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On the non-existence of time-dependent fluid spheres in general relativity obeying an equation of state

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ABSTRACT. — It is shown that there are no spherically symmetric solutions of Einstein's field equations representing a uniformly collapsing (or bouncing) fluid sphere obeying an equation of state $p = p(\rho)$, except for the trivial case $p \equiv 0$.

1. INTRODUCTION

The problem of spherically symmetric fluid in radial motion which may either be collapsing onto the central point (gravitational collapse) or be exploding from that point (spherical blast), has been studied by several authors [*1-10*]. McVittie [*1*] and Taub [*2*] consider two general classes of solutions of Einstein's field equations in which the source of the gravitational field is a perfect fluid which occupies a limited region of space-time. Taub shows that his class in the case of an equation of state $p = p(\rho)$ where p is the pressure and ρ is the energy density, is a subclass of McVittie's solutions.

Moreover, a number of explicit solutions has been given in the literature [*4-10*], none of them obeying an equation of state. Solutions with an equation of state are interesting if one likes to treat in a thermodynamically consistent way the spherically symmetric collapse of material. We will show in this paper that there can be no solution of Einstein field equations corresponding to a uniform collapse (blast) obeying an equation of state of the form $p = p(\rho)$.

In section II we will discuss the form of the metric. The field equations which are essentially given by Taub [2] will be summarized in section III. In section IV, we then discuss the field equations and show that they are inconsistent with the existence of an equation of state.

2. THE METRIC

We start from the general metric for spherically symmetric time-dependent solutions of the Einstein field equations in which the source of the gravitational field is a perfect fluid occupying a limited region of space-time. The metric can be written in the form (1)

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta}dr^2 - e^{2\mu}d\Omega^2, \tag{1}$$

where α, β and μ are functions of r and t and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{2}$$

This metric should describe the region of space-time

$$0 \leq r \leq r_b, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < t < \infty$$

filled with matter comoving with the coordinates (t, r, θ, ϕ). Therefore the four-velocity of the fluid is given by

$$u^\mu = e^{-\alpha}\delta^\mu_4 \tag{3}$$

and satisfies

$$u^\mu u_\mu = 1. \tag{4}$$

The energy-momentum tensor generally has the form [11]

$$T^{\mu\nu} = \rho u^\mu u^\nu + 2u^{(\mu}q^{\nu)} + ph^{\mu\nu} + \pi^{\mu\nu} \tag{5}$$

where ρ is the energy density, p is the pressure, $\pi^{\mu\nu}$ is the shear stress, q^ν is the heat flow, and

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu. \tag{6}$$

The shear stress and heat flow are related to the shear tensor $\sigma_{\mu\nu}$ and the temperature T in the following way [11]

$$\pi_{\mu\nu} = -\lambda\sigma_{\mu\nu} \tag{7}$$

$$q_\mu = -\kappa h^\nu{}_\mu(T_{,\nu} + Tu_{\nu;\gamma}u^\gamma) \tag{8}$$

in which λ and κ are positive constants and

$$\sigma_{\mu\nu} = \frac{1}{2}(v_{\mu\nu} + v_{\nu\mu}) - \frac{\theta}{3}h_{\mu\nu} \tag{9}$$

(1) Throughout the paper we use units in which G = c = 1. Greek indices run from 0 to 3. A comma denotes partial differentiation and a semicolon denotes covariant differentiation.

where

$$v^{\mu}{}_{\nu} = h^{\mu}{}_{\gamma} h^{\sigma}{}_{\nu} u^{\gamma}{}_{;\sigma} \tag{10}$$

and

$$\theta = v^{\mu}{}_{\mu} \tag{11}$$

is the expansion.

As we are going to consider an ideal fluid, we will demand vanishing shear stress and heat flow. Now we interpret the vanishing of these quantities not as the vanishing of the corresponding phenomenological constants λ and κ, but as the vanishing of the shear tensor $\sigma_{\mu\nu}$ and of $\frac{1}{\kappa} q_{\mu}$. The second restriction gives us a condition for the behaviour of the temperature, which is not going to interest us in this paper. But the first one gives us a condition on the metric coefficients. We will call this case of vanishing shear the non-shearing or uniform collapse.

