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Kernels and symbols of operators in quantum field theory

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Section A :
Physique théorique.

Kernels and symbols of operators in quantum field theory

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1. INTRODUCTION

The concept of symbol of differential or pseudodifferential operator (p. d. o) is useful in finite dimensional analysis. The concept of p. d. o. is implicity contained in the work of Weyl, Wigner and Moyal for the phase space formulation of quantum mechanics of systems with a finite number of degrees of freedom. It seems therefore useful to elaborate a symbolic calculus in infinite dimensional analysis in view of its application in quantum field theory (Q. F. T.).

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There exist an important discontinuity between the finite and the infinite dimensional analysis:

i) In finite dimensional analysis, a central role is played by Lebesgue measures. On the other hand, in infinite dimensional analysis, the Lebesgue measure does not exist, and we have to use Gaussian measures or their generalizations;

ii) The basic variables characterizing symbols in finite dimension are x and ξ, (or q and p). In infinite dimension, the complex variables $\bar{z}$ and z' corresponding to symbols of creation and annihilation operators respectively, are most convenient;

iii) The theory of distribution is most convenient for majority of applications in finite dimension. In infinite dimensional analysis, it seems more convenient to use analytical functionals or profunctionals.

In this paper we extend the Wigner-Weyl-Moyal formalism to systems with infinite number of degrees of freedom. The main tool used here consists on convenient triplet of spaces

$$\mathrm{D} \hookrightarrow \mathrm{F}(\mathrm{X}^c) \hookrightarrow {}'\mathrm{D}$$

centered on the Fock space $\mathrm{F}(\mathrm{X}^c)$. Using the L. Schwartz-Grothendieck kernel theory, we give an effective characterizations of an extensive class of bounded and unbounded operators in $\mathrm{F}(\mathrm{X}^c)$ with the domain D, in terms of their symbols. In addition, we give also an effective characterization of an extensive class of linears maps from D to its antidual 'D. The present formalism allows to control the regularity properties of operators in Q. F. T.: analysing merely the symbols one may easily verify when non cutoff limit of operator remain in the Fock space or when becames merely a sesquilinear form on D.

In Section 2 we develop a convenient formalism of integration theory on infinite dimensional spaces. Then in Sec. 3 we elaborate a theory of analytical functionals and profunctionals of exponential type and we discuss the properties of Borel transform in infinite dimensional spaces. In Sec. 4 we develop a theory of kernels and symbols of operators and sesquilinear forms for quantum mechanical systems with finite number of degrees of freedom: this theory allows to give a full characterization of the class of all bounded operators, unbounded operators and sesquilinear forms whose domains contain the space Exp $\mathbb{C}^n$ in terms of their kernels or symbols (cf. Propositions (4.20) and (7.1)).

In Section 5 we present a new theory of integral representations for unbounded operators and sesquilinear forms for systems with infinite number of degrees of freedom. The main difficulty in the extension of the Wigner-Weyl-Moyal theory to the interacting quantum system with infinite number of degrees of freedom consists in the lack of an effective measure theory on infinite dimensional spaces. In order to overcome this difficulty we introduce a concept of profunctionals and prokernels which generalizes a concept of a measure. Using these concepts and introducing an analogue of a Gelfand triplet $D \subset H \subset D'$ we develop a theory of integral representations of unbounded operators and sesquilinear forms, which is parallel and equally effective as the corresponding theory for systems with finite number of degrees of freedom. After experimentation with prodistributions and various spaces of profunctionals we found that most effective technique is provided by the theory of analytic functionals. Consequently we stated main results in this language.

Section 6 contains some applications of the theory of integral representations of operators in the theory of ordinary and generalized quantization of classical systems with infinite number of degrees of freedom. Finally in Section 7 we discuss interesting connections between symbols and operators. In particular we derive an effective criterion for checking, when a given classical dynamical variable Q/e. g. total hamiltonian of a physical system in Q. F. T./leads to an operator in the carrier space with $D(Q) = \text{Exp}\, X^c$ and when it leads to a sesquilinear form only. We think that this criterion will be very useful in quantization theory of classical interacting systems. We give in Section 8 several examples from quantum field theory for illustration of main results of our work see also [26] [27].

2. INTEGRATION THEORY WITH RESPECT TO A PROJECTIVE SYSTEM OF MEASURES

(2.1) NOTATION

Let X be a real Hilbert space and X^c its complexification. Let $(X_j)_{j\in J}$ be the set of all finite dimensional subspaces of X. The inclusion relation between subspaces of X induces a following order on J : for every pair $i, j \in J$ of indices there exists $k \in J$ with $k \geqslant i$ and $k \geqslant j$ and this means that $X_k \supset X_i$ and $X_k \supset X_j$. Let s_i be the orthogonal projection of X onto X_i and s_{ij} the restriction of s_i to X_j (if $i \geqslant j$). A function $\varphi : X \to \mathbb{C}$ is called cylindrical if there exists $j \in J$ and a function φ_j on X_j such that $\varphi = \varphi_j \circ s_j$; the subspace X_j is called a basis of φ. The space of cylindrical polynomial functions on X will be denoted by the symbol $\text{Pol}_{\text{cyl}}(X)$. The space of continuous cylindrical functions on X with exponential growth will be denoted by the symbol $\text{CExp}_{\text{cyl}}(X)$.

(2.2) PROMEASURES

A promeasure μ on X is a family $(\mu_j)_{j\in J}$ of bounded measures on the spaces X_j such that $s_{ij}(\mu_i) = \mu_j$ if $i \geqslant j$; μ has an exponential decay if μ_j has an exponential decay for any j :

$$(2.3) \qquad \forall m \in N, \quad \int_{X_j} \exp(m \| x \|) d |\mu_j|(x) < \infty$$

Such μ defines a linear form $\langle \mu, \varphi \rangle$ on $\mathrm{CExp}_{\mathrm{cyl}}(\mathrm{X})$ which we shall sometimes represent by the abusive symbol $\int \varphi(x)d\mu(x)$. For example the Fourier transform of the canonical gaussian promeasure ν on X is the following function on X^c, with $z = x + iy$; x and $y \in \mathrm{X}$

$$(2.4) \qquad (\mathscr{F}\nu)(z) = \int e^{-\sqrt{-1}t} d\nu(t) = \exp\left(-\frac{1}{2}z^2\right)$$

Here the symbol z^2 means in fact (z, z) i. e. the bilinear extension of the quadratic form $\| x \|^2$ on X.

A subset $\mathrm{K} \subset \mathrm{J}$ is called cofinal if for any $j \in \mathscr{J}$, there exists $k \geqslant j$, $k \in \mathrm{K}$

A promeasure $\mu = (\mu_j)_{j \in \mathrm{J}}$ is known if we know only the coherent family $(\mu_k)_{k \in \mathrm{K}}$ where K is any cofinal subset of J. For example the gaussian promeasure ν' on X^c (considered as a real space), is defined by the family $(\nu'_k)_{k \in \mathrm{K}}$ of the following gaussian measures on the finite dimensional *complex* subspaces $(\mathrm{Z}_k)_{k \in \mathrm{K}}$ of X^c:

$$(2.5) \qquad \nu'_k = \pi^{-n} \exp(-z\bar{z})d^2z, \qquad \text{with} \quad n = \dim \mathrm{Z}_k$$

(2.6) Realization of a cylindrical probability

Let $m = (m_j)_{j \in \mathrm{J}}$ be a cylindrical probability on X: m is a promeasure such that all m_j are probability measures. A realization $\{ \Omega, \tau, \mathrm{P}, (f_j)_{j \in \mathrm{J}} \}$ of m is defined as a probability space $(\Omega, \tau, \mathrm{P})$ and a family $f_j : \Omega \to \mathrm{Y}_j$ of random variables (r. v.) such that

a) $m_j = f_j(\mathrm{P})$ for every $j \in \mathrm{J}$

b) The completion $\hat{\tau}$ of the σ-field τ with respect to P is generated by the r. v. f_j.

c) $i \geqslant j$ implies $f_j = s_{ij} \circ f_i$

For any realization $\{ \Omega, \tau, \mathrm{P}, (f_j) \}$ of m, a complete description of the complex Lebesgue classes $\mathrm{L}^p(\Omega)$ and an algorithm to compute integrals uniquely in terms of m, are given below.

(2.7) For each $j \in \mathrm{J}$, let E_j be a dense subspace of the Lebesgue class $\mathrm{L}^p_{m_j}$ and let $\mathrm{E}_j = \{ \varphi_j \circ f_j \in \mathrm{L}^p(\Omega) ; \varphi_j \in \mathrm{E}_j \}$.

Then $\bigcup_j \tilde{\mathrm{E}}_j$ generates a dense subspace of $\mathrm{L}^p(\Omega)$, $1 \leqslant p < \infty$. This follows directly from (2.6.*b*).

(2.8) Definition of L^p_m for $1 \leqslant p \leqslant \infty$

Let L^p_m be the vector space of family $(\psi_j)_{j \in \mathrm{J}}$ of elements $\psi_j \in \mathrm{L}^p_{m_j}$ such that

a) $\sup_j \int | \psi_j |^p dm_j < \infty$

b) $s_{ij}(\psi_i\mu_i) = \psi_j\mu_j$ if $i \geqslant j$

c) $\int_{\{|\psi_j| > C\}} |\psi_j| dm_j \to 0$ uniformly in j, if $C \to \infty$

Remarks. — *i*) Condition *b*) means that for every Borelian subset β of X_j

$$\int_\beta \psi_j d\mu_j = \int_\beta (\psi_i \circ s_{ij}) d\mu_i$$

This means also that ψ_j is the conditional mean $\mathscr{E}(\psi_i \mid s_{ij})$ of ψ_i with respect to s_{ij}: see for example [*8*].

ii) Condition (2.8.*c*) is empty if $p > 1$, because any set of r. v. bounded in L^p is uniformly integrable if $p > 1$.

iii) The space L^p_m has a natural structure of normed space and can be defined directly with *m*, without any realization of *m*. Because conditioning is a contraction in L^p for any p $(1 \leqslant p \leqslant \infty)$ and because (2.6.*b*), the sup in (2.8.*a*) is in fact a limit for $p < \infty$.

(2.9) THEOREM([*19*] [*18*]). — Let $\{ \Omega, \tau, P, (f_j)_j \}$ be any realization of the cylindrical probability *m* on the Hilbert space X. For any *g* in $L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$ and any *j*, let $\varphi_j = \mathscr{E}(g \mid f_j)$ be the conditional mean of *g* with respect to the random variable f_j.

a) Then the map

$$L^p(\Omega) \to L^p_m$$
$$g \to (\varphi_j)_j$$

is isometric and bijective.

b) Moreover $(\varphi_j \circ f_j)_j \to g$ strongly in $L^p(\Omega)$ if $p < \infty$ and $(\varphi_j \circ f_j) \to g$ weakly for any $g \in L^\infty(\Omega)$.

This can be proved by a compacity argument or using the martingale theory. For more details and extensions, see [*18*].

