ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 25, nº 2 (1976), p. 197-230
<http://www.numdam.org/item?id=AIHPA_1976__25_2_197_0>

Ann. Inst. Henri Poincaré Section A :
Vol. XXV, n° 2, 1976, p. 197-230. *Physique théorique.*

Eigenwaves in a relativistic gas according to Marle's 14-moment description

by

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ABSTRACT. — The propagation eigenmodes in a monatomic relativistic gas with dissipation due to shear stress, bulk stress and heat conduction have been studied on the basis of the linearized relativistic 14-moment theory of Marle. Three non-trivial modes exist; two longitudinal: fast and slow and one transverse, the same number as is obtained in Chernikov theory, but there are some differencies in the dispersion laws, mainly due to the volume viscosity. Some of the wavefront speeds according to Marle's theory are slightly higher than according to Chernikov's theory. All the 12 transport coefficients (existing in the linearized case) were written in terms of the state variables and collision frequencies. The question of irreversibility of an expanding gas is briefly discussed, too.

INTRODUCTION

In accordance with the theory of relativity we believe that all possible field and/or material phenomena can propagate only with some finite signal velocity not exceeding the velocity of light in vacuo. (Fundamental postulat of relativistic causality (*).) From the mathematical point of view this requirement can be fulfilled only if the system of partial differential equations describing the appropriate physical phenomena is hyperbolic. However, this is not true for a thermodynamically stationary theory: to the category of stationary theories belong for example the phenomenological theories of Eckart (1940) and Landau-Lifshitz (1959) and the kinetic

(*) See e. g. TERLETSKII (1968), p. 71 or HAVAS (1974). Only superlight particles compatible with the special relativity are those which mass is imaginary (tachyons).

theory of Israel (1963), (1972) or of Chernikov (1964) in its original form. All these theories apply only the minimum necessary generalizations in introducing viscous stress bulk stress and heat flux by means of the Navier-Stokes and Fourier transport equations which are evidently stationary.

This difficulty with stationary theories has led some authors ([1]) to propose more satisfactory mathematical systems which must be evidently non-stationary and hyperbolic at the same time.

In kinetic theory, for a description of a relativistic gas the Boltzmann equation in convariant form is generally accepted. This equation is manifestely hyperbolic. As we do not know the exact solution of this equation we must be satisfied with some approximative solution. The most popular approximate formal solution is the Chapman-Enskog solution, also called the normal solution, which was given, also in convariant form by Israel (1963). From this and other similar non-equilibrium quasi-stationary solutions, the transport equations of dissipative processes are reduced to the traditional atemporal phenomenological laws of Fourier and Navier-Stokes which make the whole system of equations parabolic. This is evidently due to the fact that the normal solution method or iteration method contains steps which result in the elimination of temporal derivatives of higher moments.

In contrast to this, Grad's method is believed to approximate closely quite general solutions of the Boltzmann equation when a sufficient number of Hermite coefficients are included in the distribution function expansion. All the moments which are necessary to specify the Hermite coefficients retained are independent and all have separate initial conditions. Such an approximation is then represented by an explicit expression for the distribution function whose Hermite coefficients are specified by corresponding equations of moments which form a hyperbolic system.

Chernikov (1964) generalised Grad's therteen-moment method for an approximate formal solution of the Boltzmann equation to the case of the general relativistic Boltzmann equation. It is strange that he did not exploit the advantage of hyperbolicity of his formalismus and developed the moment equations no further than a stationary theory. This task of completing his equations to hyperbolic system was made by author (1972), the first relativistic transport equations which are hyperbolic were published by Marle (1969). Marle's 14-moment theory seems to be a more consistent generalisation of Grad's method to approximate the solution of the relativistic Boltzmann equation in contrast to that of Chernikov which turns out to be more special, since it does not include the description of the bulk viscosity. Namely, in a relativistic description, an monatomic gas generally

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([1]) For the phenomenological theory, see Kranyš (1966), Müller (1966), (1969), Boillat (1972) and Maugin (1974), while for the kinetic theory, see Marle (1969) or, for example, Chernikov (1964), Kranyš (1970, 1972).

possesses a bulk viscosity, the coefficient for which dissapears only in the classical and ultrarelativistic limit (i. e. for particles with mass $m = 0$). This deviation in the behaviour of an monatomic gas, which seems to be a new relativistic effect (see Israel (1963, 1970); Anderson (1970)) is included in 14-moment description of Marle.

The aim of this paper is to study the consequences of introduction of a bulk viscosity on the structure of eigenwaves. The author (1972 *a*) has studied the eigenwaves for the linearized 13-moment system both hyperbolic and parabolic. The same method (Fourier transform) can be used and the resulting dispersion equation can be also find for the 14-moment description and therefore the influence of bulk viscosity on dispersion and absorption of characteristic waves. As well, their wavefront speeds can be determined. In such a direct way the conformity of a system of starting equations with the postulate of relativistic causality can be verified.

In this study we will consider only the special relativistic case, nevertheless in general relativity the overall features of the effects remain unchanged. So this subject may be of great interest e. g. in connection with the study of some expanding cosmological models or with the study of conditions in neutron stars (See e. g. Treciokas (1971)).

I. THE LINEARIZED 14-MOMENT EQUATIONS AND THEIR FOURIER TRANSFORMS

First, we limit ourselves to the special-relativistic theory, then we will consider (as in Kranyš (1972 *a*) (2)) an immobile, unbounded space filled with a monatomic gas in thermodynamical equilibrium in which there is a disturbance of very small amplitude. Therefore we assume that the moment equations of Marle (1969) (3) may be linearized around equilibrium so all the coefficients of the differential equations will be considered as having constant values corresponding to a system in thermodynamic equilibrium (i. e. in the coefficients we set $T \rightarrow T_{eq.}$, $p \rightarrow p_{eq.}$ but $\tau^{\alpha\beta} \rightarrow 0$, $\theta \rightarrow 0$, $Q^{\alpha} \rightarrow 0$, and we will drop suffix eq.).

The moment equations in question from [I] are (3,17), (3,35), (11,23), (11,24), (11,25) and the energy balance $u_{\beta}\partial_{\alpha}T^{\alpha\beta} = 0$ (not given explicitely in [I]) have to be linearized. The nomenclature used is given in Appendix B.

From the energy momentum tensor I (3,29)

$$c^2 T^{\alpha\beta} \equiv \rho c^2 u^{\alpha} u^{\beta} - p \overset{\perp}{g}{}^{\alpha\beta} - \theta^{\alpha\beta} + \frac{1}{c}(Q^{\alpha} u^{\beta} + Q^{\beta} u^{\alpha}) + \theta u^{\alpha} u^{\beta} \qquad (1,1)$$

(2) This paper will be quoted in the following as [II].
(3) This paper will be quoted in the following as [I].

follows (the obvious non-linear terms i. e. terms containing Q^α, $\theta^{\alpha\beta}$, $\tau^{\alpha\beta}$, and θ in the coefficients were discarded immediately):

$$c^2\partial_\alpha T^{\alpha\beta} \equiv c^2 u^\beta \mathcal{D}\rho + c^2\rho u^\beta \partial_\alpha u^\alpha + c^2 \rho \mathcal{D} u^\beta - p\partial_\alpha \overset{\perp}{g}{}^{\alpha\beta} - \overset{\perp}{\partial}{}^\beta p - \partial_\alpha \theta^{\alpha\beta} + \frac{1}{c} u^\beta \partial_\alpha Q^\alpha + \frac{1}{c}\mathcal{D} Q^\beta + u^\beta \mathcal{D}\theta = 0 \quad (1,2)$$

Multiplying by u_β and contracting, and making use of $(D,3)_1$, and (D,2) we obtain

$$c^2 u_\beta \partial_\alpha T^{\alpha\beta} \equiv c^2 \mathcal{D}\rho + c^2 \rho \partial_\alpha u^\alpha + p\partial_\alpha u^\alpha - u_\beta \partial_\alpha \theta^{\alpha\beta} + \frac{1}{c}\partial_\alpha Q^\alpha + \mathcal{D}\theta = 0\,. \quad (1,3)$$

By making use of (D,12) and (D,3) we obtain

$$c^2 u_\beta \partial_\alpha T^{\alpha\beta} \equiv nc_v \mathcal{D} T + p\partial_\alpha u^\alpha + \frac{1}{c}\partial_\alpha Q^\alpha + \mathcal{D}\theta = 0. \quad (1,4)$$

This is the linearized equation for energy balance.

Linearized equation of motion one obtains from (1,2) by forming the expression:

$$c^2 \overset{\perp}{g}{}^\sigma_\beta \partial_\alpha T^{\alpha\beta} \equiv (c^2\rho + p)\mathcal{D} u^\sigma - \overset{\perp}{\partial}{}^\sigma p - \overset{\perp}{\partial}_\alpha \tau^{\alpha\sigma} - \frac{1}{3}\overset{\perp}{\partial}{}^\sigma \theta + \frac{1}{c}\mathcal{D} Q^\sigma = 0 \quad (1,5)$$

where we have used (D,4), (D,5), (D,6) and (D,10). This result can also be obtained by linearization of Eq. I (3,35).

Linearizing the transport equation I (11,23) for θ we easily obtain

$$r\left[3\mathcal{D}\left(\frac{G}{\gamma}\right) + \frac{2}{\gamma} G\partial_\sigma u^\sigma\right] + \frac{3\varphi}{c^2}\mathcal{D}\theta - \frac{10\chi}{c^3}\partial_\sigma Q^\sigma = -\frac{1}{c^3\tau}\theta \quad (1,6)$$

Multiplying this equation by c^3 and using (D,11), definitions (C,10), (C,11), and $\frac{c^2}{\gamma} = RT = \frac{p}{r}$ one obtain

$$Nc\mathcal{D}\theta + 2pG\partial_\sigma c u^\sigma + 3\bar{U} c\mathcal{D} T + 2\overset{+}{D}\partial_\sigma Q^\sigma = -\frac{1}{\tau}\theta \equiv -\lambda^+\theta\,. \quad (1,7)$$

The linearized form of the transport equation for $\tau^{\alpha\beta}$ is obtained by linearization of I (11,25), whose corrected form is given by (A,4) in Appendix A. This leaves

$$-2\overset{\perp}{g}{}^{\alpha\beta}\left[r\mathcal{D}\left(\frac{G}{\gamma}\right) + \frac{\varphi}{c^2}\mathcal{D}\theta\right] + \frac{r}{\gamma}G\left[-\frac{2}{3}\overset{\perp}{g}{}^{\alpha\beta}\partial_\gamma u^\gamma - \overset{\perp}{g}{}^{\alpha\lambda}\overset{\perp}{g}{}^{\beta\mu}(\partial_\lambda u_\mu + \partial_\mu u_\lambda)\right]$$
$$+ \frac{2\chi}{c^3}\left\{\left[\overset{\perp}{g}{}^{\beta\gamma}\overset{\perp}{g}{}^\alpha_\lambda + \overset{\perp}{g}{}^{\alpha\gamma}\overset{\perp}{g}{}^\beta_\lambda + \frac{2}{3}(4g^\gamma_\lambda + 5u^\gamma u_\lambda)\right]\right\}\partial_\gamma Q^\lambda + \frac{2\psi}{c^2}[-3\mathcal{D}\tau^{\alpha\beta}]$$
$$= +\frac{1}{c^3\tau}\tau^{\alpha\beta} \quad (1,8)$$

Making use of (D,11), (D,14), $(D,3)_2$ and $(D,14)_a$ this equation can be rewritten

$$-\frac{2}{c^2}\overset{\perp}{g}{}^{\alpha\beta}[\bar{U}\mathscr{D}T + \varphi\mathscr{D}\theta] - \frac{4}{3}\frac{rG}{\gamma}\overset{\perp}{g}{}^{\alpha\beta}\partial_\gamma u^\gamma - 2\frac{rG}{\gamma}\langle \partial^\alpha u^\beta \rangle + \frac{4\chi}{c^3}\langle \partial^\alpha Q^\beta \rangle + \frac{20}{3}\frac{\chi}{c^3}\overset{\perp}{g}{}^{\alpha\beta}\partial_\lambda Q^\lambda - \frac{6\psi}{c^2}\mathscr{D}\tau^{\alpha\beta} = \frac{1}{c^3\tau}\tau^{\alpha\beta}. \quad (1,9)$$

Multiplying by $(-c^3)$ a using (C,10), (C,11) and $rc^2/\gamma = p$ this equation reads because of $(D,5)_2$

$$\frac{2}{3}\overset{\perp}{g}{}^{\alpha\beta}\{2pG\partial_\gamma cu^\gamma + 2\overset{+}{D}\partial_\lambda Q^\lambda + 3\bar{U}c\mathscr{D}T + Nc\mathscr{D}\theta\} + 2pG\langle \partial^\alpha cu^\beta \rangle + \frac{4}{5}\overset{+}{D}\langle \partial^\alpha Q^\beta \rangle + Bc\mathscr{D}\tau^{\alpha\beta} = -\frac{1}{\tau}\tau^{\alpha\beta} \equiv -\beta^+\tau^{\alpha\beta}. \quad (1,10)$$

By the use of (1,7) the first term in (1,10) can be written as

$$-\frac{2}{3}\overset{\perp}{g}{}^{\alpha\beta}\lambda^+\theta. \quad (1,11)$$

If we contract the Eq. (1,10) we see (as $\tau^{\alpha\beta} = \langle \tau^{\alpha\beta} \rangle$) that every term is traceless (compare (D,14)) except the term (1,11) containing the tensor $\overset{\perp}{g}{}^{\alpha\beta}$. In other words, that (1,11) must be zero.

