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Section A :
Physique théorique.

Mass singularities of generic Feynman amplitudes

by

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RÉSUMÉ. — On étudie le comportement singulier de l'amplitude générique pour un graphe G de Feynman au cas où un paramètre de masse, initialement non-zéro, s'évanouit. On montre que l'amplitude se décompose localement en la somme d'une partie régulière plus des termes qui se comportent comme des puissances de la masse qui s'évanouit ; les « résidus » associés avec ces termes sont identifiés avec les amplitudes de sous-graphes de G et avec celles de graphes quotients de G.

ABSTRACT. — The singular behavior of the generic amplitude for a Feynman graph G, when one initially non-zero mass parameter vanishes, is investigated. It is shown that locally the amplitude decomposes as a sum of a regular part plus terms behaving like powers of the vanishing mass; the « residues » associated with various terms are identified with amplitudes for sub and quotient graphs of G.

I. INTRODUCTION

In this paper we investigate the analytic dependence of Feynman amplitudes on the mass variable associated with a single line. We work in the context of the *generic* amplitude defined in [*1*], i. e., the usual amplitude regularized with both the $\underline{\lambda}$ parameters of analytic renormalization and a complex dimension ν ; this avoids divergence difficulties and in addition

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leads to singular behavior which has a clear relation to the structure of the underlying Feynman graph. Our main result, described in Section II, gives a complete description of the singularities of the physical sheet of the amplitude at a point where the mass variable z vanishes. This is a corrected version of an erroneous result of [*1*] (Theorem 3.1), which in fact is correct only when all lines of the graph have non-zero masses.

In [*2*] a partial desingularization of the integration space for the Feynman amplitude was obtained, permitting a discussion of the meromorphic structure of the amplitude in λ, ν. In Section III we modify this desingularization slightly, to reach a geometry in the integration space for which the $z \to 0$ pinch is locally always of one fairly simple type. The singularity generated by this local pinch configuration is analyzed in the Appendix: in Section IV we apply this analysis to the Feynman integral to prove the main theorem.

We will follow the notation of [*2*], but have tried to define most terms as they arise. In Section III we have omitted proofs, since they involve only slight modifications of those given in [*2*].

II. TERMINOLOGY AND STATEMENT OF RESULTS

For any Feynman graph G we let Ω_G denote the set of lines of G, $\Omega_G^M \subset \Omega_G$ the set of massive lines, Θ_G the set of vertices, and Θ_G^E the set of external vertices; further, $N(G) \equiv |\Omega_G|$, $n(G) \equiv |\Theta_G|$, $c(G) \equiv$ the number of connected components of G, and $h(G) \equiv N(G) - n(G) + c(G)$, the number of loops. The *generic Feynman amplitude* $F_G(\underline{s}, \underline{z}; \underline{\lambda}, \nu)$ is a function of regularizing parameters $\nu \in \mathbb{C}$ and $\lambda_l \in \mathbb{C}$, $l \in \Omega_G$, of external invariants $s(\chi)$, $\chi \subset \Theta_G^E$ (which satisfy certain linear relations [*1*]), and of squared mass variables z_l, $l \in \Omega_G^M$, defined for G connected by

$$F_G(\underline{s}, \underline{z}; \underline{\lambda}, \nu) = \Gamma(-\pi_G) \int_{\mathscr{D}_G} \prod_{l \in \Omega_G} \alpha_l^{\lambda_l} d_G(\underline{\alpha})^{-\nu/2} D_G^*(\underline{\alpha}, \underline{s}, \underline{z})^{\pi_G} \eta. \tag{2.1}$$

Here

$$\pi_G = h(G)\nu/2 - \sum_{\Omega_G} (\lambda_l + 1) \tag{2.2}$$

$$d_G(\alpha) = \sum \prod_{l \notin T} \alpha_l \tag{2.3}$$

$$D_G(\underline{\alpha}, \underline{s}, \underline{z}) = \frac{1}{2} \sum_{X \subset \Theta^E} s(\chi) \left(\sum_{T_2} \prod_{l \notin T_2} \alpha_l \right) - \left(\sum_{\Omega_G^m} \alpha_l z_l \right) d_G(\alpha) . \tag{2.4}$$

$$D^*(\underline{\alpha}, \underline{s}, \underline{z}) = D_G(\underline{\alpha}, \underline{s}, z)/d_G(\underline{\alpha}) , \tag{2.5}$$

with sums running over all trees T in G and all 2-trees T_2 which separate χ from $\Theta^E - \chi$; $\mathscr{D}_G = \{ \underline{\alpha} \in \mathbb{P}^{N(G)-1} \mid \alpha_l \geq 0 \}$ and η is the fundamental projective differential form. The integral (1) is convergent for $(\underline{s}, \underline{z})$ in the *Symanzik region* $R_G = \{ (\underline{s}, \underline{z}) \mid s(\chi) > 0,\ z_l < 0 \}$ and $(\underline{\lambda}, \nu)$ in a suitable convergence region [2]; it is understood that a complex power of a positive quantity is defined using the principle branch of the logarithm. Analytic continuation of F_G in the $(\underline{\lambda}, \nu)$ parameters is discussed in [2].

We now consider a fixed, 2-connected graph G_0 with a distinguished massive line ω; we will drop the subscript G_0 from the line and vertex sets of this graph. Our goal is to describe the behavior of F_{G_0} when z_ω varies in a neighborhood of zero (and all other $\underline{s}$, $\underline{z}$ variables have the signs of the Symanzik region).

REMARK 2.1. — The singularity is simple to discuss if G_0 has no (or one) external vertices and ω is the only massive line. Then since

$$D(\underline{\alpha}, \underline{s}, \underline{z}) = -\alpha_\omega z_\omega d(\alpha),$$

$$F_{G_0}(\underline{s}, \underline{z}; \underline{\lambda}, \nu) = [-z_\omega]^{\pi_G} f_{G_0}(\underline{\lambda}, \nu) \tag{2.6}$$

with

$$f_{G_0} = \Gamma(-\pi_G) \int_{\mathscr{D}} \left(\prod_{l \neq \omega} \alpha_l^{\lambda_l} \right) \alpha_\omega^{\lambda_\omega + \pi_G} d(\alpha)^{-\nu/2} \eta \tag{2.7}$$

a meromorphic function of $\underline{\lambda}$, ν, i. e., F_{G_0} behaves like $z_\omega^{\pi_G}$ at $z_\omega = 0$. We will refer to this as the trivial case.

To discuss the general case we need the concepts of a *saturated* graph and a *link* (previously defined in [2]) and one additional definition, that of a *mass singularity* graph. In what follows, G_0^ω denotes the graph G_0 modified so that ω is a massless rather than massive line; G_0^∞ the graph obtained from G_0 by adding one vertex, ∞, and joining it by one line to each vertex of Θ^E. For any graphs G, H, with H a subgraph of G, G/H is the quotient graph obtained from G by contracting all lines in H, and $p_{G/H} : G \to G/H$ is the associated mapping.

DEFINITION 2.2. — *a*) Suppose that $H \subset G_0$, and that G_0^∞/H has pieces $Q_1, \ldots, Q_k$ numbered so that $\infty \notin \theta_{Q_i}$, and $\Omega^M_{Q_i} = \emptyset$, for $i > i_0$. Then $\bar{H} = H \cup p^{-1}_{G_0^\infty/H}(Q_{i_0+1} \cup \ldots \cup Q_k)$ is called the *saturation* of H; H is *saturated* if $\bar{H} = H$. *b*) A subgraph $S \subset G_0$ is called a *link* (in G_0) if (*i*) $\bar{S} = G_0$, and (*ii*) the removal of any piece of S destroys property (*i*). *c*) A subgraph $B \subset G_0$ is a *mass singularity* (MS) graph for ω if (*i*) $\omega \notin B$, (*ii*) B is a link in G_0^ω, and (*iii*) B is saturated.

We can now state the main result of this paper, to be proved in Section IV.