(2.11) COMMENTARY AND COROLLARY

a) Because $L^p(\Omega)$ is complete, L^p_m is a Banach space.

b) Part *a*) of (2.9) states that any $\phi \in L^p(\Omega)$ can be represented by the promeasure $(\varphi_j m_j)_j$. Hence an element $(\varphi_j)_j \in L^p_m \simeq L^p(\Omega)$ can be identified with the corresponding promeasure on X. Then the element $(\varphi_j)_j$ of L^p_m is written symbolically $\varphi(x)$.

c) $L^p_m \subset L^q_m$ if $q \leqslant p$. For $p < \infty$, the antidual of L^p_m is $L^{p'}_m$ with $p^{-1} + p'^{-1} = 1$. For $\varphi \in L^p_m$ and $\psi \in L^{p'}_{m'}$ the antiduality will be symbolically written $\int \bar{\varphi}(x)\psi(x)dm(x) = \langle \varphi, \psi \rangle$.

d) For any $g \in L^p(\Omega)$ and any *j*, we have

$$\int_\Omega g \, dP = \int \varphi_j dm_j \tag{2.12}$$

and this gives a simple method of computation of integral of function defined on the infinite dimensional space Ω.

e) Let K be a subset of J which is cofinal. Let $m' = (m_k)_k$ and let $L^p_{m'}$ be the space of family $(\psi_k)_{k\in K}$ of elements $\psi_k \in L^p_{m_k}$ satisfying conditions *a*), *b*), *c*) of (2.8) but only for i and j in K. Then L^p_m is isometric to $L^p_{m'}$.

f) A « function » $\varphi = (\varphi_j)_j$ in L^p_m will be called cylindrical if there exists j_0 such that $\varphi_i = \varphi_{j_0} \circ s_{ij_0}$ for any $i \geqslant j_0$. The subset K of J consisting of indeces $i \geqslant j_0$ is cofinal in J, and moreover

$$\forall i \geqslant j_0 \qquad \int |\varphi_i|^p dm_i = \int |\varphi_{j_0}|^p dm_{j_0}$$

We have also,

$$\int_X |\varphi|^p dm = \lim_{j\in K} \int |\varphi_j|^p dm_j = \int |\varphi_{j_0}|^p dm_{j_0} \tag{2.13}$$

g) With notation of *f*) let g be the element of $L^{p'}(\Omega)$ corresponding to the cylindrical functional $\varphi = (\varphi_j)_j$ if $\{\Omega, \tau, P, (f_j)_{j\in J}\}$ is any realization of m. Let $\psi = (\psi_j)_j$ in L^p_m not necessarily cylindrical and let h be the corresponding element of $L^{p'}(\Omega)$. By Hölder inequality $gh \in L^1(\Omega)$. By a well known property of conditioning we have

$$(\bar{g}h)_{j_0} = \mathscr{E}(\bar{g}h \mid f_{j_0}) = \bar{\varphi}_{j_0}\mathscr{E}(h \mid f_{j_0}) = \bar{\varphi}_{j_0}\psi_{j_0}$$

Then applying (2.12) we obtain

$$\langle \varphi, \psi \rangle = \int_X \bar{\varphi}(x)\psi(x)dm(x) = \int_{Y_{j_0}} \bar{\varphi}_{j_0}\psi_{j_0}dm_{j_0} \tag{2.14}$$

This gives a very simple formuls for computation of $\int \bar{\varphi}\psi dm$ if φ or ψ is cylindrical with a basis Y_j. The Segal space $L^2(X)$ of wave representation of free quantized scalar field is the Lebesgue class $L^2(X)$ corresponding to the canonical normal promeasure $v = (v_i)_i$ on X.

We now introduce the Segal Bargman space $F(X^c) = F^2(X^c)$ of the corpuscular representation, using the following results of infinite dimensional holomorphy.

(2.15) INFINITE DIMENSIONAL HOLOMORPHY

Let Z and Y be two complex locally convex Hausdorf spaces, and let Y be complete. The space $H_G(Z, Y)$ of Gateaux holomorphic functions on Z, valueted in Y, is the space of functions $\psi : Z \to Y$, entire on each one dimensional subspace of Z : see [*15*]. The subspace H(Z, Y) of holomorphic (or entire) functions on Z, consists of functions in $H_G(Z, Y)$ which are continuous.

We set $H_G(Z) = H_G(Z, \mathbb{C})$; and $H(Z) = H(Z, \mathbb{C})$. If Z is a Banach space, then for any $\psi \in H(G)$, and any $z_0 \in Z$, there exists $\varepsilon > 0$, such that the Taylor series of ψ at points z_0 converges to ψ in the ball $\| z - z_0 \| < \varepsilon$; if a function $\psi \in H_G(Z)$ is locally bounded, then $\psi \in H(Z)$: see [25]. In the following considerations, Z is obtained from a real locally convex Hausdorf space X by the complexification: $Z = X^c$. An antiholomorphic functional on Z is by definition a holomorphic functional on the conjugate space $\bar{Z}$ of Z; such functional is denoted in the following by $\psi = \psi(\bar{z})$.

(2.16) DEFINITION

Let $p > 1$. Let $F^p(X^c)$ be the space of Gateaux antianalytical functions ψ on X^c such that

$$\sup_i \int_{X_i^c} | \psi_i(\bar{z}) |^p dv_i'(z) < \infty \tag{2.17}$$

where ψ_i denotes the restriction of ψ to X_i^c and where v' is the canonical gaussian promeasure of X^c: see (2.2). Note that $p < p_1$ implies $F^p(X^c) \supset F^{p_1}(X^c)$.

(2.18) PROPOSITION

a) *The space $F^p(X^c)$ is isometrically embedded in the Lebesgue class $L^p_{v'}(X^c)$.*

b) *For any $\psi \in F^p(X^c)$ and any $z \in X^c$ holds the following reproducing property*

$$\psi(\bar{z}) = \int_{X^c} e^{\bar{z}z'} \psi(\bar{z}') dv'(z') \tag{2.19}$$

c) *Each ψ in $F^p(X^c)$ is entire.*

Remarks on (2.19). — *i*) Using (2.13) it can be shown that the cylindrical function (exponential)

$$e^z : z' \to e^{z\bar{z}'}$$

belongs to $L^{p'}_{v'}(X^c)$ with $p^{-1} + p'^{-1} = 1$. The integral part of (2.19) must be understood as explained in (2.11.*f*): the integral means the antiduality pairing between $\psi \in L^p$ and $e^{\bar{z}} \in L^{p'}$. Then (2.19) can also be written in the form

$$\psi(\bar{z}) = \langle e^z, \psi \rangle \tag{2.20}$$

ii) Let $\{ \Omega, \tau, P, (f_i)_i \}$ be any realization of v'. Contrarily to the finite dimensional case, the reproducing property does not hold for any z in Ω but only for z in X^c. Then the general formalism of [6] needs one improvement for its application in the infinite dimensional case.

iii) The L^2 norm of e^z is $\exp\left(\frac{1}{2} \| z \|^2\right)$. Coherent state $| z \rangle$ used in finite dimensional models for quantum optics are in fact normalized exponentials:

$$|z\rangle = e^{z\bar{z}'}\exp\left(-\frac{1}{2}\|z\|^2\right) \tag{2.21}$$

Proof of (2.18). — *a*) In view of (2.8) and of the definition of $F^p(X^c)$, it is sufficient to prove that for any ψ in $F^p(X^c)$ the family (ψ_i) is coherent. For example if s_{21} denotes the canonical projection $(z_1, z_2) \to z_1$ of $\mathbb{C}^2$ onto $\mathbb{C}$, it is necessary to prove that

$$\forall z_1 \qquad \pi^{-1}\int \psi(\bar{z}_1, \bar{z}_2, 0)e^{-z_2\bar{z}_2}d^2z_2 = \psi(\bar{z}_1, 0, 0)$$

Using polar coordinates this follows from the mean value property applied to the function $\bar{z}_2 \to \psi(\bar{z}_1, \bar{z}_2, 0)$ at point $z_2 = 0$

$$(2\pi)^{-1}\int_0^{2\pi} \psi(\bar{z}_1, \gamma e^{i\theta}, 0)d\theta = \psi(\bar{z}_1, 0, 0)$$

The same proof holds in the general case using convenient orthonormal basis and the L. Schwartz convention of multiindices.

b) Because e^z is cylindrical the integral in (2.19) can be computed using (2.14). Then the proof of (2.19) is reduced to the one dimensional case, and this case was treated by Berezin [5].

c) Using (2.15) it is sufficient to prove that ψ is uniformly bounded on any bounded subset of X^c. This follows from (2.19), using Holder inequality.

(2.22) COROLLARY

a) If $p = 2$, and if X^c is finite dimensional, a Taylor expansion gives

$$\int_{X^c} |\psi|^2 dv' = \sum_{k=0}^{\infty} \frac{\|\mathscr{D}^k\psi(0)\|^2}{k!} \tag{2.23}$$

where $\|\mathscr{D}^k\psi(0)\|$ denotes the Hilbert Schmidt norm in the symmetrical completed tensor product $\hat{\odot}\, X^c$. Then (2.23) holds also in the infinite dimensional case because

$$\int |\psi|^2 dv' = \sup_i \int |\psi_i|^2 dv'_i = \sup_i \sum_{k=0}^{\infty} \frac{\|\mathscr{D}^k\psi_i(0)\|^2}{k!} = \sum_{k=0}^{\infty} \frac{\|\mathscr{D}^k\psi(0)\|^2}{k!}$$

This proves that $F(X^c)$ is isometric to the usual Fock space.

b) For ϕ and ψ in $F(X^c)$ we have

$$\langle \phi | \psi \rangle = \int \overline{\phi(z)}\psi(z)dv'(z) = \int \langle \overline{e^{z} | \phi} \rangle \langle e^z | \psi \rangle\, dv'(z) \tag{2.24}$$

Symbolically, this means that the identity map of Fock space can be written

$$(2.25) \qquad \mathrm{Id} = \int | e^z \rangle \langle e^z | \, dv'(z)$$

c) The result of Berezin used in the proof of (2.18.*b*) can be slightly extended:

For any ϕ antientire on $\mathbb{C}^n$ such that

$$\forall a > 0 \qquad \int | \phi(\bar{z}) | \, e^{a|z|} dv'(z) < \infty$$

the following reproducing property holds

$$\forall z \in \mathbb{C}^m \qquad \phi(\bar{z}) = \int \phi(z') e^{z'\bar{z}} dv'(z')$$

(2.26) THE PROBLEM

The following isometry is known in the finite dimensional case [3]

$$\mathrm{L}^2(\mathrm{X}) \xrightarrow{\theta} \mathrm{F}(\mathrm{X}^c)$$

$$\varphi(q) \xrightarrow{\theta} \phi(\bar{z}) = \int e^{-\frac{1}{2}\bar{z}^2 + \bar{z}q} \varphi(q) dv(q)$$

This holds also in the infinite dimensional case but the integral must be understood as a sesquilinear pairing between $\varphi \in \mathrm{L}^2_v(\mathrm{X})$ and

$$q \to \exp\left(-\frac{1}{2} z^2 + zq \right); \text{ see } [18].$$

If an orthonormal basis is chosen in X, θ maps the cylindrical functions on X associated to normalized Hermite polynomials

$$(2.27) \quad \mathrm{H}_k(q) := \mathrm{H}_{k_1,\dots,k_n}(q_1, \dots, q_n) = \prod_{j=1}^{n} \mathrm{H}_{k_j}(q_j)$$

$$\text{with} \quad \mathrm{H}_l(t) = \int_{-\infty}^{+\infty} (t + iu)^l dv(u)$$

onto the cylindrical function on X^c associated to the following monomial on $\mathbb{C}^n$

$$\bar{z}^k = \bar{z}_1^{k_1} \bar{z}_2^{k_2}, \dots, \bar{z}_n^{k_n}$$

We want to analyse the properties of unbounded operators in $\mathrm{L}^2(\mathrm{X})$ or $\mathrm{F}(\mathrm{X}^c)$ and also unbounded operators of the space $\mathrm{L}^2_m(\mathrm{X})$ where m is the interacting measure of promeasure. It turns out that convenient formalism can be elaborated with the help of a certain generalized Gelfand triplet $\mathrm{D} \subset \mathrm{L}^2 \subset \mathrm{D}'$ similarly as in case $\mathrm{L}^2(\mathbb{R}^n)$. We take as the space D a certain space of analytical functions on X^c : e. g. $D = \mathrm{Exp}_{cyl}(\bar{X}^c)$. The choice of test functions in $\mathrm{Exp}_{cyl}(\bar{X}^c)$ is motivated by the formal relation corresponding to the wave representation for exponents of creation and annihilation operators

$$e^{\alpha a^*}e^{\beta a} : \psi(\bar{z}) \to e^{\alpha\bar{z}}\psi(\bar{z}+\beta)$$

and also by the following property

(2.28) PROPOSITION

The map θ defined in (2.26) is a bijection between $\mathrm{Exp}_{cyl}(\bar{X}^c)$ and $\mathrm{Exp}_{cyl}(\bar{X}^c)$.