The vanishing of the term (1,11) can be secured in three ways: (*i*) To suppose that $\overset{\perp}{g}{}^{\alpha\beta}$ must be made traceless which requires that $\overset{\perp}{g}{}^{\alpha\beta}$ be replaced by $\langle \overset{\perp}{g}{}^{\alpha\beta} \rangle$ which is zero by (D,15). (*ii*) To suppose that $\theta = 0$ holds identically, which causes the disappearance of the bulk viscosity and therefore the degeneration of the 14-moment description to a 13-moment one in the linearized case. (*iii*) To suppose only $\lambda^+ = 0$, but $\theta \neq 0$, so that the gaz expansion process itself is always reversible. The possibility (*i*), seems to be the most natural, so we do not expect the « drastically restrictive » alternatives (*ii*) and (*iii*) which eliminate the phenomenon of bulk viscosity of a relativistic monatomic gas (Note: The term (1,11) does not appear if the uncorrected Eq. I (11,25) is used). Nevertheless the choice of alternative (*i*) does not prevent us to make the different supposition later if desired.

Finally the transport equation for Q^α i. e. I (11,24) whose corrected form is (A,3) after being linearized is

$$r\left(\frac{K_4}{K_2} - \frac{G}{\gamma}\right)\mathscr{D}u^\alpha - \overset{\perp}{\partial}{}^\alpha\left(r\frac{G}{\gamma}\right) - \frac{\varphi}{c^2}\overset{\perp}{\partial}{}^\alpha\theta + \frac{2\chi}{c^3}[u_\lambda\overset{\perp}{\partial}{}^\alpha Q^\lambda - 5\overset{\perp}{g}{}^\alpha_\lambda\mathscr{D}Q^\lambda] - \frac{6\psi}{c^2}\partial_\gamma\tau^{\alpha\gamma} = -\frac{1}{c^4\tau}Q^\alpha. \quad (1,12)$$

By the use of (D,11), $(D,2)_2$ and (D,5) this equation multiplied by c^2 take the form

$$c^2 r\left(\frac{K_4}{K_2} - \frac{G}{\gamma}\right)\mathscr{D}u^\alpha - mc^2\frac{G}{\gamma}\overset{\perp}{\partial}{}^\alpha n - \bar{U}\overset{\perp}{\partial}{}^\alpha T - \varphi\overset{\perp}{\partial}{}^\alpha\theta - \frac{10\chi}{c}\mathscr{D}Q^\alpha - 6\psi\partial_\gamma\tau^{\alpha\gamma} = -\frac{1}{c^2\tau}Q^\alpha. \quad (1,13)$$

By multiplying Eq. (1,5) (in which $p + \rho c^2 = rc^2G$, and $p = nkT$ was used) by $\left(\frac{K_4}{K_3} - \frac{1}{\gamma}\right)$ and substracting from (1,13) one obtains

$$-\frac{1}{c^2}\overset{=}{\varphi}c\mathscr{D}Q^\alpha - \overset{=}{\psi}\partial_\beta\tau^{\alpha\beta} - m\overset{=}{G}\overset{\perp}{\partial}{}^\alpha n - \overset{=}{U}\overset{\perp}{\partial}{}^\alpha T - M\overset{\perp}{\partial}{}^\alpha\theta = -\frac{1}{c^2\tau}Q^\alpha, \quad (1,14)$$

where (C,17), (C,18), (C,15), (C,14) and (C,16) was used. Multiplying by c^2 and again using the same relations from App. C leaves

$$\overset{+}{Z}c\mathscr{D}Q^\alpha - RT\overset{\perp}{\partial}_\beta\tau^{\alpha\beta} + \frac{p}{n}c^2U\overset{\perp}{\partial}{}^\alpha n - \frac{n}{T}\varepsilon c^2U\overset{\perp}{\partial}{}^\alpha T - (\ M\overset{\perp}{\partial}{}^\alpha\theta = -\frac{1}{\tau}Q^\alpha \equiv -\overset{+}{\alpha}Q^\alpha. \quad (1,15)$$

Gathering together all the linearized moment equations i. e. I (3,17), (1,5), (1,4), (1,10), (1,7) and (1,15) leaves the desired system of equations:

$$c\mathscr{D}n + n\partial_\alpha cu^\alpha = 0, \quad (1,16)$$

$$rGc\mathscr{D}cu^\alpha - k[T\overset{\perp}{\partial}{}^\alpha n + n\overset{\perp}{\partial}{}^\alpha T] - \overset{\perp}{\partial}_\beta\tau^{\alpha\beta} - \frac{1}{3}\overset{\perp}{\partial}{}^\alpha\theta + \frac{1}{c^2}c\mathscr{D}Q^\alpha = 0, \quad (1,17)$$

$$nc_v.c\mathscr{D}T + p\partial_\alpha cu^\alpha + \partial_\alpha Q^\alpha + c\mathscr{D}\theta = 0, \quad (1,18)$$

$$\overset{+}{B}\langle c\mathscr{D}\tau^{\alpha\beta}\rangle + 2pG\langle\partial^\alpha cu^\beta\rangle + \frac{4}{5}\overset{+}{D}\langle\partial^\alpha Q^\beta\rangle = -\overset{+}{\beta}\tau^{\alpha\beta}, \quad (1,19)$$

$$Nc\mathscr{D}\theta + 3\bar{U}c\mathscr{D}T + 2pG\partial_\gamma cu^\gamma + 2\overset{+}{D}\partial_\gamma Q^\gamma = -\lambda^+\theta, \quad (1,20)$$

$$\overset{+}{Z}c\mathscr{D}Q^\alpha - RT\overset{\perp}{\partial}_\gamma\tau^{\alpha\gamma} + \frac{p}{n}c^2U\overset{\perp}{\partial}{}^\alpha n - \frac{n}{T}\varepsilon c^2U\overset{\perp}{\partial}{}^\alpha T - c^2M\overset{\perp}{\partial}{}^\alpha\theta = -\overset{+}{\alpha}Q^\alpha. \quad (1,21)$$

To this system belongs the supplementary conditions $(D,2)_{1,2,3}$, $\tau^{\alpha\beta} = \tau^{\beta\alpha}$, and $\tau^\alpha_\alpha = 0$. Disregarding the differences which originate in the inclusion of θ, and the fact that Marle used the Echart type of energy momentum tensor while Chernikov used the Landau-Lifchitz scheme, this system and the corresponding 13-moment system II (1,1) to (1,5) are quite similar (See nomenclature comparison in App. B), more precisely the difference is only in the structure of two transport coefficients B and $\overset{+}{Z}$.

We have introduced the collision frequencies $\overset{+}{\beta}$, $\overset{+}{\lambda}$ and $\overset{+}{\alpha}$ as distinct phenomenological quantities which need not necessarily be mutually equal as it is in [I] due to the B.-G.-C. approximation of the collision term in Boltzmann's equation, the values of which could be easily estimated by the results of Chernikov and Israel. Also the study of the dispersion curves is in no way affected by the lack of internal structure of $\overset{+}{\beta}$, $\overset{+}{\lambda}$ and $\overset{+}{\alpha}$ as the dispersion curves can be plotted using reduced collision frequencies $\overset{+}{\beta}/\omega$, etc., as one of the variables.

We will first subject the system of equations (1,16) to (1,21) to an important verification, which consists in requiring that in the low temperature limit $\gamma \equiv \dfrac{mc^2}{k\mathrm{T}} \gg 1$, called simply the classical limit, the system must go over to a corresponding system approved in classical physics, which in our case is Grad's (1949) 13-moment description. To demonstrate this, we need to apply the limit $\gamma \to \infty$ to all transport coefficients (whose values are tabled in Appendix C), taking into account in the coefficients *only* $u^\alpha = (1\,;0,0,0)$, $\overset{\perp}{g}{}^{\alpha\beta} = (0\,;-1,-1,-1)$, the consequences of which are

$$\mathrm{Q}^0 = 0 \quad (\text{by } \mathrm{Q}^\alpha u_\alpha = 0)\,; \qquad \tau^{0\alpha} = 0 \quad (\text{by } \tau^{\beta\alpha}u_\alpha = 0)\,;$$
$$\mathscr{D}u_0 = 0 \quad (\text{by } u^\alpha \mathscr{D} u_\alpha = 0)\,; \qquad \partial^\beta u_0 = 0 \quad (\text{by } u^\alpha \partial^\beta u_\alpha = 0)\,;$$

we also have

$$cu^\alpha = \left(\frac{c}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \right) \to (c, \vec{u}) \qquad (\text{or } \vec{u} = cu^a = -cu_a)\,;$$
$$\vec{\mathrm{Q}} = \mathrm{Q}^a = -\mathrm{Q}_a\,; \qquad \overset{\leftrightarrow}{\tau} = \tau^a_b\,;$$
$$\overset{\perp}{\partial}{}^\alpha = (0, -\nabla), \qquad c\mathscr{D} \to \frac{\partial}{\partial t}\,;$$
$$\text{so}: \quad c\partial^\alpha u_\alpha \to \nabla\vec{u}\,, \qquad \partial^\alpha \mathrm{Q}_\alpha \to \nabla\vec{\mathrm{Q}}\,, \qquad \partial_\beta\tau^{\alpha\beta} = \overset{\perp}{\partial}_\beta\tau^{\alpha\beta} = -\nabla\overset{\leftrightarrow}{\tau}\,;$$
$$\mathrm{Trace}\,\{\overset{\leftrightarrow}{\tau}\} = 0 \qquad (\text{by } 0 = \tau^\alpha_\alpha \equiv \tau^0_0 + \tau^a_a)\,;$$
$$x^\alpha = (ct, \vec{x}) \qquad (\text{i. e. } \vec{x} = x^a = -x_a)\,. \tag{1,21a}$$

The classical limit of our system will be done in details later, in Fourier representation.

Solution by Fourier transform

In seeking a solution to the system of 14 linear partial differential equations (1,16) to (1,21), we assume the perturbation of each of the unknown functions

$$n, \quad u^\alpha, \quad \mathrm{T}, \quad \tau^{\alpha\beta}, \quad \theta, \quad \mathrm{Q}^\alpha \qquad (\alpha, \beta = 0, 1, 2, 3) \tag{1,22}$$

to have the form of propagating *plane wave*

$$\mathrm{Q} - \hat{\mathrm{Q}}_{eq.}{}^{-i\mathrm{K}^\sigma x_\sigma}. \tag{1,23}$$

This corresponds to a Fourier transform in time and space. K^σ is the wave 4-vector in the local rest frame $\mathrm{K}^\sigma = \left(\frac{\omega}{c}, \vec{k}\right)$ and the phase velocity is defined as $\mathrm{W} = \frac{\omega}{|\vec{k}|}$. $\mathrm{K}^\sigma \mathrm{K}_\sigma \leq 0$ i. e. K^σ is a space like or null vector which is the necessary condition for having $\mathrm{W} \leq c$. Invariant decomposition of the wave 4-vector into longitudinal and transverse parts with respect to the world line of the appropriate mass element is

$$\mathrm{K}^\alpha = u^\alpha \overset{/\!/}{\mathrm{K}} + \overset{\perp}{\mathrm{K}}{}^\alpha ; \qquad \overset{\perp}{\mathrm{K}}{}^\alpha = \overset{\perp}{n}{}^\alpha \overset{\perp}{\mathrm{K}}, \qquad (\overset{\perp}{n}_\alpha \overset{\perp}{n}{}^\alpha = -1). \tag{1,24}$$

Then the frequency and 3-wave vector can be expressed in the invariant form:

$$\omega = c\overset{/\!/}{\mathrm{K}} = c\mathrm{K}^\alpha u_\alpha, \qquad |\vec{k}| = \overset{\perp}{\mathrm{K}} \equiv -\overset{\perp}{n}_\alpha \overset{\perp}{g}{}^\alpha_\beta \mathrm{K}^\beta \quad \text{and} \quad \mathrm{W} = \frac{c\overset{/\!/}{\mathrm{K}}}{\overset{\perp}{\mathrm{K}}} = \frac{\omega}{\overset{\perp}{\mathrm{K}}}. \tag{1,25}$$

In order to study the polarisation of waves in a simple manner we choose a special local rest frame with x^3 pointing in the direction of propagation of the plane wave; i. e. $\overset{\perp}{n}{}^\alpha = (0\,; 0, 0, 1)$. As we wish to investigate *forced* sound waves (description of sound propagation arising e. g. from an oscillating piston) ω will always be real while the wave vector remains complex. The real part of the refractive index of wave $\mathcal{N} = \frac{c}{\mathrm{W}} = c\frac{\overset{\perp}{\mathrm{K}}}{\omega}$ corresponds to a propagation phenomenon and $\mathrm{Im}\{\mathcal{N}\}$ to the attenuation.

