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Gauge groups and topological invariants of vacuum manifolds

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ABSTRACT. — The paper is concerned with topological properties of the vacuum manifolds in the theories with the broken gauge symmetry for the groups of the type SO(k) × U(n), SO(k) × SO(p) × U(r). For the Ginsburg-Landau theory of the superfluid ^{3}He the gauge transformations are discussed. They provide the means to indicate all possible types of the vacuum manifolds, which are likely to correspond to distinct phases of the superfluid ^{3}He. Conditions on the existence of the minimums of the Ginsburg-Landau functional are discussed.

The study of classical solutions of equations of the field theory has made clear, that the quantization conditions for the solutions are mostly of topological nature. The attention to the subject was particularly enhanced by the papers by 'tHooft and Polyakov [*1*] [*2*]. The simple condition for the existence of the non-trivial solution with compact gauge group, G, was introduced in [*3*]. At present, a considerable number of papers deals with the existence of the non-trivial solutions and their topological sense, (cf. revs. [*4*] [*5*]). The present paper is concerned with the study of gauge fields and their vacuums.

The scheme is motivated by the observation of an interesting analogy between the Ginsburg-Landau theory for the superfluid ^{3}He and a model of classical fields with several distinct vacuums. From the topological point of view, in the field theory the vacuums of different models are manifolds; the vacuum manifold of 'tHooft-Polyakov in the $\lambda\varphi^4$-theory is a two dimensional sphere S^2 the vacuum manifolds of the A- and B-phases of ^{3}He are manifolds $S^2 \times SO(3)$ and $S^1 \times SO(3)$, where SO(3) is the rotation group of the three dimensional space.

In the present paper we want to focus the attention on the existence of different (i. e. topologically non-equivalent) vacuums within a model of classical fields. Considering this problem within the framework of gauge fields we obtain a variety of topological charges, which are defined by the means of homotopy groups of the vacuum manifolds.

The coexistence of special domains of different vacuums has been considered in a number of papers (cf. [*7*] [*8*]). The specific problem of our case is topological conditions on the domains, which produce new topological charges. This situation resembles very much the possible phase transitions in ^{3}He (cf. [*9*]).

Finally we note, that the gauge groups of the familiar weak, strong and electromagnetic interactions (taking into account the colour, the charm, etc.) are of the form

$$SU(k) \times U(n), \qquad SO(p) \times SO(k) \times U(m).$$

Hence we may expect, that the sphisticated vacuum manifolds will eventually appear.

1. DEGENERATE VACUUMS AND HOMOGENOUS SPACES

We shall consider a model of a field $\Phi(x)$ with values in a linear space (real or complex). Here x is a point in the Minkovski space. The field Φ transforms according to a given representation of a compact group G. The Lagrangian density is taken to be

$$L = \frac{1}{2} | D_\mu \Phi |^2 - V(\Phi) - \frac{1}{4} \underline{\vec{F}}^2 \tag{1}$$

where $D_\mu\Phi$ are covariant derivatives

$$D_\mu \Phi = \partial_\mu \Phi - ig . \underline{\vec{A}} \vec{T} \Phi \tag{2}$$

Here $\vec{T}$ is the isospin operator, $\underline{\vec{A}}$ is a gauge field, $\underline{\vec{F}}$ is the curvature of the gauge field $\vec{A}$. The potential $V(\Phi)$ is invariant under the action of G.

The gauge symmetry is broken when the field Φ has a non-vanishing vacuum expectation, $\langle \Phi \rangle$, which must belong to a set $\mathscr{M}$ of minimums of $V(\Phi)$. Indeed, all vacuums of the model can be treated as the minimums of $V(\Phi)$. Taking into account the invariance of $V(\Phi)$ we assume, that the set $\mathscr{M}$ can be decomposed into certain subsets on which the gauge group acts transitively. Thus we decompose all vacuum states into disjoint pieces of degenerate vacuums invariant under the action of G.

We see, that the isotopic space, J, of the field Φ is foliated by the orbits of the group G. The orbits, on which the potential $V(\Phi)$ takes the minimal values, are the degenerate vacuums $V_1, V_2, \ldots$ We note, that in general the $V_1, V_2, \ldots$ are not of the same dimension.

The transitivity condition implies that the degenerate vacuum is a homogenous set of states, i. e. for every point η of the degenerate vacuum V, we have an isotropy subgroup G_η of G which leave η invariant and for two points η, η' the subgroups $G_\eta, G_{\eta'}$ are isomorph. Hence the degenerate vacuum may be identified with the manifold of the factor space G/G_η [*10*].