Proof. — Because $\varphi \in \mathrm{Exp}_{cyl}(\bar{X}^c)$ is transformed by θ into a cylindrical function with the same basis, X^c can be chosen finite dimensional and identified to $\mathbb{C}^n$:

$$\varphi = \sum_k \varphi_k \frac{H_k}{k!} \qquad \text{with} \qquad \varphi_k = \langle H_k, \varphi \rangle$$

where the multiindices notation of (2.27) is used. Using the generating function of Hermite polynomials one obtains

$$\varphi_k = \langle H_k, \varphi \rangle = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-q^2/2}\left(\left(\frac{\partial}{\partial q}\right)^k \varphi(q)\right)dq$$

Utilizing the Cauchy formula one deduces from $|\varphi(q)| \leqslant (\exp(a|q|)$ estimates for all derivatives of φ

$$\left|\left(\frac{q}{\partial q}\right)^k \varphi(q)\right| \leqslant C' \exp(b|q|).\left(\frac{ea}{k}\right)^k.k!$$

Then

$$|\varphi_k| = |\langle H_k, \varphi \rangle| \leqslant C''(k!)^{1/2}\left(\frac{ea}{k}\right)^k$$

and finally

$$|\theta\varphi(\bar{z})| = \left|\sum_k \varphi_k \frac{\bar{z}^k}{\sqrt{k!}}\right| \leqslant C''' \exp(ea|z|)$$

Conversely it must be shown that if $\phi \in \mathrm{Exp}(\bar{\mathbb{C}}^n)$ then $\varphi = \theta^{-1}\phi$ belongs also to $\mathrm{Exp}(\bar{\mathbb{C}}^n)$. This follows directly from the inversion formula for θ transform:

$$\varphi(\bar{q}) = \int_{\mathbb{R}^n} \phi(\bar{q} + it)d\nu(t) \tag{2.29}$$

3. ANALYTICAL FUNCTIONALS AND PROFUNCTIONALS OF EXPONENTIAL TYPE

To define the normal representation and the diagonal representation of an operator in $F(X^c)$ a concept of generalized measure is needed, even in the finite dimensional case. Prodistributions and distributions [*19*] [*18*] can be used in the infinite dimensional case in a way to extend well known results if dim X is finite. It seems more convenient to use a complex extension of the theory of measure, because this extension permits to pass very naturally from gaussian measures to Feynman pseudomeasures [*17*].

(3.1) BACKGROUND IN MEASURE THEORY

If Y is a vector space and if θ is a locally convex topology on Y, (Y, θ) denotes the corresponding topological vector space, and $(Y, \theta)'$ denotes its dual. If X is a completely regular space, $B^0(X)$ denotes the space of bounded continuous functions $\varphi : X \to \mathbb{C}$. Let β be the unit ball of $B^0(X)$

$$\beta = \{ \varphi \in B^0(X) ; \| \varphi \|_\infty = \sup | \varphi(x) | \leqslant 1 \}$$

The space M(X) of bounded Radon measures over X is the space of bounded complex measures of the borelian σ-field of X such that for every $\varepsilon > 0$ there exists a compact subset K of X such that $|m|(X/K) < \varepsilon$. Let t_k be the topology of uniform convergence over all compact subset of X. The strict topology τ over $B^0(X)$ is the finest locally convex topology on $B^0(X)$ which agrees with t_k on β. It can be shown [*11*] that M(X) can be defined as the dual of the locally convex space $(B^0(X), \tau)$.

(3.2) INTRODUCTION OF WEIGHTS

We shall use the strict topology on spaces of continuous functions in order to obtain a topology on spaces of entire functions. In view of the fact that any entire function φ is unbounded, we introduce a weight, in order to allow growth of φ at infinity. Let Z be a real Banach space and let m be a positive integer. The space CExp^m (Z) is the space of continuous functions φ on Z such that $\sup | \varphi(z) | \exp(- m \| z \|) < \infty$. This space has a natural unit ball β^m, a topology t_k, and it can be equipped with a strict topology τ^m. The dual MExp'^m (Z) is the space of Radon measures μ on Z such that

$$\int \exp(m \| z \|) d | \mu | (z) < \infty \tag{3.3}$$

The space $\mathrm{C\,Exp}(Z) = \bigcup_m \mathrm{C\,Exp}^m(Z)$ is the space of continuous functions on Z with exponential growth. It can be equipped with the topology $\theta = \varinjlim \tau^m$. The dual M Exp′ (Z) of (C Exp (Z), θ) is the space of measures μ with exponential decay, i. e. measures satisfying (3.3) for any m. If in the preceeding definitions the weights $\exp(m \| z \|)$ are replaced by the weights $\exp(m \| z \|^2)$ we obtain instead of C Exp (Z) and M Exp (Z) the spaces $\mathrm{C\,Exp}_2$ (Z) and $\mathrm{M\,Exp}_2$ (Z).

(3.4) ANALYTICAL FUNCTIONALS OF EXPONENTIAL TYPE

Let Z be a complex Banach space and let Exp Z be the topological subspace of (C Exp (Z), θ) consisting of entire functions. The dual Exp' (Z) of Exp (Z) is called the space of analytical functionals of exponential type over Z.

The Hahn-Banach theorem implies the following.

(3.5) CHARACTERIZATION OF EXP' (Z)

Let T be a linear form defined over a dense subspace of ($\mathrm{Exp}_{\mathrm{cyl}}$ (Z), θ). Then T $\in$ Exp' (Z) if and only if T can be represented by an exponentially decreasing Radon measure on Z.

(3.6) FOURIER TRANSFORM FT

For every $\zeta \in Z'$ the function $z \to \exp(-\sqrt{-1}\, z\zeta)$ belongs to Exp Z where $z\zeta$ denotes the bilinear duality from between Z and Z'. Then FT is defined by the following function on Z'

$$(3.7) \qquad (\mathrm{FT})(\zeta) = \int_Z e^{-\sqrt{-1}z\zeta} d\mathrm{T}(z)$$

where the integral has only a symbolic meaning. It can be shown that FT is entire of nuclear type on Z' and that the map T $\to$ FT is injective [*17*] at least if Z is separable and has the metric approximation property; these hypothesis will be assumed below.

(3.8) IMAGE BY A LINEAR MAP

Let Z and U be two complex Banach spaces, and let λ be linear continuous map Z $\to$ U. For T $\in$ Exp' (Z) the analytical functional λT on U is defined by

$$(3.9) \qquad (\lambda \mathrm{T}, \psi) = (\mathrm{T}, \psi \circ \lambda)$$

for every $\psi \in$ Exp (U). This implies the following relation between the Fourier transforms of T and λT

$$(3.10) \qquad (\mathrm{F}(\lambda\mathrm{T}))(\zeta) = (\mathrm{FT})(\lambda'\zeta) \qquad \zeta \in \mathrm{U}'$$

where λ' denotes the transpose of λ.

(3.11) Product with an element $\phi \in$ Exp (Z).

The product ϕT is defined as in the distribution theory by the formula

$$(3.12) \qquad (\phi\mathrm{T}, \psi) = (\mathrm{T}, \phi\psi)$$

for any $\psi \in$ Exp (Z).

(3.13) TENSORIAL PRODUCT

Let Z^1 and Z^2 be two complex Banach spaces and $Z = Z^1 \times Z^2$. Then the tensorial product of the two linear forms associated to $T_1 \in \mathrm{Exp}'(Z^1)$ and to $T_2 \in \mathrm{Exp}'(Z^2)$ is a linear form on the subspace $E = \mathrm{Exp}(Z^1) \otimes \mathrm{Exp}(Z^2)$ of Exp (Z). The subspace E is dense in Exp Z; indeed if $T \in \mathrm{Exp}'(Z)$ is orthogonal to E then the Fourier transform $\hat{T} = 0$ and $T = 0$. If $\mu_j \in \mathrm{M\,Exp}(Z^j)$, $j = 1, 2$ represents $\hat{T}_j$, then $\mu_1 \otimes \mu_2 \in \mathrm{M\,Exp}(Z)$ represents λ. In view of (3.5) $\lambda \in \mathrm{Exp}'(Z)$; we write $\lambda = T_1 \otimes T_2$. There is a Fubini-L. Schwartz formula

$$(3.14) \qquad \iint \phi(x, y) d(T_1 \otimes T_2) = \int_{z^1} dT^1(x) \int_{z^2} \phi(x, y) dT_2(y)$$

for any $\phi \in \mathrm{Exp}(Z)$.

(3.15) Remarks

i) The preceding theory holds for any kinds of weights on the complex Banach space Z. For example for any $p > 0$ we can define the space $\mathrm{Exp}_p(Z)$ of entire functions φ on Z satisfying the estimate

$$|\varphi(z)| \leqslant C \exp(m \| z \|^p)$$

The corresponding space of analytical functionals will be denoted by $\mathrm{Exp}'_p(Z)$. In particular, for $p = 1$ we obtain Exp′ (Z).

ii) For $T \in \mathrm{Exp}'(Z)$ the Borel transform BT of T can be defined by

$$(BT)(z) = T(e^z)$$

Comparing with (3.7) the following relation is deduced

$$(FT)(z) = BT(-\sqrt{-1}\, z)$$

Exactly as the concept of promeasure extends the concept of Radon measure the concept of analytic functionals of exponential type will be generalized in the following manner:

(3.16) Analytic profunctionals of exponential type

Let Z be a complex Hilbert space and let $(Z_j)_j$ be the family of finite dimensional complex subspaces of Z. The space $\mathrm{Exp}_{cyl}(Z)$ of cylindrical entire function on Z with exponential growth can be considered as the inductive limit of spaces $\mathrm{Exp}(Z_j)$. Then $\mathrm{Exp}_{cyl}(Z)$ can be equipped with the $\varinjlim$ topology. The space $\mathrm{Exp}'_{cyl}(Z)$ of analytical profunctionals of exponential type is defined as the dual of $\mathrm{Exp}_{cyl}(Z)$. Equivalently, $\mathrm{Exp}'_{cyl}(Z)$ is the space of linear forms T on $\mathrm{Exp}_{cyl}(Z)$ whose restrictions to each $\mathrm{Exp}(Z_j)$ are represented by an element of $\mathrm{Exp}'(Z_j)$. The result of the action of T on $\psi \in \mathrm{Exp}_{cyl}(Z)$ is symbolically written in the form

$$(3.17) \qquad T(\psi) = \int \psi(z) dT(z)$$

(3.18) LEMMA

a) *The natural injection*

$$(\mathrm{Exp}_{\mathrm{cyl}}(\mathrm{Z}), \underrightarrow{\lim}) \xrightarrow{\mathrm{J}} (\mathrm{Exp}(\mathrm{Z}), \theta)$$

is continuous with a dense range.

b) *By transposition of* J *we obtain a canonical injection of* Exp' (Z) *into* $\mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z})$.

Proof of a). — The continuity of J follows from general properties of generalized inductive limit [*11*]. Density follows by a natural polarity argument. Usual operations can be now defined in $\mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z})$. Fourier transform of $\mathrm{T} \in \mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z})$ can be defined by (3.7) because the function $z \to \exp(-\sqrt{-1}\, z\zeta)$ is cylindrical on Z.

