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Discontinuities in an arbitrarily moving gas in special relativity

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1. INTRODUCTION

The study of hydrodynamics in special relativity has received considerable attention in recent years. Such a study and, in particular, that of wave propagation indicates several applications in astrophysics [*1*, *2*, *3*]. Many authors have used singular surface theory to study shock wave propagation in relativistic fluids [*4*, *5*, *6*, *7*]. In [*8*, *9*] ray theory has also been used to study the propagation of discontinuities and Alfven waves in a relativistic gas.

In this paper, we have studied, using the singular surface theory and ray theory, the propagation of a weak discontinuity in an *arbitrarily* moving gas within the framework of special relativity. We have obtained a differential equation describing the variation of the strength of the discontinuity along the rays and discussed its integration.

2. THE BASIC EQUATIONS, COMPATIBILITY CONDITIONS AND RAY THEORY

Consider a coordinate system x^{A} with $x^{0} = t$ as time and x^{i} as spatial coordinates in the flat spacetime of special relativity with the fundamental metric tensor h_{AB} as $h_{00} = c^{2}$, $h_{ij} = -\delta_{ij}$, $h_{AB} = 0$ (A $\neq$ B), where c is the velocity of light in vacuum. Latin capital indices range over 0, 1, 2, 3 and the lower case Latin indices assume the values 1, 2, 3 only. Usual summation convention has been followed.

The equations of motion of a perfect fluid in special relatively, described by the stress-energy tensor [10]

$$T^{AB} = \rho\mu U^A U^B - \frac{p}{c^2} h^{AB} \tag{2.1}$$

can be written as

$$\rho U^B(\mu U^A)_{,B} = \frac{1}{c^2} p_{,B} h^{AB}, \tag{2.2}$$

$$(\rho U^B)_{,B} = 0, \tag{2.3}$$

where ρ is the proper mass density, p is the pressure, U^A is the unit, four-dimensional, fluid-velocity vector,

$$\mu = 1 + \frac{1}{c^2}\left(e + \frac{p}{\rho}\right), \tag{2.4}$$

and

$$e = e(p, \rho) \tag{2.5}$$

is the proper specific internal energy of the fluid. Equation (2.5) is referred to as the caloric equation of state of the fluid. In the above equations comma followed by a Latin index denotes partial differentiation.

Consider a propagating, time-like, singular hypersurface Σ represented by either of the equations

$$f(x^A) = 0, \qquad x^A = x^A(u^\alpha), \tag{2.6}$$

where u^α (Greek indices assume the values 0, 1, 2) are the coordinates on Σ. The latter set of equations in (2.6) form a parametric representation of the surface Σ. Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be, respectively, the components of the first and second fundamental covariant tensors of Σ. Let N^A be the space-like unit normal vector to Σ. Now let us note the following formulae [11].

$$\begin{aligned} a_{\alpha\beta} &= h_{AB} x^A_{,\alpha} x^B_{,\beta}, \qquad N_A N^A = -1, \qquad N_A x^A_{,\alpha} = 0, \\ x^A_{,\alpha\beta} &= b_{\alpha\beta} N^A, \qquad N^A_{,\alpha} = b^\beta_\alpha x^A_{,\beta}, \qquad N^A_{,A} = a^{\alpha\beta} b_{\alpha\beta} = b^\alpha_\alpha, \\ a^{\alpha\beta} x^A_{,\alpha} x^B_{,\beta} &= h^{AB} + N^A N^B, \quad x_{A,\alpha} = h_{AB} x^B_{,\alpha} \text{ (notation).} \end{aligned} \tag{2.7}$$

In the above formulae, comma followed by a Greek index denotes covariant derivative with respect to $a_{\alpha\beta}$; since x^A are scalar functions of u^α, we have

$$x^A_{,\alpha} = \partial x^A / \partial u^\alpha.$$

The surface Σ, in our study, is interpreted as the wave front of a propagating weak discontinuity by which we mean that all the field variables describing the fluid motion are continuous across it but at least some of their first partial derivatives are discontinuous across Σ. We now note the compatibility conditions, derived by using Hadamard's Lemma, which must be satisfied across Σ by the partial derivatives of the field variables [9]. For the first and second partial derivatives, under the assumption that Σ is a weak discontinuity, these are