Thus from the viewpoint of the theory of Lie groups the degenerate vacuum is a homogenous space. There is a useful proposition (the so-called E. Cartan theorem) which states that a compact symmetric space $\mathcal{S}$ can be imbedded into its group $G\mathcal{S}$ of isometries as a totally geodesic submanifold, i. e. a submanifold which contains all geodesics of the group $G\mathcal{S}$ tangent of $\mathcal{S}$ at some point (for the proof and the discussion cf. [*11*]).

We note that the degenerate vacuums are compact manifolds since they are factor spaces of compact gauge groups and at least for quite a few gauge groups the isometrics groups are identical with the universal covering spaces of these groups [*11*]. E. Cartan's theorem provides a geometrical picture of the degenerate vacuums lying as totally geodesic submanifolds in the gauge group.

Let us consider, for example, the gauge group SU(2) and its representation in the three dimensional real space take the potential

$$V(\varphi) = \alpha . |\varphi|^2 + \beta |\varphi|^4 \tag{3}$$

where φ is a 3-dimensional real vector. The degenerate vacuum is a two dimensional sphere, which is an equator in the 3-dimensional sphere S^3 which is identical to $SU(2)$.

A more sophisticated example is the two dimensional complex projective space, $\mathbb{C}P(2)$, which is a totally geodesic submanifold of SU(3). It can be adequately represented as a factor space SU(3)/U(2). The manifold $\mathbb{C}P(2)$ provides some degenerate vacuums with rather unusual properties concerning the topological charges (there are two of them), the instanton solutions, etc.

2. BOUNDARY CONDITIONS AND SOPHISTICATED VACUUMS

To determine a solution of the Euler equations of the Lagrangian density (1) we need assymptotic conditions at infinity. They are usually of two types. The radial condition for the stationary problem is a field symmetric under the action of the rotation group SO(3). This field is defined on the 2-dimensional sphere of an infinite radius and a center at the point (0, 0, 0) in the 3-dimensional space. If we assume that the center is a point where the solution equals to zero we obtain the monopole solution of 'tHooft and Polyakov. The radial condition for the instanton problem is a field symmetric under the action of the rotation group SO(4) of the euclidean theory [*12*] [*13*]. This field is defined on the 3-dimensional sphere of an infinite radius and a center at the point (0, 0, 0, 0) in the four dimensional euclidean space. We see that in both cases assymptotic conditions are defined by a submanifold (were shall call it the assymptotic submanifold) in the euclidean space and the values taken by the field at the submanifold. We assume that these values are located in a degenerate vacuum manifold in the isotopic space, as in the case of 'tHooft-Polyakov solution. Since the field is defined in the whole interior Q bounded by the assymptotic submanifold P we have a map of the interior Q, which is a manifold with boundary, into the isotopic space of the field and such that the boundary is mapped into the vacuum manifold

$$f : (\mathrm{Q}, \mathrm{P}) \rightarrow (\mathrm{J}, \mathrm{V})$$

Still another form of the assymptotic conditions is to require that the field takes a constant value at infinity. Then we may consider the underlying physical space completed at infinity with a point at which the assymptotic value is taken. Then the whole space is a sphere of dimension 3 for the stationary problem and of dimension 4 for the instanton solution.

The similar situation arises for the 1-dimensional $\lambda\varphi^4$-theory. Indeed, the familiar kink solution requires the condition $\varphi(+\infty) = +1, \varphi(-\infty) = -1$ i. e. we have to complete the line R^1 by two end points.

Now we may introduce a more general boundary problem, the assymptotic conditions at infinity being incorporated. To this end we consider a manifold in the physical space (3-dimensional for the stationary problem $n = 3$, time variable t is fixed, and 4-dimensional for the instanton solution, $n = 4$) with a boundary consisting of several pieces $\mathrm{P}_1^{n-1}, \ldots, \mathrm{P}_\mathrm{K}^{n-1}$ which are manifolds of dimension $n - 1$. The field defined at Q^n yields a continuous map f of Q into the isotopic space. We require that the components of the boundary should be mapped by f into the degenerate vacuum manifolds of J, i. e. we have a bordism

$$f : (\mathrm{Q}^n ; \mathrm{P}_1^{n-1}, \mathrm{P}_2^{n-1}, \ldots, \mathrm{P}_\mathrm{K}^{n-1}) \rightarrow (\mathrm{J} ; \mathrm{V}_1, \mathrm{V}_2, \ldots) \qquad (4)$$

We note that the topological requirements yield necessary conditions for the existence of this map ([1]).