(3.19) IMAGE BY A LINEAR CONTINUOUS MAP

Let Z and U be two complex Hilbert spaces and λ be a linear continuous map $\mathrm{Z} \to \mathrm{U}$. We wish to define the image by λ of $\mathrm{T} = (\mathrm{T}_i)_i \in \mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z})$. Let U_j be any finite dimensional subspace of U and t_j be the corresponding orthogonal projection of U. If Z_j is the subspace of Z orthogonal to $\ker(t_j \circ \lambda) = \lambda^{-1}(\mathrm{U}_j^{\perp})$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Z} & \xrightarrow{\lambda} & \mathrm{U} \\ \downarrow{s_j} & & \downarrow{t_j} \\ \mathrm{Z}_j & \xrightarrow{\lambda_j} & \mathrm{U}_j \end{array} \tag{3.20}$$

where s_j is the projector on Z_j and where λ_j is injective. Then $\lambda\mathrm{T}$ is defined by

$$(\lambda\mathrm{T})_j = \lambda_j(\mathrm{T}_j) \tag{3.21}$$

This implies (3.9) for any $\psi \in \mathrm{Exp}_{\mathrm{cyl}}(\mathrm{U})$. Then formula (3.10) connects the Fourier transforms of T and $\lambda\mathrm{T}$.

The product of $\mathrm{T} \in \mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z})$ with $\phi \in \mathrm{Exp}_{\mathrm{cyl}}(\mathrm{Z})$ is defined by (3.12) for any $\psi \in \mathrm{Exp}_{\mathrm{cyl}}(\mathrm{Z})$.

(3.22) TENSOR PRODUCT

Let $(\mathrm{Z}_j^1)_j$ and $(\mathrm{Z}_k^2)_k$ be a family of finite dimensional complex subspaces of Hilbert spaces Z^1 and Z^2 respectively. Let $\mathrm{T} = (\mathrm{T}_j) \in \mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z}^1)$, $\mathrm{U} = (\mathrm{U}_k) \in \mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z}^2)$ and $\mathrm{Z} = \mathrm{Z}^1 \times \mathrm{Z}^2$. To define $\mathrm{T} \otimes \mathrm{U} \in \mathrm{Exp'}_{\mathrm{cyl}}(\mathrm{Z})$ it is sufficient to define $\langle \mathrm{T} \otimes \mathrm{U}, \psi \rangle$ for any $\psi \in \mathrm{Exp}_{\mathrm{cyl}}(\mathrm{Z})$. Because any basis of ψ belongs to some $\mathrm{Z}_j^1 \times \mathrm{Z}_k^2$, ψ admits a basis of the type $\mathrm{Z}_j^1 \times \mathrm{Z}_k^2$ and

$$\psi = (\psi_{jk}) \circ (s_j \times \sigma_k) \quad \text{with} \quad \psi_{jk} \in \mathrm{Exp}(\mathrm{Z}_j^1 \times \mathrm{Z}_k^2) \tag{3.23}$$

where s_j and σ_k are orthogonal projectors on Z_j^1 and Z_k^2 respectively. We set

$$(3.24) \qquad (T \otimes U, \psi) = (T_j \otimes U_k, \psi_{jk})$$

(3.25) CONVOLUTION

The convolution $T * U$ of $T \in \mathrm{Exp}'_{\mathrm{cyl}}(Z)$ and $U \in \mathrm{Exp}'_{\mathrm{cyl}}(Z)$ is the image of $T \otimes U$ by the addition map: $Z \times Z \to Z$. This means

$$\forall \psi \in \mathrm{Exp}_{\mathrm{cyl}}(Z) \qquad (T * U, \psi) = (T_y \otimes U_z, \psi(y + z))$$

(3.26) EXAMPLES

a) Let P be a polynomial on a complex Hilbert space Z identified with its dual. For any finite dimensional complex subspace Z_j of Z the restriction P_j of P to Z_j is the Fourier transform of a distribution T_j supported by the origin of Z_j. Cauchy formula proves that $T_j \in \mathrm{Exp}'(Z_j)$. For any pair (i, j) of indices such that $i > j$ (3.10) permits to show that $T_j = s_{ij}(T_i)$ and $T = (T_j)$ is an analytical profunctional of exponential type. For example $T = \delta_0$ if $P \equiv 1$.

b) Let X be a real Hilbert space, and $X^c = X + \sqrt{-1}\, X$ be complexified space. Let zz' be the complex extension of the scalar product $x, y \to \langle x, y \rangle$ on X. To any $m \in \mathrm{M\,Exp}_{\mathrm{cyl}}(X)$ can be associated its natural extension to X^c

$$m(x) \otimes \delta_0(y) \in \mathrm{M\,Exp}_{\mathrm{cyl}}(X^c)$$

By (3.5) this measure defines an analytical profunctional T of exponential type on X^c and FT is the analytic extension of Fm. For example, if $m = \nu$, then $\mathrm{FT}(\zeta) = \exp\left(-\frac{1}{2}\zeta^2\right)$.

c) For any real θ, the rotation

$$X^c \overset{R_\theta}{\to} X^c$$

$$z = x + \sqrt{-1}\, y \to ze^{i\theta} = (x + \sqrt{-1}\, y)e^{\theta\sqrt{-1}}$$

transforms m into an analytical profunctional $R_\theta T = T_\theta$. For example $\nu(x) \otimes \delta_0(y)$ is transformed in ν_θ such that

$$(\mathrm{F}\nu_\theta)(z) = \exp\left(-\frac{e^{2i\theta}}{2}z^2\right)$$

For example, if $\theta = \pi/4$, we obtain the Feynman pseudomeasure w on X.

(3.27) FIRST MODIFICATION OF THE THEORY : S-ANALYTICAL PROFUNCTIONALS

Let S be a dense vector subspace of the complex Hilbert space Z. In some cases it is useful to replace the family $(Z_i)_{i \in J}$ of all finite dimensional subspaces of S by the subfamily $(Z_j)_{j\in J}$ of all finite dimensional subspaces of S. The space $\mathrm{Exp}_{S\text{-cyl}}(Z) = D$ is the subspace of $\mathrm{Exp}_{cyl}(Z)$ consisting of functions admitting a basis contained in S. The space

$$\mathrm{Exp}_{S\text{-cyl}}(Z) = \bigcup_j \mathrm{Exp}(Z_j) = D$$

can be equipped with the inductive limit topology. The dual $\mathrm{Exp}'_{S\text{-cyl}}(Z)$ of $\mathrm{Exp}_{S\text{-cyl}}(Z)$ is the set of coherent family $T = (T_j)_j$ with $T_j \in \mathrm{Exp}'(Z_j)$.

The Borel transform $\{T\}$ of T is the following function defined on S

$$\{T\}(f) = \int e^{tf} dT(t) = T(e^f)$$

The theory of usual operations is the natural extension of the corresponding theory for $\mathrm{Exp}'_{cyl}(Z)$.

(3.28) CONNECTION WITH WORKS CONCERNING ANALYTICAL FUNCTIONALS ON $\mathbb{C}^n$

An analytical functional on $\mathbb{C}^n$ (see [*10*] [*23*] [*22*]) is usually defined as an element of the dual $H'(\mathbb{C}^n)$ of $(H(\mathbb{C}^n), t_k)$; the Borel transform realizes a linear bijection of $H'(\mathbb{C}^n)$ onto $\mathrm{Exp}(\mathbb{C}^n)$. Martineau in his thesis has equipped $\mathrm{Exp}^m(\mathbb{C}^n)$ with the topology t^m_∞ corresponding to the supremum norm

$$\varphi \to \sup |\varphi(z)| \exp(-m|z|)$$

and $\mathrm{Exp}(\mathbb{C}^n)$ with the topology $\lim t^m_\infty$. Using an argument of Grothendieck thesis [*13*] Martineau [*23*] has proved that the Borel transform realizes the following homeomorphism

$$\begin{aligned} H'(\mathbb{C}^n)\ \text{strong} &\xrightarrow{\beta} (\mathrm{Exp}(\mathbb{C}^n), \varinjlim_{m} t^m_\infty) \\ T &\to (z \to (e^{zt}, T_t)) \end{aligned} \tag{3.29}$$

The following proposition gives the connection between this natural topology and the topology θ:

(3.30) PROPOSITION

On $\mathrm{Exp}(\mathbb{C}^n)$ the two topology $\lim t^m_\infty$ and $\theta = \lim \tau^m$ agree.

Proof. — Because the inclusion $\theta \subset \lim t^m_\infty$ is evident, it is sufficient to prove the continuity of the identity map

$$(\mathrm{Exp}(\mathbb{C}^n), \varinjlim \tau^m) \xrightarrow{\mathrm{Id}} (\mathrm{Exp}(\mathbb{C}^n), \varinjlim t^m_\infty)$$

Using the universal property of topology defined by inductive limit it is sufficient to prove the continuity of the injection

$$(\mathrm{Exp}^m(\mathbb{C}^n), \tau^m) \to (\mathrm{Exp}(\mathbb{C}^n), \varinjlim t^m_\infty)$$

With the same argument, it is sufficient to prove the continuity of the restriction to the unit ball β^m of $\text{Exp}^m(\mathbb{C}^n)$. Finally, using a lemma of Grothendieck *[14]* Chapter II, n° 14 it is sufficient to prove continuity at the origin of $\text{Exp}^m(\mathbb{C}^n)$. The problem is reduced to proof of the following implication concerning a filter (φ_j) in β^m:

$$\left.\begin{array}{l} a)\ \sup_j |\varphi_j(z)| \exp(-m|z|) \leqslant 1 \\ b)\ (\varphi_j) \to 0 \text{ for } t_k \end{array}\right\} \Rightarrow (\varphi_j)_j \to 0 \text{ for } \varinjlim t_\infty^m$$

But an arbitrary neighbourhood of the origin for $\varinjlim t_\infty^m$ contains a ball $V = \{ \varphi ; \sup |\varphi(z)| \exp(-2m|z|) < \infty \}$. Because if $|z| \to \infty$, then $e^{m|z|}/e^{2m|z|} \to 0$, hypothesis *a)* implies that $|\varphi_j(z)| \exp(-2m|z|) < \varepsilon$ if $|z| > R$. Now hypothesis *b)* implies $|\varphi_j(z)| \exp(-2m)z|) \leqslant \varepsilon$ if $|z| \leqslant R$ if j is big enough; this implies the assertion of Proposition.

Let us note that specialists on finite dimensional holomorphy work with more general analytical functionals than elements of $H'(\mathbb{C}^n)$. For example in *[10]* *[23]* elements of the dual $\text{Exp}'(\mathbb{C}^n)$ of $(\text{Exp}(\mathbb{C}^n), \varinjlim t_\infty^m)$ are studied. By transposition of the homeomorphism (3.29) and because $H(\mathbb{C}^n)$ is reflexive the following proposition can be obtained.

(3.31) PROPOSITION

The Borel transform realizes the following isomorphism

$$\text{Exp}'(\mathbb{C}^n) \text{ strong} \xrightarrow{\beta} (H(\mathbb{C}^n), t_k) \tag{3.32}$$

(3.33) CONNECTION WITH THE SPACE Z ($\mathbb{C}^n$) OF GELFAND-EHRENPREIS *[10]*

A theorem of Paley-Wiener type asserts that β maps bicontinuously the Schwartz space $\mathcal{D}(\mathbb{R}^n)$ onto some space $\hat{\mathcal{D}} = Z(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$, By transposition of the injections with dense range

$$Z(\mathbb{C}^n) \to \text{Exp}(\mathbb{C}^n) \to H(\mathbb{C}^n)$$

the following injections with dense range are deduced

$$H'(\mathbb{C}^n) \to \text{Exp}'(\mathbb{C}^n) \to Z'(\mathbb{C}^n)$$

For applications in physics $H'(\mathbb{C}^n)$ is in general too small because any $T \in H'(\mathbb{C}^n)$ is represented by a measure with compact support; and for example gaussian measure has no compact support. In confront to $Z'(\mathbb{C}^n)$ the smaller space $\text{Exp}'(\mathbb{C}^n)$ has the following interesting properties:

— Fourier transforms (or Borel transforms) of elements of $\text{Exp}'(\mathbb{C}^n)$ are analytical functions (and not distributions).