THEOREM 2.3. — Let G_0 be a non-trivial graph with $\omega \in \Omega^M_{G_0}$. Then for $s(\chi)$ and $z_l (l \neq \omega)$ restricted to a compact subset of the Symanzik region for G_0^ω, F_{G_0} may be analytically continued in z_ω to a fixed punctured neighborhood $\{ 0 < | z_\omega | < \varepsilon \}$ of $z_\omega = 0$. In this neighborhood,

$$F_{G_0}(\underline{s}, \underline{z} ; \underline{\lambda}, \nu) = H(\underline{s}, \underline{z} ; \underline{\lambda}, \nu) + \sum_B (- z_\omega)^{\pi_{G_0/B}} K_B(\underline{s}, \underline{z} ; \underline{\lambda}, \nu), \quad (2.8)$$

where the sum is over all MS graphs B. H and K_B are analytic at $z_\omega = 0$, in fact,

$$H |_{z_\omega = 0} = F_{G_0^\omega} ; \quad (2.9)$$

$$K_B |_{z_\omega = 0} = f_{G_0/B} \prod_{i=1}^{r} F_{B_i}, \quad (2.10)$$

if B has connected components $B_1, \ldots, B_r$.

Thus near $z_\omega = 0$, F_{G_0} decomposes into a regular piece together with pieces which behave like a power of z; one piece for each of a certain class of subgraphs of G_0. Both the power and the « residue » $K_B |_{z_\omega = 0}$ are simply characterized in terms of the subgraph B and quotient graph G_0/B.

EXAMPLE 2.4. — *a*) If G_0 is massive (i. e., $\Omega^M = \Omega$) there is a unique MS graph B containing all lines except ω. In this case the singularity structure

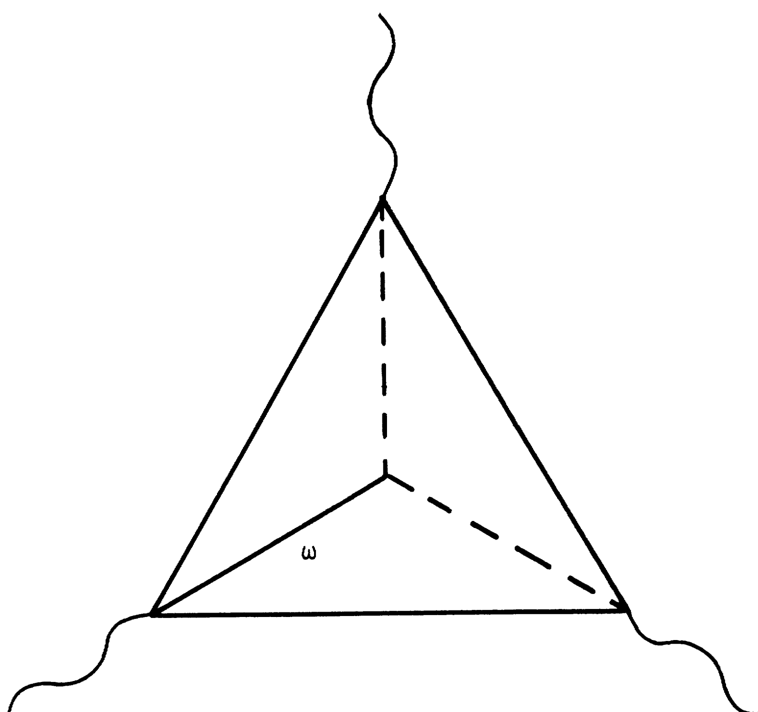

FIG. 1. — Feynman graph G_0.

given by Theorem 1.3 is the same as that described in [*1*]. *b*) In the graph of figure 1, in which dotted lines are massless, solid lines massive, and wavy lines denote external vertices, there are two MS graphs, shown in figure 2. Thus the corresponding sum in (2.8) has two terms.

FIG. 2. — MS subgraphs of G_0.

REMARK 2.5. — If G^0 is trivial (Remark 2.1), the singular behavior described by (2.6) may in fact be regarded as a special case of Theorem 1.3; we must take the empty graph to be the unique MS graph in G_0 and make the convention that $F_G = 0$ whenever $\Theta_G^E = \Omega_G^M = \emptyset$ unless G is the empty graph, in which case $F_G = 1$.

III. DESINGULARIZATION

In order to discuss the singular behavior at $z_\omega = 0$ of the integral (2.1), it is necessary to analyze the pinch which occurs in the integration space. For this purpose we introduce here a desingularization of the integration space which reduces the pinch geometry to a relativity simple form, analyzed in the appendix. The desingularization is described by *s-families* of sub and quotient graphs of G_0, which label necessary blow-ups and blow-downs of the boundary of the integration region in (2.1). Our procedure is a modification of that used in [*2*]; the *s*-families used here differ from those of [*2*] primarily by the inclusion of mass singularity graphs. The important properties of these *s*-families are summarized in Lemma 3.2; the proof of this Lemma is quite similar to the proofs given in [*2*] and is therefore omitted.

Let $\mathscr{H} = \{ H \subset G_0 \mid H$ is saturated and irreducible, or H is an MS graph $\}$. $\mathscr{Q} = \{ Q = G_0/S \mid S$ is a link, and Q is irreducible $\}$. If $H \in \mathscr{H}$, a *link in H* is a subgraph $H_1 \subset H$ such that $\bar{H}_1 = H$ and H_1 is either irreducible or a link in G^ω. If $Q \in \mathscr{Q}$ with $Q = G_0/S$, a subgraph $S_1 \subset Q$ is a *link in Q* if $S \cup S_1$ is a link. For any $\mathscr{E} \subset \mathscr{Q} \cup \mathscr{H}$ we write $\mathscr{E}_q = \mathscr{E} \cap \mathscr{Q}$, $\mathscr{E}_h = \mathscr{E} \cap \mathscr{H}$, $\mathscr{E}^0 = \{ K \in \mathscr{E} \mid \Omega_K$ is maximal $\}$, and, for $K \in \mathscr{E}$, $\mathscr{E}(K) = \{ K' \in \mathscr{E} \mid \Omega_{K'} \subsetneq \Omega_K \}$. We write $\mathscr{E}_q^0(K) \equiv (\mathscr{E}(K)^0)_q$, etc.

DEFINITION 3.1. — An *s-family* $\mathscr{E} \subset \mathscr{Q} \cup \mathscr{H}$ is a maximal family satisfying

1) $G_0 \in \mathscr{E}$;
2) The sets Ω_K, $K \in \mathscr{E}$, are non-overlapping;
3) If $K \in \mathscr{E}$, and $\mathscr{E}_q^0(K) = \{ K/S_1, \ldots, K/S_r \}$, then $S = \bigcap_{i=1}^{r} S_i$ is a link in K, and the pieces of K/S are precisely the elements of $\mathscr{E}_q^0(K)$;
4) If $K \in \mathscr{E}$, $\mathscr{F} \subset \mathscr{E}_h^0(K)$ with $|\mathscr{F}| \geq 2$, and $H_1 = \bigcup_{H \in \mathscr{F}} H$, then (*a*) H_1 is not irreducible and is not a link in G_0^ω, and (*b*) if $K = G_0/S \in \mathscr{Q}$, the pieces of $H_1 \cup S$ are precisely the pieces of H_1 together with the pieces of S.

LEMMA 3.2. — *a*) If $\mathscr{E}$ is an *s*-family and $K \in \mathscr{E}$, there is precisely one line, denoted $\sigma(K)$, in $\left[\Omega_K - \bigcup_{K' \in \mathscr{E}(K)} \Omega_{K'} \right]$; in particular, this implies that $|\mathscr{E}| = |\Omega| = N$.

b) For any *s*-family $\mathscr{E}$, define

$$\mathscr{D}(\mathscr{E}) = \{ \alpha \in \mathbb{P}^{N-1} \mid \alpha_l \geq 0, \quad \alpha_{\sigma(H)} \leq \alpha_{\sigma(K)} \leq \alpha_{\sigma(Q)} \text{ whenever } H \in \mathscr{E}_h^0(K), \quad Q \in \mathscr{E}_q^0(K) \}.$$

Then $\mathscr{D} = \cup \mathscr{D}(\mathscr{E})$, and if $\underline{\alpha} \in \mathscr{D}(\mathscr{E}) \cap \mathscr{D}(\mathscr{E}')$, for $\mathscr{E} \neq \mathscr{E}'$, then $\alpha_l = \alpha_{l'}$ for some $l \neq l'$.

c) For any *s*-family $\mathscr{E}$ there is a distinguished tree T in G_0; T consists of all lines $\sigma(H)$, where $H \in \mathscr{E}_H$ and $\sigma(H)$ is a piece of the graph formed by adjoining $\sigma(H)$ to $\bigcup_{\mathscr{E}_h^0(H)} H'$. $T \cap \Omega_K$ is a (spanning) tree in K for each $K \in \mathscr{E}$, moreover, either (*i*) $\sigma(G_0) \in \Omega^M$, (*ii*) $\sigma(G_0) \in T$, and the 2-tree $T - \sigma(G_0)$ separates Θ^E into two non-empty subsets ψ, $\Theta^E - \psi$, or (*iii*) both.