For example, let us consider the stationary problem with the manifold Q^3 of the form $\mathrm{S}^2 \times \mathrm{I}$ a product of the 2-dimensional sphere and a segment, the isotopic space having the degenerate vacuum $\mathrm{S}_1^2 \times \mathrm{S}_2^2$, the product of two 2-dimensional spheres. Then from the requirement that the boundary

([1]) All topological notations used in this paper may be found in [*14*] [*15*] [*16*].

values produce a map of $S^2 \times [0]$ into S_1^2 and $S^2 \times [1]$ into S_2^2 follows that the maps

$$f_i : S^2 \to S_1^2 \times S_2^2, \quad f_i(S^2) = S_i^2, \qquad i = 1, 2$$

are homotopic, since $f_i(x) = F(x, i)$, where F is the map

$$F : Q^3 \to S_1^2 \times S_2^2$$

defined by the field. But the maps belong to the elements of the distinct subgroups of the homotopy group $\pi_2(S_1^2 \times S_2^2) = Z \oplus Z$. Therefore the boundary problem has no solution.

In the 3- or 4-dimensional euclidean spaces, R^3, R^4, the submanifolds may be associated with the domains containing the vacuums. Then the concept of the boundary problem via bordism corresponds to the concept of the domain structure of the space (cf. [7] [8]). Indeed, we may consider it as a generalisation of the 1-dimensional kink, when the solution takes on some vacuum values (the analogues of $\varphi(+\infty)$, $\varphi(-\infty)$ of the 1-dimensional case) in the interior of the domains (the analogues of the infinite regions $\pm\infty$ of the line R^1).

It is clear, that this generalisation introduces topological selection rules.

The topological types of the boundaries of the vacuum domains may be different, for example, in the 3-dimensional case some of them may be 2-dimensional spheres and others 2-dimensional tori, $S_1^2 \times S_2^2$, i. e. products of two circles.

It is interesting to have some means to compare different sets of boundaries of the vacuum domains. Suppose we have a number of parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$ of the problem. Then we may take an extension of the physical space considering the λ_i as additional, if necessary dummy, variables. For the sake of simplicity we shall take only the 3-dimensional case. Let us consider a boundary problem having some set of the parameters

$$\lambda_i = a_i, \qquad i = 1, 2, \ldots, N \tag{5}$$

Then we have the 3-dimensional manifold Q^3 of the problem lying in the hypersurface $\mathscr{H}_\lambda$, defined by these equations. If we take the different values of the parameters

$$\lambda_i = a_i', \qquad i = 1, 2, \ldots, N \tag{5'}$$

we have to consider the different manifolds Q'^3, lying in the hypersurface $\mathscr{H}_{\lambda'}$, defined by the equations. In general, the manifolds Q^3 and Q'^3 are topologically different, since the choice of the values for the λ_i influence the possible choice of the boundaries, for which the boundary problem has a solution.

Now let the λ_i be smooth functions of a parameter τ. Then in the space $R^3 \times R^N$ we have the family of the manifolds, depending on the parameters τ. In general for some values of τ the manifolds Q_τ^3 may have singularities. Hence, we may only expect the set of all Q^3_τ to form a 4-dimensional manifold W^4. If this requirement is fulfilled, the one parameter set of boundary problems corresponds to one bordism class.

We note, that here the formalism of smooth manifolds is rather restrictive. Indeed, some parts of the boundaries may contract and produce singularities, when the parameters λ_i change. Therefore some concepts of the modern Plato problem may turn out useful here. Indeed, for the solutions of the Plato problem one has to work with minimal compacts rather than with smooth manifolds (cf. [*17*]).

The topological invariants of bordism classes provide generalisations of the topological charges (cf. [*4*] [*5*]). Usually topological charge is defined as an element of a homotopy group of a sphere. In this form the topological charge appeared in the papers by 'tHooft and Polyakov [*1*] [*2*], where the vacuum manifold was a sphere. A natural generalisation of this situation is to consider it as an element of homotopy group of some vacuum manifold [*3*]. Then in general we obtain a number of topological charges for the vacuum. For example, taking the vacuum $S^2 \times S^2$, we have

$$\pi_2(S^2 \times S^2) = Z \oplus Z$$

and at least two charges.

Still further generalisation implied by the discussed boundary problem and vacuum domains, is the definition of topological charges as invariants of bordism classes. Here we may expect some connections between topological charges and characteristic classes of smooth manifolds. The most simple vacuum manifold, which could provide some interesting examples, is the 2-dimensional complex projective space, $\mathbb{C}P(2)$. It has non-trivial Chern classes, C_1, C_2, and the Pontrjagin class p_1, [*14*] [*16*]. We note, that the homotopy groups of $\mathbb{C}P(2)$ in the low dimensions are

$$\pi_2(\mathbb{C}P(2)) = Z, \quad \pi_i(\mathbb{C}P(2)) = 0, \qquad i = 1, 3, 4$$

The topological charge, corresponding to the elements of $\pi_2(\mathbb{C}P(2))$ is identical with the charge generated by the first Chern class. But we have also the second charge, corresponding to the second Chern class, C_2. This charge is absent with usual interpretation with homotopy groups.