In our case we have for the expansion and the shear

$$\theta = e^{-\alpha}(\beta_{,t} + 2\mu_{,t}) \tag{12}$$

$$\sigma = \sqrt{\frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu}} = \sqrt{1/3}\, e^{-\alpha}(\beta_{,t} - \mu_{,t}). \tag{13}$$

Therefore the vanishing of $\sigma_{\mu\nu}$ or σ means

$$\beta_{,t} = \mu_{,t}. \tag{14}$$

This is just the restriction which was introduced by Thompson-Whitrow [5] and was used by Taub [2]. We see that this is the condition of uniform expansion (contraction). As a consequence of the equation (14) we may write

$$\mu = \beta + \log f(r), \tag{15}$$

where $f(r)$ is an arbitrary function of r. Hence the line element given by equation (1) may be written as

$$ds^2 = e^{2\alpha}dt^2 - \frac{R^2}{f^2}(dr^2 + f^2 d\Omega^2) \tag{16}$$

where

$$R = e^{\mu}, \tag{17}$$

or as

$$ds^2 = e^{2\alpha}dt^2 - R^2(d\bar{r}^2 + d\Omega^2) \tag{18}$$

where

$$d\bar{r} = \frac{dr}{f}. \tag{19}$$

This metric with the restriction (14) has been studied by Taub [2], who also shows that in the case when an equation of state $p = p(\rho)$ exists, the solutions satisfying equation (14) in a comoving coordinate system must be of the McVittie subclass [1]. Without going into detail we take over from

Taub's paper [2] the field equations for the case when an equation of state exists. In what follows we will use the metric (18) writing it in the unbarred r coordinate to avoid unnecessary complications.

3. THE FIELD EQUATIONS

The Einstein field equations together with the requirement that the coordinate system in which equation (1) holds be a comoving one and the restriction (14) lead to the condition that

$$e^{\alpha} = \frac{R_{,t}}{R}\frac{P}{P_{,t}} \tag{20}$$

where $P = P(t)$ is a function of t alone.

The conservation laws $T^{\mu\nu}{}_{;\nu} = 0$ imply

$$R\rho_{,t} + 3R_{,t}(\rho + p) = 0 \tag{21}$$

$$p_{,r} + (\rho + p)\alpha_{,r} = 0. \tag{22}$$

If one defines a function $\sigma(\rho)$ by the equation

$$\frac{d\sigma}{\sigma} = \frac{d\rho}{p + \rho} \tag{23}$$

then this function is determined up to a constant of integration. We shall determine this constant such that

$$\sigma(\rho_0) = \rho_0, \tag{24}$$

where ρ_0 is the value of energy density at the boundary of the fluid sphere, that is

$$p(\rho_0) = 0. \tag{25}$$

Here we have assumed that an equation of state $p = p(\rho)$ exists.

As a consequence of equations (22) and (23) one gets

$$e^{\alpha} = \frac{\sigma}{\rho + p} = \frac{R_{,t}}{R}\frac{P}{P_{,t}} \tag{26}$$

where we have used the following normalization of the coordinate t:

$$\alpha(r_b, t) \equiv 0. \tag{27}$$

In view of equation (23) it follows from equation (21) that

$$R^3\sigma = h^3(r) \tag{28}$$

where $h(r)$ is a function of r alone. As a consequence of equations (26) and (21) one gets the following relations

$$\rho = \rho(x), \qquad p = p(x) \tag{29}$$

$$x\rho_{,x} = 3\sigma \tag{30}$$

$$\alpha = \alpha(x) \tag{31}$$

where

$$x = \frac{Q(r)}{P(t)} \tag{32}$$

and $Q(r)$ is a function of r alone. It then follows from equation (28) that

$$R = \frac{f(r).Q(r)}{X(x)} \tag{33}$$

where we have defined

$$f(r) = \frac{h(r)}{Q(r)} \tag{34}$$

and

$$X(x) = \sigma^{1/3}(x). \tag{35}$$

These are just the conditions imposed by McVittie in his discussion of similarity solutions of Einstein's field equations. The condition that the stresses be isotropic, that is the condition that

$$T^1{}_1 = T^2{}_2 = T^3{}_3$$

with the restriction (14) leads to the following differential equation for R:

$$\left(\frac{1}{R}\right)_{,rr} = \frac{1}{R} - \frac{B(r)}{R^2} \tag{36}$$

where $B(r)$ is an arbitrary function of its argument.