Proof. — Omitted; see [2].

We may now use this lemma to rewrite the integral (2.1) defining $F = F_{G_0}$. For by Lemma 3.2 (*b*),

$$F(\underline{s}, \underline{z}; \underline{\lambda}, \nu) = \Sigma F_{\mathscr{E}}(\underline{s}, \underline{z}; \underline{\lambda}, \nu), \tag{3.1}$$

the sum running over all *s*-families, with

$$F_{\mathscr{E}}(\underline{s}, \underline{z}; \underline{\lambda}, \nu) = \Gamma(-\pi_{G_0}) \int_{\mathscr{D}(\mathscr{E})} \Pi \alpha_l^{\lambda_l} d^{-\nu/2} (D^*)^{\pi_{G_0}} \eta \tag{3.2}$$

In (3.2) we make the variable change

$$\alpha_l = \prod_{H \in \mathscr{E}_h} t_H \prod_{Q \in \mathscr{E}_q} t_Q^{-1}, \tag{3.3}$$

and normalize by setting $\alpha_{\sigma(G_0)} = t_{G_0} = 1$; the integration region $\mathscr{D}(\mathscr{E})$ then becomes the cube $\{ \underline{t} \mid 0 \leq t_K \leq 1, K \in \mathscr{E}(G_0) \}$, and

$$\eta = \prod_{\mathscr{E}_h(G_0)} t_H^{N(H)-1} dt_H \prod_{\mathscr{E}_q} t_Q^{-N(Q)-1} dt_Q . \tag{3.4}$$

Any tree in G_0 must intersect each $H \in \mathscr{E}_h(G_0)$ in at most $n(H) - c(H)$ lines, and each $Q \in \mathscr{E}_q$ in at least $n(Q) - c(Q) = n(Q) - 1$ lines; from Lemma 3.2 (*c*), these numbers are exact for the tree T. Thus the definition (2.3) of $d_{G_0}(\underline{\alpha})$ becomes

$$d(\underline{\alpha}) = \prod_{\mathscr{E}_h(G_0)} t_H^{h(H)} \prod_{\mathscr{E}_q} t_Q^{-h(Q)} (1 + e_{\mathscr{E}}(t)) , \tag{3.5}$$

with $e_{\mathscr{E}}$ a polynomial having positive coefficients. Similarly,

$$D^*(\underline{\alpha}, \underline{s}, \underline{z}) = \zeta_{\mathscr{E}} + g_{\mathscr{E}}(\underline{t}, \underline{s}, \underline{z}) , \tag{3.6}$$

where $\zeta_{\mathscr{E}}$ depends on the different cases of Lemma 3.2 (*c*):

$$\zeta_{\mathscr{E}} = \begin{cases} - z_{\sigma(G_0)} , & \text{case } (i), \\ s(\psi)(1 + e_{\mathscr{E}}(t))^{-1} , & \text{case } (ii), \\ s(\psi)(1 + e_{\mathscr{E}}(t))^{-1} - z_{\sigma(G_0)} , & \text{case } (iii); \end{cases} \tag{3.7}$$

$g_{\mathscr{E}}$ is continuous and non-negative for $t_K \geq 0$ and $(\underline{s}, \underline{z})$ in the Symanzik region, and is independent of $z_{\sigma(G_0)}$ in cases (*i*) and (*iii*). Thus

$$F_{\mathscr{E}}(\underline{s}, \underline{z} ; \underline{\lambda}, \nu) = \Gamma(-\pi_{G_0}) \int_0^1 \dots \int_0^1 \prod_{\mathscr{E}_h(G_0)} t_H^{-\pi_H - 1} dt_H$$
$$\prod_{\varepsilon_q} t_Q^{\pi_Q - 1} dt_Q (1 + e_{\mathscr{E}})^{-\nu/2} (\zeta_{\mathscr{E}} + g_{\mathscr{E}})^{\pi_{G_0}}. \tag{3.8}$$

This is our desingularized form of the integral defining $\mathscr{F}_{\mathscr{E}}$.

IV. PROOF OF MAIN RESULT

We now investigate the behavior of F for $s(\chi)$ and z_l, $l \neq \omega$, restricted to a compact subset of the Symanzik region $R_{G_0^{(\omega)}}$, as z_ω varies in $\{ \operatorname{Re} z_\omega < 0 \} \cup \{ \mid z_\omega \mid < \varepsilon \}$. Consider a single term $F_{\mathscr{E}}$ in the decomposition (3.1). From the representation (3.8) $F_{\mathscr{E}}$ is singular in this region only if the term $(\zeta_{\mathscr{E}} + g_{\mathscr{E}})$ vanishes for some $\underline{t}$ in the integration region. According to (3.7) this implies that, for ε sufficiently small, $F_{\mathscr{E}}$ is analytic in the region unless $\zeta_{\mathscr{E}} = - z_\omega$. We will now study *s*-families which satisfy this condition. For any s-family $\mathscr{E}$, we let $\mathscr{E}_m \subset \mathscr{E}_h$ denote the set of MS graphs belonging to $\mathscr{E}$.

LEMMA 4.1. — For any s-family $\mathscr{E}$, $\mathscr{E}_m$ is totally ordered by inclusion. If $\mathscr{E}_m = \{ B_1, \ldots, B_r \}$ with $B_i \subset B_{i+1}$, then $B_i \in \mathscr{E}_h^0(B_{i+1})$ and $B_r \subset \mathscr{E}_h^0(G_0)$.

Proof. — Since elements of $\mathscr{E}$ are non-overlapping, the first statement will follow if we show that no two elements of $\mathscr{E}_m$ are disjoint. Now note that, if $B \in \mathscr{E}_m$ and $H \in \mathscr{E}_h$ with $H \supset B$, then either $H \in \mathscr{E}_m$ or $\omega \in H$, in which case $H = \bar{H} = G_0$. Moreover, we cannot have $B \subset Q$ for any $Q \in \mathscr{E}_q$. If $|\Omega^M| \geq 2$ this is true because a line $l \in \Omega^M - \{\omega\}$ must lie in B but not in Q; if $|\Theta^E| \geq 2$ because by choosing Q to be the minimal element of $\mathscr{E}_q$ containing B and B′ a maximal element of $\mathscr{E}_h$ with $B \subset B' \subset Q$ we will have $B' \in \mathscr{E}_h^0(Q)$, and setting $Q = G_0/S$ we see that any two external vertices are connected by distinct paths in S and B, contradicting Definition 3.1 (4).

Suppose then that $B_1, B_2 \in \mathscr{E}_m$ satisfy $B_1 \cap B_2 = \emptyset$. Let H be the minimal element of $\mathscr{E}_h$ containing B_1 and B_2; then there exist $B_1', B_2' \in \mathscr{E}_m$ with $B_1', B_2' \in \mathscr{E}_h^0$. $B_1 \cup B_2$ will then be a link in G_0^ω, again contradicting Definition 3.1 (4). This proves the first part of the Lemma; the second follows from the observation above that if $B \in \mathscr{E}_m$ and $H \supset B$ with $H \in \mathscr{E}_h$, then $H \in \mathscr{E}_m$ or $H = G_0$.

LEMMA 4.2. — If $\mathscr{E}$ is an s-family, then $\zeta_E = -z_\omega$ if and only if $\mathscr{E}_m \neq \emptyset$.

Proof. — Suppose that $\zeta_E = -z_\omega$; we will show that $\mathscr{E}_h^0(G_0)$ contains an MS graph. For certainly, if we define

$$G = \bigcup_{\mathscr{E}_h^0(G_0)} H, \tag{4.1}$$

then $\Omega_G \supset \Omega^M - \{\omega\}$, since $\Omega_{G_0} = \Omega_G \cup \bigcup_{\mathscr{E}_q^0(G_0)} \Omega_Q \cup \{\omega\}$ and $\Omega_Q \cap \Omega^M = \emptyset$ for any $Q \in \mathscr{Q}$. Moreover, G must connect all external vertices, since otherwise $\zeta_{\mathscr{E}}$ would have the form of (3.5), case (*iii*). Thus the saturation of G in G_0^ω is G_0^ω. By discarding those elements from the union (4.1) which are not necessary to make this last statement true, we may find an $\mathscr{F} \subset \mathscr{E}_h^0(G_0)$ with $\bigcup_{\mathscr{F}} H$ a link in G_0^ω. Then Definition 3.1 (4) implies that $|\mathscr{F}| = 1$, i. e., $\mathscr{E}_h^0$ contains a graph which is a link in G_0^ω and hence an MS graph.