We note, that the complex projective space $\mathbb{C}P(2)$ is a vacuum manifold of the gauge group SU(3). Indeed, it can be put into the form of the factor space, SU(3)/U(2). Let us consider a situation, where the vacuum manifold $\mathbb{C}P(2)$ appears [*3*] [*2*]). Take the adjoint representation of SU(3). Then the fields variables take the values in the space of 3×3-hermitian matrixes with the trace zero. The action of the gauge group is given by the formula

$$\Phi \to S\Phi S^+, \qquad \Phi = \| \Phi_{ij} \|$$

Consider the potential $V(\Phi)$ of the form

$$\alpha \mathrm{Tr}\,(\phi\phi^+) + \beta[\mathrm{Tr}\,(\phi\phi^+)]^2$$

Then the set of all vacuums of $V(\Phi)$ is the sphere S^7, defined by the equation

$$\mathrm{Tr}\,(\phi\phi^+) = \mathrm{const}$$

This sphere is foliated by the orbits of the gauge group. If an element Φ_0 of S^7 is a hermitian matrix, having two equal eigenvalues, then the stationary subgroup $\mathscr{H}_0$ of Φ_0 is isomorph to U(2) and the orbit, containing Φ_0 is the manifold $\mathbb{C}P(2)$.

This construction can be easily expanded to obtain the vacuum manifolds of the type

$$SU(m+n)/U(m) \times SU(n)$$

Hence for sufficiently big gauge groups the vacuum manifolds may turn out to be Grassman manifolds, having considerable topological structure with non-trivial characteristic classes (topological charges).

3. COMMENTS ON THE GINSBURG-LANDAU THEORY FOR ^{3}He

To some extent the ideas of the two preceding sections can be illustrated by the example of the Ginsburg-Landau theory for ^{3}He [*6*].

It is alleged to be known, that the superfluid phases of ^{3}He are triplet superfluids with the condensate amplitude

$$\Delta_{\alpha\beta}(k) = \sum_{pi} A_{pi}\,.\,k_i\,.\,w_p(\alpha\beta) \tag{6}$$

where $k_i, i = 1, 2, 3$ are components of the wave vector ; w_p, $p = 1, 2, 3$ are the spin wave functions, w_p being an eigenfunction of the pair spin operator S_p with eigenvalue zero, $S_p w_p = 0$. The nine complex variables A_{pi} involved in Eq. (6) define the order parameter of the system.

The analog of the Lagrangian density (1) is the free-energy density of the form

$$\mathscr{F} = \sum \left\{ \frac{K_L}{2} |\,\mathrm{div}\,\vec{A}_p|^2 + \frac{K_T}{2} |\,\mathrm{rot}\,\vec{A}_p|^2 + \frac{Q}{2}[\vec{A}_p.\mathrm{rot}\,\overset{*}{\vec{A}}_p + \overset{*}{\vec{A}}_p.\mathrm{rot}\,\vec{A}_p] \right\} + V(A)\,; \tag{7}$$

where the potential V(A) is of the form

$$V(A) = \alpha\mathrm{Tr}\,(AA^+) + \beta_1 |\,\mathrm{Tr}\,(AA^T)|^2 + \beta_2 |\,\mathrm{Tr}\,(AA^+)|^2 + \beta_3\mathrm{Tr}\,\{(AA^T)(AA^T)^*\} + \beta_4\mathrm{Tr}\,\{(AA^+)^2\} + \beta_5\mathrm{Tr}\,\{(AA^+)(AA^+)^*\}, \tag{8}$$

$$(A^T)_{ij} = A_{ji}, \quad (A^*)_{ij} = A^*_{ij}, \quad A^+ = A^{*T} \tag{9}$$

The fields variables are the components of the order parameter. The

Ginsburg-Landau method of expansion of the free energy is acceptable to ^{3}He, because of the low temperature coherence length is much longer than the interparticle distance.

It is immediate, that the potential V(A), is invariant under the gauge transformations of the first kind

$$A_{pi} \rightarrow R_{pm} R_{in} e^{i\varphi} A_{mn} \tag{10}$$

where the repeated indices imply the summation.