— the space $\text{Exp}'(\mathbb{C}^n)$ is invariant by any complex rotation $z \to e^{i\theta}z$; this is useful for instance in the deduction (3.26.*c*). The preceding considerations imply the following corollaries:

(3.34) COROLLARY

All spaces $H'(\mathbb{C}^n)$, $\mathrm{Exp}'(\mathbb{C}^n)$, $\mathrm{Exp}(\mathbb{C}^n)$ *are complete, nuclear, and reflexive.*

This follows from the reflexivity and the nuclearity of $H(\mathbb{C}^n)$ [*13*] and from theorem 7 of [*13*] chapter 2.

(3.35) COROLLARY

Let Z *be any complex Hilbert space identified with its antidual and let* S *be any dense subspace of* Z. *Then the Borel transform realizes a linear bijection between* $\mathrm{Exp}'_{S\text{-cyl}}(Z)$ *and* $H_G(S)$.

Moreover, this bijection is bicontinuous: in order that a filter (T_j) in $\mathrm{Exp}'_{S\text{-cyl}}(Z)$ converge to zero it is necessary and sufficient that the family $(\hat{T}_j)$ of their Borel transforms converge to zero on every finite dimensional compact subset of Z.

We now prove a useful technical lemma.

(3.36) LEMMA

For any $\psi \in \mathrm{Exp}(\mathbb{C}^n)$ *there exists a sequence* $(\psi^k)_k$ *of linear combinations of exponentials such that*

$$\exists C, \ \exists \alpha, \ \forall k, \ \forall z \qquad |\psi^k(z)| \leqslant Ce^{\alpha|z|}$$

$(\psi^k) \to \psi$ uniformly on any compact subset of $\mathbb{C}^n$.

Proof. — The inverse Borel transform T of ψ can be represented by a measure μ with a compact support $K \subset \mathbb{C}^n$. Because K is separable, there exists a sequence $(\mu^k)_k$ of finite linear combinations of Dirac measures on K, converging weakly to μ in M(K). Then the sequence $(\hat{\mu}^k) = (\psi^k)$ satisfies the conditions of Lemma (3.36).

(3.37) SECOND MODIFICATION OF THE THEORY

In general, if E and F are two locally convex spaces, the antidual 'E of E is the space of all antilinear continuous forms on E. If $\bar{E}$ denotes the conjugate space of E, then $'E = (\bar{E})' \simeq (\bar{E}')$. If $A : E \to F$ is a linear continuous map, the adjoint $A^* : {}'F \to {}'E$ of A is defined by

$$(3.38) \qquad \forall e \in E \quad \forall \lambda \in {}'F \qquad \langle \lambda, Ae \rangle = \langle A^*\lambda, e \rangle$$

the bracket being linear with respect to ket, and antilinear with respect to bra. Let Z be a complex Hilbert space. Since the Riesz theorem gives an isomorphism $Z \simeq {}'Z$, it is more convenient to work in the complex case, with antilinear rather than linear functionals. Then if the Borel transform β of $T \in {}'\mathrm{Exp}(\bar{\mathbb{C}}^n)$ is defined by

$$(3.39) \qquad \{T\}(\bar{z}) := T(e^{z}) = \langle e^{z}, T \rangle$$

then we have two conjugate isomorphisms

$$(3.40)\qquad \begin{array}{c} H(\bar{\mathbb{C}}^n) \overset{\beta}{\leftarrow} {}'\mathrm{Exp}\,(\bar{\mathbb{C}}^n) \\ {}'H(\bar{\mathbb{C}}^n) \overset{\beta}{\rightarrow} \mathrm{Exp}\,(\bar{\mathbb{C}}^n) \end{array}$$

In the same manner we obtain

$$(3.41)\qquad \begin{array}{c} H(\mathbb{C}^n) \overset{\beta}{\leftarrow} {}'\mathrm{Exp}\,(\mathbb{C}^n) \\ {}'H(\mathbb{C}^n) \overset{\beta}{\rightarrow} \mathrm{Exp}\,(\mathbb{C}^n) \end{array}$$

By taking tensor products on obtain

$$(3.42)\qquad \begin{array}{l} H(\mathbb{C}^n \times \bar{\mathbb{C}}^n) \simeq H(\mathbb{C}^n) \hat{\otimes} H(\bar{\mathbb{C}}^n) \\ \qquad\qquad \overset{\beta}{\leftarrow} \mathrm{Exp}\,(\bar{\mathbb{C}}^n \times \bar{\mathbb{C}}^n) \simeq \mathrm{Exp}\,(\mathbb{C}^n) \hat{\otimes} \mathrm{Exp}\,(\bar{\mathbb{C}}^n) \\ {}'H(\mathbb{C}^n \times \bar{\mathbb{C}}^n) \simeq {}'H(\mathbb{C}^n) \hat{\otimes} {}'H(\bar{\mathbb{C}}^n) \\ \qquad\qquad \overset{\beta}{\rightarrow} \mathrm{Exp}\,(\mathbb{C}^n \times \bar{\mathbb{C}}^n) \simeq \mathrm{Exp}\,(\mathbb{C}^n) \hat{\otimes} \mathrm{Exp}\,(\bar{\mathbb{C}}^n) \end{array}$$

The action of $T \in {}'\mathrm{Exp}\,(\mathbb{C}^n \times \bar{\mathbb{C}}^n)$ on $\varphi(z, \bar{z}') \in \mathrm{Exp}\,(\mathbb{C}^n \times \bar{\mathbb{C}}^n)$ is denoted by

$$(3.43)\qquad \langle \varphi, T \rangle = \langle \varphi(z, \bar{z}'), T(z, \bar{z}') \rangle = \int \overline{\varphi(z, \bar{z}')} dT(z, \bar{z}')$$

By interchange of variables, there is an isomorphism of $H(\mathbb{C}^n \times \overline{\mathbb{C}^n})$ to the space $H(\overline{\mathbb{C}^n} \times \mathbb{C}^n)$ of sesquiholomorphic functions on $\mathbb{C}^n \times \mathbb{C}^n$; and so the Borel transform of any T in $'\mathrm{Exp}\,(\mathbb{C}^n \times \overline{\mathbb{C}^n})$ is written

$$(3.44)\qquad \{T\}(\bar{z}, z') = \langle \overline{e^{\bar{z}'}} \otimes e^{z}, T \rangle = \int e^{\overline{\alpha\bar{z}' + \bar{z}\alpha'}} dT(\alpha, \bar{\alpha}')$$

4. KERNELS AND SYMBOLS (FINITE DIMENSIONAL CASE)

In this case $X = \mathbb{R}^n$ and $X^c = \mathbb{C}^n$.

(4.1) Extension of the map θ

The L. Schwartz theory of Fourier transform of real tempered distributions can be illustrated by the following scheme:

$$(4.2)\qquad \begin{array}{ccc} {}'S(\mathbb{R}^n) & \overset{F}{\longrightarrow} & {}'S(\mathbb{R}^n) \\ \uparrow & & \uparrow \\ L^2(\mathbb{R}^n) & \overset{F}{\longrightarrow} & L^2(\mathbb{R}^n) \\ \uparrow & & \uparrow \\ S(\mathbb{R}^n) & \overset{F^{-1}}{\longleftarrow} & S(\mathbb{R}^n) \end{array}$$

We now extend this scheme to the complex case. Since $\theta : L^2_v(R^n) \to F(\mathbb{C}^n)$ is an isometry, and by virtue of (2.28) we have

(4.3) $\forall\varphi \in \operatorname{Exp}(\mathbb{C}^n)$, $\forall\psi \in \operatorname{Exp}(\bar{\mathbb{C}}^n)$, $\langle \theta^{-1}\varphi \mid \psi \rangle = \langle \varphi \mid \theta\psi \rangle$

Then, the adjoint of the linear bicontinuous map $\theta^{-1} : \operatorname{Exp}(\bar{\mathbb{C}}^n) \to \operatorname{Exp}(\bar{\mathbb{C}}^n)$ is a linear bicontinuous map: $'\operatorname{Exp}(\bar{\mathbb{C}}^n) \to {}'\operatorname{Exp}(\bar{\mathbb{C}}^n)$ extending the map θ defined by (2.27). The extension is called θ and we have the scheme

(4.4)
$$\begin{array}{ccc} '\operatorname{Exp}(\bar{\mathbb{C}}^n_{j*}) & \xrightarrow{\theta} & '\operatorname{Exp}(\bar{\mathbb{C}}^n_{k*}) \\ \uparrow & & \uparrow \\ L^2_v(\mathbb{R}^n) & \xrightarrow{\theta} & F(\mathbb{C}^n_k) \\ \uparrow & & \uparrow \\ \operatorname{Exp}(\bar{\mathbb{C}}^n) & \xrightarrow{\theta} & \operatorname{Exp}(\bar{\mathbb{C}}^n) \end{array}$$

where j and k are the canonical injections; the spaces $L^2_v(\mathbb{R}^n)$ and $F(\mathbb{C}^n)$ are identified with their antiduals, and j^* and k^* are the adjoints of j and k respectively.

(4.5) THE REPRODUCING PROPERTY (2.19) MEANS THAT ANY ϕ IN $F(\mathbb{C}^n)$ COINCIDES WITH THE BOREL TRANSFORM OF THE ANTIANALYTICAL FUNCTIONAL $\phi v'$

$$\operatorname{Exp}(\bar{\mathbb{C}}^n) \xrightarrow{\phi v'} \mathbb{C}$$
$$\psi \to \langle \psi, \phi v' \rangle = \int \overline{\psi(z')}\phi(z')dv'(z')$$

This property is very important and can be extended to other elements of $'\operatorname{Exp}(\bar{\mathbb{C}}^n)$:

(4.6) LEMMA

Let $T \in {}'\operatorname{Exp}(\mathbb{C}^n)$ *be such that*

(4.7)
$$\forall a > 0 \quad \int_{\mathbb{C}^n} |\{T\}(\bar{z})| e^{a|z|} dv' |z| < \infty$$

Then the action of T *on* $\psi \in \operatorname{Exp}(\bar{\mathbb{C}}^n)$ *is given by the following formula*

(4.8)
$$\langle \psi, T \rangle = \int_{\mathbb{C}^n} \overline{\psi(z)} \{T\}(\bar{z}) dv'(z)$$

Proof. — This formula holds it ψ is any exponential, by the definition of $\{T\}$. By linearity, the formula holds for any linear combination of exponentials. Using (3.36) the formula holds for any $\psi \in \operatorname{Exp}(\bar{\mathbb{C}}^n)$.

(4.9) THE SPACE (Q) OF LINEAR OPERATORS

Let (Q) be the space of linear continuous operators $\operatorname{Exp}(\bar{\mathbb{C}}^n) \to {}'\operatorname{Exp}(\bar{\mathbb{C}}^n)$. This space is equipped with the topology of uniform convergence of all bounded subsets of $\operatorname{Exp}(\bar{\mathbb{C}}^n)$.

