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Schrödinger Operators in the Uniform Norm

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ABSTRACT. — We study Schrödinger operators in $L^\infty(E^n)$. We show how to find closed extensions with essential spectrum equal to the non-negative real axis. The potentials are allowed to have singularities almost as strong as those permitted in L^2 case.

1. INTRODUCTION

The purpose of the present paper is to create a theory for the Schrödinger operator

$$- \Delta + V(x) \tag{1.1}$$

on $L^\infty = L^\infty(E^n)$, where Δ is the Laplacian in E^n and $V(x)$ is a real valued function. When one commences studying (1.1) in L^∞, several problems arise. Firstly, most of the Hilbert space techniques that are used in L^2 cannot be applied here. But more seriously, in L^∞ there arises the difficulty of defining the domain of a differential operator. Not even the continuous functions are dense, and consequently the closure of the Laplacian is not densely defined. Another difficulty becomes apparent when V is allowed to have singularities. The domain of the multiplication operator resulting from this function is also not dense in L^∞, since functions it contains must vanish on the singularities of V.

This problem seems to be of importance in the algebraic formulation of quantum mechanics (see [*1*]). In the study of momentum states which correspond to plane wave states of the usual quantum mechanics, it is found that the representation given by the G. N. S. construction is not the usual L^2 representation but the set of almost periodic functions with the scalar product given by the mean. It is rather difficult to study partial differential operators in this space. However, since it is contained in L^∞, one is led to consider such operators there.

Our hypotheses on V will be given in terms of the functions

$$M_{\alpha,p,\delta,x}(V) = \int_{|x-y|<\delta} |V(y)|^p \omega_\alpha(x-y)dy,$$

where

$$\begin{aligned} \omega_\alpha(x) &= |x|^{\alpha-n}, & 0<\alpha<n \\ &= -\log|x|, & \alpha = n \\ &= 1, & \alpha > n. \end{aligned}$$

We shall put

$$M_{\alpha,p,\delta}(V) = \sup_x M_{\alpha,p,\delta,x}(V),$$

$$M_{\alpha,p}(V) = M_{\alpha,p,1}(V),\ M_{\alpha,p,x}(V) = M_{\alpha,p,1,x}(V),$$

and we shall say that $V \in M_{\alpha,p}$ if $M_{\alpha,p}(V) < \infty$. Our main theorem is

THEOREM 1.1. — *Assume that* $V \in M_{2,1}$ *and that*

(1.2) $$M_{2,1,\delta}(V) \to 0 \quad as \quad \delta \to 0 \quad if \quad n \geq 2,$$

(1.3) $$M_{2,1,x}(V) \to 0 \quad as \quad |x| \to \infty.$$

Then the operator (1.1) *has a closed realization* H *in* L^∞ *such that*

$$\sigma_e(H) = [0, \infty).$$

Since we are not in a Hilbert space, the essential spectrum $\sigma_e(H)$ of H is not uniquely defined. However, our theorem holds for most of the definitions (cf. [2, p. 241]).

Note that the same theorem holds in L^2 (in fact a stronger result is given in Theorem 9.1, ch. 7 of [*2*]). On the other hand, Theorem 1.1 is unknown in L^p for $p \neq 2$ or $p \neq \infty$ (cf. Theorem 5.1, ch. 6, of [*2*] for a weaker statement). We shall prove Theorem 1.1 in Section 2 after we prove several lemmas. January 28, 1976.

2. SOME LEMMAS

Put

(2.1) $$G(r, \varkappa) = -\frac{i}{4}(\varkappa/2\pi r)^\nu H_\nu^{(1)}(\varkappa r),$$

where $\nu = \frac{1}{2}n - 1$ and $H_\nu^{(1)}(z)$ is the Bessel function of the third kind. It is the Green's function for the operator $\Delta + \varkappa^2$ and satisfies the following estimates (cf. [3])

$$(2.2)\qquad \begin{aligned} |G(r, \varkappa)| &\le c_n\omega_2(r), & r|\varkappa| \le 1 \\ &\le c_n r^{-\nu-\frac{1}{2}} \exp(-\operatorname{Im} \varkappa r), & r|\varkappa| > 1, \end{aligned}$$