On the contrary, Lagrangian density (7) is not since the gradient terms are not invariant under the action of (10). In the sequel we shall restrict ourselves to the study of solutions defined only by the properties of the potential, i. e. we shall cancel out the gradient terms of (7), (cf. the similar approach in [*22*]). Physically, this implies uniform systems of the infinite characteristic length with respect to the coherence length. The corresponding solutions may have singularities at some points lines or surfaces.

The matrixes $R_{pm} . R_{in}$ constitute the gauge group

$$\mathscr{H} = SO(3)_1 \times SO(3)_2 \times U(1)$$

where the first subgroup $SO(3)_1$ of acts on the left-hand indices of the order parameter A_{pi} and corresponds to the representation of the total momentum $l = 1$ of the spinor group SU(2), whereas the second $SO(3)_2$ subgroup of $\mathscr{H}$ acts on the right-hand indices of A_{pi} and corresponds to the transformations in the x, y, z-space.

In special cases gauge transformations (10) have been discussed in a number of papers [*8*] [*18*] [*23*]) ([2]).

Following the lines of n° 1 we shall consider the minimums (or the vacuums) of the potential V(A). They correspond to the different phases of ^{3}He. It is not clear that the condition of the transitivity of the gauge group $\mathscr{H}$ on the vacuum manifold is verified. But it is natural to chose the minimums for which this condition is fulfilled, since we can split the minimum into the orbits of the gauge group, each orbit being the vacuum manifold with the transitivity condition. In that case we can easily find

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([2]) It is tentative to introduce a gauge field for the group $\mathscr{H}$. Such an attempt to follow the analogy with superconductors theory was done for ^{4}He [*20*]. In the present case we would like to substitute the covariant derivatives

$$D_K A_{pi} = \partial_K A_{pi} - \mu S^l_K \varepsilon^{plm} A_{mi} - \nu B^l_K \varepsilon^{ilm} A_{pm} - ieh_K A_{pi}$$

for the derivatives $\partial_K = \partial/\partial x_K . K = 1, 2, 3$ into Eq. (7). Here $\underline{\vec{S}}$, $\underline{\vec{B}}$ are the gauge fields corresponding to the subgroups $SO(3)_1$, $SO(3)_2$, If we require, that the fields $\underline{\vec{S}}$, $\underline{\vec{B}}$ should be the gauge fields, i. e. they should be transformed by the formulae [*10*],

$$\vec{T}\underline{\vec{S}}' = R^{-1}(\vec{T}\underline{\vec{S}})R + 1/\mu R^{-1}\underline{\partial} R, \quad \vec{T}\underline{\vec{B}}' = R^{-1}(\vec{T}\underline{\vec{B}})R + 1/\nu R^{-1}\underline{\partial} R,$$
$$h'_K = h_K + 1/e\partial_K g, \tag{12}$$

we still lack the gauge equations for the gauge fields, i. e. the analogues of the Maxwell equations for the superconductors case.

all the possible minimums by the Cartan theorem (cf. n° 1). They turn out to be the alleged A- and B-phases of ^3He and a certain minimum (C-phase) for which the physical meaning is not clear (cf. below).

The vacuum manifolds for the gauge group $\mathscr{H}$ are as follows

$$S^2 \times SO(3),\ SO(3) \times U(1),\ S^2 \times S^2 \times U(1),\ S^2 \times S^2, \quad S^2 \times U(1),\ SO(3),\ U(1), \text{ etc.} \tag{13}$$

The first two of them are the A- and B-phases, the third one is the C-phase. There are explicit formulae for the order parameter for the A-, B-, C-phases:

A-phase; $A_{pi} = \text{const } V_p\Delta_i e^{i\varphi}$; V_p is a real vector, $V_p^2 = 1$,

$\vec{\Delta} = \vec{\Delta}' + i\vec{\Delta}''$, $\vec{\Delta}'\vec{\Delta}'' = 0$; is a complex vector.

B-phase; $A_{pi} = \dfrac{\Delta}{\sqrt{3}} R_{pi}$ Δ is a complex number,

R_{pi} is a rotation matrix;

C-phase; $A_{pi} = \text{const } V_p\Delta_i e^{i\varphi}$; $\vec{V}, \vec{\Delta}$ are real unit vectors.

It is easy to prove by direct computation, that the A, B, C-phases verify the necessary conditions for the extremum

$$\frac{\partial V}{\partial A_{ij}^+} = \alpha A_{ij} + 2\beta_1 \text{Tr}\,(AA^T)A_{ij}^+ + 2\beta_2 \text{Tr}\,(AA^+)A_{ji} + 2\beta_3(AA^+)_{js}A_{si} + \beta_4\{(AA^T)_{rj}A_{ir}^+ + (AA^T)_{js}A_{is}^+\} + 2\beta_5(AA^+)_{rj}A_{ri} = 0, \tag{14}$$

The similar equation is valid for the case of $\partial V/\partial A_{ij}$.

