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Section A :
Physique théorique.

Unbounded derivations of von Neumann algebras

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ABSTRACT. — We define the notion of an unbounded derivation of a von Neumann algebra and discuss various necessary and sufficient conditions that ensure that the derivation is the generator of a weakly continuous one-parameter group of *-automorphisms. Some of these criteria are of a general nature, while some apply in the case that the derivation is implemented by a self-adjoint operator H in a representation with a cyclic and separating vector Ω such that $H\Omega = 0$.

RÉSUMÉ. — Nous définissons la notion d'une dérivation non bornée d'une algèbre de von Neumann et nous discutons des différentes conditions nécessaires et suffisantes, qui assurent que la dérivation est le générateur d'un groupe à un paramètre faiblement continu de *-automorphismes. Certains de ces critères sont de nature générale, tandis que d'autres ne sont valables que dans le cas où la dérivation est implantée par un opérateur self-adjoint H dans une représentation avec un vecteur cyclique et séparateur Ω tel que $H\Omega = 0$.

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1. INTRODUCTION

In two previous papers [1] [2] we analyzed the general structure of unbounded derivations of C*-algebras. In the present paper this analysis is extended to W*-algebras with special emphasis on spatial derivations.

In simple models of statistical mechanics, e. g. quantum spin systems, the dynamics is almost directly defined in terms of an unbounded derivation. The time development of the system is defined whenever these derivations generate a one-parameter group of automorphisms of a suitable algebra of observables. There appear to be at least two levels at which this is possible. Either the derivations generate automorphisms of a C*-algebra $\mathfrak{A}$ and hence lead to a unique description of the time-development of every state ω over $\mathfrak{A}$, or, the derivations generate automorphisms of the W*-algebras associated with special classes of states over $\mathfrak{A}$. In this latter case it is even possible that the derivations are only defined at the W* level. An illustration of the first possibility is given by quantum spin systems with « short range » interactions [3] and the second possibility is illustrated by the same systems but with « long range » interactions [4] [5]. An example where the dynamical derivation is only specifiable at the W*-algebra level is provided by the non-interacting Bose gas [6] [7].

The analysis of the present paper is aimed at characterizing structural properties of automorphisms and derivations which occur at the W* level.

2. DERIVATIONS OF VON NEUMANN ALGEBRAS

A derivation δ of a von Neumann algebra $\mathfrak{M}$ is defined to be a linear mapping from a weakly dense *-subalgebra $D(\delta) \subseteq \mathfrak{M}$ into $\mathfrak{M}$ such that the identity element* $\mathbb{1}$ of $\mathfrak{M}$ is contained in the domain $D(\delta)$ of δ and, further,

$$\delta(AB) = \delta(A)B + A\delta(B) \quad , \quad A, B \in D(\delta)$$

A derivation is called symmetric if

$$\delta(A^*) = \delta(A)^*$$

The principal interest of symmetric derivations is that they arise as infinitesimal generators of weakly continuous one parameter groups of *-automorphisms of $\mathfrak{M}$. If $\mathfrak{M}_*$ denotes the predual of $\mathfrak{M}$, i. e. the linear span of the normal states over $\mathfrak{M}$, [8], then a one-parameter group $t \to \tau_t$ of *-automorphisms of $\mathfrak{M}$ is said to be weakly continuous if

$$t \to \varphi(\tau_t(A))$$

(*) Note that if δ is σ-weakly closed then the assumption $\mathbb{1} \in D(\delta)$ may be replaced by the condition that $D(\delta)$ contains a positive invertible element. A slight adaptation of the analysis of [2], Section 2, establishes that these assumptions are equivalent.

is continuous for all $A \in \mathfrak{M}$ and $\varphi \in \mathfrak{M}_*$. For such a group define $D(\delta)$ as the set of $A \in \mathfrak{M}$ such that there exists a $B \in \mathfrak{M}$ with the property that

$$\varphi(B) = \lim_{t \to 0} \varphi(\tau_t(A) - A)/t$$

for all $\varphi \in \mathfrak{M}_*$. For $A \in D(\delta)$ define $\delta(A) = B$, i. e.

$$\varphi(\delta(A)) = \lim_{t \to 0} \varphi(\tau_t(A) - A)/t \quad , \quad \varphi \in \mathfrak{M}_*$$

Using the easily derived formula

$$\tau_t(A) - A = \int_0^t ds \tau_s(\delta(A)) \quad , \quad A \in D(\delta),\, t \in \mathbb{R}$$

and the isometric property of *-automorphisms one checks that it is sufficient to assume that φ is a vector state in the above definitions because the vector states span a norm dense subset of $\mathfrak{M}_*$ [8].