Inserting into the set of equation (1,16) to (1,21) and (D,2) for each of the unknown functions (1,22) a plane wave solution (1,23), one obtains $\left(\text{as } \partial^\beta \mathrm{Q} \to \frac{1}{i}\mathrm{K}^\beta \hat{\mathrm{Q}},\ c\mathscr{D}\mathrm{Q} \to \frac{1}{i}(c\overset{/\!/}{\mathrm{K}}) = \frac{\omega}{i}\hat{\mathrm{Q}},\ \overset{\perp}{\partial}{}^\beta \mathrm{Q}^{\cdot} \to \frac{1}{i}\overset{\perp}{\mathrm{K}}{}^\beta \hat{\mathrm{Q}}\right)$

$$\omega\hat{n} + n\mathrm{K}_\alpha c\hat{u}^\alpha = 0, \tag{1,26}$$

$$r\mathrm{G}\omega c\hat{u}^\alpha - k[\mathrm{T}\overset{\perp}{\mathrm{K}}{}^\alpha \hat{n} + n\overset{\perp}{\mathrm{K}}{}^\alpha \hat{\mathrm{T}}] - \overset{\perp}{\mathrm{K}}_\beta \hat{\tau}^{\alpha\beta} - \frac{1}{3}\overset{\perp}{\mathrm{K}}{}^\alpha \hat{\theta} + \frac{1}{c^2}\omega\hat{\mathrm{Q}}^\alpha = 0, \tag{1,27}$$

$$nc_v\omega\hat{\mathrm{T}} + p\mathrm{K}_\alpha c\hat{u}^\alpha + \mathrm{K}_\alpha \hat{\mathrm{Q}}^\alpha + \omega\hat{\theta} = 0, \tag{1,28}$$

$$\mathrm{B}\langle \omega\hat{\tau}^{\alpha\beta} \rangle + 2p\mathrm{G}\langle \mathrm{K}^\alpha c\hat{u}^\beta \rangle + \frac{4}{5}\overset{+}{\mathrm{D}}\langle \mathrm{K}^\alpha \hat{\mathrm{Q}}^\beta \rangle = -i\overset{+}{\beta}\hat{\tau}^{\alpha\beta}, \tag{1,29}$$

$$\mathrm{N}\omega\hat{\theta} + 3\bar{\mathrm{U}}\omega\hat{\mathrm{T}} + 2p\mathrm{G}\mathrm{K}_\gamma c\hat{u}^\gamma + 2\overset{+}{\mathrm{D}}\mathrm{K}_\gamma \hat{\mathrm{Q}}^\gamma = -i\lambda^+\hat{\theta}, \tag{1,30}$$

$$Z\omega\hat{Q}^\alpha - RTK_\gamma \hat{\tau}^{\alpha\gamma} + \frac{p}{n} c^2 U \overset{\perp}{K}{}^\alpha \hat{n} - \frac{n}{T} \varepsilon c^2 U \overset{\perp}{K}{}^\alpha \hat{T} - c^2 M \overset{\perp}{K}{}^\alpha \hat{\theta} = - i \overset{+}{\alpha} \hat{Q}^\alpha, \quad (1,31)$$

$$u_\alpha \hat{u}^\alpha = 0, \quad u_\alpha \hat{Q}^\alpha = 0, \quad u_\alpha \hat{\tau}^{\alpha\beta} = 0, \quad \hat{\tau}^{\alpha\beta} = \hat{\tau}^{\beta\alpha}, \quad \hat{\tau}^\sigma_\sigma = 0 \ \ (^4). \quad (1,32)$$

The equations (1,26) to (1,31) for a set of twenty one homogeneous equations in the unknowns (1,22) of which only fourteen are independent because of (1,32). With $\overset{\perp}{n}{}^\alpha = (0\,;0,0,1)$ i. e. $\overset{\perp}{K}{}^\alpha = (0\,;0,0,\overset{\perp}{K})$; $\overset{\perp}{K}_\alpha = (0\,;0,0,-\overset{\perp}{K})$ and $u^\alpha = (1\,;0,0,0)$ (immobile medium) $g_{\alpha\beta} = (1\,;-1,-1,-1)$ and $\overset{\perp}{g}_{\alpha\beta} = (0\,;-1,-1,-1)$ conditions $(1,32)_{1,2,3}$ requires

$$\hat{u}^0 = 0, \qquad \hat{Q}^0 = 0, \qquad \hat{\tau}^{0\alpha} = 0. \quad (1,33)$$

Further, $\langle \omega\hat{\tau}^{\alpha\beta} \rangle = \omega \langle \hat{\tau}^{\alpha\beta} \rangle = \omega\hat{\tau}^{\alpha\beta}$ and following (D,13)

$$\langle K^\alpha \hat{Q}^\beta \rangle = \frac{1}{2}(\overset{\perp}{K}{}^\alpha\hat{Q}^\beta + \overset{\perp}{K}{}^\beta\hat{Q}^\alpha) - \frac{1}{3}\overset{\perp}{g}{}^{\alpha\beta}\overset{\perp}{K}_\lambda\hat{Q}^\lambda$$

so that

$$\langle K^1\hat{Q}^1 \rangle = 0 - \frac{1}{3}(-1)(-\overset{\perp}{K})\hat{Q}^3 = -\frac{1}{3}\overset{\perp}{K}\hat{Q}^3, \qquad \langle K^1\hat{Q}^2 \rangle = 0,$$

$$\langle K^2\hat{Q}^2 \rangle = -\frac{1}{3}\overset{\perp}{K}\hat{Q}^3, \qquad \langle K^1\hat{Q}^3 \rangle = \frac{1}{2}\overset{\perp}{K}\hat{Q}^1, \quad (1,34)$$

$$\langle K^3\hat{Q}^3 \rangle = \frac{2}{3}\overset{\perp}{K}\hat{Q}^3, \qquad \langle K^2\hat{Q}^3 \rangle = \frac{1}{2}\overset{\perp}{K}\hat{Q}^2.$$

If we replace $\hat{Q}^\alpha$ by $\hat{u}^\alpha$ in (1,34) we obtain relations valid for $\langle K^\alpha \hat{u}^\beta \rangle$. Taking all those formulæ into account, equations (1,26) to (1,31) can be put in matrix form (1,35) (Due to the constraint

$$0 = \hat{\tau}^\alpha_\alpha \equiv g_{\alpha\beta}\hat{\tau}^{\alpha\beta} = -\hat{\tau}^{11} - \hat{\tau}^{22} - \hat{\tau}^{33},$$

$\hat{\tau}^{11} = -\hat{\tau}^{22} - \hat{\tau}^{33}$ is not an independent quantity and will not be considered in our system of equations).

where

$$\bar{B} = B + i\frac{\overset{+}{\beta}}{\omega}, \qquad \bar{N} = N + i\frac{\overset{+}{\lambda}}{\omega}, \qquad \bar{Z} = \overset{+}{Z} + i\frac{\overset{+}{\alpha}}{\omega}. \quad (1,36)$$

The system (1,35) admits a non-trivial solution if and only if its determinant vanishes, i. e.

$$\Delta_{14} = 0. \quad (1,37)$$

This homogeneous algebric equation of degree 14 in $\dfrac{\omega}{\overset{\perp}{K}} = W_0 . c$ is the dis-

(4) As u_α is a time-like vector, $\hat{u}^\alpha$ and $\hat{Q}^\alpha$ must be space-like and $\hat{\tau}^{\alpha\beta}$ is space-like in both indices.

$$
\left(\begin{array}{cccc|cccc|cccccc}
rG\omega & \cdot & \overset{\perp}{K} & \frac{\omega}{c^2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bar{B}\omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
pG\overset{\perp}{K} & \cdot & \bar{B}\omega & \frac{2}{5}\overset{+}{D}\overset{\perp}{K} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & R\bar{T}K & \bar{Z}\omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & rG\omega & \cdot & \overset{\perp}{K} & \frac{\omega}{c^2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \bar{B}\omega & \cdot & \cdot & \cdot & -\frac{2}{3}pG\overset{\perp}{K} & \cdot & \cdot & \cdot & -\frac{4}{15}\overset{+}{D}\overset{\perp}{K} \\
\cdot & \cdot & \cdot & \cdot & pG\overset{\perp}{K} & \cdot & \bar{B}\omega & \frac{2}{5}\overset{+}{D}\overset{\perp}{K} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & RT\overset{\perp}{K} & \bar{Z}\omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega & -n\overset{\perp}{K} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -kT\overset{\perp}{K} & rG\omega & -kn\overset{\perp}{K} & \overset{\perp}{K} & -\frac{1}{3}\overset{\perp}{K} & \frac{\omega}{c^2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -p\overset{\perp}{K} & nc_v\omega & \cdot & \omega & -\overset{\perp}{K} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{4}{3}pG\overset{\perp}{K} & \cdot & \bar{B}\omega & \cdot & \frac{8}{15}\overset{+}{D}\overset{\perp}{K} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2pG\overset{\perp}{K} & 3\bar{U}\omega & \cdot & \bar{N}\omega & = 2\overset{+}{D}\overset{\perp}{K} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{p}{n}c^2U\overset{\perp}{K} & \cdot & -\frac{n}{T}\varepsilon c^2U\overset{\perp}{K} & RT\overset{\perp}{K} & -c^2M\overset{\perp}{K} & \bar{Z}\omega
\end{array}\right)
\left(\begin{array}{c}
c\hat{u}^1 \\ \hat{\tau}^{12} \\ \hat{\tau}^{13} \\ \hat{Q}^1 \\ \hline c\hat{u}^2 \\ \hat{\tau}^{22} \\ \hat{\tau}^{23} \\ \hat{Q}^2 \\ \hline \hat{n} \\ c\hat{u}^3 \\ \hat{T} \\ \hat{\tau}^{33} \\ \hat{\theta} \\ \hat{Q}^3
\end{array}\right) = 0 \qquad (1,35)
$$

persion equation. The 14 roots of Eq. (1,37) correspond to the 14 possible particular waves.

As is evident from (1,35), Δ_{14} is equal to the product of three lower order determinants: $\Delta_{14} = \Delta_4\Delta_4\Delta_6$. Hence instead of the dispersion Eq. (1,37), we need investigate only the two much simpler equations

$$\Delta_4 = 0\,, \tag{1,38}$$

$$\Delta_6 = 0\,. \tag{1,39}$$

Evidently Eq. (1,38) corresponds to waves with the transverse polarization directed along the axes x^1 or x^2, while Eq. (1,39) corresponds to a wave with longitudinal polarization directed along x^3. It is usefull to put the equation (1,37) to the form when all the elements of determinant are dimensionless (See Appendix F).

II. TRANSVERSE WAVES

The possible phase velocities $W_0 = \frac{W}{c} = \frac{\omega}{c\overset{\perp}{K}}$ with a transverse polarisation are given using (1,38) and (1,35) (in dimensionless form, see App. F) by the equation

$$\Delta_4 \equiv \begin{vmatrix} GW_0 & \cdot & \frac{1}{\gamma} & \frac{1}{\gamma}W_0 \\ \cdot & \bar{B}W_0 & \cdot & \cdot \\ G & \cdot & \bar{B}W_0 & \frac{2}{5}\overset{+}{D} \\ \cdot & \cdot & \frac{1}{\gamma} & \bar{Z}W_0 \end{vmatrix} = 0\,. \tag{2,1}$$

This equation can be reduced to the form

$$\Delta_4 \equiv \text{const.}\,(W_0^2 - W_{0T}^2)W_0^2 = 0\,, \tag{2,2}$$

where

$$W_{0T}^2 = \frac{\left(\bar{Z} + \frac{2}{5}\overset{+}{D}\right) - \frac{1}{\gamma}}{\gamma\bar{Z}\bar{B}} = \frac{2}{5}\frac{RT}{c^2}\frac{\overset{+}{D}}{\bar{B}\bar{Z}} + \frac{RT}{c^2}\frac{1}{\bar{B}} - \frac{(RT)^2}{c^4}\frac{1}{\bar{B}\bar{Z}}\,. \tag{2,3}$$

The first two terms in this expression have the same form as the corresponding ones in Chernikov's 13-moment theory (II (4,13)), however the structure of D and Z is now different; while the new third term is very small. The result (2,3) is independent of the inclusion of θ in the system and remains unchanged if the reduction to 13-moments is made. The reduction to 11-moment description (i. e. absence of heat conduction in the problem) can be done by the limit transition $\bar{Z} \to \infty$ (see II (2,29)):

$$W_{0T}^2 = \frac{RT}{c^2} \cdot \frac{1}{\bar{B}}; \qquad (2,4)$$

this result remains unchanged in the 10-moment approach and is identical with that of the 10-moment system of Chernikov.

The wavefront speed (i. e. the signal speed) is defined as

$$V = \lim_{\omega \to \infty} W(\omega) \qquad (= W(\infty)). \qquad (2,5)$$

In the ultrarelativistic limit $\gamma \ll 1$ (i. e. $\gamma \to 0$ or equivalently $m \to 0$) one obtains for the wavefront speed of W_T by virtue of (1,36), (C,9), (C,10) and (C,17):

$$\frac{V_T^2}{c^2} \equiv \lim_{\substack{\gamma \to 0 \\ \omega \to \infty}} \frac{W_T^2}{c^2} \equiv \frac{Z_0 + \frac{2}{5}D_0 - \frac{1}{\gamma}}{\gamma B_0 Z_0} = \frac{\frac{5}{\gamma} + \frac{2}{5}\frac{5}{\gamma} - \frac{1}{\gamma}}{\gamma \frac{6}{\gamma}\frac{5}{\gamma}} = \frac{1}{5} = (0{,}4472)^2, \quad (2,6)$$

which is the same value as is obtained from the 13-moment theory II (4,15).

The classical limit $\gamma \gg 1$ (i. e. $\gamma \to \infty$ or equivalently $c^2 \to \infty$) for the transverse modes defined by (2,3) is

$${}_\infty W_\perp^2 = \lim_{\gamma \to \infty} \cdot \frac{c^2}{\gamma} \frac{\left(\bar{Z} + \frac{2}{5}\overset{+}{D}\right) - \frac{1}{\gamma}}{\bar{B}\bar{Z}} = RT \frac{\bar{Z}_\infty + \frac{2}{5}}{\bar{B}_\infty \bar{Z}_\infty} \qquad (2,7)$$

by virtue of (C,10) and definitions (3,11). We notice that the result is independent of any parameter characterising the bulk viscosity and at the same time is identical with the result one obtains from the classical 13-moment approach discussed in [II]. The same is also true of course for the wavefront speed (see II (4,14)).