Conversely, suppose that $\mathscr{E}_m \neq \emptyset$. By Lemma 4.1 we may find a $B \in \mathscr{E}_m$ with $B \in \mathscr{E}_h^0(G_0)$. Let T be the tree of Lemma 3.2 (*c*). B connects all external vertices and, since $T \cap B$ is a tree in B, $T \cap B$ does also. Then even if $\sigma(G_0) \in T$, $T - \{\sigma(G_0)\} \supset T \cap B$ cannot separate the external vertices, so that $\mathscr{E}$ must belong to case (*i*) of Lemma 3.2 (*c*). Then $\sigma(G_0) \in \Omega^M$, $\sigma(G_0) \notin \Omega_B$, and $\Omega_B \supset \Omega^M - \{\omega\}$ imply $\sigma(G_0) = \omega$, completing the proof.

LEMMA 4.3. — If $\mathscr{E}$ is an s-family with $\mathscr{E}_m \neq \emptyset$, then

$$g_{\mathscr{E}}(\underline{t}, \underline{s}, \underline{z}) = \prod_{B \in \mathscr{E}_m} t_B h_{\mathscr{E}}(\underline{t}, \underline{s}, \underline{z}), \tag{4.2}$$

with $h_{\mathscr{E}}$ analytic and nonvanishing for $0 \leq t_K \leq 1$, $s(\chi) > 0$, and $z_l < 0$ $(l \neq \omega)$.

Proof. — Since $\zeta_{\mathscr{E}} = - z_\omega$, (3.6) and (2.3)-(2.5) imply

$$g_{\mathscr{E}} = \left[\frac{1}{2} d(\alpha)^{-1} \sum_{\chi} s(\chi) \Big(\sum_{T_2} \prod_{l \notin T_2} \alpha_l \Big) - \sum_{\substack{l \neq \omega \\ l \in \Omega^m}} \alpha_l z_l \right]_{\alpha_l = \Pi t_H \Pi t_Q^{-1}} \tag{4.3}$$

We make the indicated substitutions of the $\underline{t}$ variables in (4.3), and use the known factorization (3.5) of $d(\alpha)$. Observe that since $B \in \mathscr{E}_m$ connects the external vertices, a 2-tree T_2 which separates them can intersect B in at most $n(B) - c(B) - 1$ lines, and hence the first term in (4.3) contains a factor t_B. Similarly, if $l \in \Omega^M - \{ \omega \}$, then $l \in \Omega_B$, and hence the second term also contains a factor t_B; this proves (4.2).

Now let B_1 be the minimal element of $\mathscr{E}_m$, which exists by Lemma 4.1, and let T be the tree of Lemma 3.2 (*c*). Then (as in that Lemma) either (*i*) $\sigma(B_1) \in \Omega^M$, (*ii*) $\sigma(B_1) \in T$ and $T - \sigma(B_1)$ separates Θ^E into non-empty subsets ψ and $\Theta^E - \psi$, or (*iii*) both; the essential idea of the proof is that otherwise $\mathscr{E}_h^0(B_1)$ would contain an MS graph, contradicting the minimality of B_1. Consider case (*i*): then $\alpha_{\sigma(B_1)} = \prod_{H \ni \sigma(B_1)} t_H = \prod_{\mathscr{E}_m} t_B$ by Lemma 4.1, so that

$$h_{\mathscr{E}} = - z_{\sigma(B_1)} + \text{non-negative terms}$$

and hence is nonvanishing. Cases (*ii*) and (*iii*) are similar.

Proof of Theorem 2.3. — According to the discussion at the beginning of this section, and Lemma 4.2, the only terms in $F = \Sigma F_{\mathscr{E}}$ which are singular at $z_\omega = 0$ are those for which $\mathscr{E}_m \neq \emptyset$. For such an $\mathscr{E}$, we have by Lemma 4.3 that

$$F_{\mathscr{E}} = \Gamma(-\pi_{G_0}) \int_0^1 \dots \int_0^1 \prod_{\mathscr{E}_h(G_0)} t_H^{-\pi_H - 1} \prod_{\mathscr{E}_q} t_Q^{\pi_Q - 1} (1 + e_{\mathscr{E}})^{-\nu/2} \left(\prod_{B \in \mathscr{E}_m} t_B h_{\mathscr{E}} - z_\omega \right)^{\pi_{G_0}} \tag{4.4}$$

Now the pinch for $z_\omega \to 0$ in (4.4) is precisely that analyzed in Theorem A.1; specifically, the variables t_B, $B \in \mathscr{E}_m$, correspond to $u_1, \dots u_n$ of that theorem, and $\underline{s}$, $z_l(l \neq \omega)$, and t_K, $K \notin \mathscr{E}_m$, to the variables $\underline{w}$. Thus for z_ω near 0, (A.2) implies that

$$F_{\mathscr{E}} = H_{\mathscr{E}} + \sum_{B \in \mathscr{E}_b} (-z_\omega)^{\pi_G - \pi_B} K_{\mathscr{E}B},$$

and since $\pi_G - \pi_B = \pi_{G/B}$, (2.8) is proved, with

$$\begin{aligned} H &= \sum_{\mathscr{E}_m = \emptyset} F_{\mathscr{E}} + \sum_{\mathscr{E}_m \neq \emptyset} H_{\mathscr{E}}, \\ K_B &= \sum_{\mathscr{E}_m \supset B} K_{\mathscr{E}B}. \end{aligned} \tag{4.5}$$

From (A.3), $H_{\mathscr{E}}|_{z_\omega = 0} = F_{\mathscr{E}}|_{z_\omega = 0}$, so

$$H|_{z_\omega = 0} = \sum_{\mathscr{E}} F_{\mathscr{E}}|_{z_\omega = 0} = F_{G_0}|_{z_\omega = 0} = F_{G_0^\omega},$$

proving (2.9). There remains to prove (2.10), i. e.,

$$K_B|_{z_\omega = 0} = f_Q \prod_{i=1}^{r} F_{B_i}, \tag{4.6}$$

for B any MS graph, $B_1, \ldots, B_r$ the connected components of B, and $Q = G_0/B$.

We introduce the following notation. For any sub or quotient graph K, let $\mathbb{P}_K = \mathbb{P}^{N(K)-1}$ be projective space with homogeneous coordinates indexed by Ω_K; let $\mathscr{D}_K = \{ \alpha \in \mathbb{P}_K \mid \alpha_l \geq 0, \text{ all } l \}$, and let $\mathscr{D}_K^0$ be the interior of $\mathscr{D}_K$. There is a natural map $\psi_K : \mathbb{P}_{G_0} \to \mathbb{P}_K$ with $(\psi_K(\underline{\alpha}))_l = \alpha_l$, $l \in \Omega_K$. For a fixed s-family $\mathscr{E}$ and MS graph $B \in \mathscr{E}_m$, let

$$J_{\mathscr{E}} = \{ t_K, K \in \mathscr{E}^0(G_0) \mid 0 < t_K \leq 1 \},$$

and let $J_{\mathscr{E}B}$ be the (half-open) face of $J_{\mathscr{E}}$ on which $t_B = 0$. Now (3.3) defines an invertible map $\phi_{\mathscr{E}} : J_{\mathscr{E}} \to \mathscr{D}_{G_0}^0 \cap \mathscr{D}_{\mathscr{E}}$; moreover, if H is the minimal element of $\mathscr{E}$ containing B (see Lemma 4.1) then

$$t_B(\underline{\alpha}) \equiv [\phi_{\mathscr{E}}^{-1}(\underline{\alpha})]_B = \alpha_{\sigma(B)}/\alpha_{\sigma(H)}$$

is actually well defined for all $\underline{\alpha} \in \mathscr{D}_{G_0}^0$. Thus we have the diagram

$$\begin{array}{ccccc} & & \mathscr{D}_{G_0}^0 & & \\ & \phi_{\mathscr{E}} \nearrow & & \searrow \psi_B \times \psi_Q \times (\phi_{\mathscr{E}}^{-1})_B = \chi & \\ J_{\mathscr{E}} = J_{\mathscr{E}B} \times (0,1] & & \xrightarrow[\phi_{\mathscr{E}B} \times i]{} & & \mathbb{P}_B \times \mathbb{P}_Q \times \mathbb{R} \end{array}$$

Here $Q = G_0/B$, and $i : (0,1] \to \mathbb{R}$ is the natural inclusion. The indicated factorization of the composition map, with $\phi_{\mathscr{E}B} : J_{\mathscr{E}B} \to \mathbb{P}_B \times \mathbb{P}_Q$, follows from (3.3) since the ratio $\phi_{\mathscr{E}}(\underline{t})_l/\phi_{\mathscr{E}}(\underline{t})_{l'}$ is independent of t_B if $l, l' \in \Omega_B$ or $l, l' \in \Omega_Q$.

