is interpreted as the electromagnetic field and $\alpha$ is the gauge independent potential.

The manifold S is the configuration space of a particle with mass $m$ and charge $e$. The phase space is $\mathrm{R} = \mathrm{T}^*\mathrm{S}$. The dynamics of the particle is generated by a zero Hamiltonian defined on the first class constraint submanifold

$$\mathrm{K} = \{ r \in \mathrm{R} ; \langle \mathrm{Z}, r \rangle = e, \langle g, \varkappa(r) \otimes \varkappa(r) \rangle = m^2 \} , \quad (9.32)$$

where Z and $\varkappa$ are objects defined as in Example *c*). The constraint submanifold can be considered as the intersection of two submanifolds of codimension 1:

$$K_e = \{ r \in R ; \langle Z, r \rangle = e \} \tag{9.33}$$

and

$$K_m = \{ r \in R ; \langle g, \varkappa(r) \otimes \varkappa(r) \rangle = m^2 \} . \tag{9.34}$$

Reduction with respect to $K_e$ leads to the reduced phase space $(P, \hat{\omega})$, where $P = T^*M$ and $\hat{\omega} = \omega_M - e\pi_M^* F$. The reduced constraint is

$$\hat{K} = \{ p \in P ; \langle \tilde{g}, p \otimes p \rangle = m^2 \} . \tag{9.35}$$

As in Example *b*) we introduce the hamiltonian field $\hat{X} = -(G \lrcorner d\bar{H}) \mid K$, where

$$\bar{H} : P \to \mathbb{R} : p \to \langle \tilde{g}, p \otimes p \rangle^{\frac{1}{2}} . \tag{9.36}$$

The field $\hat{D}$ satisfies

$$\langle \operatorname{hor}(\hat{X}), \operatorname{ver}(u) \rangle = \frac{1}{m} \langle \tilde{g}, \operatorname{ver}(u) \otimes p \rangle \tag{9.37}$$

for each vertical vector $u \in T_pP$, and

$$\langle \operatorname{hor}(v), \operatorname{ver}(\hat{X}) \rangle = e \langle \operatorname{hor}(\hat{X}) \wedge \operatorname{hor}(v), F \rangle \tag{9.38}$$

for each horizontal vector $v \in T_pP$. In terms of coordinates $(x^\varkappa, p_\lambda)$; $\varkappa, \lambda = 1, 2, 3$ the corresponding differential equations are

$$\frac{dx^\varkappa}{ds} = \frac{1}{m} g^{\varkappa\lambda} p_\lambda \tag{9.39}$$

and

$$\frac{Dp_\lambda}{ds} = -\frac{1}{m} g^{\mu\nu} F_{\lambda\mu} p_\nu , \tag{9.40}$$

or

$$p_\varkappa = m g_{\varkappa\lambda} \frac{dx^\lambda}{ds} \tag{9.41}$$

and

$$m g_{\varkappa\lambda} \frac{D}{ds} \frac{dx^\lambda}{ds} = -eF_{\varkappa\lambda} \frac{dx^\lambda}{ds} . \tag{9.42}$$

These equations are gauge independent.

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In the second part of the proof of Theorem 8.1 we omitted the case of the vector $w$ tangent to the characteristic foliation of $\omega \mid K$. The proof of integrability in this case is obvious.