III. LONGITUDINAL WAVES

A. *General case (14-moment description).* — The possible phase velocities $W_0 = \frac{W}{c} = \frac{\omega}{c\overset{\perp}{K}}$ with the longitudinal polarization are defined by (1,39), together with (1,35) (or in dimensionless form by the equation (see App. F, in which the definition (C,6), (C,12) and (C,8) was adopted):

$$\Delta_6 \equiv \begin{vmatrix} W_0 & -1 & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{\gamma} & GW_0 & -\frac{1}{\gamma} & \frac{1}{\gamma} & -\frac{1}{3\gamma} & \frac{1}{\gamma}W_0 \\ \cdot & -1 & VW_0 & \cdot & W_0 & -1 \\ \cdot & \frac{4}{3}G & \cdot & \bar{B}W_0 & \cdot & \frac{8}{15}\overset{+}{D} \\ \cdot & -2G & 3\tilde{U}W_0 & \cdot & \bar{N}W_0 & -2\overset{+}{D} \\ U & \cdot & -EU & \frac{1}{\gamma} & -M & \bar{Z}W_0 \end{vmatrix} = 0. \qquad (3,1)$$

This equation can be reduced to the form

$$\Delta_6 \equiv \text{const.}\,(W_0^2 - W_{0I}^2)(W_0^2 - W_{0II}^2)W_0^2 = 0, \qquad (3,2)$$

where

$$W_{0I,II}^2 = \frac{1}{2\tilde{A}}\{\tilde{B} \pm \sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}}\} \qquad (3,3)$$

and

$$\tilde{A} \equiv \gamma^2 G\{V\bar{B}\bar{N}\bar{Z} - 3\tilde{U}\bar{B}\bar{Z}\}, \qquad (3,4)$$

$$\begin{aligned} \tilde{B} \equiv\ & \gamma(1+V)\bar{B}\bar{N}\bar{Z} - \gamma U(E - \gamma GE - V)\bar{N}\bar{B} + \frac{4}{3}\gamma GV\bar{N}\bar{Z} \\ & - \gamma\left\{4\tilde{U} + 2G\left(1 - \frac{V}{3}\right)\right\}\bar{B}\bar{Z} \\ & - \gamma\bar{B}\{3\tilde{U}[U + \gamma GM - M] + 2G[MV - EU] + 2\gamma G\overset{+}{D}[EU - MV]\} \\ & - 4\gamma G\tilde{U}\bar{Z} - \frac{4}{3}GV\left(1 - \frac{2}{3}\gamma\overset{+}{D}\right)\bar{N} + 4G\tilde{U}\left(1 - \frac{2}{5}\gamma\overset{+}{D}\right), \end{aligned} \qquad (3,5)$$

$$\begin{aligned} \tilde{C} \equiv\ & \gamma U(1+E)\bar{N}\bar{B} + \left\{\frac{8}{15}\overset{+}{D}[\gamma U(V-E) + (V+1)] + \frac{4}{3}G[\gamma UE - 1]\right\}\bar{N} \\ & + \left\{2\gamma\overset{+}{D}\left[\frac{U}{3}(V-E) + M(V+1) - U(1+E)\right] - \gamma\tilde{U}(3M+U) \right. \\ & \left. + \frac{2}{3}\gamma G(UE - 3M)\right\}\bar{B} + \frac{4}{3}G\tilde{U}(1 - 3\gamma M) \\ & + \frac{8}{5}\gamma\overset{+}{D}\left[G(MV - UE) + \tilde{U}\left(M - U - \frac{4}{3\gamma}\right) + \frac{G}{\gamma}\left(1 - \frac{V}{3}\right)\right]. \end{aligned} \qquad (3,6)$$

By virtue of the relations (C,19) to (C,22) and some definitions from App. C the coefficients $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ can be rewritten in the form:

$$\tilde{A} \equiv \frac{P}{U}\bar{B}\bar{Z}\{V\bar{N} - 3\tilde{U}\}, \qquad (3,7)$$

$$\frac{\tilde{B}}{\gamma} \equiv \bar{B}\bar{Z}\left\{ P\bar{N} - \left[4\tilde{U} + 2G\left(1 - \frac{V}{3}\right)\right]\right\}$$
$$+ \bar{B}\{ (V + E^2)U\bar{N} - [3\tilde{U}(U + EM) + 2G(\gamma\overset{+}{D} - 1)(UE - VM)]\}$$
$$+ \frac{4}{3}G\left[\left(\bar{Z} + \frac{2}{5}\overset{+}{D}\right) - \frac{1}{\gamma}\right]\{ V\bar{N} - 3\tilde{U}\}, \tag{3,8}$$

$$\tilde{C} \equiv \bar{B}\left\{ P\bar{N} - \left[2\gamma P\overset{+}{D}\left(\frac{4}{3\gamma} - M - \frac{U}{3}\right) + \gamma\tilde{U}(3M + U) + \frac{2}{3}\gamma G(3M - EU)\right]\right\}$$
$$+ \left[\frac{8}{15}\gamma UP\overset{+}{D} + \frac{4}{3}(GP - \tilde{U})\right]\bar{N}$$
$$+ \frac{8}{5}\gamma\overset{+}{D}\left[G(MV - EU) + \tilde{U}\left(M - U - \frac{4}{3\gamma}\right) + \frac{G}{\gamma}\left(1 - \frac{V}{3}\right)\right]$$
$$+ \frac{4}{3}G\tilde{U}[1 - 3\gamma M]. \tag{3,9}$$

The zero root of eq. (3,2) has to be associated with the mass flow velocity along streamlines. Because $W_{0I}^2 \geq W_{0II}^2$ we call the I-wave a fast longitudinal wave (or sound wave) and the II-wave a slow wave (or « thermal wave »). The complex phase velocities $W_{I,II}$, depending on the wave frequency through the expression $\bar{B}(\omega)$, $\bar{Z}(\omega)$ and $\bar{N}(\omega)$ give us information on both the effective phase speed $^{+}W = \omega/\operatorname{Re}\{\overset{\perp}{K}\}$ or refraction index $\operatorname{Re}\{\mathcal{N}\}$ and the coefficient of absorption, $\operatorname{Im}\{\mathcal{N}\}\left(\mathcal{N} = \frac{c}{W}\right)$.

B. *The classical limit for the phase velocities* $W_{I,II}$ can be obtained easily by applying the limit $\gamma \gg 1$ to formula (3,3) i. e. to the coefficients $c^2\tilde{B}/\tilde{A}$ and $c^2\tilde{C}/\tilde{A}$. For that purpose, let us apply this limit to the expressions (3,7) to (3,9), using the formulæ of Appendix C. We find that

$$\frac{\tilde{A}_\infty}{c^2\gamma} \equiv \lim_{\gamma\to\infty} \frac{P}{c^2\gamma U}\bar{B}\bar{Z}\{ V\bar{N} - 3\tilde{U}\}$$
$$= \frac{5}{2}\frac{2}{5}\frac{\gamma^2}{c^2\gamma}\left\{\frac{3}{2}\bar{N} - 3\right\}\bar{B}_\infty\bar{Z}_\infty = \frac{3}{2}\frac{\bar{B}_\infty\bar{Z}_\infty}{RT}(\bar{N}_\infty - 2) \tag{3,10}$$

where

$$\bar{N}_\infty = 3 + i\frac{\lambda_\infty^+}{\omega}, \qquad \bar{B}_\infty = 1 + i\frac{\beta_\infty^+}{\omega}, \qquad \bar{Z}_\infty = 1 + i\frac{\alpha_\infty^+}{\omega} \tag{3,11}$$

by virtue of (1,36), (C,11), (C,9) and (C,17).

Furthermore,

$$\frac{\tilde{B}_\infty}{\gamma} = \bar{B}_\infty \bar{Z}_\infty \left\{ \frac{5}{2}\bar{N}_\infty - \left[4 + 2\left(1 - \frac{1}{2}\right)\right]\right\} + \bar{B}_\infty \left\{ \left(\frac{3}{2} + \gamma^2 + \frac{25}{4} + 5\gamma\right)\frac{5}{2\gamma^2}\bar{N}_\infty \right.$$
$$\left. - \left[3\left(1 + \frac{5}{\gamma}\right)\left(0 + \gamma\left[\frac{2}{3} + \frac{8}{3\gamma}\right]\right) + 2\left(\gamma\left[1 + \frac{7}{2\gamma}\right] - 1\right)\left(0 - \frac{3}{2}\left[\frac{2}{3} + \frac{8}{3\gamma}\right]\right)\right]\right\}$$
$$+ \frac{4}{3}\left[\left(\bar{Z}_\infty + \frac{2}{5}\left[1 + \frac{7}{2\gamma}\right]\right) - \frac{1}{\gamma}\right]\left\{\frac{3}{2}\bar{N}_\infty - 3\right\}$$
$$= \frac{5}{2}\bar{B}_\infty \bar{Z}_\infty(\bar{N}_\infty - 2) + \frac{5}{2}\bar{B}_\infty(\bar{N}_\infty - 2) + 2\left(\bar{Z}_\infty + \frac{2}{5}\right)(\bar{N}_\infty - 2), \quad (3,12)$$

$$\tilde{C}_\infty = \bar{B}_\infty \left\{ \frac{5}{2}\bar{N}_\infty - \left[\!\left[2\gamma\frac{5}{2}\left(\frac{4}{3\gamma} - \left[\frac{2}{3} + \frac{8}{3\gamma}\right] - 0\right) + \gamma\left(3\left[\frac{2}{3} + \frac{8}{3\gamma}\right] + 0\right) \right.\right.\right.$$
$$\left.\left.\left. + \frac{2}{3}\gamma\left(3\left[\frac{2}{3} + \frac{8}{3\gamma}\right] - \gamma\frac{5}{2\gamma^2}\right)\right]\!\right]\right\} + \left[\frac{8}{15}\gamma\frac{5}{2}\frac{1}{\gamma^2}\cdot\frac{5}{2} + \frac{4}{3}\left(\frac{5}{2} - 1\right)\right]\bar{N}_\infty$$
$$+ \frac{8}{5}\gamma\left[\left(\frac{2}{3} + \frac{8}{3\gamma}\right)\frac{3}{2} - \gamma\frac{5}{2\gamma^2} + \left(\frac{2}{3} + \frac{8}{3\gamma}\right) - 0 - \frac{4}{3\gamma} + \frac{1}{\gamma}\left(1 - \frac{1}{2}\right)\right]$$
$$+ \frac{4}{3}\left(1 - 3\gamma\left[\frac{2}{3} + \frac{8}{3\gamma}\right]\right) = \frac{5}{2}\bar{B}_\infty(\bar{N}_\infty - 2) + 2(\bar{N}_\infty - 2). \quad (3,13)$$

So the limit of the coefficients in question are

$$c^2\frac{\tilde{B}_\infty}{\tilde{A}_\infty} = \mathrm{RT}\frac{\left[\frac{5}{2}\bar{B}_\infty(\bar{Z}_\infty + 1) + 2\left(\bar{Z}_\infty + \frac{2}{5}\right)\right](\bar{N}_\infty - 2)}{\frac{3}{2}\bar{B}_\infty\bar{Z}_\infty(\bar{N}_\infty - 2)} \quad (3,14)$$

$$c^4\frac{\tilde{C}_\infty}{\tilde{A}_\infty} = (\mathrm{RT})^2\frac{\left[\frac{5}{2}\bar{B}_\infty + 2\right](\bar{N}_\infty - 2)}{\frac{3}{2}\bar{B}_\infty\bar{Z}_\infty(\bar{N}_\infty - 2)} \quad (3,15)$$

We see first that the factor $(\bar{N}_\infty - 2)$ representing the volume viscosity influence simplified itself, which means absence of volume viscosity for ideal gas in non-relativistic limit, and secondly the expression (3,14) and (3,15) are identical with the results one obtains from the classical 13-moment approach II (2,19). This together with (2,7) proves that the starting equations (1,16) to (1,21) under this same (low temperature) limit go realy over into the appropriate classical description. Therefore the classical limits for signal velocities V_I, V_{II} and for all other quantities are identical with those derived from Grad's 13-moment approach characterized by (2,7), (3,14) and (3,15), and which can be found in [II], so we do not need to take the classical limit here for every case separately.

C. *The ultrarelativistic limit for the phase velocities* $W_{I,II}$ can be found by applying the asymptotic expression valid for $\gamma \ll 1$ (or possibly $\gamma \to 0$, or equivalently $m \to 0$) (See Appendix C) to expressions (3,7) to (3,9). Doing this, with the nomenclature

$$\lim_{\gamma\to 0} \bar{B} \equiv \bar{B}_0 \equiv B_0 + i\frac{\beta_0^+}{\omega} \approx \frac{1}{\gamma}\left(6 + i\frac{\beta^*}{\omega}\right) \equiv \frac{\overset{*}{B}}{\gamma}; \qquad (\overset{*}{\beta} = \lim_{\gamma\to 0} \gamma\overset{+}{\beta}) \tag{3,16}$$

$$\bar{N}_0 \equiv N_0 + i\frac{\lambda_0^+}{\omega} \approx \frac{1}{\gamma}\left(12 + i\frac{\lambda^*}{\omega}\right) \equiv \frac{\overset{*}{N}}{\gamma}, \tag{3,17}$$

$$\bar{Z}_0 \equiv Z_0 + i\frac{\alpha_0^+}{\omega} \approx \frac{1}{\gamma}\left(5 + i\frac{\alpha}{\omega}\right) \equiv \frac{\overset{*}{Z}}{\gamma} \tag{3,18}$$

we obtain

$$\frac{\tilde{A}_0}{\gamma} = \frac{1}{\gamma}4\gamma\left\{3\frac{\overset{*}{N}}{\gamma} - 3\frac{8}{\gamma}\right\}\frac{\overset{*}{B}}{\gamma}\frac{\overset{*}{Z}}{\gamma} = 12\frac{\overset{*}{B}\overset{*}{Z}}{\gamma^3}(\overset{*}{N} - 8), \tag{3,19}$$

$$\begin{aligned}\frac{\tilde{B}_0}{\gamma} &= \frac{\overset{*}{B}\overset{*}{Z}}{\gamma^2}\left\{4\frac{\overset{*}{N}}{\gamma} - \left[4\frac{8}{\gamma} + \frac{8}{\gamma}(1-1)\right]\right\} \\ &\quad + \frac{\overset{*}{B}}{\gamma}\left\{(3+9)\frac{1}{\gamma}\frac{\overset{*}{N}}{\gamma} - \left[3\cdot\frac{8}{\gamma}\left(\frac{1}{\gamma} + \frac{7}{\gamma}\right) + \frac{8}{\gamma}(5-1)\left(\frac{3}{\gamma} - \frac{7}{\gamma}\right)\right]\right\} \\ &\quad + \frac{4}{3}\cdot\frac{4}{\gamma}\left[\frac{\overset{*}{Z}}{\gamma} + \frac{2}{5}\frac{5}{\gamma} - \frac{1}{\gamma}\right]\left\{3\frac{\overset{*}{N}}{\gamma} - 3\frac{8}{\gamma}\right\} \\ &= 4\frac{\overset{*}{B}\overset{*}{Z}}{\gamma^3}(\overset{*}{N} - 8) + 12\frac{\overset{*}{B}}{\gamma^3}\left(\overset{*}{N} - \frac{16}{3}\right) + \frac{16}{\gamma^3}(\overset{*}{Z} + 1)(\overset{*}{N} - 8).\end{aligned} \tag{3,20}$$

$$\begin{aligned}\frac{\tilde{C}_0}{\gamma} &= \frac{1}{\gamma}\frac{\overset{*}{B}}{\gamma}\left\{4\frac{\overset{*}{N}}{\gamma} - \left[2\gamma 4\frac{5}{\gamma}\left(\frac{4}{3\gamma} - \frac{7}{3\gamma} - \frac{1}{3\gamma}\right) + \gamma\frac{8}{\gamma}\left(\frac{7}{\gamma} + \frac{1}{\gamma}\right) + \frac{2}{3}\gamma\frac{4}{\gamma}\left(\frac{7}{\gamma} - \frac{3}{\gamma}\right)\right]\right\} \\ &\quad + \frac{1}{\gamma}\left[\frac{8}{15}\gamma\frac{1}{\gamma}4\frac{5}{\gamma} + \frac{4}{3}\left(\frac{4}{\gamma}4 - \frac{8}{\gamma}\right)\right]\frac{\overset{*}{N}}{\gamma} \\ &\quad + \frac{1}{\gamma}\frac{8}{5}\gamma\frac{5}{\gamma}\left[\frac{4}{\gamma}\left(\frac{7}{\gamma} - \frac{3}{\gamma}\right) + \frac{8}{\gamma}\left(\frac{7}{3\gamma} - \frac{1}{\gamma} - \frac{4}{3\gamma}\right) + \frac{4}{\gamma^2}(1-1)\right] + \frac{1}{\gamma}\frac{4}{3}\frac{4}{\gamma}\frac{8}{\gamma}[1-7] \\ &= \frac{4}{\gamma^3}\overset{*}{B}\left(\overset{*}{N} - \frac{16}{3}\right) + \frac{64}{3}\frac{1}{\gamma^3}(\overset{*}{N} - 6).\end{aligned} \tag{3,21}$$