where c_n is a constant depending only on n, If f is in L^2, it is easily checked that for $\varkappa^2 = \lambda$, $\operatorname{Im} \varkappa > 0$,

$$(2.3)\qquad R(\lambda)f(x) = \int G(|x - y|, \varkappa)f(y)dy$$

is the unique solution in L^2 of

$$(2.4)\qquad (\Delta + \lambda)u = f.$$

First we note

LEMMA 2.1. — *If* $\operatorname{Im} \varkappa > 0$, *then* $R(\lambda)$ *is a bounded operator on* L^∞.

Proof. — This follows from the estimate

$$(2.5)\quad |R(\lambda)f(x)| \le \Omega_n c_n \|f\|_\infty\left(\frac{1}{2}|\varkappa|^{-2} + |\varkappa|^{\nu-\frac{1}{2}}\int_{|\varkappa|^{-1}}^{\infty} r^{\nu+\frac{1}{2}} \exp(-\operatorname{Im} \varkappa r)dr\right),$$

where Ω_n denotes the surface of the unit sphere in E^n. Inequality (2.5) follows from (2.2). □

Next we put

$$(2.6)\qquad T(b)f(x) = \int G(|x - y|, bi)V(y)f(y)dy, \qquad b > 0.$$

We have

LEMMA 2.2. — *If* $V \in M_{2,1}$, *then* $T(b)$ *is a bounded operator in* L^∞ *with*

$$(2.7)\qquad \|T(b)\| \le c_n M_{2,1,1/b}(V) + C_n b^{-2} M_{n+1,1}(V),$$

where C_n *depends only on* n.

Proof. — We have by (2.2)

$$\begin{aligned} |T(b)f(x)| \le c_n\|f\|_\infty\Bigg(&\int_{b|x-y|<1} |x - y|^{2-n}|V(y)|dy \\ &+ b^{\nu-\frac{1}{2}}\sum_{k=1}^{\infty}\int_{k<b|x-y|<k+1} |x - y|^{-\nu-\frac{1}{2}} \exp(-b|x - y|)|V(y)|dy\Bigg) \\ \le c_n\|f\|_\infty&\left(M_{2,1,1/b}(V) + Cb^{-2}M_{n+1,1}(V)\sum_{k=1}^{\infty} k^{\nu+\frac{1}{2}}e^{-k}\right), \end{aligned}$$

where C is a constant depending only on n such that the shell between the spheres of radius k/b and $(k+1)/b$ can be covered by $Ck^{n-1}b^{-n}$ spheres of radius 1. This gives (2.7). □

COROLLARY 2.3. — $\|T(b)\| \to 0$ *as* $b \to \infty$.

Next we put

$$V_m(x) = V(x), \qquad |V(x)| \leq m,$$
$$= 0, \qquad \text{otherwise.}$$

We have

LEMMA 2.4. — $M_{2,1}(V - V_m) \to 0$ *as* $m \to \infty$.

Proof. — First we note that

$$M_{2,1,x}(V - V_m) \leq 2M_{2,1,x}(V) \to 0 \quad \text{as} \quad |x| \to \infty$$

by (1.3). Let $\varepsilon > 0$ be given, and take R so large that

$$\text{(2.8)} \qquad M_{2,1,x}(V - V_m) < \varepsilon \quad \text{for} \quad |x| > R.$$