Elements of $\mathfrak{M}$ may be regularized in the following manner

$$A_f = \int dt \tau_t(A) f(t) \quad , \quad f \in L^1(\mathbb{R}),\, A \in \mathfrak{M}$$

because the integrals make sense as σ-weak integrals.
This follows because $\mathfrak{M}$ is the dual of the Banach space $\mathfrak{M}_*$ [8] [9] and the dual action τ_t^T of τ_t on $\mathfrak{M}_*$ is weakly, thus strongly continuous [10]. For $A \in \mathfrak{M}$ given one may thus form the elements

$$A_\alpha = \frac{1}{\sqrt{\pi}} \int dt \tau_{\alpha t}(A) e^{-t^2} \quad , \quad \alpha > 0 .$$

It is straightforward to deduce that $A_\alpha \in D(\delta^n)$ for all n and that

$$\sum_n \frac{t^n}{n!} \| \delta^n(A_\alpha) \| < +\infty$$

for all $t > 0$, i. e. A_α is analytic (entire) for δ. Since

$$w\text{-}\lim_{\alpha \to 0} A_\alpha = A ,$$

δ has a weakly dense set of analytic (entire) elements, and, in particular, $D(\delta)$ is weakly dense. A little computation then shows that δ is a symmetric derivation in the sense defined above.

A derivation δ is said to be σ-weakly closed if its graph is a σ-weakly closed subspace of $\mathfrak{M} \oplus \mathfrak{M}$. A derivation is said to be a generator if it is the infinitesimal generator of a weakly continuous group of *-automorphisms in the sense described in the foregoing.

In Theorems 1 and 2 we give some necessary and sufficient conditions for a symmetric derivation to be a generator. The necessity of some of these conditions has already been derived in [*11*].

THEOREM 1. — *Let δ denote a symmetric derivation of a von Neumann algebra $\mathfrak{M}$. The following conditions are equivalent*

1. *δ is the infinitesimal generator of a weakly continuous one-parameter group of *-automorphisms of $\mathfrak{M}$.*

2. *δ is σ-weakly closed,* $(\alpha\delta + 1)(D(\delta)) = \mathfrak{M}$ *for all* $\alpha \in \mathbb{R}\backslash\{0\}$ *and*

either a) $\quad \|(\alpha\delta + 1)(A)\| \geqslant \|A\| \quad , \quad \alpha \in \mathbb{R}, \quad A \in D(\delta)$

or b) $\quad (\alpha\delta + 1)(A) \geqslant 0$ implies $A \geqslant 0, \quad \alpha \in \mathbb{R}, \quad A \in D(\delta)$

Proof. — 1 $\Rightarrow$ 2. Introduce the mappings $R_\alpha : \mathfrak{M} \to \mathfrak{M}$ by

$$R_\alpha(A) = \int_0^\infty dt e^{-t}\tau_{-\alpha t}(A) \quad , \quad \alpha \in \mathbb{R}\backslash\{0\}, \quad A \in \mathfrak{M}$$

The adjoint mappings $R_\alpha^T : \mathfrak{M}_* \to \mathfrak{M}_*$ given by

$$R_\alpha^T(\varphi) = \int_0^\infty dt e^{-t}\tau_{-\alpha t}^T(\varphi) \quad , \quad \varphi \in \mathfrak{M}_*$$

are well defined and bounded. Thus the R_α are σ-weakly continuous. Proceeding as in the Hille-Yosida theory of one-parameter groups on Banach space one verifies that

$$R_\alpha : \mathfrak{M} \to D(\delta)$$
$$(\alpha\delta + 1)(R_\alpha(A)) = A \quad , \quad A \in \mathfrak{M}$$
$$R_\alpha((\alpha\delta + 1)(A)) = A \quad , \quad A \in D(\delta)$$