We will need certain properties of the map $\phi_{\mathscr{E}B}$, described in

LEMMA 4.4. — *a)* $\bigcup_{\{\mathscr{E}|B\in\mathscr{E}_m\}} \phi_{\mathscr{E}B}(J_{\mathscr{E}B}) = \mathscr{D}_B^0 \times \mathscr{D}_Q^0$. *b)* If $\mathscr{E} \neq \mathscr{E}'$, then

$$\phi_{\mathscr{E}B}(J_{\mathscr{E}B}) \cap \phi_{\mathscr{E}'B}(J_{\mathscr{E}'B})$$

has measure 0.

Proof. — If $(\beta, \gamma) \in \mathscr{D}_B^0 \times \mathscr{D}_Q^0$, we normalize by setting $\gamma_l = \beta_{l'} = 1$ for some $l \in \Omega_Q$, $l' \in \Omega_B$. For $x > 0$ define $\underline{\alpha}(x) \in \mathbb{P}_{G_0}$ by $\alpha_l = x\beta_l$, $l \in \Omega_B$, $\alpha_l = \gamma_l$, $l \in \Omega_Q$, and suppose x is small enough so that $\alpha_l < \alpha_{l'}$ for all $l \in \Omega_B$, $l' \in \Omega_Q$. By Lemma 3.2 (*b*) there is an *s*-family $\mathscr{E}$ with $\underline{\alpha}(x) \in \mathscr{D}_{\mathscr{E}}$; and $\mathscr{E}$ is unique if $\beta_l \neq \beta_{l'}$, $l, l' \in \Omega_B$, and $\gamma_l \neq \gamma_{l'}$, $l, l \in \Omega_Q$. Moreover, it follows from the construction method for *s*-families described in [2] that $B \in \mathscr{E}$. Now if $\underline{t}(x) = \phi_{\mathscr{E}}^{-1}(\alpha(x)) \in J_{\mathscr{E}}$, $(t(x))_K$ is independent of x for $K \neq B$; then the point $\hat{t} \in J_{\mathscr{E}B}$ for which $\hat{t}_K = (t(x))_K$, $K \neq B$, satisfies $\phi_{\mathscr{E}B}(\hat{t}) = (\beta, \gamma)$. This proves (*a*); (*b*) follows from the uniqueness of $\mathscr{E}$ noted above.

We now continue with the verification of (4.6). Let us write

$$f_Q(\hat{\underline{\lambda}}, \nu) = \Gamma(-\pi_Q)\int_{\mathscr{D}_Q} \theta_Q ,$$

with θ_Q the differential form given in (2.7), and

$$\prod_{i=1}^{r} F_{B_i} = \prod_i \left\{ \Gamma(-\pi_{B_i})\int_{\mathscr{D}_{B_i}} \prod_{l\in\Omega_{B_i}} \beta_l^{\lambda_l} d_{B_i}^{-\nu/2}(D_{B_i}^*)^{-\pi_{B_i}}\eta \right\} = \Gamma(-\pi_B)\int_{\mathscr{D}_B} \theta_B ,$$

with

$$\theta_B = \prod_{\Omega_B} \beta_l^{\lambda_l} \prod_{i=1}^{r} d_{B_i}^{-\nu/2} \left[\sum_{i=1}^{r} D_{B_i}^*\right]^{-\pi_B} \eta ;$$

the last equality is a projection-space variant of Feynman's formula for the combination of denominators. Thus by Lemma 4.4,

$$f_Q \Pi F_{B_i} = \Gamma(-\pi_Q)\Gamma(-\pi_B)\int_{\mathscr{D}_B\times\mathscr{D}_Q} \theta_B \wedge \theta_Q$$
$$= \Gamma(-\pi_Q)\Gamma(-\pi_B) \sum_{\{\mathscr{E}|B\in\mathscr{E}_m\}} \int_{J_{\mathscr{E}B}} \phi_{\mathscr{E}B}^*(\theta_B \wedge \theta_Q) , \quad (4.8)$$

with $\phi^*_{\mathscr{E}B}$ the standard pullback map on differential forms. However, from (4.4) and (A.4),

$$K_{\mathscr{E}B}|_{z_\omega=0} = \Gamma(-\pi_Q)\Gamma(-\pi_B)\int_{J_{\mathscr{E}B}} \theta_{\mathscr{E}B} \tag{4.9}$$

with

$$\theta_{\mathscr{E}B} = \prod_{\substack{K\in\mathscr{E}^0(G_0)\\ K\neq B}} t_K^{\pm\pi_K-1}dt_K\left\{[1+e_{\mathscr{E}}(\underline{t})]^{-\nu/2}\left[h_{\mathscr{E}}\prod_{B'\neq B}t_{B'}\right]^{-\pi_B}\right\}\Bigg|_{t_B=0} \tag{4.10}$$

where the exponent is $+\pi_K(-\pi_K)$ if $K\in\mathscr{E}_q(K\in\mathscr{E}_h)$]. Comparing (4.5) and (4.9) with (4.8), we see that (4.6) will follow if we can show that

$$\theta_{\mathscr{E}B} = \phi^*_{\mathscr{E}B}(\theta_B \wedge \theta_Q). \tag{4.11}$$

Let

$$\theta = \prod_{l\in\Omega}\alpha_l^{\lambda_l}d(\alpha)^{-\nu/2}D^*(\underline{\alpha},\underline{s},\underline{z})^{\pi_{G_0}}\eta$$

be the form whose integral defines F_{G_0} ((2.1)). Then the calculation leading to (3.4) says that

$$\phi^*_{\mathscr{E}}(\theta) = \rho \wedge t_B^{-(\pi_B+1)}dt_B, \tag{4.12}$$

where

$$\rho = \prod_{K\neq B} t_K^{\pm\pi_K-1}dt_K(1+e_{\mathscr{E}})^{-\nu/2}(g_{\mathscr{E}}-z_\omega)^{\pi_{G_0}}.$$

From (4.2) and (4.10),

$$\theta_{\mathscr{E}B} = (-z_\omega)^{-\pi_{G_0}}\left\{\left[\frac{\partial}{\partial t_B}\phi^*_{\mathscr{E}}(D^*)^{\pi_B}\rho\right\}\right|_{t_B=0} \tag{4.13}$$

(note $\phi^*_{\mathscr{E}}(D^*) = g_{\mathscr{E}} - z_\omega$).