Substituting (33) into (36) and taking account of the independence of r and x, we get the following differential equations for $X(x)$, $f(r)$ and $Q(r)$:

$$\frac{Q_{,r}^2}{Q^2} = \nu \frac{B}{fQ} \tag{37}$$

$$\frac{Q_{,rr}}{Q} - 2\frac{f_{,r}}{f} \cdot \frac{Q_{,r}}{Q} - 2\frac{Q_{,r}^2}{Q^2} = \mu \frac{B}{fQ} \tag{38}$$

$$\frac{f_{,rr}}{f} - 2\frac{f_{,r}^2}{f^2} + 1 + \frac{Q_{,rr}}{Q} - 2\frac{f_{,r}}{f}\frac{Q_{,r}}{Q} - 2\frac{Q_{,r}^2}{Q^2} = -\lambda \frac{B}{fQ} \tag{39}$$

$$\nu x^2 X_{,xx} + \mu x X_{,x} + \lambda X + X^2 = 0 \tag{40}$$

We can integrate equation (38) and eliminate the differentials of Q in (39) to get

$$B = \gamma A^2 f^5 Q^{2\varepsilon+3} \tag{41}$$

$$\frac{Q_{,r}}{Q^{\varepsilon+3}} = A.f^2 \tag{42}$$

$$\frac{f_{,rr}}{f} - 2\frac{f_{,r}^2}{f^2} + 1 = -\frac{\delta+\varepsilon}{\gamma} \cdot \frac{B}{fQ}, \tag{43}$$

where A is an integration constant and

$$\varepsilon = \frac{\mu}{\nu} \tag{44}$$

$$\delta = \frac{\lambda}{\nu} \tag{45}$$

$$\gamma = \frac{1}{\nu} \tag{46}$$

From equations (23), (30) and (35) we get

$$\frac{x\mathrm{X}_{,x}}{\mathrm{X}} = \frac{\sigma}{p + \rho} \tag{47}$$

Hence

$$e^{\alpha} \equiv y(x) = \frac{x\mathrm{X}_{,x}}{\mathrm{X}} = \frac{\sigma}{p + \rho} \tag{48}$$

Equation (40) can now be written in the form

$$x\mathrm{Y}_{,x} + \mathrm{Y}^2 + (\varepsilon - 1)\mathrm{Y} + \delta + \gamma\mathrm{X} = 0 \tag{49}$$

which is a first integral of the equation

$$x^2\mathrm{Y}_{,xx} + (\varepsilon + \mathrm{Y})x\mathrm{Y}_{,x} - [\delta + (\varepsilon - 1)\mathrm{Y} + \mathrm{Y}^2]\mathrm{Y} = 0 \tag{50}$$

To calculate p and ρ we proceed as follows [3]. We define a function $m(r, t)$ in the following way

$$m(r, t) = \frac{1}{2}\mathrm{R}\left[1 + \mathrm{R}^2\left(\frac{\mathrm{P}_{,t}}{\mathrm{P}}\right)^2 - \left(\frac{\mathrm{R}_{,r}}{\mathrm{R}}\right)^2\right] \tag{51}$$

The Einstein field equations then imply that

$$m_{,r} = 4\pi\rho\mathrm{R}^2\mathrm{R}_{,r} \tag{52}$$

$$m_{,t} = -\,p\mathrm{R}^2\mathrm{R}_{,t} \tag{53}$$

If we define $\mathrm{M} = m(r_b, t)$, we see that as a consequence of the condition $p(r_b) = 0$ and (53)

$$\mathrm{M} = \text{const.}$$

This condition and the condition of vanishing of pressure at the boundary are just the junction conditions for matching the interior solution to the exterior Schwarzschild space-time [12].