Hence the $\alpha\delta + 1$ are invertible with inverses R_α. Since the R_α are σ-weakly continuous, δ is σ-weakly closed. Further, as R_α is everywhere defined, $(\alpha\delta + 1)(D(\delta)) = \mathfrak{M}$. Finally one has

$$\|R_\alpha(A)\| \leqslant \int_0^\infty dt e^{-t}\|\tau_{-\alpha t}(A)\| = \|A\|$$

Thus estimate 2 *a* is valid. Condition 2 *b* follows by noting that the integral representation of R_α establishes that $A \geqslant 0$ implies $R_\alpha(A) \geqslant 0$.

2 $\Rightarrow$ 1. Note first that $(\alpha\delta + 1)(D(\delta)) = \mathfrak{M}$ and condition 2 *b* immediately imply 2 *a* by Lemma 2 of [2]. Hence we adopt condition 2 *a* and assume that δ is σ-weakly closed, the resolvent $(\alpha\delta + 1)^{-1}$ is everywhere defined, and $\|(\alpha\delta + 1)^{-1}\| \leqslant 1$.

Since δ is σ-weakly closed, it is the adjoint of a densely defined norm closed operator δ^T on $\mathfrak{M}_*$. As

$$(\alpha\delta^T + 1)^{-1} = (\alpha\delta + 1)^{-1T}$$

one deduces that

$$\| (\alpha\delta^{\mathrm{T}} + 1)^{-1} \| \leqslant 1$$

Hence, by the Hille-Yosida theory, δ^{T} is the infinitesimal generator of a strongly continuous one-parameter group τ_t^{T} of isometries of $\mathfrak{M}_*$. By duality, τ_t^{T} defines a σ-weakly continuous group τ_t of isometries of $\mathfrak{M}$. For $\mathrm{A} \in \mathfrak{M}$ and $\varphi \in \mathfrak{M}_*$ one has

$$\begin{aligned}\varphi\left(\int_0^\infty dte^{-t}\tau_{-\alpha t}(\mathrm{A})\right) &= \int_0^\infty dte^{-t}\varphi(\tau_{-\alpha t}(\mathrm{A}))\\ &= \left(\int_0^\infty dte^{-t}\tau^{\mathrm{T}}_{-\alpha t}(\varphi)\right)(\mathrm{A})\\ &= ((\alpha\delta^{\mathrm{T}} + 1)^{-1}\varphi)(\mathrm{A})\\ &= \varphi((\alpha\delta + 1)^{-1}(\mathrm{A}))\end{aligned}$$

Thus

$$(\alpha\delta + 1)^{-1}(\mathrm{A}) = \int_0^\infty dte^{-t}\tau_{-\alpha t}(\mathrm{A})$$

Hence δ is the infinitesimal generator of the group τ. It remains to show that each τ_t is a *-automorphism. This follows from the computation

$$\begin{aligned}\frac{d}{dt}(\tau_{-t}(\tau_t(\mathrm{A})\tau_t(\mathrm{B}))) &= \tau_{-t}(-\,\delta(\tau_t(\mathrm{A})\tau_t(\mathrm{B})) + \delta(\tau_t(\mathrm{A}))\tau_t(\mathrm{B}) + \tau_t(\mathrm{A})\delta(\tau_t(\mathrm{B})))\\ &= 0\end{aligned}$$

which is valid for $\mathrm{A}, \mathrm{B} \in \mathrm{D}(\delta)$. This implies

$$\tau_t(\mathrm{AB}) = \tau_t(\mathrm{A})\tau_t(\mathrm{B}) \quad , \quad \mathrm{A}, \mathrm{B} \in \mathrm{D}(\delta)\,.$$

Since $\tau_t = (\tau_t^{\mathrm{T}})^*$, τ_t is σ-weakly continuous, so this last relation extends to all of $\mathfrak{M}$. The symmetry of δ also implies that $\tau_t(\mathrm{A}^*) = \tau_t(\mathrm{A})^*$ by similar reasoning.