On the other hand, the map χ of (4.7) is a diffeomorphism onto its range, so we may calculate $(\chi^{-1})^*(\theta)$. In order to make the factors d, D^*, etc. well defined functions, we normalize coordinates in $\mathbb{P}_{G_0}$ by $\alpha_\omega = 1$, and in $\mathbb{P}_B \times \mathbb{P}_Q$ by $\beta_{\sigma(B)} = \gamma_\omega = 1$. (This normalization was adopted in calculating $\phi^*_{\mathscr{E}}(D^*)$ above.) Then

$$(\chi^{-1})(\theta) = [(\chi^{-1})^*d]^{-\nu/2}[(\chi^{-1})^*D^*]^{\pi_{G_0}}[(\chi^{-1})^*(\Pi d_l^{\lambda_l}\eta)],$$

and since $\eta = \prod_{l\neq\omega} d\alpha_l$ with our normalization,

$$(\chi^{-1})^*\left(\prod_{\Omega}\alpha_l^{\lambda_l}\eta\right) = \prod_{\Omega_B}\beta_l^{\lambda_l}\prod_{\Omega_Q}\gamma_l^{\lambda_l}\eta_B\eta_Q t_B^{\sum_{\Omega_B}(\lambda_l+1)-1}dt_B.$$

Now it is shown in [1], Lemma 4.3.4, that

$$[t_B^{-h(B)}(\chi^{-1})^*(d)]_{t_B=0} = d_Q\Pi d_{B_i}$$

$$(\chi^{-1})^*(D^*)\,|_{t_B=0} = -\,z_\omega$$

$$\left\{\frac{\partial}{\partial t_B}[(\chi^{-1})^*(D^*)]\right\}\Big|_{t_B=0} = \sum_{i=1}^{r} D^*_{B_i}$$

and hence

$$(\chi^{-1})^*(\theta) = \hat{\rho} \wedge t_B^{-(\pi_B+1)}dt_B \tag{4.14}$$

with

$$\theta_B \wedge \theta_Q = (-\,z_\omega)^{-\pi_{G_0}}\left\{\left[\frac{\partial}{\partial t_B}((\chi^{-1})^*(D^*))\right]^{\pi_B}\hat{\rho}\right\}\Big|_{t_B=0} \tag{4.15}$$

Since $\phi_{\mathscr{E}}^*(\theta) = (\phi_{\mathscr{E}B} \times i)^*\,[(\chi^{-1})^*(\theta)]$ from (4.7), (4.12) and (4.14) imply that

$$\rho \wedge t_B^{-(\pi_B+1)}dt_B = (\phi_{\mathscr{E}B} \times i)^*\left[\hat{\rho} \wedge t_B^{-(\pi_B+1)}dt_B\right] = [(\phi_{\mathscr{E}B} \times i)^*(\hat{\rho})] \wedge t_B^{-(\pi_B+1)}dt_B$$

(i is essentially the identity map.) But then

$$\rho\,|_{t_B=0} = [(\phi_{\mathscr{E}B} \times i)^*(\hat{\rho})]\,|_{t_B=0} = \phi_{\mathscr{E}B}^*(\hat{\rho}\,|_{t_B=0}),$$

and (4.13) and (4.15) imply (4.11), since

$$(\phi_{\mathscr{E}B} \times i)^*\left[\frac{\partial}{\partial t_B}(\chi^{-1})^*(D^*)\right] = \frac{\partial}{\partial t_B}[(\phi_{\mathscr{E}B} \times i)^*(\chi^{-1})^*(D^*)] = \frac{\partial}{\partial t_B}\phi_{\mathscr{E}}^*(D^*)$$

and hence

$$\phi_{\mathscr{E}B}^*\left\{\left[\frac{\partial}{\partial t_B}(\chi^{-1})^*(D^*)\right]_{t_B=0}\right\} = \left[\frac{\partial}{\partial t_B}\phi_{\mathscr{E}}^*(D^*)\right]_{t_B=0}.$$

This completes the proof of Theorem 2.3.

APPENDIX

In this appendix we discuss the behavior of a certain analytic function, defined by a multiple integral, near one of its singular points. The singularity is due to a (non-simple) pinch in the integration space which may be described as follows: there are n singular surfaces for the integrand which are in general position; in the pinch configuration, an additional singular surface degenerates into the union of these. The integrand is multiple-valued and infinitely ramified around each singular variety.

Suppose then that W is a compact subset of $\mathbb{R}^m$, that $J \subset \mathbb{R}^n$ is the unit cube $\{ \underline{u} \mid 0 \leq u_i \leq 1 \}$, and that $h(\underline{u}, \underline{w})$ and $g(\underline{u}, \underline{w})$ are real analytic on an open neighborhood of $J \times W$, with $g > 0$. For $z < 0$, $\underline{w} \in W$, and $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{C}^{n+1}$ with $\operatorname{Re} \alpha_i > -1$ for $i = 1, 2, \ldots, n$, define

$$G(\underline{\alpha}, \underline{w}, z) = \int_J \prod_{i=1}^n u_i^{\alpha_i} du_i h(\underline{u}, \underline{w}) \left[g(\underline{u}, \underline{w}) \prod_{i=1}^n u_i - z \right]^{\alpha_0}. \tag{A.1}$$

[In (A.1), and throughout this appendix, it is understood that a complex power of a positive quantity is defined using the principle branch of the logarithm.] We note that G may be analytically continued to a meromorphic function of $\underline{\alpha} \in \mathbb{C}^{n+1}$ by an integration by parts. Here we study the behavior of G under analytic continuation in the variable z throughout a punctured disc $D_\varepsilon = \{ z \mid 0 < |z| < \varepsilon \}$. We will prove

THEOREM A.1. — For sufficiently small ε, G may be analytically continued along any path in D_ε. For generic $\underline{\alpha}$ there is a decomposition

$$G(\underline{\alpha}, \underline{w}, z) = G_0(\underline{\alpha}, \underline{w}, z) + \sum_{i=1}^n (-z)^{\alpha_0 + \alpha_i + 1} G_i(\underline{\alpha}, \underline{w}, z), \tag{A.2}$$

where G_0, G_i are analytic at $z = 0$. Moreover, we have the formulae

$$G_0 |_{z=0} = G |_{z=0} = \int_J \Pi u_i^{\alpha_i + \alpha_0} du_i h g^{\alpha_0}, \tag{A.3}$$

valid for $\operatorname{Re} \alpha_i + \alpha_0 > -1$, and

$$G_i |_{z=0} = \frac{\Gamma[-(\alpha_0 + \alpha_i + 1)]\Gamma(\alpha_i + 1)}{\Gamma(-\alpha_0)} \int_0^1 \ldots \int_0^1 \prod_{j \neq i} u_j^{\alpha_j - \alpha_i - 1} du_j \times [h(\underline{u}, \underline{w}) g(\underline{u}, \underline{w})^{-\alpha_i - 1}] |_{u_i = 0}, \tag{A.4}$$

valid for $\operatorname{Re} \alpha_j > \operatorname{Re} \alpha_i$, $j \neq i$.

We will prove Theorem A.1 by a sequence of lemmas;our approach will necessitate an explicit construction of the analytic continuation of G throughout D_ε. Let S_θ denote the operation of analytic continuation clockwise around $z = 0$ by angle θ, so that if $F(z)$ is the germ of some analytic function defined for $\arg z = \phi$, $S_\theta f$ is a germ defined for $\arg z = \phi + \theta$. We write $T \equiv S_{2\pi}$.

Formula (A.1) defines G for $\operatorname{Re} z < 0$. Then clearly

$$(S_\pi - S_{-\pi})G = (e^{\pi i \alpha_0} - e^{-\pi i \alpha_0})H, \tag{A.5}$$

with H defined for $z > 0$ and $\alpha_i > 0$, $i = 0, 1, \ldots, n$, by

$$H(\underline{\alpha}, \underline{w}, z) = \int_{J_0} \prod_{i=1}^n u_i^{\alpha_i} du_i h(\underline{u}, \underline{w})[z - g(\underline{u}, \underline{w})\Pi u_i]^{\alpha_0} \qquad J_0 = J \cap \left\{ \underline{u} \mid g \prod_{i=1}^n u_i \leq z \right\}. \tag{A.6}$$

LEMMA A.2. — Let $X_i = \exp 2\pi i(\alpha_0 + \alpha_i)$, $i = 1, \ldots, n$. Then

$$(T - X_1)(T - X_2) \ldots (T - X_n)H = 0.$$

Proof. — *Case 1.* We first consider the special case $g(\underline{u}, \underline{w}) \equiv 1$, and construct explicitly the contour deformation necessary for the analytic continuation of H. For $R \geq 0$, $\theta \geq 0$, and $A \subset \{1, \ldots, n\}$, $A \neq \emptyset$, the contour $C(A, \theta, R) \subset \mathbb{C}^n$ is defined in terms of parameters r, $v_i (i \notin A)$, and $\phi_i (i \in A)$ as follows:

$$u_i = \begin{cases} v_i(1 - r) + r & i \notin A, \\ re^{i\phi_i}, & i \in A, \end{cases}$$

with $0 \leq v_i \leq 1$, $\phi_i \geq 0$ and $\Sigma\phi_i = \theta$, and $0 \leq r \leq \rho(\underline{v}, R)$, where $\rho(\underline{v}, R)$ is the smallest positive root of

$$|\Pi u_i| \equiv r^{|A|} \prod_{i \notin A} [v_i(1 - r) + r] = R.$$

Note that, for fixed θ and R, the contours $C(A, \theta, R)$ form a polyhedral complex on whose boundary either $u_i = 0$ for some i, $u_i = 1$ for some i, or $\Pi u_i = Re^{i\theta}$. Moreover, for $\theta = 0$ this complex reduces to J_0. Thus

$$S_\theta H = \sum_A \int_{C(A,\theta,|z|)} \Pi u_i^{\alpha_i} du_i h(\underline{u}, \underline{w})(z - \Pi u_i)^{\alpha_0} \qquad (A.7)$$

where on $C(A, \theta, |z|)$, $u_i^{\alpha_i}$ is defined using $\arg u_i = \phi_i$, $i \in A$, and $(z - \Pi u_i)^{\alpha_0}$ is defined using $\arg (z - \Pi u_i) = \theta$.