$$\begin{aligned} [F_{,A}] &= \lambda N_A, \\ [F_{,AB}] &= K N_A N_B + a^{\alpha\beta}\lambda_{,\alpha}(N_A x_{B,\beta} + N_B x_{A,\beta}) + \lambda b^{\alpha\beta} x_{A,\alpha} x_{B,\beta} \end{aligned} \tag{2.8}$$

where F is any field variable, the square bracket denotes the jump in the quantity across Σ *i. e.*, $[F] = F_2 - F_1$; the subscript 1(2) on F denotes the value of F just ahead of (behind) the surface Σ, and λ and K denote the discontinuities in the normal derivatives *i. e.*,

$$[F_{,A}]N^A = -\lambda, \; [F_{,AB}]N^A N^B = K.$$

Finally, we note a few results from the *ray theory*. Let U_A be the unit, time-like velocity vector of the medium just ahead of Σ. Then the speed of propagation u or the frequency ($n = c/u$) of the wave Σ, in relativity, is defined as

$$lu = cL, \qquad L = U_A N^A, \qquad l^2 = 1 + L^2. \tag{2.9}$$

Let $\phi_A = f_{,A}$ (see eq. (2.6)) denote the gradient of Σ so that $\phi_A = \phi N_A$, $\phi^2 = -\phi_A \phi^A$. Now rewrite (2.9) as

$$2H \equiv h^{AB}\phi_A\phi_B + (n^2 - 1)(\phi_A U^A)^2 = 0. \tag{2.10}$$

Regarding eq. (2.10) as a first order partial differential equation for the determination of Σ, it can be solved by obtaining the solutions of the equations

$$\frac{dx^A}{dw} = \frac{\partial H}{\partial \phi_A}, \qquad \frac{d\phi_A}{dw} = -\frac{\partial H}{\partial x^A}, \tag{2.11}$$

where w is a curve parameter and x^A and ϕ_A are to be regarded as independent in obtaining the right hand members of these equations. While the first set of equations in (2.11) determine the curves known as rays, the second set describes the variation of the normal, to the wave front, along the rays.

3. VELOCITY OF PROPAGATION

As noted earlier, let Σ be a weak discontinuity propagating into an arbitrarily moving perfect gas. Let us denote the jumps in the normal derivatives of the mass density ρ and the fluid velocity vector U^A by

$$[\rho_{,A}]N^A = -\xi, \qquad [U^A_{,B}]N^B = -\lambda^A. \tag{3.1}$$

It follows from eq. (2.2) that $S_{,A}U^A = 0$ where S is the specific entropy of the fluid. We will, however, assume that the fluid motion is isentropic. In that case we obtain $p = p(\rho)$. Since U^A is a unit vector, we have the relation

$$U_A U^A_{,B} = 0. \tag{3.2}$$

Let prime denote differentiation with respect to ρ and let $p' = a^2$. Now take jumps in (2.2), (2.3) and (3.2) by using the first set of equations in (2.8) and equ. (3.1) to obtain

$$(\rho\mu' L U^A - a^2/c^2 N^A)\xi + \rho\mu L\lambda^A = 0, \tag{3.3}$$

$$L\xi + \rho\lambda_N = 0, \tag{3.4}$$

$$U_A\lambda^A = 0, \tag{3.5}$$

where $L = U^A N_A$, and *the field variables in these equations describe the medium ahead of* Σ. The suffix N in eq. (3.4) denotes the normal component; *it is reserved to denote only normal component*, it is not a tensor index. Now multiplying (3.3) by U_A and N_A and summing over A we obtain

$$(\rho\mu' - a^2/c^2) L\,\xi = 0, \tag{3.6}$$

$$(\rho\mu' L^2 + a^2/c^2)\xi + \rho\mu L \lambda_N = 0, \tag{3.7}$$

where eq. (3.5) has been used in obtaining (3.6). Since ξ and L are non-vanishing, it follows from (3.6) that

$$\rho\mu' = a^2/c^2. \tag{3.8}$$

By using (3.4) and (3.8) in (3.7), and noting that Σ is a weak discontinuity, we obtain

$$\frac{L^2}{l^2} = \frac{a^2}{\mu c^2}, \tag{3.9}$$

where $l^2 = 1 + L^2$. We obtain the unit, ray velocity vector of propagation of Σ, from the first of eq. (2.11) as