The limiting values of coefficients in (3,3) are then

$$\frac{\tilde{B}_0}{\tilde{A}_0} = \frac{[\overset{*}{B}\overset{*}{Z} + 4(\overset{*}{Z} + 1)](\overset{*}{N} - 8) + 3\overset{*}{B}\left(\overset{*}{N} - \frac{2}{3}8\right)}{3\overset{*}{B}\overset{*}{Z}(\overset{*}{N} - 8)}, \tag{3,22}$$

$$\frac{\tilde{C}_0}{\tilde{A}_0} = \frac{\overset{*}{B}\left(\overset{*}{N} - \frac{2}{3}8\right) + \frac{16}{3}(\overset{*}{N} - 6)}{3\overset{*}{B}\overset{*}{Z}(\overset{*}{N} - 8)}. \tag{3,23}$$

From (3,22) and (3,23) we see that the factor $(\overset{*}{N} - 8)$ does not simplify itself as did the factor $(\bar{N}_\infty - 2)$ in (3,14) and (3,15). One might expect that such cancelation would be based on Israel's result that the volume viscosity coefficient disappear in the ultrarelativistic limit in the same manner as in classical limit. However the situation in the limit $\gamma \to 0$ is different and the detailed explanation is given in Section V.

The signal speeds of longitudinal modes can be calculated from (3,3) by applying the limit $\omega \to \infty$ (see (2,5)) to coefficients (3,22) and (3,23) which by virtue of (3,16) to (3,18) (i. e. $\overset{*}{B} \to 6$, $\overset{*}{N} \to 12$, $\overset{*}{Z} \to 5$) leads to

$$\frac{V_I}{c} = \sqrt{\frac{3}{5}} = 0{,}7746\,, \qquad \frac{V_{II}}{c} = \sqrt{\frac{1}{3}} = 0{,}5773 \left(\frac{\tilde{B}_0}{\tilde{A}_0} = \frac{14}{15}\ \frac{\tilde{C}_0}{\tilde{A}_0} = \frac{1}{5}\right). \tag{3,24}$$

These values are slightly higher but in fair agreement with the results coming out of the competitive description of Chernikov (compare II (2,26) and the table in App. E).

In order to better understand the results obtained up to now, let us turn to some special cases.

D. *The 13-moment description* can be deduced easily from the preceeding general results by taking the limit $\bar{N} \to \infty$, which is after all connected to an annulation of the coefficient of bulk viscosity, and elimination of θ from the system (compare II (2,29)). Doing this with (3,7) to (3,9) one obtains

$$\tilde{A} \equiv \frac{P}{U} V\bar{B}\bar{Z} \underset{\gamma \to 0}{\longrightarrow} \frac{12}{\gamma}\overset{*}{B}\overset{*}{Z} \tag{3,25}$$

$$\frac{\tilde{B}}{\gamma} \equiv P\bar{B}\bar{Z} + (V + E^2)U\bar{B} + \frac{4}{3}GV\left[\bar{Z} + \frac{2}{5}\overset{+}{D} - \frac{1}{\gamma}\right] \underset{\gamma \to 0}{\longrightarrow} \frac{4}{\gamma^2}\{\overset{*}{B}\overset{*}{Z} + 3\overset{*}{B} + 4[\overset{*}{Z} + 1]\} \tag{3,26}$$

$$\tilde{C} \equiv P\bar{B} + \left[\frac{8}{15}\gamma UP\overset{+}{D} + \frac{4}{3}(GP - \tilde{U})\right] \underset{\gamma \to 0}{\longrightarrow} \frac{4}{\gamma}\left\{\overset{*}{B} + \frac{16}{3}\right\}. \tag{3,27}$$

The signal speeds in this case, making use of (3,25) to (3,27) together with $\overset{*}{B} = 6$, $\overset{*}{N} = 12$ and $\overset{*}{Z} = 5$ comes out from (3,3) as follows:

$$\frac{V_I}{c} = 0{,}7646\,, \qquad \frac{V_{II}}{c} = 0{,}4641\,; \qquad \left(\frac{\tilde{B}_0}{\tilde{A}_0} = \frac{4}{5}\ \frac{\tilde{C}_0}{\tilde{A}_0} = \frac{17}{135}\right). \tag{3,28}$$

These values are lower than predicted by 14-moment description but are very close to those of the Chernikov description II (2,26) (compare Appendix E).

E. *The 11-moment description* can be deduced from the proceeding general results by taking the limit $\bar{Z} \to \infty$, which means elimination of heat conduction effects from the description. Doing this with (3,7) to (3,9) one obtain: $\tilde{c}/\tilde{A} = 0$ and therefore

$$W_{0I}^2 = \frac{\tilde{B}}{\tilde{A}}$$

$$= \frac{\gamma\bar{B}\left\{P\bar{N} - \left[4\tilde{U} + 2G\left(1 - \frac{V}{3}\right)\right]\right\} + \frac{4}{3}\gamma G(V\bar{N} - 3\tilde{U})}{\frac{P}{U}\bar{B}(V\bar{N} - 3\tilde{U})}, \qquad W_{0II} = 0. \quad (3,29)$$

We see that the absence of heat conduction has as a consequence the disappearance of the slow longitudinal propagation mode II (« thermal dissipation wave »). The phase velocity of fast longitudinal mode I (« true acoustical wave ») may be put in the form (see (C,7) and $c^2 = \gamma RT$)

$$W_I^2 = \frac{4}{3}RT\frac{1}{\bar{B}} + \frac{RT}{G}\frac{\left\{P\bar{N} - \left[4\tilde{U} + 2G\left(1 - \frac{V}{3}\right)\right]\right\}}{(V\bar{N} - 3\tilde{U})}. \qquad (3,30)$$

While the first term describes the dispersion dependence due to the shear viscosity which is the same as in the 10-moment approach the second term expresses the dispersion dependence due to the volume viscosity.

The ultrarelativistic limit of the fast longitudinal propagation mode (3,29) is

$$W_{0I}^2 = \frac{W_I^2}{c^2} = \frac{(\overset{*}{B} + 4)(\overset{*}{N} - 8)}{3\overset{*}{B}(\overset{*}{N} - 8)} = \frac{4}{3}\frac{1}{\overset{*}{B}} + \frac{1}{3}, \qquad \text{and therefore} \qquad \frac{V_I}{c} = \sqrt{\frac{5}{9}} = 0{,}7454 \quad (3,31)$$

which its wavefront speed. In this case the influence of bulk stress does not appear due to the cancelation of factor $(\overset{*}{N} - 8)$.

F. *The 10-moment description* can be obtained from (3,30) in the limit $\bar{N} \to \infty$, which means that the shear viscosity is the only kind of dissipation which is covered by this description. Doing this one obtains

$$W_I^2 = \frac{4}{3}RT\frac{1}{\bar{B}} + \frac{P}{V}\frac{RT}{G} \underset{\gamma \to 0}{\longrightarrow} \frac{4}{3}\frac{c^2}{\overset{*}{B}} + \frac{c^2}{3}; \qquad \text{therefore} \qquad \frac{V_I}{c} = \sqrt{\frac{5}{9}}, \quad (3,32)$$

i. e. in the ultrarelativistic limit $\gamma \to 0$ the same results as from 11-moment approximation (3,31).

G. *The 6-moment description* can be deduced from (3,30) by a subsequent limit $\bar{B} \to \infty$, which means that the shear viscosity effects are excluded too and the only kind of dissipation taken in consideration is bulk viscosity. In that case we obtain,

$$W_I^2 = \frac{P}{V}\left(\frac{RT}{G}\right)\frac{\bar{N} - \left[4\tilde{U} + 2G\left(1 - \frac{V}{3}\right)\right]\frac{1}{P}}{\bar{N} - 3\tilde{U}\frac{1}{V}} \underset{\gamma\to 0}{\longrightarrow} \frac{c^2}{3}\frac{(\overset{*}{N} - 8)}{(\overset{*}{N} - 8)} = V_I^2, \quad (3,33)$$

which means that in the ultrarelativistic case the dispersion dissapears and the phase velocity of this propagation mode is identical with that of addabatic sound (Synge (1957)).

H. *Mode for heat conduction only without a sound wave* (or Cattaneo-Vernotte mode; see II (2,47)) while shear and bulk viscosity are disregarded i. e. $\overset{+}{\lambda} \to \infty$, $\overset{+}{\beta} \to \infty$ but $\infty > \overset{+}{\alpha} > 0$ is retained which strictly from the point of view of kinetic theory is not possible because of the existence of unique relations between $\overset{+}{\lambda}$, $\overset{+}{\beta}$ and $\overset{+}{\alpha}$; but which is used sometimes in a phenomenological approach. Therefore we will call this a « heuristic case ». In this case we obtain

$$W_{0I}^2 = 0, \quad W_{0II}^2 = \frac{EU}{V\bar{Z}} \underset{\gamma\to 0}{\longrightarrow} \frac{1}{\overset{*}{Z}} \quad \text{and therefore} \quad \frac{V_{II}}{c} = \sqrt{\frac{1}{5}} = 0{,}4472 \quad (3,34)$$

this value is higher than the one obtained in II (2,53) which reads $\sqrt{\frac{1}{7}}$.

All the considerations made here and in preceding chapter were based on the least restrictive supposition (*i*) concerning the expression (1,11), which we consider to be the most suitable. If we accept supposition (*ii*) then $\bar{N} = \infty$ (by 1,36) and therefore there is no bulk viscosity at all and the 14, 11, and 6-moment descriptions reduce themselves to 13, 10, and 5-moment descriptions respectively. And finally, accepting supposition (*iii*) means that $\bar{N} = N$, i. e. bulk stress itself produces neither dispersion nor an increase in entropy, but only modifies the frequency-independent factors in expressions for phase velocities of the propagation modes.

IV. THE HYPERBOLICITY OF THE THEORY

As the characteristic equation of the system under consideration (1,16) to (1,21) is in our case (1,37) in the limit $\omega \to \infty$ (see (2,2) and (3,2))

$$\lim_{\omega\to\infty} \Delta_{14} \equiv \lim_{\omega\to\infty} \text{const.}\,(W^2 - W_T^2)^2 (W^2 - W_I^2)(W^2 - W_{II}^2)W^6 = 0. \quad (4,1)$$

Direct computation performed for $\gamma \gg 1$ and $\gamma \ll 1$ has shown that the complete characteristic polynomial (4,1) has neither *complex*, nor imaginary nor *infinite* roots for $\omega \to \infty$. Thus we can conclude that the system is hyperbolic (at least for $\gamma \gg 1$ and $\gamma \ll 1$).

For arbitrary γ this can be seen from the fact (which can be shown graphically) that the wavefront speed of each eigenmode start from zero or some finite value and then monotomically increase if $\frac{1}{\gamma}$ ranges from 0 to ∞ and reaches a maximum for $\frac{1}{\gamma} = \infty$ which is the ultrarelativistic signal speed. Therefore signal speeds always remain less than c as they must, according to the fundamental postulate of relativistic causality.

V. IRREVERSIBILITY OF GAS EXPANSION

Let me discuss briefly the question of entropy production due to the expansion of a relativistic gas. This question is closely related to the fact (see Israel-Vardalas (1970) and Anderson (1970)) that a relativistic gas generally possesses a bulk viscosity (in contrast to the classical case where this phenomenon is essentially found only in polyatomic gases) but coefficient of bulk viscosity vanish in both the classical and the ultrarelativistic limit (i. e. for zero rest mass particles).