Put

$$\tau_m(x) = \int_{|x-y|<1} \omega_2(x-y)\,|V_m(y)|\,dy$$

and

$$\tau(x) = \int_{|x-y|<1} \omega_2(x-y)\,|V(y)|\,dy.$$

Since the V_m are bounded, it is easily checked that the functions τ_m are continuous. Moreover, we have

$$0 \leq \tau_m(x) \leq \tau_{m+1}(x) \leq M_{2,1}(V),$$

and for each fixed x

$$\omega_2(x-y)\,|V_m(y)| \to \omega_2(x-y)\,|V(y)| \quad \text{as} \quad m \to \infty$$

pointwise. Since they are majorized by the limit, we have $\tau_m(x) \to \tau(x)$ for each x. By Dini's theorem, this convergence is uniform on $|x| \leq R$. Since

$$|V(y)| - |V_m(y)| = |V(y) - V_m(y)|,$$

we have

$$\tau(x) - \tau_m(x) = \int_{|x-y|<1} \omega_2(x-y)\,|V(y) - V_m(y)|\,dy = M_{2,1,x}(V - V_m).$$

Hence $M_{2,1,x}(V - V_m) \to 0$ as $m \to \infty$ uniformly in $|x| \leq R$. Take N so large that

$$M_{2,1,x}(V - V_m) < \varepsilon, \quad m > N, \quad |x| \leq R.$$

Combining this with (2.8), we obtain the lemma. □

LEMMA 2.5. — $\sup\limits_f \dfrac{|T(b)f(x)|}{\|f\|_\infty} \to 0$ *as* $|x| \to \infty$.

Proof. — Put $V^N(x) = V(x)$ for $|x| > N$ and 0 otherwise. By Lemma 2.2

$$|T(b)f(x)| \leq \|f\|_\infty \left(\int_{|y|<N} G(|x-y|, bi)\,|V(y)|\,dy + CM_{2,1}(V^N) \right).$$

By Lemma 3.2 of [5]

$$M_{2,1}(V^N) \to 0 \quad \text{as} \quad N \to \infty .$$

Let $\varepsilon > 0$ be given, and take N so large that $M_{2,1}(V^N) < \varepsilon$. Now by (2.2)

$$(2.9) \quad \int_{|y|<N} G(|x-y|, bi)|V(y)|\,dy \le c_n \int_{b|x-y|<1} |x-y|^{2-n}|V(y)|\,dy + c_n b^{\nu-\frac{1}{2}} \int_{\substack{b|x-y|>1 \\ |y|<N}} |x-y|^{-\nu-\frac{1}{2}} e^{-b|x-y|}|V(y)|\,dy \le c_n M_{2,1,1/b,x}(V) + c_n b^{2\nu} e^{bN-b|x|} \int_{|y|<N} |V(y)|\,dy .$$

For fixed N, both of these terms tend to 0 as $|x| \to \infty$. Thus we can make the left hand side of (2.9) $< \varepsilon$. This gives the lemma. □

Next we put

$$T_m(b)f(x) = R(-b^2)V_m f.$$

we have

LEMMA 2.6. — *For each m and $b > 0$, $T_m(b)$ is a compact operator on* L^∞.

Proof. — First we show that

$$(2.10) \qquad |T_m(b)f(x) - T_m(b)f(x')| \le C \| f \|_\infty |x - x'| ,$$

where the constant depends only on b, V and m. To prove this we make use of the estimates

$$(2.11) \quad |G(|x-y|, bi) - G(|x'-y|, bi)| \le C|x-x'|(|x-y|^{-\nu-\frac{1}{2}}e^{-b|x-y|} + |x'-y|^{-\nu-\frac{1}{2}}e^{-b|x'-y|}), \quad |x-y| > r_0,$$
$$\le C|x-x'| \sum_{k=1}^{n-2} |x-y|^{1+k-n}|x'-y|^{-k}, \qquad |x-y| \le r_0,$$

which hold for $|x - x'| < \frac{1}{2}$ (cf. [4]). Thus we have

$$|T_m(b)f(x) - T_m(b)f(x')| \le C|x-x'| \| f \|_\infty \Big(\int_{b|x-y|<1} \sum_{k=1}^{n-2} |x-y|^{1+k-n}|x'-y|^{-k} dy + \int_{b|x-y|>1} [|x-y|^{-\nu-\frac{1}{2}}e^{-b|x-y|} + |x'-y|^{-\nu-\frac{1}{2}}e^{-b|x'-y|}]\,dy \Big) \le C'|x-x'| \| f \|_\infty \Big(\int_0^{1/b} r\,dr + \int_{1/b}^{\infty} e^{-br}dr \Big).$$