The linear map transforming any $T \in (\text{Exp } \bar{\mathbb{C}}^n)'$ in the element $\psi \to T(\psi)$ of $'\text{Exp}\ (\bar{\mathbb{C}}^n)$ is an isomorphism of locally convex spaces. Then using (3.42) and the theorem 6, chapter II of [13],

$$(4.10)\quad (Q) \simeq (\text{Exp } \bar{\mathbb{C}}^n)' \bar{\otimes}\ '\text{Exp } (\bar{\mathbb{C}}^n) \simeq ('\text{Exp } \mathbb{C}^n) \hat{\otimes}\ '\text{Exp } (\bar{\mathbb{C}}^n) \simeq\ '\text{Exp } (\mathbb{C}^n \times \bar{\mathbb{C}}^n)$$

(4.11) DEFINITION OF THE KERNEL Q^K OF Q IN (Q)

The antianalytical functional of exponential type associated by (4.10) to any $Q \in (\mathcal{Q})$ is called the kernel of Q. This kernel will be denoted by the symbol Q^K.

(4.12) SOME RELATIONS

a) For any ψ and $\varphi \in \text{Exp } (\bar{\mathbb{C}}^n)$

$$\langle \psi, Q\varphi \rangle = \langle \bar{\varphi} \otimes \psi, Q^K \rangle = \langle \overline{\varphi(\bar{z})}\psi(\bar{z}'), Q^K(z, \bar{z}') \rangle$$

b) In particular, the Borel transform $\{ Q^K \}$ of Q^K is given by

$$\{ Q^K \}(\bar{z}, z') = \langle e^z, Qe^{z'} \rangle$$

For the sake of simplicity, this can be also denoted by $Q(z, z')$: see § 8. From (3.42) it follows that the map $Q \to \{ Q^K \}$ is a homeomorphism of Q onto $H(\bar{\mathbb{C}}^n \times \mathbb{C}^n)$.

c) The Borel transform of the kernel of the adjoint Q^* of Q is

$$\{ Q^{*K} \} (\bar{z}, z') = \langle e^z \mid Q^* e^{z'} \rangle = \langle \overline{e^{z'}, Qe^{z}} \rangle$$

so

$$\{ Q^{*K} \} (\bar{z}, z') = \overline{\{ Q^K \} (\bar{z}, z')}$$

It is evident that Q is symmetric if and only if $\{ Q^K \} (\bar{z}, z') = \overline{\{ Q^K \} (\bar{z}, z')}$.

(4.13) PROPOSITION

Any operator $Q \in (Q)$ *admits the following* R *representation*

$$(4.14)\qquad Q = \iint | e^z \rangle \langle e^{z'} | \, dQ^K(\bar{z}, z')$$

More precisely, for any $\psi \in \text{Exp } (\bar{\mathbb{C}}^n)$, the Borel transform of $Q\psi$ is given by the formula

$$(4.15)\qquad \{ Q\psi \} (\bar{z}) = \iint \psi(\bar{z}_1) e^{\bar{z}z_1} dQ^K(z_1, \bar{z}'_1)$$

This follows directly from (4.12.*a*) because $\{ Q\psi \} (\bar{z}) = \langle e^z, Q\psi \rangle$.

(4.16) PROPOSITION

Suppose that $Q \in (Q)$ *satisfies the following condition:*

$$(4.17)\qquad \forall z \in \mathbb{C}^n ;\ \exists a > 0 ; \quad \int | \{ Q^K \} (\bar{z}, z') | \, e^{a|z|} dv'(z) < \infty$$

then for any $\psi \in \operatorname{Exp}(\bar{\mathbb{C}}^n)$, *the Borel transform of* $Q\psi$ *has the following integral representation*

$$(4.18) \qquad \forall z \in \mathbb{C}^n \qquad \{ Q\psi \}(\bar{z}) = \int \{ Q^K \}(\bar{z}, z')\psi(\bar{z}')dv'(z')$$

Proof. — For $\psi(\bar{z}') = e^{\bar{z}'z}$, relation (4.18) follows from (4.12.*b*). By linearity, (4.18) holds for any linear combination of exponentials. And using (3.36), relation (4.18) holds for any $\psi \in \operatorname{Exp}(\bar{\mathbb{C}}^n)$.

(4.19) WICK SYMBOL AND QUANTIZATION

a) The Wick symbol $Q^w(\bar{z}, z')$ of $Q \in (Q)$ is the following function

$$Q^w(\bar{z}, z') = e^{-\bar{z}z'} \{ Q^K \}(\bar{z}, z')$$

b) The symbol map : $Q \to Q^w$ is an isomorphism from (Q) to $H(\bar{\mathbb{C}}^n \times \mathbb{C}^n)$. The inverse map is called a quantization.

A quantization Ω of S is a map $F(\bar{z}, z') \to \Omega(F)$ of symbols of classical dynamical variables into operators in the Segal-Bargman space $F(C^n)$, which satisfies the following conditions

1° $\Omega(\lambda_1 F_1 + \lambda_2 F_2) = \lambda_1 \Omega(F_1) + \lambda_2 \Omega(F_2)$, $\quad \lambda_1, \lambda_2 \in \mathbb{C}$
2° $\Omega(\lambda) = \lambda I$, $\quad \lambda \in C$.

The following Proposition gives a method of prescribing to any classical dynamical variable $F(\bar{z}, z')$ an operator Q in Bargman-Segal space of quantum states.

(4.20) PROPOSITION

Let $\tilde{Q}^w$ *be the inverse Borel transform of the Wick symbol* Q^w *of* $Q \in (Q)$. *Then* Q *admits the following normal representation*

$$(4.21) \qquad Q = \iint e^{\alpha a^*} e^{\bar{\beta} a} d\tilde{Q}^w(\beta, \bar{\alpha})$$

More precisely, for any $\psi \in \operatorname{Exp}(\bar{\mathbb{C}}^n)$, *the Borel transform of* $Q\psi$ *is given by the following formula*

$$(4.22) \qquad \{ Q\psi \}(\bar{z}) = \iint \psi(\bar{z} + \beta) e^{\alpha \bar{z}} d\tilde{Q}^w(\beta, \bar{\alpha})$$

Proof. — By the density argument and because $\tilde{Q}^w$ is an antilinear continuous form on $\operatorname{Exp}(\mathbb{C}^n \times \bar{\mathbb{C}}^n)$, it is sufficient to prove (4.22) for $\psi = e^{\bar{z}'}$. But in this case (4.22) follows directly from the definition of the Wick symbol.

Proposition (4.20) allows for defining the Wick product of $Q_1, ..., Q_n \in (Q)$ by quantization of $Q_1^w \times \ldots \times Q_n^w$.

In order to clarify the meaning of introduced concepts we consider the following example. Let Q be a formal operator given by the formula

(4.23) $$Q = \sum_{k,l=0}^{\infty} q_{k,l} a^{*k} a^{l}$$

where $k = (k_1, \ldots, k_n)$, $l = (l_1, l_2, \ldots, l_n)$ are multiindexes.

Its Borel transform $\{ Q^K \}$ by (4.12.b) has the form

$$\{ Q^K \} (\bar{z}, z') = \langle e^z, Qe^{z'} \rangle = \sum_{k,l=0}^{\infty} q_{kl} \langle a^k e^z, a^l e^{z'} \rangle$$
$$= \langle e^z, e^{z'} \rangle \sum_{k,l=}^{\infty} q_{k,l} \bar{z}^k z'^{l_1} = e^{\bar{z}z'} \sum^{\infty} q_{kl} \bar{z}^k z'^l$$

Hence the Wick symbol (4.19.a) of Q is

$$Q^w(\bar{z}, z') = \sum_{k,l=0}^{\infty} q_{k,l} \bar{z}^k z'^l$$

Consequently, if $\sum^{\infty} q_{k,l} \bar{z}^k z'^{l_1} \in H(\bar{C}^n \times C^n)$ then by virtue of (4.19.b) the formal operator Q represents the map from Exp $\bar{C}^n$ into 'Exp C^n. Most operators considered in Quantum Optics are given in the form (4.23) with

$$Q^w(\bar{z}, z') \in H(\bar{C}^n \times C^n).$$

To verify when (4.23) maps Exp ($\mathbb{C}^n$) in F($\mathbb{C}^n$): see (7.4.a).

5. WICK SYMBOLS AND KERNELS (INFINITE DIMENSIONAL CASE)

We shall now extend the results of previous section to the case of infinite number of degrees of freedom. We shall carry it out using the formalism of analytical profunctionals of section 3. Let S be a dense subspace of $Z = X^c$, which is the complexification of $S \cap X$. Consider the natural continuous injection with a dense range $D \to L^2(X)$ and $D \to F(Z)$, where D is defined in (3.27). Taking the adjoints, two triplets are obtained.

$$D \to L^2(X) \to D' \quad \text{and} \quad D \to F(Z) \to D'$$

Since θ maps a cylindrical function with exponential growth into a cylindrical function with the same basis, θ can be extended to 'D. More explicitely, the diagram (4.4) is generalized to the following one:

(5.1) $$\begin{array}{ccc} 'D & \overset{\theta\simeq}{\to} & 'D \\ \uparrow & & \uparrow \\ L^2(X) & \overset{\theta\simeq}{\to} & F(Z) \\ \uparrow & & \uparrow \\ D & \overset{\theta\simeq}{\to} & D \end{array}$$

In what follows we shall work only with the Fock representation.

(5.2) DEFINITION OF (Q_S)

Let Q be a linear map: $D \to 'D$. For any $j \in J$, let $\tilde{Q}_j$ be the restriction of Q to Exp (X_j^c). Then for $\varphi = \varphi_j \circ s_j^c$ in D, we have:

$$Q\varphi = \tilde{Q}_j\varphi = ((Q\varphi)_k)_{k\in J} \qquad \text{with} \qquad (Q\varphi)_k \in {}'\text{Exp } X_k^c$$

The following map

$$\text{Exp}\,(X_j^c) \to {}'\text{Exp}\,(X_j^c)$$
$$\varphi_j \to (Q\varphi)_j$$

is called Q_j. It is easy to see that the set $(Q_j)_j$ characterizes Q. Then (Q_s) can be defined as the space of linear maps $Q = (Q_j)$ from D into 'D such that for any j, Q_j belongs to the space (Q_j) of linear continuous maps Exp $(X_j^c) \to$ 'Exp (X_j^c).

(5.3) PROKERNEL Q^K

Any map Q_j by virtue of (4.10) has a kernel $Q_j^K \in {}'\text{Exp}\,(X_j^c \times \bar{X}_j^c)$. The family $(Q_j^K)_j$ $(j \in J)$ satisfies the following coherence condition

$$(5.4) \qquad i \geqslant j \Rightarrow Q_j^K = (s_{ij}^c \times s_{ij}^c)(Q_i^K)$$

Hence this family defines an antianalytical S-profunctional Q_K of exponential type on $\bar{Z} \times Z$. This profunctional is called the prokernel of Q.

(5.5) BOREL TRANSFORM $\{Q^K\}$ OF Q^K. WICK SYMBOL OF OPERATOR

By analogy with (3.44) we define the Borel transform $\{Q^K\}(\bar{z}, z')$ and the Wick symbol $Q^w(z, z')$ on $S \times S$ in the following manner

$$(5.6) \qquad \{Q^K\}(\bar{z}, z') = \langle \overline{e^{\bar{z}'}} \otimes e^{z'}, Q^K \rangle = \langle e^{\bar{z}'z_1 + z\bar{z}'_1}, Q^K(z_1, \bar{z}'_1) \rangle$$

$$(5.7) \qquad Q^w(\bar{z}, z') = e^{-zz'}\{Q^K\}(\bar{z}, z')$$

We note that for any $j \in J$, the restriction of $\{Q^K\}$ (resp. Q^w) to $X_j^c \times X_j^c$ coincides with $\{Q_j^K\}$ (resp. Q_j^w).

The symbol map $Q \to Q^w$ is a bijection between (Q_s) and the space $H_G(S \times S)$ of Gateaux sesquiholomorphic functions on $S \times S$. This follows directly from (3.42).