$$V^A \equiv \frac{dx^A}{ds} = \frac{1}{l}(LN^A + U^A), \tag{3.10}$$

where s is the ray parameter defined by $Lds = \phi l dw$. The eq. (3.10) is the same as obtained in [*8*]. Note that the ray velocity is tangent to Σ and the ray derivative of any quantity F is given by

$$\frac{dF}{ds} = F_{,A}V^A = F_{,\alpha}V^\alpha.$$

The second set of equations in (2.11) and eq. (3.9) lead to

$$\frac{dN_A}{ds} = \frac{L}{l}\left\{ l^2\left(\log\frac{L}{l}\right)_{,C} - \frac{l}{L} N_B U^B_{,C} \right\}(h_A{}^C + N_A N^C) \tag{3.11}$$

Now let us note that the eq. (3.3) can be written as

$$\rho\mu L\lambda^A = (\xi a^2/c^2)(N^A - LU^A)$$

which suggests that the discontinuity vector λ^A is parallel to the vector $N^A - LU^A$. Therefore, let us define the strength of the discontinuity, denoted by ψ, by the equation

$$\lambda^A = \psi M^A, \tag{3.12}$$

where M^A is the vector defined by

$$lM^A = N^A - LU^A,\quad M^A M_A = -1,\quad M^A N_A = -l,\quad M^A U_A = 0. \tag{3.13}$$

Then ξ is given by

$$\xi = -\frac{\rho}{L}\lambda_N = \frac{\rho l \psi}{L}. \tag{3.14}$$

4. GROWTH EQUATION

In this section we obtain a differential equation, along the rays, governing the strength of the discontinuity ψ. Let the discontinuities in the second partial derivatives along the normal vector in the fluid velocity and mass density be denoted as

$$[U^A_{,BC}]N^BN^C = \bar{\lambda}^A, \quad [\rho_{,BC}]N^BN^C = \bar{\xi}. \tag{4.1}$$

Now differentiate the equations (2.2) and (2.3) with respect to x^C, multiply by N^C and then take jumps by using (4.1) and the second set of equations in (2.8) to obtain

$$(\rho\mu' LU^A - a^2/c^2 N^A)\bar{\xi} + \rho\mu L\bar{\lambda}^A = Q^A - P^A, \tag{4.2}$$

$$L\bar{\xi} + \rho\bar{\lambda}^A N_A = W, \tag{4.3}$$

where

$$\begin{aligned} Q^A = {}& \xi(a^2/c^2)'(U^AU^B - h^{AB})\{ -\rho_{,B} + N_B\rho_{,C}N^C - \xi N_B \} \\ & + (a^2/c^2)\{ \xi N_B(U^AU^B)_{,C}N^C - \rho_{,B}(U^A\lambda^B + U^B\lambda^A) - \xi(U^A\lambda_N + 2L\lambda^A) \\ & - \xi U^A_{,B}U^B + \rho_{,C}N^C L\lambda^A \} + \mu\{ -U^A_{,B}(\xi U^B + \rho\lambda^B) + \lambda^A N_B N^C(\rho U^B)_{,C} \}. \\ P^A = {}& a^{\alpha\beta}x_{B,\beta}\{ \rho\mu U^B\lambda^A_{,\alpha} + a^2/c^2\xi_{,\alpha}(U^AU^B - h^{AB}) \}, \\ W = {}& \xi(N_BU^B_{,C}N^C - 2\lambda_N - U^B_{,B}) - \rho_{,B}(\lambda^B - \lambda_N N^B) - a^{\alpha\beta}x_{B,\beta}(U^B\xi_{,\alpha} + \rho\lambda^B_{,\alpha}). \end{aligned}$$

Note that all the field variables which appear in the equations (4.2) and (4.3) describe the medium ahead of the surface Σ. Note the comparison of equations (4.2), (4.3) with the corresponding homogeneous equations (3.3), (3.4). Multiplying (4.2) by N_A and then eliminating $\bar{\lambda}^A N_A$ by using (4.3), we obtain

$$(a^2l^2/c^2 - \mu L^2)\bar{\xi} = (Q^A - P^A)N_A - \mu LW. \tag{4.4}$$

In view of eq. (3.9) the coefficient of $\bar{\xi}$ in (4.4) vanishes and thus we obtain the growth equation as