Now suppose $\{ f_k \}$ is a sequence satisfying $\| f_k \|_\infty = 1$, and let $\varepsilon > 0$ be given. By Lemma 2.5 there is an R so large that

$$(2.12) \qquad | T_m(b) f_k(x) | < \varepsilon , \qquad | x | > R .$$

On the other hand (2.10) implies that there is a subsequence (also denoted by $\{ f_k \}$) such that $T_m(b) f_k(x)$ converges uniformly on each compact subset of E^n. Thus there is an N so large that

$$(2.13) \qquad | T_m(b)(f_j - f_k)(x) | < \varepsilon , \qquad | x | \leq R , \qquad j, k > N .$$

Inequalities (2.12) and (2.13) imply

$$\| T_m(b)(f_j - f_k) \|_\infty < \varepsilon , \qquad j, k > N .$$

Hence the sequence $\{ T_m(b) f_k \}$ converges in L^∞. Thus $T_m(b)$ is a compact operator. □

LEMMA 2.7. — $T_m(b) \to T(b)$ *in norm on* L^∞.

Proof. — By Lemmas 2.2 and 2.4

$$(2.14) \qquad \| T(b) - T_m(b) \| \leq CM_{2,1}(V - V_m) \to 0 . \qquad \square$$

COROLLARY 2.8. — $T(b)$ *is compace on* L^∞.

Proof. — Apply Lemmas 2.6 and 2.7. □

Next we take b so large that $\| T(b) \| < 1$. Let

$$(2.15) \qquad W(- b^2) = [I - T(b)]^{-1} R(- b^2) .$$

We have

LEMMA 2.9. — $W(- b^2)$ *is a bounded operator on* L^∞ *and is the resolvent of a closed operator* $H = - b^2 - W(- b^2)^{-1}$ *which is an extension of the operator* (1.1) *defined on the set of those twice continuously differentiable functions* u *with compact support such that* $V\ u \in L^\infty$.

Proof. — Fist we consider the case $V = 0$. Thus $T(b) = 0$ and $W(- b^2) = R(- b^2)$. Now

$$(2.16) \qquad R(\lambda) - R(\lambda_1) = (\lambda_1 - \lambda) R(\lambda) R(\lambda_1)$$

and $R(\lambda)$ is injective. Hence there is a closed operator H_0 on L^∞ such that $R(\lambda) = (\lambda - H_0)^{-1}$ (cf. [6], p. 185). If $u \in C^2$ with compact support, then $f = (\Delta + \lambda) u \in L^2$. Hence $R(\lambda)$ is a solution of (2.4), and consequently $u = R(\lambda) f$. But this shows that $u \in D(H_0)$ and $(\lambda - H_0) u = f$. Thus H_0 satisfies the requirements of the lemma for the case $V = 0$. For the general case, put

$$(2.17) \qquad W_m(- b^2) = [I - T_m(b)]^{-1} R(- b^2) .$$

We note that $[I - T_m(b)]^{-1}$ takes $D(H_0)$ into itself. For it $w \in D(H_0)$ and $w = [I - T_m(b)]g$, then $g = w + R(-b^2)V_m g$ is in $D(H_0)$. Thus the range of $W_m(-b^2)$ is $D(H_0)$. This implies

$$\begin{aligned} W_m(-b^2)^{-1} &= (-b^2 - H_0)(I - R(-b^2)V_m) \\ &= -b^2 - H_0 - V_m . \end{aligned}$$

Consequently

$$W_m(-b^2) - W_m(-b_1^2) = W_m(-b^2)(b^2 - b_1^2)W_m(-b_1^2).$$

Taking the limit as $m \to \infty$, we have by Lemma 2.7

$$W(-b^2) - W(-b_1^2) = (b^2 - b_1^2)W(-b^2)W(-b_1^2). \tag{2.18}$$

Again since $W(-b^2)$ is injective we see that it is the resolvent of a closed operator H. Finally, suppose $u \in C^2$ has compact support and $Vu \in L^\infty$. Put $f = (\Delta - V - b^2)u$. Then $f \in L^2$ and $R(-b^2)f = u - T(b)u$. Hence $u = W(-b^2)f$, showing that $u \in D(H)$ and $(-b^2 - H)u = f$. This proves the Lemma. □