The first theorem which is expressed by the equation (1,18) can be rewritten as (by the use of (1,16)),

$$n\left\{ c_v c\mathscr{D}\mathrm{T} + pc\mathscr{D}\left(\frac{1}{n}\right)\right\} + c\mathscr{D}\theta + \partial_\alpha \mathrm{Q}^\alpha = 0\,. \tag{5,1}$$

Here the terms $c\mathscr{D}$... means the changement of state variables ... along the world line and $\partial_\alpha \mathrm{Q}^\alpha$ is the passive source of heat energy due to dissipation. In (5,1) is missing the heating effect due to shear viscosity because it is only the non-linear effect. The non-linearized form of first law of course includs such terms as:

$$n\left\{ c_v c\mathscr{D}\mathrm{T}' + \left(p + \frac{4}{3}\theta\right)c\mathscr{D}\left(\frac{1}{n}\right)\right\} + c\mathscr{D}\theta + \partial_\alpha \mathrm{Q}^\alpha + \frac{1}{c^2}\, cu_\beta c\mathscr{D}\mathrm{Q}^\beta + \tau^{\alpha\beta}\partial_\alpha c u_\beta = 0\,. \tag{5,2}$$

The explicit form of the second theorem (H-theorem), is not derived in [I] but it is not difficult to predict the form of the expression for production of entropy in 14-moment theory viz (compare III (3,18))

$$\sigma \sim [\lambda^+\theta^2 + |\,\mathrm{C}_1\,|\,\overset{+}{\beta}\tau^{\alpha\beta}\tau_{\alpha\beta} - |\,\mathrm{C}_2\,|\,\overset{+}{\alpha}\mathrm{Q}^\alpha \mathrm{Q}_\alpha] \geq 0\,. \qquad (\sigma_\theta \sim \lambda^+\theta^2) \tag{5,3}$$

Now, we will try to show how the above-mentioned limiting values of bulk viscosity coefficients are connected to the present results.

As we do not know the temperature dependence of $\lambda^+(\gamma)$ we do not know whether the bulk viscosity coefficient limits

$$\Lambda_\infty = \lim_{\gamma\to\infty} \Lambda(\gamma) \qquad \text{and} \qquad \Lambda_0 = \lim_{\gamma\to 0} \Lambda(\gamma) \qquad \left(\text{where } \Lambda \equiv \frac{2pG}{\lambda^+}\right) \quad (5,5)$$

dissapear or not; therefore the question arises whether without this information we are able to deduce from our equations the absence of bulk stress (i. e. $\theta = 0$) in the limits $\gamma \to \infty$ and $\gamma \to 0$ (Compare the text following Eq. (5,8)). As we will see, it is, except some cases, possible.

Combining together the equations (1,18) and (1,20) in their classical limit leads to the relation

$$\frac{\partial\theta}{\partial t} + \lambda^+\theta = 0$$

whose Fourier picture is $\frac{\omega}{i}(\bar{\mathrm{N}}_\infty - 2)\hat{\theta} = 0$ (see (1,23) and (3,11)). (5,4)

As in general $(\bar{\mathrm{N}}_\infty - 2) \neq 0$ we obtain $\hat{\theta} = 0$, meaning that the only oscillating (wave) solution for θ is $\theta = 0$. Of course this reduces the 14-moment equations to the 13-moment equations of Grad. On the other hand it is visible from Eqs. (3,14) and (3,15) that the 14th quantity has no influence on the eigenmodes. The contribution of θ to the entropy production is zero ($\sigma_\theta = 0$ by (5,3)). Doing the same manipulations with Equations (1,18) and (1,20) but in their ultrarelativistic limit we obtain the relation

$$4c\mathscr{D}\theta + \lambda^*\theta + 2\partial_\alpha Q^\alpha = 0$$

which Fourier picture is $\omega(\overset{*}{\mathrm{N}} - 8)\hat{\theta} - 2\overset{\perp}{\mathrm{K}}\hat{\mathrm{Q}}^3 = 0.$ (5,6)

Now the situation is different. In this case the condition necessary to conclude that $\hat{\theta} = 0$ is the independent vanishing of $\overset{\perp}{\mathrm{K}}\hat{\mathrm{Q}}^3$. So supposing

$$\partial_\alpha Q^\alpha = 0 \qquad \text{or} \qquad \overset{\perp}{\mathrm{K}}\hat{\mathrm{Q}}^3 = 0\,, \qquad (5,7)$$

we may conclude $\theta = 0$, and then we obtain no contribution from gas expansion to entropy production as above. It is evident from our Equations (3,22) and (3,23), which are not subjected to the extracondition (5,7), that there exists an influence of bulk viscosity on dispersion; and as $\theta \neq 0$ there is also a positive contribution to σ. So in that case the gas's expansion is irreversible even in the ultrarelativistic limit. Here we are speaking of the contribution to the entropy production due to bulk stress. However, when the condition (5,7) is fulfilled which occures in the 11- and 6-moment approximations (lower order approximation) the same expansion is qualified as reversible (for $\gamma \to 0$) which is reflected by the formulæ (3,31) and (3,33). However the condition (5,7) is sufficient for the linearized theory considered here; for non-linear theory more restrictive conditions will be necessary to ensure that the gas expansion in ultrarelativistic limit is reversible, as is evident also from (5,2).

According to the 14-moment parabolic theory (Israel-Vardalas, 1970) the transport equation for bulk stresses is

$$\Lambda(\gamma)\partial^{\alpha}cu_{\alpha} = -\theta . \qquad (\Lambda \text{ bulk viscosity coefficient}) \qquad (5,8)$$

For the case $\gamma \to \infty$ was derived $\Lambda_{\infty} = 0$ so one may immediately conclude from (5,8) that $\theta = 0$ and $\sigma_{\theta} = 0$ (For this case we came to the same conclusion). For the case $\gamma \to 0$, $\Lambda_0 = 0$ holds and therefore from (5,8) follows $\theta = 0$ and $\sigma_{\theta} = 0$; saying that expansion motion alone of the gas is reversible. But this is true only on the level of this parabolic approximation which is based on transport equation of type (5,8) while the hyperbolic theory what we have considered uses the more precise equation (1,20) instead of (5,8). Our prediction $\sigma_{\theta} \neq 0$ was obtained as a consequence of hyperbolic 14-moment theory which is more general than the above-mentioned parabolic 14-moment description. Thus the parabolic theory prediction must be considered as less exact, as it was obtained from the hyperbolic one by application of the normal solution method, which adds one more approximation on the top. This is another reason in favor of hyperbolic theory.

VI. CONCLUSION

The propagation eigenmodes in an ideal relativistic gas with dissipation due to bulk stress shear stress and heat conduction have been studied on the basis of the linearized relativistic 14-moment theory of Marle.

It was found that the following non-trivial modes exist: two for longitudinal waves (I-mode (fast) and II-mode (slow)) and one for transverse waves (T-mode) whose multiplicity is obvious from (4,1). Thus the number of non trivial modes and their multiplicity is the same as in the Chernikov 13-moment theory. However there are some differencies in dispersion dependence mainly due to the bulk stresses. Once the bulk stress is neglected (13- and 10-moment descriptions) the two theories lead to almost identical results as far as can be judged from the dispersion formulæ and from the signal speeds of the modes (which are tabled in Appendix E) and which in Marle's theory are either identical to or slightly higher than in Chernikov's theory. The wavefront speeds are important because they represent the speed of propagation of characteristic surfaces on which a discontinuity of some quantities can occur in other words weak shock-waves fronts.

Stewart (1971) found for the signal speed of an « arbitrary non-adiabatic perturbation » using the 14-moment theory (perhaps identical with that of [I]) the value $c\sqrt{\frac{3}{5}} = 0{,}7746c$. Although this value coincides with our $(3{,}24)_1$, its derivation is not clear. At least two points make me spectical: first, the « derivation » was made under the supposition $u^\alpha = \text{const.}$ which admiting only locally rigid motion eliminates mechanical sound vibration. Therefore instead of dealing with 6 independent quantities for description of longitudinal waves n, u^3, T, τ^{33}, θ and Q^3 he deals only with 5 quantities. Second: In his derivation temperature dependent functions $K_n(\gamma)$ are involved which at the end cancel out giving a temperature independent result. But we know that the wave front speed must be temperature dependent and for $\gamma \gg 1$ go over to the corresponding classical result. The ultra-relativistic signal speed $c\sqrt{3/5}$ of mode 1 is obtained only in the limit $\gamma \to 0$ from a signal speed of arbitrary perturbation which depends on temperature.

Some partial results about wave propagation based on parabolic theory were published e. g. by Guichellaar *et al.* (1972), (1973), Weinberg (1971) and de Groot (1973 *a*), (1973 *b*). The parabolic theories of course lead to an infinite signal speed for all propagation modes. More precisely, due to frequency range restrictions ω are allowed only up to frequencies which are much less than the collision frequencies, so the wavefronts and their neighbourhoods are byond the reach of validity of this theory. *The Chapman-Enskog method can at best be a time asymptotic theory; this together with its incapability to describe also the propagation of discontinuity surface (shock wave) and therefore its incompatibility with relativistic causality seems to be enough reasons to reject it as a valid approach for this problem.*

The entropy production due to the expansion of the gas which is described by the hyperbolic 14-moment equations in the ultrarelativistic limit could be estimated positive in general, which is in contradiction with the estimate that follows from Israel's (1970) less general description.

All the 12-coefficients in transport equations (1,19) to (1,21) were expressed in terms of state variables and collision frequencies for the general case, in particular in the ultrarelativistic limit as well as in the classical limit. In the classical limit these coefficients coincide with those of Grad. In the corresponding parabolic theory one needs only 3 instead of 12 transport coefficients, however this is of course to the detriment of the quality of this approximation already mentioned.

Following Marle's linearized moment equations the particular solutions (eigenmodes) with non-vanishing bulk stress ($\theta \neq 0$) do not seems to be the only possibility (see text after (1,11)).

ACKNOWLEDGMENT

The authors dedicates this paper to the memory of tragically deceased M. LAPORTE who first studied and tabeled the coefficients ψ, χ, and φ. For helpful exchanges of information the author is grateful to Dr. MARLE. The financial support from the National Research Council of Canada for this research programme is gratefully acknowledged.

APPENDIX A

SOME CORRECTIONS WHICH WAS DONE BY DR. C. MARLE (1975)

The text furnished to me follows: coefficient φ defined by I (11,17) reads correctly:

$$\frac{3}{4}\varphi = \frac{5\mathrm{K}_4 - 3\dfrac{\mathrm{K}_2\mathrm{K}_3}{\mathrm{K}_1} - \left(\dfrac{\mathrm{K}_3}{\mathrm{K}_2} - \dfrac{\mathrm{K}_2}{\mathrm{K}_1}\right)(2\gamma\mathrm{K}_4 + 3\mathrm{K}_3)}{3\gamma\mathrm{K}_4 + 2\mathrm{K}_3 - 3\gamma\dfrac{\mathrm{K}_2\mathrm{K}_3}{\mathrm{K}_1} - 8\gamma\mathrm{K}_3\left(\dfrac{\mathrm{K}_3}{\mathrm{K}_2} - \dfrac{\mathrm{K}_2}{\mathrm{K}_1}\right)}. \tag{A,1}$$

It is possible to prove that the coefficients ψ, φ and χ have the asymptotical form as follows:

coefficient	for $\gamma \to \infty$	for $\gamma \to 0$
ψ	$\frac{1}{6}\left(1 + \frac{7}{2\gamma}\right)$	$\simeq \frac{1}{\gamma}$
$-\chi$	$\frac{1}{5}\left(1 + \frac{7}{2\gamma}\right)$	$\simeq \frac{1}{\gamma}$
φ	$1 + \frac{7}{2\gamma}$	$\simeq \frac{4}{\gamma}$

(A,2)

This results are confirmed by the numerical calcul which was made by M. LaPorte. Further in equation I (11,24) the last but one term on RHS reads:

$$-\frac{\mathrm{Q}^\alpha}{c^4\tau} = \dots + \frac{2}{c^3}\{[\overset{\perp}{g}{}^{\alpha\gamma}u_\lambda - 5\overset{\perp}{g}{}^{\alpha}_{\lambda}u^\gamma]\nabla_\gamma(\chi\mathrm{Q}^\lambda) - 6\chi(\mathrm{Q}^\alpha\nabla_\gamma u^\gamma + \mathrm{Q}^\gamma\nabla_\gamma u^\alpha)\} - \dots \tag{A,3}$$

and I (11,25) reads correctly:

$$\begin{aligned}\frac{\tau^{\alpha\beta}}{c^3\tau} = &- 2\overset{\perp}{g}{}^{\alpha\beta}u^\gamma\left[r\partial_\gamma\left(\frac{\mathrm{K}_3}{\gamma\mathrm{K}_2}\right) + \frac{1}{c^2}\partial_\gamma(\varphi\theta)\right] - 2\frac{\varphi\theta}{c^2}\overset{\perp}{g}{}^{\alpha\beta}\nabla_\gamma u^\gamma \\ &+ \left(\frac{r\mathrm{K}_3}{\gamma\mathrm{K}_2} + \frac{\varphi\theta}{c^2}\right)\left[-\frac{2}{3}\overset{\perp}{g}{}^{\alpha\beta}\nabla_\gamma u^\gamma - \overset{\perp}{g}{}^{\alpha\lambda}\overset{\perp}{g}{}^{\beta\mu}(\nabla_\lambda u_\mu + \nabla_\mu u_\lambda)\right] \\ &+ \frac{2}{c^3}\left\{\left[\overset{\perp}{g}{}^{\beta\gamma}\overset{\perp}{g}{}^{\alpha}_{\lambda} + \overset{\perp}{g}{}^{\alpha\gamma}\overset{\perp}{g}{}^{\beta}_{\lambda} + \frac{2}{3}\overset{\perp}{g}{}^{\alpha\beta}(4g^\gamma_\lambda + 5u^\gamma u_\lambda)\right]\nabla_\gamma(\chi\mathrm{Q}^\lambda) - 6\chi u^\gamma(\mathrm{Q}^\alpha\nabla_\gamma u^\beta + \mathrm{Q}^\beta\nabla_\gamma u^\alpha)\right\} \\ &+ \frac{2}{c^2}\{-3\mathscr{D}(\psi\tau^{\alpha\beta}) + \psi[-2\overset{\perp}{g}{}^{\alpha\beta}\tau^{\lambda\gamma} - 3(u^\alpha\tau^{\lambda\beta} + u^\beta\tau^{\lambda\alpha})u^\gamma]\nabla_\gamma u_\lambda \\ &- 3\psi(\tau^{\alpha\beta}\nabla_\gamma u^\gamma + \tau^{\beta\gamma}\nabla_\gamma u^\alpha + \tau^{\alpha\gamma}\nabla_\gamma u^\beta)\}.\end{aligned} \tag{A,4}$$

APPENDIX B

TABLE OF NOMENCLATURE COMPARISON

(The sign $\sim$ means the equivalent notion but not necessarily identical).