All properties and formulas of Section 3 and 4 can be now extended to the infinite dimensional case. For example for any $\psi \in D$, the Borel transform of $Q\psi$ is given by

$$(5.8) \qquad \forall z \in S \qquad \{Q\psi\}(\bar{z}) = \iint \psi(\bar{z}_1)e^{\bar{z}z'_1}dQ^K(z_1, z'_1)$$

where the integral means the action of Q^K on the cylindrical functional $z_1, z'_1 \to \psi(\bar{z}_1)\exp(\bar{z}z'_1)$.

In particular also Propositions (4.13) and (4.20) extend to infinite dimensional case. Since the concept of prokernel reduces the proofs to the finite dimensional case we left them as an exercise for the reader.

(5.9) USE OF NUCLEAR PROPERTY OF S

In conclusion without any assumption on the dense subspace S of X^c, it is possible to quantize any Gateaux sesquiholomorphic function on S × S, using a cylindrical formalism reducing the infinite dimensional case to the finite one.

However, if S is a Fréchet nuclear space, using some recents results of infinite dimensional holomorphy and some topological and bornological arguments, it can be show [20] that mains of results of § 4 can be extended in the infinite dimensional case, replacing $\mathbb{C}^n$ by the nuclear triplet:

$$T = (S \subset X^c \subset S)$$

The space Exp 'S is defined as the space of entire functions ψ on 'S such that there exists a continuous semi-norm p on 'S with

$$\sup_z |\psi(z)| \exp(-p(z)) < \infty .$$

Then Exp 'S is an inductive limite of Banach spaces B_p, and we have a triplet

$$\text{(5.10)} \qquad \text{Exp}('S) \subset F(X^c) \subset '\text{Exp}('S)$$

where 'Exp ('S) is equipped with the topology of uniform convergence of all ball of all the spaces B_p: This is a complet nuclear space and its strong dual is Exp ('S). The space $0p$ T of linear continuas maps Exp ('S) → 'Exp ('S) is equipped with the topology of uniform convergence on all balls of all spaces B_p. Then we have three isomorphism

$$\text{(5.11)} \quad \begin{array}{ccccccc} 0p\,T & \xrightarrow[\text{map}]{\text{Kernel}} & '\text{Exp}('S \times S') & \xrightarrow[\text{transform}]{\text{Borel}} & H(S \times S) & \xrightarrow[\text{by } e^{-\bar{z}z'}]{\text{multiplication}} & H(S \times S) \\ Q & \longrightarrow & Q^K & \longrightarrow & \{Q^K\} & \longrightarrow & Q^w \end{array}$$

So in this particular case, any element of $H(\bar{S} \times S)$ can be quantize, without using the cylindrical formalism.

6. REMARKS ON GENERALIZED QUANTIZATIONS

Let Z be the complexification of the real separable Hilbert space X. Let $\text{Exp}_2(\bar{Z} \times Z)$ be the space of continuous sesquiholomorphic functions $f(\bar{z}, z')$ on Z × Z, satisfying for some $n > 0$ the following growth condition:

$$\sup |f(\bar{z}, z')| \exp[-n(||z|| + ||z'||)^2] \qquad \text{bounded.}$$

This space has a natural inductive limit topology given by (3.2). The Borel transform of any $T \in {}'\text{Exp}_2(\bar{Z} \times Z)$ is the function on Z × Z defined by (3.44). Denote by the symbol Ω a fixed nonzero element of $\text{Exp}_2(\bar{Z} \times Z)$. Then the quantization rule (4.19) can be generalized in the following manner

(6.1) GENERALIZED QUANTIZATION $\Phi \to \hat{\Phi}$

(6.2) Suppose that $\Phi(\bar{z}, z') \in \mathrm{H}(\bar{\mathrm{Z}} \times \mathrm{Z})$ is the Borel transform of some $\tilde{\Phi} \in {}'\mathrm{Exp}_2\,(\bar{\mathrm{Z}} \times \mathrm{Z})$.

Then for any $\psi \in \mathrm{Exp}_{\mathrm{cyl}}\,\bar{\mathrm{Z}}$, the function

$$\bar{z} \to \iint \psi(\bar{z} + \bar{\beta}) e^{\alpha \bar{z}} e^{\Omega(\bar{\alpha}, \beta)} d\tilde{\Phi}(\beta, \alpha) \tag{6.3}$$

is Gateaux holomorphic on $\bar{\mathrm{Z}}$. The linear operator $\hat{\Phi}$:

$$\mathrm{Exp}_{\mathrm{cyl}}\,(\bar{\mathrm{Z}}) \to {}'\mathrm{Exp}_{\mathrm{cyl}}\,(\bar{\mathrm{Z}})$$

transforming ψ into $\Phi\psi$, whose Borel transform is given by (6.3), may be written symbolically in the form

$$\hat{\Phi} = \iint e^{\alpha a^*} e^{\beta a} e^{\Omega(\bar{\alpha}, \beta)} d\tilde{\Phi}(\bar{\alpha}, \beta) \tag{6.4}$$

Let us note that using the definition (3.12) of the product of an analytical functional by an analytical function we can write (6.4) in the form

$$\hat{\Phi} = \int e^{\alpha a^*} e^{\bar{\beta} a} d(e^{\Omega} \tilde{\Phi})(\bar{\alpha}, \beta)$$

This shows that the theory of a generalized quantization Ω may be reduced to the theory of Wick ordering with a weight. The normal (or the Wick) quantization rule corresponds to $\Omega \equiv 0$. The antinormal quantization rule corresponds to $\Omega = \bar{\alpha}\beta$. The Weyl quantization rule corresponds to $\Omega = \frac{1}{2}\bar{\alpha}\beta$. Any quantization rule used so far in Quantum Theory (e. g. standard, Born-Jordan, Rivers, etc.) in case of finite number of degrees of freedom can be written down with the help of a function Ω in the space $\mathrm{Exp}\,\bar{\mathrm{C}}^n$. Therefore in this case we obtain a natural extension of results of [*6*]. In case of infinite number of degrees of freedom, contrary to common believe, there exists a restriction on the set of sesquiholomorphic functions which can be quantized. This restriction is intrinsic and not technical. For instance the energy operator at a free scalar classical field $\hat{\Phi}(t, x)$ has the symbol

$$\mathrm{P}_0 = \int d^3\vec{p}\, \bar{z}(\vec{p})\, z(\vec{p})$$

were $z(\vec{x}) = \frac{1}{\sqrt{2}}[1/2\Phi + i\omega^{-\frac{1}{2}}\partial_t \Phi](0, \vec{x})$, $\omega = \sqrt{m^2 - \Delta}$ and $z(\vec{p})$ is the Fourier transform of $z(\vec{x})$. The normal quantization of P_0 gives

$$\hat{\mathrm{P}}_0^w = \int d^3\vec{p} a^*(p) a(p)$$

which is a well defined self-adjoint operator in $F(\bar{Z})$ whereas antinormal quantization leads to

$$\hat{P}_0^A = \int d^3\vec{p}a(\vec{p})a^*(\vec{p}) = -\int d^3\vec{p}a^*(\vec{p})a(p) + \delta(0)\int d^3\vec{p}$$

The same phenomenon occurs for many other classical dynamical variables as well as for other generalized quantizations. The reason for this difficulty lies in the fact that other than Wick quantization rules give expression for dynamical variables in which $a^*(p)$ proceeds $a(p)$: in this case since for $\psi \in \text{Exp } \bar{Z}$, $a^*(p)\psi \notin \text{Exp } \bar{Z}$ in general, the action of not normally ordered operators cannot be defined in Exp $\bar{Z}$. It seems therefore that in Quantum Field Theory the Wick quantization is more suitable for analysis of properties of various models.

7. PROPERTIES OF OPERATORS CONNECTED WITH THEIR SYMBOLS

The preceding analysis shows that, starting with a nuclear triplet $T = (S \to X^c \to 'S)$ any sesquiholomorphic function on $S \times S$ is the Wick symbol of an linear continuous operator Q defined on Exp S, with values in the space 'Exp S′, which is bigger than the Fock space. Because in constructive quantum field theory, one usually starts with cutoff operators in the Fock space, it is interesting to have an easily verifable criterion concerning Wick symbol Q^w (or Borel transform of the kernel), assuring that Q maps continuously Exp S′ into $F(X^c)$. This criterion which describes the regularity properties of operators is given by the following theorem.

(7.1) THEOREM

Let Q *be in* $0p$ T. *The map:* $Q \to \{Q^K\}$, *induces a topological isomorphism*

$$L(\text{Exp } S', F(X^c)) \to H(S, F(X^c))$$

More explicitely an operator $Q \in 0p$ T maps continuously Exp S′ into the Fock space, if and only if for any fixed $z' \in S$, the function $z \to \{Q^K\}(\bar{z}, z')$ belongs to the Fock space, and if the map $z' \to Q^w(., z')$ is continuous from S into $F(X_c)$.

Proof. — By L. Schwartz, A. Grothendieck kernel theory [*13*] the map $Q \to Q^K$ induce an isomorphism

$$L(\text{Exp } S', F(X^c)) \to '\text{Exp } ('S) \hat{\otimes} F(X^c) \quad (7.2)$$

In turn the Borel transform induces an isomorphism 'Exp S $\to$ H(S) (see [*20*]); and by (2.20), coincides with the identity map on $F(X^c)$. By tensor product of these two isomorphisms, it follows that Borel transform $Q^K \to \{ Q^K \}$ induces the topological isomorphism

$$(7.3) \qquad '\mathrm{Exp}\,('\hat{S}) \otimes F(X^c) \to H(S) \hat{\otimes} F(X^c).$$

By standard arguments of [*13*], this last space is topologically isomorphic to the space $H(S, F(X^c))$ of holomorphic functions defined on S, with values in $F(X^c)$ which map continuously $z' \in S$ into $F(X^c)$. Theorem (7.1) follows by composition of the isomorphisms (7.2) and (7.3).

If we consider in particular the nuclear triplet $T = (\mathbb{C}^n = \mathbb{C}^n = \mathbb{C}^n)$ we obtain the following.

(7.4) COROLLARY

The map $Q \to \{ Q^K \}$ *induces a topological isomorphism*

a) $L(\mathrm{Exp}\,(\mathbb{C}^n), F(\mathbb{C}^{\bar{n}})) \to H(\mathbb{C}^n, F(\mathbb{C}^n))$

In addition we have:

b) $L(\mathrm{Exp}\,(\overline{\mathbb{C}^n}), \mathrm{Exp}\,(\mathbb{C}^n)) \simeq H(\mathbb{C}^n, \mathrm{Exp}\,(\mathbb{C}^n))$

c) $L('\mathrm{Exp}\,(\overline{\mathbb{C}^n}), \mathrm{Exp}\,(\mathbb{C}^n)) \simeq \mathrm{Exp}\,(\mathbb{C}^{\bar{n}} \times \mathbb{C}^n)$

According to our knowledge this kind of effective characterization of an extended class of unbounded operators in the Fock space, was never obtained even in the finite dimensional case.

We present now some other properties connecting operators and symbols which are usefull in applications.

Let us observe that any unbounded symmetric operator Q in F whose domain D(Q) contains the linear envelope $\tilde{D}$ of coherent states vectors e^z, $z \in S$ has the Wick symbol $Q^w(\bar{z}, z')$ which is the sesquiholomorphic functional on $S \times S$ of order two. Indeed since $\psi = Q(e^{z'}) \in F(X^c)$ the function

$$\psi_{z'}(\bar{z}) = \langle e^z, Qe^{z'} \rangle$$

by (2.18) is antiholomorphic on S of order two. Now because

$$\langle e^z, Qe^{z'} \rangle = \langle Qe^z, e^{z'} \rangle = \bar{\psi}_z(\bar{z}') \in F(X^c),$$

the Borel transform $Q(\bar{z}, z')$ is holomorphic in z'. Since $Q^w(\bar{z}, z') = e^{-\bar{z}z'} Q(\bar{z}, z')$ the assertion follows.