As in the proof of Lemma 2.9, we let H_0 be the operator $-b^2 - R(-b^2)^{-1}$. The domains of H and H_0 need not have much in common. However, we have

Lemma 2.10. — *$W(-b^2) - R(-b^2)$ is a compact operator on* L^∞.

Proof. — We have by (2.15)

$$W(-b^2) - R(-b^2) = T(b)W(-b^2).$$

Since $T(b)$ is compact (Corollary 2.8) and $W(-b^2)$ is bounded, the result follows. □

We are now ready for the proof of Theorem 1.1. We let H be the operator given by Lemma 2.9. By Lemma 2.10 we see that

$$(-b^2 - H)^{-1} - (-b^2 - H_0)^{-1}$$

is compact. By Theorem 1.6, ch. 11 of [2] we have

$$\sigma_{ek}(H) = \sigma_{ek}(H_0), \qquad k = 2, 3, 4, \alpha . \tag{2.19}$$

On the other hand, it follows from Theorem 3.1, ch. 4, of [2] that

$$[0, \infty) \subset \sigma_{ek}(H_0).$$

Moreover, by Lemma 2.1, all complex λ not in $[0, \infty)$ are in $\rho(H_0)$. Hence we have

$$\sigma_{ek}(H_0) = \sigma(H_0) = [0, \infty). \tag{2.20}$$

The conclusion of Theorem 1.1 now follows from (2.19) and (2.20). □

3. FURTHER OBSERVATIONS

We shall prove

THEOREM 3.1. — *$\sigma(H) - \sigma_e(H)$ consists of isolated, negative, finite dimensional eigenvalues having 0 as their only possible accumulation point. All nonreal points are in $\rho(H)$.*

Proof. — First assume that V is bounded. Put $H_N = H_0 + V - V^N$, where V^N is defined as in the proof of Lemma 2.5. Now H_N has no nonreal eigenvalues. For if z is not real and $(z - H_N)\, u = 0$, then

$$u = R(z)[(V - V^N)u]. \tag{3.1}$$

Since $(V - V^N)\, u \in L^2$, this implies that $u \in L^2$ as well. But any solution of (3.1) in L^2 must vanish. Thus H_N has no nonreal eigenvalues. Since it has no nonreal essential spectrum (Theorem 1.1), it has no nonreal spectrum at all. Note next that

$$W(-b^2) - (-b^2 - H_N)^{-1} = [I - R(-b^2)(V - V^N)]^{-1}$$
$$[T(b) - R(-b^2)(V - V^N)]W(-b^2).$$

This is bounded in norm by

$$CM_{2,1}(V^N)\, \| [I - T(b)]^{-1} \| \, \| W(-b^2) \| \to 0 \quad \text{as} \quad N \to \infty .$$

Thus H_N tends to H in the generalized sense (cf. [*7*], p. 206). This shows that H cannot have any nonreal isolated points in its spectrum (*ibid.*, p. 212). On the other hand, the complement of $[0, \infty)$ in the complex plane is in the Fredholm set of H ([*2*], p. 15) and contains points of its resolvent (Lemma 2.9). Thus it can only contain isolated eigenvalues ([*8*], p. 206). Hence H cannot have nonreal spectrum. If V is not bounded, put $H_m = H_0 + V_m$. Since V_m is bounded, we see that H_m cannot have nonreal spectrum by what we have just proved. Moreover, $W_m(-b^2) \to W(-b^2)$ in norm as $m \to \infty$ by Lemma 2.7. Hence H_m approaches H in the generalized sense, and consequently H cannot have nonreal isolated points in its spectrum. It cannot have nonisolated points in its spectrum for the reasons given above. Hence its spectrum must be real. □

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