Quantity	Dimens.	Present text	Marle [I]	Chernikov & [II], [III]
temperature	grad	T	T	θ
pressure	$\frac{\text{erg}}{\text{cm}^3}$	$p = nk\mathrm{T} = r\mathrm{RT}$	$p = r\mathrm{RT}$	$p = nk\theta$
Boltzmann constant	$\frac{\text{erg}}{\text{grad}}$	$k = m\mathrm{R}$	k	k
$\frac{1}{\text{effective temperature}}$	1	$\gamma = \frac{mc^2}{k\mathrm{T}}$	ζ	γ
rest mass density	$\frac{g}{\text{cm}^3}$	$r = nm$	r	nm
internal energy per particle	$\frac{\text{erg}}{\text{partic.}}$	$m\varepsilon_{\mathrm{M}} = \varepsilon - mc^2$	$m\varepsilon$	$\varepsilon - mc^2 = (mc^2\mathrm{G} - k\mathrm{T}) - mc^2$ $= k\mathrm{T}[\gamma\mathrm{G} - 1 - \gamma]$
energy density	$\frac{\text{erg}}{\text{cm}^3}$	$\rho c^2 = rc^2 + r\varepsilon_{\mathrm{M}} = n\varepsilon$	$rc^2 + r\varepsilon$	$n\varepsilon$
enthalpy density	$\frac{\text{erg}}{\text{cm}^3}$	$ri_{\mathrm{M}} = r\varepsilon_{\mathrm{M}} + p$ I (12,3) $= (\rho - r)c^2 + p$	$ri = r\varepsilon + p$ $= (f - 1)rc^2$	$n(w - mc^2) = n\varepsilon + p - nmc^2$ $= nmc^2(\mathrm{G} - 1)$
energy flux (heat)	$\frac{\text{erg}}{\text{cm}^2\,\text{sec}}$	Q^α	Q^α	$\sim - mc^3 v^\alpha$
metrical tensor	1	$\beta, \alpha = 0, 1, 2, 3$ $g_{\alpha\beta} \to (1 ; -1, -1, -1)$	$g_{\alpha\beta} \to (1 ; -1, -1 -1)$	$g_{\alpha\beta} \to (1 ; -1, -1, -1)$
stream 4-velocity	1	$u^\alpha \quad (u^\alpha u_\alpha = 1)$	$u^\alpha \quad (u^\alpha u_\alpha = 1)$	$u^\alpha \quad (u^\alpha u_\alpha = 1)$
projection tensor	1	$\overset{\perp}{g}{}^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$	$g^{\alpha\beta} - u^\alpha u^\beta$	$- h^{\alpha\beta} \quad h^\alpha_\alpha = - 3$
pressure tensor	$\frac{\text{erg}}{\text{cm}^3}$	$p\overset{\perp}{g}{}^{\alpha\beta} + \theta^{\alpha\beta}$; where	*I* (3,32) $\theta^{\alpha\beta} = \tau^{\alpha\beta} + \frac{1}{3}\theta\overset{\perp}{g}{}^{\alpha\beta}$	$\sim p\overset{\perp}{g}{}^{\alpha\beta} - cr^{\alpha\beta} = - ph^{\alpha\beta} - cr^{\alpha\beta}$
shear stress tensor	$\frac{\text{erg}}{\text{cm}^3}$	$\tau^{\alpha\beta}$	$\tau^{\alpha\beta}$	$\sim - cr^{\alpha\beta}$
bulk stress (compressional stress)	$\frac{\text{erg}}{\text{cm}^3}$	$\theta = \theta^\alpha_\alpha$	θ	—
energy momentum tensor	$\frac{\text{erg}}{\text{cm}^3}$	$c^2\mathrm{T}^{\alpha\beta}$	$c^2\mathrm{T}^{\alpha\beta}$ I (3,29)	$\sim c\mathrm{A}^{\alpha\beta}$ III (1,22)
space-like 4-gradient (transverse derivative)	$\frac{1}{\text{cm}}$	$\overset{\perp}{\partial}_\alpha = \overset{\perp}{g}{}^\beta_\alpha \partial_\beta$	$g^\beta_\alpha \partial_\beta - \mathrm{U}_\alpha \mathrm{U}^\beta \partial_\beta$	$\overset{\perp}{\partial}_\alpha = - h^\beta_\alpha \partial_\beta$
longitudinal derivative (convective)	$\frac{1}{\text{cm}}$	$\mathscr{D} = u^\alpha \partial_\alpha$	$\mathrm{U}^\alpha \partial_\alpha$	$\mathscr{D}$

Quantity	Dimens.	Present text	Marle [I]	Chernikov & [II], [III]
4-gradient (decomposition)	$\frac{1}{\text{cm}}$	$\partial_\alpha = u_\alpha \mathscr{D} + \overset{+}{\partial}_\alpha$	$U_\alpha(U^\beta \partial_\beta) + (g^{\alpha\beta} - U^\alpha U^\beta)\partial_\beta$	$u_\alpha \mathscr{D} - h^\beta_\alpha \partial_\beta$
heat capacity coefficients	$\frac{\text{erg}}{\text{grad}}$	$c_v = \frac{\partial(m\varepsilon_M)}{\partial T} = \frac{\partial(\varepsilon - mc^2)}{\partial T}$		$c_v \equiv \frac{\partial \varepsilon}{\partial T} = -k(\gamma^2 G' + 1)$ $= k(\gamma^2 GU - 1)$
	$\frac{\text{erg}}{\text{grad}}$	$c_p = \frac{\partial(mi_M)}{\partial T} = \frac{\partial(w - mc^2)}{\partial T}$		$c_p \equiv \frac{\partial w}{\partial T} = -k\gamma^2 G'$ $= k\gamma^2 GU = c_v + k$
	$\frac{\text{erg}}{\text{g.cm}}$	$\mathscr{D}\varepsilon_M = \frac{c_v}{m} \mathscr{D}T$	$\mathscr{D}\varepsilon$	$\frac{c_v}{m} \mathscr{D}\theta$
index of fluid	1	$G = \frac{w}{mc^2} = \frac{1}{mc^2}\left(\varepsilon + \frac{p}{n}\right)$	$f = \frac{1}{r}\left(\rho + \frac{p}{c^2}\right)$ I (12,3)	$G = \frac{K_3}{K_2}$

Some formulæ from [II] *concerning the Chernikov 13-moment description*

The homologues of coefficients (3,7) to (3,9), or more precisely, of coefficients (3,25) to (3,27) are II (1,32) to II (1,34) which can be rewritten in terms of terminology of Appendix C as follows

$$\tilde{A} \equiv \frac{PV}{U} \bar{B} \overset{\times}{\bar{Z}} \underset{\gamma \to 0}{\longrightarrow} \frac{12}{\gamma^2} \overset{*}{B} \overset{*}{Z}, \tag{B,1}$$

$$\frac{\tilde{B}}{\gamma} \equiv P\bar{B}\overset{\times}{\bar{Z}} + GU(V + E^2)\bar{B} + \frac{4}{3} GV \left[\overset{\times}{\bar{Z}} + \frac{2}{5} \overset{\times}{D} \right] \underset{\gamma \to 0}{\longrightarrow} \frac{4}{\gamma^3} \left[\overset{*}{B}\overset{*}{Z} + 12\overset{*}{B} + 4\overset{*}{Z} + \frac{2}{5} 56 \right] \ (^5), \tag{B,2}$$

$$\tilde{C} \equiv G \left\{ P\bar{B} + \left[\frac{8}{15} \frac{\gamma UP}{G} \overset{\times}{D} + \frac{4}{3} (GP - \tilde{U} + PU) \right] \right\} \underset{\gamma \to 0}{\longrightarrow} \frac{16}{\gamma^2} \left[\overset{*}{B} + \frac{88}{15} \right]. \tag{B,3}$$

where the coefficients $\overset{\times}{\bar{Z}}$ and $\overset{\times}{D}$ are defined by II (1,30), II (D,7) and II (D,4) (so that $\lim\limits_{\gamma \to 0} \overset{\times}{D} = \frac{14}{\gamma^2}$; $\lim\limits_{\substack{\gamma \to 0 \\ \omega \to \infty}} \overset{\times}{\bar{Z}} = \frac{28}{\gamma^2}$) are different from $\bar{Z}$ and $\overset{+}{D}$ utilized here in the main text.

(5) In II (2,23) please correct the misprint (replace 61 by 56).

APPENDIX C

TABLE OF COEFFICIENTS AND SOME OTHER QUANTITIES,

their definitions and classical and ultrarelativistic limit

Quantity (coefficient)	Its definition	$f(\gamma, G)$ or $f(\gamma, G, U)$	Classical limit $\gamma \gg 1$	Ultrarelativistic $\gamma \ll 1$ limit	E_q.
$\frac{5}{3}\frac{c^2}{c_0^2} = \gamma$	$\frac{mc^2}{kT}$	γ	$\gamma \to \infty$	$\gamma \to 0$	(C,1)
$\frac{w}{mc^2} = G$	$\frac{K_3}{K_2}$	G	$1 + \frac{5}{2\gamma} \to 1$	$\frac{4}{\gamma}$	(C,2)
G'	$\frac{dG}{d\gamma}$	$-\frac{5G}{\gamma} - 1 + G^2 = -GU$	$-\frac{5}{2\gamma^2} \to 0$	$-\frac{4}{\gamma^2}$	(C,3)
$\frac{c_p T}{w}\left(\frac{kT}{mc^2}\right) = U$	$-\frac{G'}{G}$	$\frac{5}{\gamma} + \frac{1}{G} - G = U$	$\left(1 - \frac{5}{2\gamma}\right)\frac{5}{2\gamma^2} \to 0$	$\frac{1}{\gamma}$	(C,4)
$\frac{c_p T}{kT} = P$	$-\gamma^2 G'$	$5\gamma G + \gamma^2(1 - G^2) = \gamma^2 GU$	$\frac{5}{2}$	4	(C,5)
$\frac{c_v T}{kT} = V$	$-(\gamma^2 G' + 1)$	$= (\gamma^2 GU - 1)$	$\frac{3}{2}$	3	(C,6)
$\frac{w}{kT} = E + 1$	γG	γG	$\gamma + \frac{5}{2}$	4	(C,7)
$\frac{\varepsilon}{kT} = E$	$\gamma G - 1$	$\gamma G - 1$	$\gamma + \frac{3}{2}$	3	(C,8)
$6\psi \equiv B$	$\frac{K_4}{K_3}$ I (11,17)	$\frac{6}{\gamma} + \frac{1}{G} = \frac{1}{\gamma} + U + G$	$1 + \frac{7}{2\gamma} \to 1$	$\frac{6}{\gamma}$	(C,9)
$-5\chi \equiv \overset{+}{D}$	$\frac{-5}{2\gamma G'}\left(\frac{K_4}{K_2} - G^2\right)$	$= 5\frac{U + \frac{1}{\gamma}}{2\gamma U}$	$1 + \frac{7}{2\gamma} \to 1$	$\frac{5}{\gamma}$	(C,10)
$\varphi \equiv \frac{N}{3}$	(A,1)	—	$1 + \frac{7}{2\gamma} \to 1$	$\frac{4}{\gamma}$	(C,11)
$\frac{w + c_p T}{mc^2} = \tilde{U}$	$\frac{\bar{U}}{nk} = \frac{c^2}{R}\frac{\partial}{\partial T}\left(\frac{G}{\gamma}\right)$	$= G(\gamma U + 1)$	$1 + \frac{5}{\gamma} \to 1$	$\frac{8}{\gamma}$	(C,12)
$\frac{K_4}{K_3} - \frac{1}{\gamma}$		$\frac{1}{G}\left(\frac{K_4}{K_2} - \frac{G}{\gamma}\right) = U + G$	$1 + \frac{5}{2\gamma} \to 1$	$\frac{5}{\gamma}$	(C,13)
$\frac{n}{T}\varepsilon c^2 U = c^2\bar{\bar{U}}$	$c^2[\bar{U} - nk(U + G)]$	$= nkc^2 U(\gamma G - 1)$	$\frac{5}{2}pR$	$3nk\frac{c^2}{\gamma} = 3pR$	(C,14)
$\frac{p}{n}c^2 U = -mc^2\bar{\bar{G}}$	$kTc^2\left[\left(\frac{K_4}{K_3} - \frac{1}{\gamma}\right) - G\right]$	$= kTc^2 U$	$\frac{5}{2}kTRT\frac{1}{\gamma} \to 0$	$kT\frac{c^2}{\gamma}$	(C,15)
M	$\varphi - \frac{1}{3}\left(\frac{K_4}{K_3} - \frac{1}{\gamma}\right)$	$= \varphi - \frac{U + G}{3}$	$\frac{2}{3} + \frac{8}{3\gamma} \to \frac{2}{3}$	$\frac{7}{3}\frac{1}{\gamma}$	(C,16)
$-\bar{\bar{\chi}} = \overset{+}{Z}$	$-\left[10\chi + \left(\frac{K_4}{K_3} - \frac{1}{\gamma}\right)\right]$	$= -[10\chi + U + G]$	$1 + \frac{9}{2\gamma} \to 1$	$\frac{5}{\gamma}$	(C,17)
$\bar{\bar{\psi}}$	$6\psi - \left(\frac{K_4}{K_3} - \frac{1}{\gamma}\right)$	$\frac{1}{\gamma}$	$\frac{1}{\gamma} \to 0$	$\frac{1}{\gamma}$	(C,18)

Some useful relations which can be found between the tabled quantities.