It is not hard to note, that the sufficient condition of the positive second variation is not true in general for these phases. Let us consider for example the B-phase. A variation transversal to the vacuum submanifold V_B may be taken in the form

$$A_{pi} = \frac{\Delta}{\sqrt{3}} e^{i\varphi} R_{ps}\Lambda_{si} \tag{15}$$

where Λ_{si} is a real symmetric non-degenerate matrix. Then we may require, that the matrix of the second variation should be positive

$$\|\,\partial^2 V/\partial\lambda_i\partial\lambda_j\,\| > 0 \tag{16}$$

where the entries of the matrix are of the form

$$\partial^2 V/\partial\lambda_i\partial\lambda_j = 2/3\,|\Delta|^2 \left\{ \left(\alpha + \frac{2}{3}|\Delta|^2[(\beta_1 + \beta_2).\text{Tr}\,\Lambda^2 + (\beta_3 + \beta_4 + \beta_5).3\lambda_i^2] \right)\delta_{ij} + 2(\beta_1 + \beta_2)\lambda_i\lambda_j \right\} \tag{17}$$

Thus we need certain condition on the coefficients $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \alpha$, that the B-phase should be a minimum of the potential V(A). Indeed, condition (16) implies, that the coefficients of potential density (8) must satisfy the conditions

$$\beta_1 + \beta_2 > 0 \tag{18}$$

$$\beta_1 + \beta_2 > \frac{\alpha}{4\,|\,\Delta\,|^2}\left(1 - \frac{|\,\Delta\,|^4}{12\alpha^2}\right) \tag{19}$$

where $|\,\Delta\,|^2$ is the function of the coefficients $\beta_1, \ldots, \beta_5, \alpha$

$$|\,\Delta\,|^2 = -\frac{3}{2}\alpha[3(\beta_1 + \beta_2) + \beta_3 + \beta_4 + \beta_5] \tag{20}$$

The function $|\,\Delta\,|^2$ of Eq. (19) is just the square of modulus of the constant Δ of Eq. (15). Indeed, it is defined by necessary condition (14) for the B-phase.

Of course, Eqs. (18), (19) are not a set of sufficient conditions for the minimum, since we have taken only a special variation (15). However, they indicate that without certain conditions on the coefficients, the B-phase, defined by Eq. (14) is not a minimum manifold and therefore it is not a vacuum manifold.

A similar kind of arguments may be applied to the A- and C- phases. Following the lines of n° 2 we shall show, how the topological concepts may turn out useful for the treatment of coexistent phases of ^{3}He.

Let us consider a vessel containing the superfluid ^{3}He near the transition point. Then we may expect that in different regions of the vessel ^{3}He exists in different phases. These regions are likely to be of a rather complicated nature; they may be drops, fibres, etc. The order parameter A_{pi} defines a map of the domain bounded by the vessel, the regions $\Omega_A, \Omega_B, \Omega_C$ containing the various phases of ^{3}He being mapped into the corresponding vacuum manifolds V_A, V_B (and, probably, V_C). If we suggest, that the nature of the regions should not be too intricate and they should be bounded by the walls which can be treated as smooth surfaces, then we may apply the speculations of n° 2. Thus we obtain the map

$$(\Omega\,;\,\Omega_A, \Omega_B, \Omega_C) \rightarrow (J\,;\,V_A, V_B, V_C)$$

For a special case we can introduce the quantization condition for the phase transition. Suppose we have a fibre of the A-phase, surrounded by the B-phase. Let us take a surface Π bounded by a closed path L, lying in the region of the B-phase. Let Π intersect the fibre of the A-phase. Supposing Π to be a disk D^2 we obtain the map

$$f : (D^2\,;\,S^1) \rightarrow (J\,;\,V_B)$$

which defines an element χ of the relative homotopy group $\pi_2(J\,;\,V_B)$. The element χ may be considered as a topological index (or charge) of the phase transition between the A- and B-phases.