Finally as a consequence of equations (36), (48) and (49) we have

$$\mathrm{B}(r) = 3m - 4\pi\mathrm{R}^3\rho \tag{54}$$

Now we have all the field equations and other relations which we need and shall go over to the question of consistency of these field equations.

4. DISCUSSION OF THE FIELD EQUATIONS

We have seen that the state functions σ, ρ and p are functions of x alone. That is

$$p = p(x) = p\left(\frac{Q(r)}{P(t)}\right).$$

At the boundary of the fluid the pressure vanishes, that means

$$p(r = r_b, t) = p\left(\frac{Q(r = r_b)}{P(t)}\right) = 0. \tag{55}$$

Therefore, assuming the pressure doesn't vanish identically ($p \not\equiv 0$), we must have

$$Q(r = r_b) = 0 \tag{56}$$

otherwise (55) would lead to an equation for $R(t)$, which means that $R(t) = \text{const.}$, corresponding to the static case. Hence we are led to (56), which means that

$$x_b = \frac{Q(r = r_b)}{P(t)} = 0. \tag{57}$$

We now evaluate the equation (50) at the boundary $x = 0$. Because of the junction condition at the boundary $x = 0$, $Y_{,x}$ and $Y_{,xx}$ must be bounded, therefore we have at the boundary

$$xY_{,x} = x^2 Y_{,xx} = 0 \tag{58}$$

Taking into account also equation (27), we see that equation (50) when evaluated at the boundary leads to

$$\delta + \varepsilon = 0. \tag{59}$$

Evaluating equation (49) at the boundary and taking account of (59) we find that

$$\gamma X(x = 0) = 0 \quad \text{or} \quad X(x = 0) = 0 \tag{60}$$

because $\gamma = \dfrac{1}{v} \neq 0$, otherwise this would lead to $B(r) = 0$ and this would mean uniform density and no equation of state [*2*]. From (35) and (24) we find that

$$\sigma_0 = \sigma(x = 0) = 0 \tag{61}$$

$$\rho_0 = \rho(x = 0) = 0. \tag{62}$$

Evaluating now equation (54) at the boundary and taking account of the relation $m(r = r_b, t) = M \neq 0$, and (41), one gets as a necessary condition

$$2\varepsilon + 3 = 0 \tag{63}$$

and therefore

$$B(r) = \gamma A^2 f^5 \tag{64}$$

Differentiating now equation (54) with respect to r and substituting for $B(r)$ from (64) we get

$$ff_{,r} = -\frac{12\pi}{5\gamma A^2}\cdot Q^2 Q_{,r} \tag{65}$$

and after integrating

$$f^2 = c - \frac{8\pi}{5\gamma A^2} Q^3 \tag{66}$$

where c is an integration constant.

Now, taking account of the relations (59) and (63) we can integrate the equations (42) and (43) to find $f(r)$ and $Q(r)$. We get

$$f(r) = \frac{c_1 e^r}{1 + c_2 e^{2r}}$$

$$Q(r) = \begin{cases} \left(c_3 - \dfrac{A^2 c_1^2}{c_2} \dfrac{1}{1 + c_2 e^{2r}}\right)^2 & c_2 \neq 0 \\ (c_3 + A^2 c_1^2 e^{2r})^2 & c_2 = 0 \end{cases}$$

where c_1, c_2 and c_3 are integration constants.

It is easily seen that these functions don't satisfy the relation (66).

5. CONCLUSION

We have seen that in the case of uniform collapse (blast) of an ideal fluid the equation of state $p = p(\rho)$ is not consistent with the Einstein field equations, except for the trivial case $p \equiv 0$ where solutions are already known [*13*]. One has to consider more general relations between p and ρ such as $p = p(\rho, t)$, $p = p(\rho, r)$ or $p = p(\rho, r, t)$. This would probably mean that one has to take into account the effect of expansion and its contribution to the pressure. This question and a discussion of various equations of state which have been treated in the literature will be considered in a later note.

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