$$V + 1 = P \qquad \text{by (C,5) and (C,6)} \tag{C,19}$$

$$E + 1 = \frac{P}{\gamma U} \qquad \text{by (C,7) and (C,5)} \tag{C,20}$$

$$V - E = P - \gamma G = P - \frac{P}{\gamma U} \qquad \text{by (C,19) and (C,20)} \tag{C,21}$$

$$\gamma UE - 1 = P - \frac{\tilde{U}}{G} \qquad \text{by (C,20) and (C,12)} \tag{C,22}$$

$$\tilde{U} = G + \frac{P}{\gamma} \qquad \text{by (C,5) and (C,12)} \tag{C,23}$$

APPENDIX D

SOME AUXILIARY FORMULAE AND DEFINITIONS
which are necessary to derive the linearized moment equations

First from the conditions (see I (3,13), I (3,33), etc.)

$$u^\alpha u_\alpha = 1\,, \qquad u_\alpha Q^\alpha = 0\,, \qquad u_\alpha \tau^{\alpha\beta} = 0\,, \qquad u_\alpha \theta^{\alpha\beta} = 0\,, \tag{D,1}$$

by differentiation and linearization one obtains (for example $u_\alpha Q^\alpha = 0 \Rightarrow Q^\alpha \partial^\lambda u_\alpha + u_\alpha \partial^\lambda Q^\alpha = 0$ but the term $Q^\alpha \partial^\lambda u_\alpha$ has to be neglected as non-linear one)

$$u_\alpha \partial^\lambda u_\alpha = 0\,, \qquad u_\alpha \partial^\lambda Q^\alpha = 0\,, \qquad u_\alpha \partial^\lambda \tau^{\alpha\beta} = 0\,, \qquad u_\alpha \partial^\lambda \theta^{\alpha\beta} = 0\,. \tag{D,2}$$

Evidently from the same reason also holds

$$u_\alpha \mathscr{D} u^\alpha = 0\,, \qquad u_\alpha \mathscr{D} Q^\alpha = 0\,, \qquad u_\alpha \mathscr{D} \tau^{\alpha\beta} = 0\,, \qquad u_\alpha \mathscr{D} \theta^{\alpha\beta} = 0\,. \tag{D,3}$$

Further

$$\overset{\perp}{g}{}^{\alpha\beta} u_\beta \equiv g^{\alpha\beta} u_\beta - u^\alpha u^\beta u_\beta = 0 \quad \text{by (D,1)}_1\,; \qquad \overset{\perp}{g}{}^\alpha_\mu \overset{\perp}{g}{}^{\beta\mu} = \overset{\perp}{g}{}^{\beta\alpha}\,; \qquad \overset{\perp}{g}{}^\sigma_\sigma = 3\,, \tag{D,4}$$

$$\overset{\perp}{g}{}^\alpha_\lambda \mathscr{D} Q^\lambda \equiv g^\alpha_\lambda \mathscr{D} Q^\lambda - u^\alpha u_\lambda \mathscr{D} Q^\lambda = \mathscr{D} Q^\alpha \quad \text{by (D,3)}_2\,; \quad \text{also} \quad \overset{\perp}{\partial}_\lambda Q^\lambda = \partial_\lambda Q^\lambda\,, \tag{D,5}$$

$$\overset{\perp}{g}{}^\sigma_\beta \partial_\alpha \overset{\perp}{g}{}^{\alpha\beta} \equiv (\delta^\sigma_\beta - u^\sigma u_\beta)(-u^\beta \partial_\alpha u^\alpha - u^\alpha \partial_\alpha u^\beta) = -\mathscr{D} u^\sigma \quad \text{by (D,3)}_1 \tag{D,6}$$

$$\overset{\perp}{g}{}^{\alpha\lambda} \partial_\sigma u_\lambda = g^{\alpha\lambda} \partial_\sigma u_\lambda = \partial_\sigma u^\alpha \quad \text{by (D,2)}_1 \tag{D,7}$$

$$\overset{\perp}{g}{}^\beta_\lambda \overset{\perp}{\partial}{}^\alpha u^\lambda = \overset{\perp}{\partial}{}^\alpha u^\beta \quad \text{by (D,2)}_1 \tag{D,8}$$

$$\overset{\perp}{\partial}{}^\beta (\overset{\perp}{g}{}^\alpha_\lambda Q^\lambda) \equiv \overset{\perp}{\partial}{}^\beta (\delta^\alpha_\lambda Q^\lambda - u^\alpha u_\lambda Q^\lambda) = \overset{\perp}{\partial}{}^\beta Q^\alpha\,, \quad \text{by (D,1)}_2\,; \tag{D,9}$$

$$\overset{\perp}{g}{}^\sigma_\beta \partial_\alpha \theta^{\alpha\beta} \equiv (\delta^\sigma_\beta - u^\sigma u_\beta)\left(\partial_\alpha \tau^{\alpha\beta} + \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \partial_\alpha \theta + \frac{\theta}{3} \partial_\alpha \overset{\perp}{g}{}^{\beta\alpha}\right) = \partial_\alpha \tau^{\alpha\beta} + \frac{1}{3} \overset{\perp}{\partial}{}^\sigma \theta\,. \tag{D,10}$$

In (D,10) besides the definition $\theta^{\alpha\beta} \equiv \tau^{\alpha\beta} + \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \theta$, (D,4) has been used and the non-linear term $\frac{1}{3} \theta \partial_\alpha \overset{\perp}{g}{}^{\beta\alpha}$ has been dropped.

$$r\mathscr{D}\left(\frac{G}{\gamma}\right) \equiv r \frac{d}{d\gamma}\left(\frac{G}{\gamma}\right)\left(-\frac{\gamma}{T} \mathscr{D} T\right) = \frac{\bar{U}}{c^2} \mathscr{D} T \qquad \text{by (C,12).} \tag{D,11}$$

Relation

$$c^2 \mathscr{D} \rho = -n\varepsilon \partial_\alpha u^\alpha + n c_v \mathscr{D} T \tag{D,12}$$

follows from the definition of $c^2 \rho = n\varepsilon$, and c_v (App. B) and from the use of the eq. of continuity $\partial_\alpha (n u^\alpha) = 0$.

We introduce the definition of symmetrized traceless tensor

$$\langle D_{\mu\nu} \rangle \equiv \frac{1}{2} \overset{\perp}{g}{}^\alpha_\mu \overset{\perp}{g}{}^\beta_\nu (D_{\alpha\beta} + D_{\beta\alpha}) - \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \overset{\perp}{g}_{\mu\nu} D_{\alpha\beta}\,; \qquad \langle D^\mu_\mu \rangle = 0 \qquad \text{by (D,4)}_{2,3}. \tag{D,13}$$

Following the definition (D,13) we have

$$\langle \partial^\alpha u^\beta \rangle = \frac{1}{2} \overset{\perp}{g}{}^\alpha_\sigma \overset{\perp}{g}{}^\beta_\lambda (\partial^\sigma u^\lambda + \partial^\lambda u^\sigma) - \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \partial_\gamma u^\gamma$$
$$= \frac{1}{2} (\overset{\perp}{g}{}^\beta_\lambda \overset{\perp}{\partial}{}^\alpha u^\lambda + \overset{\perp}{g}{}^\alpha_\sigma \overset{\perp}{\partial}{}^\beta u^\sigma) - \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \partial_\gamma u^\gamma = \frac{1}{2} (\overset{\perp}{\partial}{}^\alpha u^\beta + \overset{\perp}{\partial}{}^\beta u^\alpha) - \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \overset{\perp}{\partial}_\gamma u^\gamma , \quad \text{(D,14)}$$

because $\overset{\perp}{\partial}{}^\alpha \equiv \overset{\perp}{g}{}^\alpha_\sigma \partial^\sigma$, (D,7) and (D,8). A similar formula holds for $\langle \partial^\alpha Q^\beta \rangle$ . . . (D,14*a*). Also holds

$$\langle \overset{\perp}{g}{}^{\alpha\beta} \rangle \equiv \overset{\perp}{g}{}^\alpha_\sigma \overset{\perp}{g}{}^\beta_\lambda \overset{\perp}{g}{}^{\sigma\lambda} - \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \overset{\perp}{g}_{\sigma\lambda} \overset{\perp}{g}{}^{\sigma\lambda} = \overset{\perp}{g}{}^\alpha_\sigma \overset{\perp}{g}{}^{\beta\sigma} - \frac{1}{3} \overset{\perp}{g}{}^{\alpha\beta} \overset{\perp}{g}{}^\sigma_\sigma = 0 \quad \text{by } (D,4)_{2,3} . \quad \text{(D,15)}$$

APPENDIX E

COMPARISON TABLE OF THE ULTRARELATIVISTIC FRONTWAVE SPEEDS

of propagation modes according to different moment approaches.
(Cases designated as « heuristic » are not allowed
by the kinetic approach.)

Number of moments	Marle's theory (present text)	Propagation modes: longitudinal $V_{I/C}$ (fast)	Propagation modes: longitudinal $V_{II/C}$ (slow)	Propagation modes: transverse $V_{T/C}$	Chernikov's theory [II]
14	$\overset{*}{N} = 12$, $\overset{*}{B} = 6$, $\overset{*}{Z} = 5$	0,775	0,577	0,447	–
		–	–	–	
13	$\overset{*}{N} \to \infty$	0,765	0,464	0,447	$\overset{*}{B} = 6$
		0,762	0,403	0,447	$\overset{*}{Z} = 28$
11	$\overset{*}{Z} \to \infty$	0,745	0	0,408	–
		–	–	–	
10	$\overset{*}{N} \to \infty$	0,745	0	0,408	$\overset{*}{Z} \to \infty$
	$\overset{*}{Z} \to \infty$	0,745	0	0,408	
9 heuristic (heat and bulk stress)	$\overset{*}{B} \to \infty$	0,577	0,577	0	–
		–	–	–	
8 heuristic (heat)	$\overset{*}{N} \to \infty$	0,577	0,447	0	$\overset{*}{B} \to \infty$
	$\overset{*}{B} \to \infty$	0,577	0,378	0	
6 (bulk stress)	$\overset{*}{B} \to \infty$	0,577	0	0	–
	$\overset{*}{Z} \to \infty$	–	–	–	
5 (adiabatic)	$\overset{*}{N} \to \infty$, $\overset{*}{B} \to \infty$, $\overset{*}{Z} \to \infty$	0,577	0	0	$\overset{*}{B} \to \infty$
		0,577	0	0	$\overset{*}{Z} \to \infty$
3 heuristic Cattaneo-Vernotte (heat-no sound)	$\overset{*}{N} \to \infty$, $\overset{*}{B} \to \infty$	0	0,447	0	$\overset{*}{B} \to \infty$
		0	0,378	0	

APPENDIX F

THE DISPERSION DETERMINANT IN A FORM INVOLVING ONLY DIMENSIONLESS ELEMENTS

The determinant (1,37) and (1,35) was transformed by multiplication of lines by factors indicated on the right-hand side, and of columns by factors indicated under the determinant. Doing this, some abbreviations given in App. C were included $\left(W_0 = \frac{W}{c} = \frac{\omega}{c\overset{\perp}{K}}\right)$.

$p\hat{u}^1$	$\hat{\tau}^{12}$	$\hat{\tau}^{13}$	$\frac{1}{c}\hat{Q}^1$	$p\hat{u}^2$	$\hat{\tau}^{22}$	$\hat{\tau}^{23}$	$\frac{1}{c}\hat{Q}^2$	$\frac{p}{n}\hat{n}$	$p\hat{u}^3$	$kn\hat{T}$	$\hat{\tau}^{33}$	$\hat{\theta}$	$\frac{1}{c}\hat{Q}^3$	← variables
GW_0	·	$\frac{1}{\gamma}$	$\frac{1}{\gamma}W_0$	·	·	·	·	·	·	·	·	·	·	$\frac{1}{\gamma}$
·	$\bar{B}W_0$	·	·	·	·	·	·	·	·	·	·	·	·	$\frac{1}{c}$
G	·	$\bar{B}W_0$	$\frac{2}{3}\overset{+}{D}$	·	·	·	·	·	·	·	·	·	·	$\frac{1}{c}$
·	·	$\frac{1}{\gamma}$	$\bar{Z}W_0$	·	·	·	·	·	·	·	·	·	·	$\frac{1}{\gamma RT} = \frac{1}{c^2}$
·	·	·	·	GW_0	·	$\frac{1}{\gamma}$	$\frac{1}{\gamma}W_0$	·	·	·	·	·	·	$\frac{1}{\gamma}$
·	·	·	·	·	$\bar{B}W_0$	·	·	·	$-\frac{2}{3}G$	·	·	·	$-\frac{4}{15}\overset{+}{D}$	$\frac{1}{c}$
·	·	·	·	G	·	$\bar{B}W_0$	$\frac{2}{5}\overset{+}{D}$	·	·	·	·	·	·	$\frac{1}{c}$
·	·	·	·	·	·	$\frac{1}{\gamma}$	$\bar{Z}W_0$	·	·	·	·	·	·	$\frac{1}{c^2}$
·	·	·	·	·	·	·	·	W_0	-1	·	·	·	·	$\frac{kT}{c}$
·	·	·	·	·	·	·	·	$-\frac{1}{\gamma}$	GW_0	$-\frac{1}{\gamma}$	$\frac{1}{\gamma}$	$-\frac{1}{3\gamma}$	$\frac{1}{\gamma}W_0$	$\frac{1}{\gamma}$
·	·	·	·	·	·	·	·	·	-1	VW_0	·	W_0	-1	$\frac{1}{c}$
·	·	·	·	·	·	·	·	·	$\frac{4}{3}G$	·	$\bar{B}W_0$	·	$\frac{8}{15}\overset{+}{D}$	$\frac{1}{c}$
·	·	·	·	·	·	·	·	·	$-2G$	$3\tilde{U}W_0$	·	$\bar{N}W_0$	$-2\overset{+}{D}$	$\frac{1}{c}$
·	·	·	·	·	·	·	·	U	·	$-EU$	$\frac{1}{\gamma}$	$-M$	$\bar{Z}W_0$	$\frac{1}{c^2}$
$\frac{c}{p}$	1	1	c	$\frac{c}{p}$	1	1	c	$\frac{n}{p}$	$\frac{c}{p}$	$\frac{1}{kn}$	1	1	c	↑ factors ←

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(Manuscrit reçu le 30 juin 1975)