We now partially evaluate the integral in a typical term of (A.7). Since $\rho(\underline{v}, R) \to 0$ as $R \to 0$ we may, by choosing ε and hence $|z|$ sufficiently small, expand $h(\underline{u}, \underline{w})$ in a power series converging uniformly on $C(A, \theta, |z|)$:

$$h(\underline{u}, \underline{w}) = \sum_{\underline{i}} h_{\underline{i}}(\underline{u}_A, \underline{w}) \prod_{j \in A} u_j^{i_j},$$

the sum running over multi-indices $\underline{i} = (i_j)$, $j \in A$, and $\underline{u}_A$ denoting the variables (u_j), $j \notin A$. The Jacobean of the variable change on $C(A, \theta, |z|)$ from $\underline{u}$ to $(\underline{v}, \underline{\phi}, r)$ is of the form $F(r)e^{i\theta}$, so that (A.7) becomes

$$S_\theta H = \sum_A \sum_{\underline{i}} H_{A,|z|,\underline{i}} \left\{ \int \cdots \int_{\Sigma\phi_j = \theta} e^{i\phi_j(\alpha_j + i_j)} e^{i(\alpha_0 + 1)\theta} \Pi d\phi_j \right\} \qquad (A.8)$$

with

$$H_{A,|z|,\underline{i}} = \int dr \prod_j dv_j F(r) h_{\underline{i}}(\underline{u}_A, w) \prod_{j \in A} r^{\alpha_j} \prod_{j \in A} u_j^{\alpha_i}.$$

The only θ dependence in (A.8) is in the bracketed term, which may be evaluated to give

$$(-i)^{|A|-1} \sum_{j \in A} \frac{e^{i\theta(\alpha_j + i_j + \alpha_0 + 1)}}{\prod_{k \neq j} (\alpha_j + i_j - \alpha_k - i_k)} \qquad (A.9)$$

(We assume that the α variables are chosen generically so that the denominators do not vanish). If $\theta = 2\pi l$, (A.9) reduces to the form

$$\sum_{j \in A} f_j(\underline{\alpha}) X_j^l.$$

Thus

$$\prod_{k=1}^{n}(\mathrm{T} - \mathrm{X}_k)\mathrm{H} = \sum_{l=0}^{n} \sum_{\substack{\mathrm{B} \subset \{1,\dots,n\} \\ |\mathrm{B}| = l}} \left(\prod_{k \notin \mathrm{B}} \mathrm{X}_k\right)(\mathrm{T}^l\mathrm{H})$$

$$= \sum_{\mathrm{A},\underline{i}} \mathrm{H}_{\mathrm{A},|z|,\underline{i}} \sum_{j \in \mathrm{A}} f_j(\alpha) \sum_{l=0}^{n} \mathrm{X}_j^l \sum_{\mathrm{B}} \prod_{k \notin \mathrm{B}} \mathrm{X}_k .$$

But

$$\sum_{l=0}^{n} \mathrm{X}_j^l \sum_{\substack{\mathrm{B} \subset \{1,\dots,n\} \\ |\mathrm{B}| = l}} \prod_{k \notin \mathrm{B}} \mathrm{X}_k = \prod_{k=1}^{n} (\mathrm{Y} - \mathrm{X}_k) |_{\mathrm{Y} = \mathrm{X}_j} = 0 ,$$

so that the lemma is proved in the case $g = 1$.

Case 2. — We now assume that

$$\left| \frac{\partial g}{\partial u_i} \right| < g/n \qquad (\mathrm{A}.10)$$

on $\mathrm{J} \times \mathrm{W}$. Let $\mathrm{U} \subset \mathbb{R}^n$ be a convex open neighborhood of $\mathrm{J} - \{ (1, 1, \dots, 1) \}$ on which g is analytic, (A.10) holds, and

$$a_i(u) = (1 - u_i) \Big/ \left[\sum_{j=1}^{n} (1 - u_j) \right]$$

satisfies $| u_i a_i(\underline{u}) | \leq 1$ for all i. For $\underline{u} \in \mathrm{U}$, define

$$x_i(\underline{u}, \underline{w}) = u_i g(\underline{u}, \underline{w})^{a_i(\underline{u})} \qquad (\mathrm{A}.11)$$

Then for fixed $\underline{w}$, (A.11) is $1 - 1$ on U, since if $\underline{u}_1$ and $\underline{u}_2 = \underline{u}_1 + \underline{s}$ are in U, and i is chosen so that $| s_i | \geq \frac{1}{\sqrt{n}} | s |$, an easy calculation using (A.10) shows that $\sum_j \frac{\partial x_i}{\partial u_j} s_j$ has the same sign as s_i, and hence

$$x_i(\underline{u}_2, \underline{w}) - x_i(\underline{u}_1, \underline{w}) = \int_0^1 \sum_j s_j \frac{\partial x_i}{\partial u_j} (\underline{u}_1 + t\underline{s}) dt \neq 0 .$$

Choosing ε small enough so that $\mathrm{J}_0 \subset \mathrm{U}$, we may rewrite (A.6) as

$$\mathrm{H} = \int_{\mathrm{J}'_0} \prod_{i=1}^{n} \mathrm{X}_i^{\alpha_i} dx_i h(\underline{u}(\underline{x}), \underline{w}) g(\underline{u}(\underline{x}), \underline{w})^{(-\Sigma a_i \alpha_i)} j(\underline{x}, \underline{w}) \left(z - \prod_{i=1}^{n} x_i \right)^{\alpha_0} , \qquad (\mathrm{A}.12)$$

where

$$\mathrm{J}'_0 = \left\{ \underline{x} \,|\, 0 \leq x_i \leq 1, \prod_{i=1}^{n} x_i \leq z \right\} ,$$

and $j(\underline{x}, \underline{w})$ is the Jacobean of (A.11). Applying Case 1 to (A.12) completes the proof of Case 2.

We now discuss the general case. Since g is positive on $\mathrm{J} \times \mathrm{W}$, there is an $\mathrm{M} > 0$ such that

$$\left| \frac{\partial g}{\partial u_i} \right| \leq \mathrm{M} g \qquad (\mathrm{A}.13)$$

on $\mathrm{J} \times \mathrm{W}$, for all i. Taking N a positive integer with $\mathrm{N} > n\mathrm{M}$, we subdivide the cube J into subcubes of side 1/N; thus $H = \sum_K H_K$, where K denotes a subcube and H_K is the integral (A.6) taken over $J_0 \cap K$. By choosing ε (and hence $|z|$) sufficiently small we may guarantee that J_0 intersects only those cubes $K = \{ \underline{u} \mid j_i/N \leq u_i \leq (j_i + 1)/N \}$ for which at least one j_i is zero. Suppose that K is such a cube, with $j_i = 0$ for $i \in A \subset \{ 1, \ldots, n \}$. In the integral for H_K we introduce new variables by $u'_i = N u_i$, $i \in A$; $\underline{w}' = (\underline{w}, \underline{u}_A)$, where $\underline{u}_A = (u_i)$, $i \notin A$. Then H_K is itself of the form (A.6) [with an additional integration over some w variables, which does not affect the argument], but

$$\left| \frac{\partial g}{\partial u'_i} \right| = N^{-1} \left| \frac{\partial g}{\partial u_i} \right| < g$$

for $i \in A$, by (A.13). Case 2 then implies that $\prod_{i \in A} (T - X_i) H_K = 0$, from which the lemma follows.