We note, that there exists an isomorphism

$$\pi_2(J\,;\,V_B) \cong \pi_1(V_B)$$

which permits, in the situation of this example, to define the index χ by the map of the closed path L into V_B

$$L \to V_B$$

This method was used by G. Volovik and V. Mineev [22]. In general, the relative homotopy groups are necessary. Consider « the biological looking object » of fig. 1. It was discussed by N. D. Mermin at the Sussex Symposium [9]. It consists of a bulk of the B-phase containing a fibre of the superfluid ^{3}He. Within the fibre there is a nuclear or a drop of the

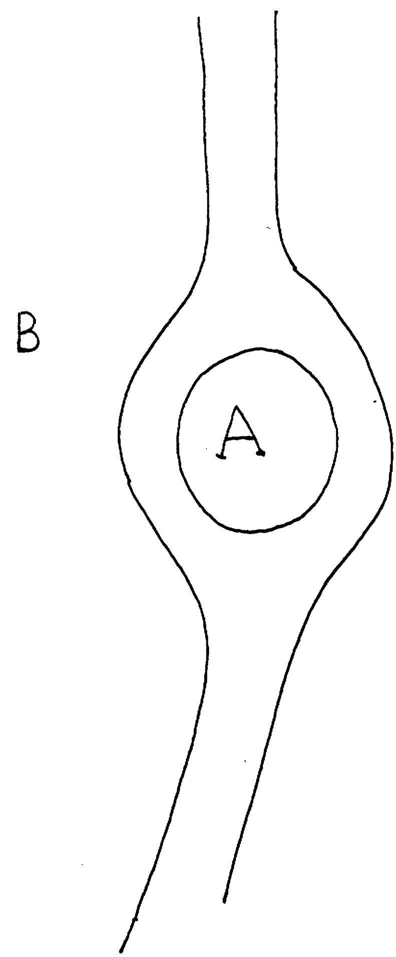

FIG. 1.

A-phase. Taking a closed path S^1 inside the region of the B-phase around the fibre and a sphere inside the nuclear of the A-phase, we obtain a map

$$(\Omega\,;\, S^1, S^2) \to (J\,;\, V_B, V_A) \tag{21}$$

where Ω is the whole bulk of ^{3}He. We may consider Ω to be the whole x, y, z-space. It is convenient to substitute map (21) by the map

$$f : (\tilde{\Omega}\,;\, S^1 \times R^1, S^2) \to (J\,;\, V_B, V_A)$$

where $\tilde{\Omega}$ is a cylinder or the product $R^1 \times D^2$; $S^1 \times R^1$ is a surface of the cylinder and S^2 is the boundary of the ball cut out of the cylinder. This map is determined by the maps of the boundaries $S^1 \times R^1$ and S^2 of the cylinder into V_B and V_A. The topological index is a pair of numbers (m, n) corresponding to the elements of the groups $\pi_1(B)$, $\pi_2(V_A)$, which provide the quantization conditions for the superfluid ^{3}He.

Finally we want to make some remarks concerning the Goldstone modes. The familiar quantization condition for ^{4}He uses the integral of the phase gradient along the closed path. The isotropy subgroup for that case being zero, we infer from it that the quantization condition consists in the integration of the Goldstone modes. Within the framework of the Kibble formalism (cf. n° 2) the Goldstone modes correspond to the phases of the field at the vacuum manifold.

In the London limit for ^{3}He the field variables (i. e. the order parameter) may be put into the form

$$A(\vec{r}) = g(\vec{r})A_0$$

where g is an element of the gauge group $\mathscr{H}$, and A_0 is a fixed value of the order parameter. Then the analog of the gradient of the phase (or the superfluid velocity) is the element of the Lie algebra of the gauge group

$$\underline{v} = g^{-1}(\vec{r}).\underline{\partial} g(\vec{r})$$

It is a matrix vector

$$\underline{v} = g^{-1}.\underline{\partial} g = \left(R_1^{-1}\underline{\partial}R_1,\, R_2^{-1}\underline{\partial}R_2,\, g^{-1}\underline{\partial} g\right)$$

where R_1, R_2 are elements of the subgroups $SO(3)_1$, $SO(3)_2$ and g of $U(1)$. This form of the phase of the superfluid is also suggested by the gauge transformations (10).

The quantization condition defined by the vector $g^{-1}\underline{\partial} g$ requires, that the integral along the closed path

$$I = \oint_L g^{-1}\underline{\partial} g d\vec{r}$$

should be independent of the choice of L in its homotopic class. Note, that I takes the values in the Lie algebra of the gauge group.

Using the Stokes formula we obtain the equation

$$I_i = \oint_L R_i^{-1}\underline{\partial}R_i d\vec{r} = \iint_\Pi \{ \partial_l(R_i^{-1}\partial_K R_i) - \partial_K(R_i^{-1}\partial_l R_i) \} dx^K dx^l$$
$$= \iint_\Pi [R_i^{-1}\partial_K R_i ; R_i^{-1}\partial_l R_i] dx^K dx^l$$

where the brackets [;] denote the commutator. We see, that the integral I_i is independent of the path, if the equations

$$[R_i^{-1}\partial_K R_i ; R_i^{-1}\partial_l R_i] = 0, \qquad K, l = 1, 2, 3$$

are fulfilled at the surface bounded by the path. In the particular case, when the rotations R_i have the same axis along all the path, we have the familiar condition for the phase, since then the equations are valid

$$R_i^{-1} . \underline{\partial}R_i = \underline{\partial}\varphi_i . \left\| \begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right\|$$

On integrating along the path we obtain the phase gradient, which must be an integer multiple of 2π.