(7.5) REMARK

The linear continuous operator $Q : \mathrm{Exp}\,(S') \to {'\mathrm{Exp}}\,(S')$ coïncide with its adjoint $Q^* : \mathrm{Exp}\,(S') \to {'\mathrm{Exp}}\,(S')$ if and only if its Wick symbol has the following hermitian symmetry property

$$Q^w(z, z') = \overline{Q(\bar{z}', z)}$$

(7.6) LEMMA

The coherent state $|\alpha\rangle$ *tends weakly to zero in* $F(X^c)$ *if* $\|\alpha\| \to \infty$.

Proof. — It is sufficient to prove that for any

$$\varphi \in F(X^c) \qquad \langle \alpha \mid \varphi \rangle = \varphi(\bar{\alpha}) \exp - \left(\frac{\alpha\bar{\alpha}}{2}\right)$$

tends to zero. For any $\varepsilon > 0$, there exists a cylindrical polynomial function ψ such that $\| H \| < \varepsilon$ with $H = \varphi - \psi$. Then

$$| \langle \alpha \mid \varphi \rangle | = | \langle \alpha \mid \psi \rangle + \langle \alpha \mid H \rangle | \leqslant | \psi(\alpha) | e^{-\alpha\bar{\alpha}/2} + \| H \|$$

Hence $| \langle \alpha \mid \varphi \rangle | \leqslant 2\varepsilon$ if $\| \alpha \|$ is sufficiently large because

$$| \psi(\alpha) | \exp \left(- \frac{\alpha\bar{\alpha}}{2} \right) \to 0 \qquad \text{for } \| \alpha \| \to \infty .$$

(7.7) PROPOSITION

If Q is a compact operator in the Fock space, then $Q^w(\bar{z}, z)$ *tends to zero if* $\| z \| \to \infty$.

Proof. — In fact.

$$Q^w(\bar{z}, z) = e^{-z\bar{z}} \langle e^z, Qe^z \rangle = \langle z \mid Q \mid z \rangle \tag{7.8}$$

Using the preceding lemma, and because Q is compact, if $\| z_n \|$ tends to infinity, then $Q \mid z_n \rangle$ tends to zero strongly. And because $\mid z_n \rangle$ has norm one, then $Q^w(\bar{z}_n, z_n) = \langle z_n \mid Q \mid z_n \rangle$ tends to zero.

(7.9) PROPOSITION

Let $T = (S \subset X^c \subset {}'S)$ *be a nuclear triplet. Suppose that* $Q \in 0p\, T$ *admits an integral kernel of the type*

$$Q^K(\bar{t}, t) = \emptyset(E, t) v'(E, t)$$

where $\emptyset$ *defined on* S′, *belongs to some Lebesgue class* L^p, $p \geqslant 1$. *Let*

$$\operatorname{Im} Q^w = \{ | Q^w(\bar{z}, z) |, \ z \in S \}$$
$$\rho = \sup \text{ess} \{ | \emptyset(\bar{z}, z) |, \ z \in S \}$$
$$\operatorname{Dom} Q = \{ \langle \psi, Q\varphi \rangle, \ \| \psi \| \ \textit{and} \ \| \psi \| \leqslant 1, \ \varphi \ \textit{and} \ \psi \in \operatorname{Exp} S'$$

Then $\operatorname{Im} Q^w \subset \operatorname{Dom} Q \subset D(\rho)$ *where* $D(\rho)$ *is the disk of the complex plane where* $| z | \leqslant \rho$.

Proof. — The first inclusion follows form (7.8). The second inclusion follows from

$$| \langle \psi, Q\varphi \rangle | = | \int \varphi(\bar{z}) \overline{\psi(\bar{z})} \emptyset(\bar{z}, z) dv'(\bar{z}, z) | \leqslant \rho \| \varphi \| . \| \psi \|$$

(7.10) COROLLARY

a) If ρ is finite, then Q is bounded and

$$\sup_{z \in S} | Q^w(\bar{z}, z) | \leqslant \| Q \| \leqslant \rho$$

b) If $z \to Q^w(\bar{z}, z)$ is not bounded on S, then Q cannot be bounded.
c) If Q is self adjoint in $F(X^c)$, then

$$\operatorname{Im} Q^w \subset \text{Convex Hull of Sp } Q \subset D(\ )$$

8. EXAMPLES

We shall illustrate now the main theorem 7.1 by several examples from quantum field theory.

Let $m > 0$; $N = 2, 3, 4, \ldots$; $x = (x_0, \vec{x})$ and

$$p = (p_0, \vec{p}) \in \mathbb{R}^N ; \quad px = p_0 x_0 - \vec{p}.\vec{x}.$$

Let $\Gamma^+ = \{ p \in \mathbb{R}^N ; p_0 > 0, p^2 = m^2 \}$.

We use the real nuclear triplet $T = (S_r \subset L^2(\Gamma^+) \subset S'_r)$, where S_r is the L. Schwartz space of Γ^+. Let $\Phi(x)$ be a free scalar massive quantum field on R^N:

$$(8.1) \qquad \Phi(x) = C_N \int_{p \in \Gamma^+} [e^{ipx} a^*(p) + e^{-ipx} a(p)] d\tau(p)$$

where $C_N = (2\pi)^{-\frac{N+1}{2}}$; $d\sigma(p) \dfrac{d^{N-1}\vec{p}}{\sqrt{\vec{p}^2 + m^2}}$

The Borel transform of the kernel of $\Phi(x)$ is the following function on $S \times S$

$$(8.2) \quad (z, z') \to \varnothing(x)(\bar{z}, z') = C_N e^{\bar{z}z'} \int_{p \in \Gamma^+} [e^{ipx} \bar{z}(p) + e^{-ipx} z'(p)] d\tau(p)$$
$$= e^{\bar{z}z'} [\bar{z}(x) + z'(x)]$$

Because this function belongs to $H(\bar{S} \times S)$, $\Phi(x)$ belongs to $0p$ T for any $x \in \mathbb{R}^N$. For any $f \in S(\mathbb{R}^N)$. the usual Wick monomial: $\Phi^n : (f)$ has for Wick symbol the following sesquiholomorphic functional:

$$(8.3) \qquad z, z' \to \langle e^z : \Phi^n : (f) e^{z'} \rangle e^{-\bar{z}z'} = \sum_{k=0}^{n} \binom{n}{k} \bar{z}^k z'^{n-k}(f)$$

Then : $\Phi^n : (f)$ belongs to $0p$ T. Now, we prove that : $\Phi^n : (f)$ is a linear continuous operator Exp (S) $\to F(X^c)$. We have : $\Phi^n : (f) = \sum_{k=0}^{n} \binom{n}{k} Q_k$ with $Q_k^w(\bar{z}, z') = \bar{z}^k z'^{n-k}(f)$. Then

$$Q_k^w(\bar{z}, z') = \int_{R^n} \bar{z}^k(x) g(x) dx ;$$

with $g(x) = f(x) z'^{n-k}(x)$. It is sufficient to prove that $Q_k \in L(\operatorname{Exp}(S'), F(X^c))$

Using Fourier transform on $\mathbb{R}^N$:

$$Q_k^w(\bar{z}, z') = \int g(\Sigma p^l) \prod_{l=1}^{k} \bar{z}(p^l) d\sigma(p^l)$$

Because $A(\bar{z}) = Q_k^w(\bar{z}, z')$ is a polynomial of degree k on S, this function belongs to $F(X^c)$ if and only if

$$(8.4) \qquad \| A \|^2 = \int \left| g\left(\sum_{l=1}^{k} p^l \right) \right|^2 d\sigma(p^1), \ldots, d\sigma(p^k) < \infty$$

This inequality follows from $g(p) \in S(\mathbb{R}^N)$, and from the existence of a strictly convex cone in $(\mathbb{R}^N)^k$ containing $\prod_{l=1}^{k} \Gamma^+$. Now we have

$$\psi(z') = e^{\bar{z}z'} A(\bar{z}) = \sum_{l=0}^{\infty} \frac{(\bar{z}z')^l}{l!} A(\bar{z})$$

Taking norm in the Fock space

$$\| \psi(z') \|^2 \leqslant \sum_l \frac{\| (\bar{z}z')^l A(\bar{z}) \|^2}{l!} \leqslant \left(\sum_l \frac{\| z' \|^{2l}}{l!} \right) \| A \|^2$$

By a similar argument the continuity of the map $z' \to \psi(z')$ from S to $F(X^c)$ can be proved. Theorem 7.2 implies that $Q_k \in L\,(\operatorname{Exp} S', F(X^c))$. Because (7.5), $: \Phi^n : (f)$ is a symmetric operator of $F(X^c)$. Now if $f \to 1$ in $S'(\mathbb{R}^N)$, then the Wick symbol of $: \Phi^n : (f)$ tends to the Wick symbol of $: \Phi^n : (1)$ in $H(\bar{S} \times S)$. Using the topological isomorphisms (5.11), this means that $: \Phi^n : (f)$ tends to $: \Phi^n : (1)$ in $0p$ T. We see therefore that the present formalism allows to control the properties of operators when cutoffs are removed. This will allow to elaborate an effective technique for introduction of proper counter terms in order to return back the operator properties of noncutoff sesquilinear forms.

Consider now the sharp time Wick polynomial $: \Phi^n : (0, g)$, for $g \in S(\mathbb{R}^{N-1})$. The similar analysis as previous prove that $: \Phi^n : (0, g)$ definies a linear continuous operator $\operatorname{Exp}(S') \to F(X^c)$ if and only if for $k = 1, \ldots, n$

$$(8.5) \qquad \int_{\mathbb{R}^{N-1}} \left| g\left(\prod_{l=1}^{k} p^l \right) \right|^2 \prod_{l=1}^{k} \frac{d^{N-1}\vec{p}^l}{\sqrt{(p^l)^2 + m^2}} < \infty$$

The elementary computation show that (8.5) is finite if and only if $N = 2$.

Hence the sharp time Wick polynomials are defined as operators in two dimensional space time only.

In the same manner we see that if $N = 2$, and if $g \to 1$ in $S'(R)$, then the unbounded operator $:\Phi^n:(0, g)$ of $F(X^c)$ tends to $:\Phi^n:(0, 1)$ in the space $0p$ T. Consider now the cutoff hamiltonien H_g in Φ_2^4 quantum field theory, given by the formula

$$(8.5)\qquad H_g = H_0 + \frac{\lambda}{4}\int :\Phi^4:(\vec{x})g(\vec{x})d\vec{x}\,; \qquad \vec{x} \in R$$

where H_0 is the free hamiltonian. It was shown in [12] that H_g is essentially self adjoint in the Fock space $F(X^c)$, and for weak coupling the vacum Ω_g of H_g is non degenerate.

We show now using symbolic calculus that H_g is unbounded symmetric operator $\text{Exp}\, S' \to F(X^c)$ whose non cutoff limit exists as a map from $\text{Exp}\, S'$ to $'\text{Exp}\, S'$. Indeed, the Wick symbol of H_g is:

$$H_g^w(z, \bar{z}') = \int \bar{z}(\vec{p})z'(\vec{p})d\vec{p} + \frac{\lambda}{4}\,(:\Phi^4:(0, g))^w(\bar{z}, z')$$

The first term satisfies all condition of theorem (7.1). Hence it is sufficient to apply the preceding analysis to the second term.

Let us note that corollary (7.10) implies that all operators $:\Phi^n(f):$ and $:\Phi^n:(0, g)$ are not bounded.

Further applications of symbolic calculus to the theory of interacting quantum fields are in preparation.

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