Lemma A.3. — For ε sufficiently small, and $z \in D_\varepsilon = \{ z \mid 0 < |z| < \varepsilon \}$,

$$H(\underline{\alpha}, \underline{w}, z) = \sum_{i=1}^{n} z^{\alpha_0 + \alpha_i + 1} H_i(\underline{\alpha}, \underline{w}, \underline{z}) \tag{A.14}$$

$$G(\underline{\alpha}, \underline{w}, z) = G_0(\underline{\alpha}, \underline{w}, z) + \sum_{i=1}^{n} (-z)^{\alpha_0 + \alpha_i + 1} G_i(\underline{\alpha}, \underline{w}, z) \tag{A.15}$$

with G_0, G_i and H_i single valued in D_ε.

Proof. — We take α a generic point for which $X_i \neq X_j \neq 1$, for any i, j. Then

$$f(y) \equiv \sum_{i=1}^{n} \prod_{j \neq i} \left(\frac{Y - X_j}{X_i - X_j} \right) = 1$$

for all Y, since the left hand side is a polynomial in Y of degree $n - 1$, and equality holds at the n points $Y = X_i$. Hence (A.14) will be satisfied with

$$H_i = z_i^{-(\alpha_0 + \alpha_i + 1)} \prod_{j \neq i} \left(\frac{T - X_j}{X_i - X_j} \right) H . \tag{A.16}$$

Lemma (A.2) then implies that $TH_i = H_i$, i. e., H_i is single valued.
Now from (A.15),

$$(T - 1)G = (e^{\pi i \alpha_0} - e^{-\pi i \alpha_0}) S_\pi H .$$

Applying $\Pi(T - X_i)$ to this equation gives $(T - 1)\prod_{i=1}^{n} (T - X_i) G = 0$, and an argument as above yields (A.15).

Remark A.4. — If we insert (A.15) into (A.5), we find

$$(e^{\pi i \alpha_0} - e^{-\pi i \alpha_0}) H = \sum_{i=1}^{n} [e^{\pi i(\alpha_0 + \alpha_i + 1)} - e^{-\pi i(\alpha_0 + \alpha_i + 1)}] z^{\alpha_0 + \alpha_i + 1} S_\pi G_i .$$

Comparison with (A.14) shows that $\sin(\alpha_0\pi)H_i = \sin[(\alpha_0 + \alpha_i + 1)\pi]S_\pi G_i$. Since these functions are single valued the operator S is redundant, and

$$G_i = \frac{\sin \alpha_0 \pi}{\sin(\alpha_0 + \alpha_i + 1)\pi} H_i . \tag{A.17}$$

Proof of Theorem A.1. — We first show that the functions G_0, G_i, H_i of Lemma A.3 have removable singularities at $z = 0$. By a straightforward calculation it may be shown that the area of the contour $C(A, \theta, |z|)$ need to define analytic continuations of H (Lemma A.2), Case 1) is bounded by a multiple of $|z|^{1/n}$, if θ is bounded. Moreover, if $\operatorname{Re}\alpha_i \geq 0$ for $i = 0, 1, \ldots, n$, the integrand in (A.7) is uniformly bounded on $C(A, \theta, |z|)$; hence, for bounded θ,

$$|S_\theta H| \leq K |z|^{1/n} .$$

If $\operatorname{Re}(\alpha_0 + \alpha_i) < 1/n$, (A.16) implies that $\lim_{z\to 0} zH_i = 0$, i. e., H_i has a removable discontinuity for $\underline{\alpha}$ in the above range. Since H is meromorphic in $\underline{\alpha}$, the discontinuity is removable for all $\underline{\alpha}$. Then (A.17) shows that G_i is also analytic at $z = 0$; an argument similar to the above implies the same conclusion for G_0.

To verify (A.3), we note from (A.2) that, if $\operatorname{Re}(\alpha_i + \alpha_0) > -1$ for all i,

$$G_0 |_{z=0} = \lim_{z\to 0} G .$$

(A.3) follows from the Lebesque dominated convergence theorem.

It remains to verify (A.4) which, by (A.17), is equivalent to

$$H_i(\underline{\alpha}, \underline{w}, 0) = \frac{\Gamma(\alpha_i + 1)\Gamma(\alpha_0 + 1)}{\Gamma(\alpha_i + \alpha_0 + 2)} \int_0^1 \cdots \int_0^1 \prod_{j\neq i} u_j^{\alpha_j - \alpha_i - 1} du_j \times [h(\underline{u}, \underline{w}) g(\underline{u}, \underline{w})^{\alpha_i + 1}] |_{u_i = 0} . \tag{A.18}$$

On the other hand, (A.14) implies that if $\operatorname{Re}\alpha_j > \operatorname{Re}\alpha_i$ for all $j \neq i$,

$$H_i(\underline{\alpha}, \underline{w}, 0) = \lim_{z\to 0} z^{-(\alpha_i + \alpha_0 + 1)} H(\underline{\alpha}, \underline{w}, z) . \tag{A.19}$$

We will show that (A.18) and (A.19) agree for

$$\begin{gathered} \operatorname{Re}(\alpha_j + 1) > n\operatorname{Re}(\alpha_i + 1) > 0, \qquad (j = 1, \ldots, \hat{i}, \ldots, n), \\ \operatorname{Re}\alpha_k > 0, \qquad (k = 0, 1, \ldots, n), \end{gathered} \tag{A.20}$$

the result then follows whenever $\operatorname{Re}\alpha_j > \operatorname{Re}\alpha_i$ by analytic continuation. We assume $z \geq 0$.

Let us write $H = \int_{J_0} \theta$, with θ the n-form given in (A.6). For fixed $\underline{w}$, decompose J_0 as $J_a \cup J_b$. where $J_a = J_0 \cap \left\{ \underline{u} \mid g(\underline{u}, \underline{w}) \prod_{j\neq i} u_j \geq z \right\}$, and $J_b = J_0 - J_a \subset \bigcup_{j\neq i} J_j$, where $J_j = \{ u \mid 0 \leq u_k \leq 1,\ u_j \leq z^{1/n} \}$. Now

$$\left| z^{-(\alpha_0 + \alpha_i + 1)} \int_{J_j} \theta \right| \leq |z^{-(\alpha_i + 1)}| \int_0^{z^{1/n}} |u_j^{\alpha_j}| du_j = \frac{1}{(\operatorname{Re}\alpha_j + 1)} z^{\operatorname{Re}[\alpha_j + 1 - n(\alpha_i + 1)]/n}$$

and hence by (A.20)

$$\lim_{z\to 0} \left| z^{-(\alpha_0 + \alpha_i + 1)} \int_{J_j} \theta \right| = 0 . \tag{A.21}$$

On the other hand,

$$z^{-(\alpha_0 + \alpha_i + 1)} \int_{J_a} \theta = \int_K \prod_{j\neq i} u_j^{\alpha_j - \alpha_i - 1} du_j \int_0^1 U^{\alpha_i}(1 - U)^{\alpha_0} dU \tilde{h} \tilde{g}^{-(\alpha_i + 1)} ,$$

where we have made the substitution

$$U = u_i g(\underline{u}, \underline{w}) \prod_{j \neq i} u_j / z \,,$$

and

$$I = \left\{ (u_1, \ldots, \hat{u}_i, \ldots, u_n) \mid g \prod_{j \neq i} u_j \geq z \right\},$$

$$\tilde{h}(U, \underline{u}, \underline{w}) = h(\underline{u}, \underline{w}) \big|_{u_i = Uz/g \prod_{j \neq i} u_j},$$

etc. For α satisfying (A.20) we may apply the Lebesgue dominated convergence theorem to find

$$\lim_{z \to 0} z^{-(\alpha_0 + \alpha_i + 1)} \int_{J_\alpha} \theta = \frac{\Gamma(\alpha_i + 1)\Gamma(\alpha_0 + 1)}{\Gamma(\alpha_i + \alpha_0 + 2)} \int_0^1 \cdots \int_0^1 \prod_{j \neq i} u_j^{\alpha_j - \alpha_i - 1} du_j [h(\underline{u}, \underline{w}) g(\underline{u}, \underline{w})^{-(\alpha_i + 1)}] \big|_{u_i = 0} \quad \text{(A.22)}$$

(A.19), (A.21) and (A.22) imply (A.18), completing the proof.

REFERENCES

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