CONCLUSION

Finally we want to emphasize considerable difficulties of finding the vacuum manifolds, i. e. the manifolds, which provide the minimum of the potential V(A). Even the enumeration of these manifolds by the variational Euler equations for the necessary conditions of the extremum is non-trivial. On applying the gauge group we reduce this problem to the algebraic problem of the enumeration of the symmetric spaces for the given gauge group. Thus we obtain the unified approach to the different theories like the Jang-Mills fields, the superfluid ^{3}He, etc. We note, that for the superfluid ^{3}He the results of this paper suggest the existence of a phase different from the A- and B-phases. We note also, that the sufficient condition for the symmetric space for the given gauge group to be a minimum of the potential defines the conditions on the coefficients of the Ginsburg-Landau equations (cf. Eqs. (18) (19)).

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APPENDIX

It is worth reminding the reader of the concept of bordism used in this paper.

A closed orientable manifold M^n is bordant to zero if there exists a compact orientable manifold B^{n+1} which has the boundary diffeomorph to M^n.

Let M^n be a closed orientable manifold. Denote by $-M^n$ the manifold M^n considered with the opposite orientation. Two manifolds M_1^n, M_2^n are topologically bordant if there exists a compact orientable manifold B^{n+1} with the boundary

$$\partial B^{n+1} = M_1^n \cup (-M_2^n)$$

Examples.

1) The sphere S^n defined by the equation

$$\sum_{i=1}^{n+1} x_i^2 = 1 .$$

is bordant to zero, the manifold B^{n+1} being the ball defined by the equation

$$\sum_{i=1}^{n+1} x_i^2 \leqq 1$$

2) Two circles can be bordant to a circle (cf. fig. 2).

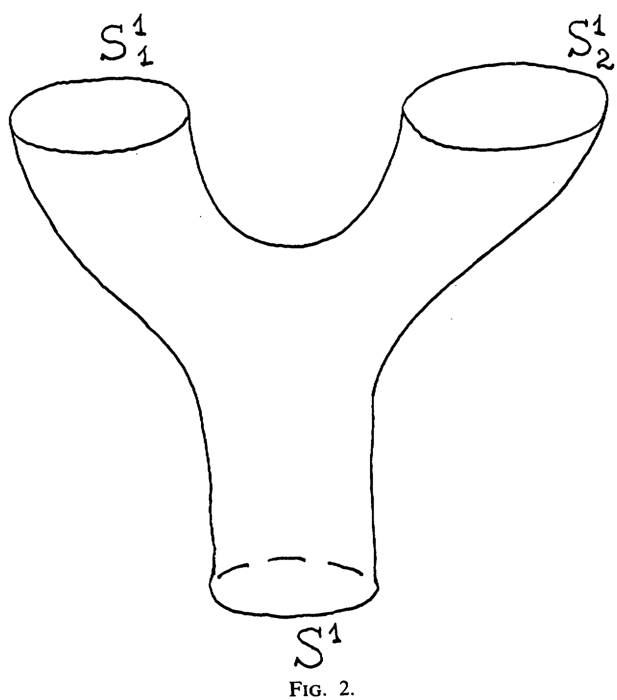

FIG. 2.

3) To prove that the 2-dimensional sphere is bordant to the torus, let us take the non-intersecting torus T^2 and the sphere S^2 in the x, y, z-space, R^3. Then we cut the inside of T^2, S^2 and complete the remaining part of R^3 with the point at infinity. We obtain the compact orientable manifold B^3, which provides the necessary bordism.

4) The complex projective space $\mathbb{C}P(2)$ is not bordant to zero [*15*].

The bordism properties are fully defined by the invariants constructed with the help of characteristic classes (cf. [*15*]). For the 4-dimensional manifolds like $\mathbb{C}P(2)$, one can construct a simple invariant of bordisms by the formula [*12*],

$$\chi(M^4) = \int_{M^4} F_{ij}F^{ij}d^4x$$

where F_{ij} is the Riemann curvature form [*14*]. An orientable 4-dimensional manifold M^4 is not bordant to zero if $\chi(M^4) \neq 0$.

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