$$(Q^A - P^A)N_A - \mu LW = 0. \tag{4.5}$$

In order to simplify the growth equation we note the following result [*12*] viz., the divergence of the ray velocity four-vector is the ray-derivative of the logarithm of the expansion ratio E, *i. e.*,

$$\frac{d}{ds}(\log E) = V^A_{,A}. \tag{4.6}$$

By straight forward calculation from (3.10), we obtain

$$lV^A_{,A} = -\frac{dl}{ds} + L_{,A}N^A + Lb^\alpha_\alpha + U^A_{,A}, \tag{4.7}$$

$$\begin{aligned} lV^\alpha_{,\alpha} &= -\frac{dl}{ds} + U^A_{,B}N_AN^B + Lb^\alpha_\alpha + U^A_{,A} \\ &= lV^A_{,A} - N_{A,B}U^AN^B. \end{aligned} \tag{4.8}$$

The last term in the right hand member of (4.8) can be expressed as a ray derivative as follows. Consider (cf. Sec. 2)

$$\phi N_A = f_{,A}. \tag{4.9}$$

Differentiating (4.9) with respect to x^B, we obtain

$$\phi_{,B} N_A + \phi N_{A,B} = f_{,AB}. \tag{4.10}$$

Since the right hand member of (4.10) is symmetric in A and B, we must have

$$\phi_{,B} N_A + \phi N_{A,B} = \phi_{,A} N_B + \phi N_{B,A}. \tag{4.11}$$

Multiplying (4.11) by the unit vector N^B, we get

$$\begin{aligned} \phi N_{A,B} N^B &= -\phi_{,B} h_{AC}(h^{BC} + N^B N^C) \\ &= -\phi_{,B} a^{\alpha\beta} x_{A,\alpha} x^B_{,\beta} \\ &= -a^{\alpha\beta} x_{A,\alpha} \phi_{,\beta}, \end{aligned} \tag{4.12}$$

where we have used a formula given in (2.7) in the above derivation. Now multiplying (4.12) by U^A, we get

$$\begin{aligned} U^A N_{A,B} N^B &= -\frac{1}{\phi} \phi_{,\beta} U^\beta = -l(\log \phi)_{,\beta} V^\beta \\ &= -l \frac{d}{ds}(\log \phi). \end{aligned} \tag{4.13}$$

By using (4.6) and (4.13) in (4.8) we get

$$V^\alpha_{,\alpha} = \frac{d}{ds}(\log E\phi). \tag{4.14}$$

Now by using the various relations obtained so far, in particular eqns (2.7), (3.8)-(3.14) and (4.6)-(4.14) the growth equation (4.5) becomes

$$\frac{d\psi}{ds} + \psi \frac{d}{ds}(\log X) - \frac{(\rho L)'}{L}\psi^2 = 0, \tag{4.15}$$

where

$$X = (\rho\mu E/(Ll\phi^3))^{1/2}. \tag{4.16}$$

The integration of eq. (4.15) subject to the initial condition $\psi = \psi_0$ at $s = s_0$ yields the solution

$$\frac{1}{\psi X} - \frac{1}{\psi_0 X_0} = -\int_{s_0}^{s} \frac{(\rho L)'}{LX} ds, \tag{4.17}$$

where X_0 denotes the initial value of X.

It is clear from (4.17) that the solution breaks down when

$$\psi_0^{-1} = X_0 \int_{s_0}^{s} \frac{(\rho L)'}{LX} ds. \tag{4.18}$$

In otherwords, the weak discontinuity terminates in a shock. Given the medium ahead of Σ, now one would like to enquire the time at which the shock formation takes place. It can be calculated by evaluating the integral in (4.18). However, the presence of $\phi = (-h^{AB}f_{,A}f_{,B})^{1/2}$ in X seems to suggest that, in order to be able to find when the shock occurs, it is necessary to integrate the equations (2.10) (*i. e.*, (3.10) and (3.11)). When the medium ahead of Σ is one of constant state, then ϕ is a constant along the rays and in that case it is enough if the medium ahead is known to evaluate the integral in (4.18). But one may note an interesting case viz. that when the normal trajectories to Σ are *geodesics i. e.*, when

$$N_{A,B}N^{B} = 0.$$

In this case again one can evaluate the integral in (4.18) when the medium ahead, not necessarily one of constant state, is given.

In the linear problem, it is easy to see that ψX remains constant along the rays.

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