COROLLARY 1. — *If δ is a symmetric derivation of a von Neumann algebra $\mathfrak{M}$ then δ is a generator if, and only if, $(\alpha\delta + 1)^{-1}$ exists and is a positive normal mapping for all $\alpha \in \mathbb{R}$.*

Proof. — This result follows from Theorem 1, the definition of positive normal maps, and the observation that the σ-weak continuity of $(\alpha\delta + 1)^{-1}$ was established in the proof of Theorem 1.

THEOREM 2. — *Let δ be a symmetric derivation of a von Neumann algebra $\mathfrak{M}$ and let $\mathfrak{M}_\alpha$ denote the elements of $\mathfrak{M}$ which are analytic for δ. The following conditions are equivalent*

1. *δ is the infinitesimal generator of a weakly continuous one-parameter group of *-automorphisms of $\mathfrak{M}$.*

2. *δ is σ-weakly closed, δ is the σ-weak closure of its restriction to $\mathfrak{M}_a$, and*

either a) $\quad \| (\alpha\delta + 1)(A) \| \geqslant \| A \| \quad , \quad A \in \mathfrak{M}_a , \quad \alpha \in \mathbb{R} .$

or b) $\quad (\alpha\delta + 1)(A) \geqslant 0 \quad \text{implies} \quad A \geqslant 0 , \quad A \in \mathfrak{M}_a , \quad \alpha \in \mathbb{R} .$

Proof. — 1 $\Rightarrow$ 2. If δ is a generator it follows from Theorem 1 that δ is σ-weakly closed and satisfies conditions 2 *a* and 2 *b*. From the comments at the beginning of this section it follows that $\mathfrak{M}_a$ is σ-weakly dense in $\mathfrak{M}$. From the relation

$$\delta^n(\alpha\delta + 1)^{-1} = (\alpha\delta + 1)^{-1}\delta^n$$

and the fact that $\| (\alpha\delta + 1)^{-1} \| \leqslant 1$ one deduces that $(\alpha\delta + 1)^{-1}\mathfrak{M}_a \subseteq \mathfrak{M}_a$ i. e. $\mathfrak{M}_a \subseteq (\alpha\delta + 1)\mathfrak{M}_a$. Thus it follows from Corollary 1 that δ is the σ-weak closure of its restriction to $\mathfrak{M}_a$.

2 $\Rightarrow$ 1. As δ is a symmetric derivation it easily follows that the set $\mathfrak{M}_a$ is a weakly dense *-subalgebra of $\mathfrak{M}$ and clearly δ maps $\mathfrak{M}_a$ into $\mathfrak{M}_a$. Let $\mathfrak{A}$ denote the C*-algebra generated by $\mathfrak{M}_a$ and $\hat{\delta}$ the norm closure of δ restricted to $\mathfrak{M}_a$. It follows that $\hat{\delta}$ is a restriction of δ and δ may be viewed as a derivation of $\mathfrak{A}$. As such it satisfies 2 *a*, or 2 *b*, and has a norm dense set of analytic elements. Thus by Theorems 4 and 5 of [*2*], $\hat{\delta}$ is the generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$. In particular $(\alpha\hat{\delta} + 1)(D(\hat{\delta})) = \mathfrak{A}$ for $\alpha \in \mathbb{R}\backslash\{ 0 \}$, whence

$$\mathfrak{M}_a \subseteq \mathfrak{A} \subseteq (\alpha\delta + 1)(D(\delta)) \quad , \quad \alpha \in \mathbb{R}\backslash\{ 0 \} .$$

Now for a given $A \in \mathfrak{M}$ there exists, by Kaplanskys density theorem [*8*], a net $A_\beta \in \mathfrak{M}_a$ such that $\| A_\beta \| \leqslant \| A \|$ and

$$s\text{-}\lim_\beta A_\beta = A$$

From the foregoing argument, one may choose B_β such that

$$A_\beta = (\alpha\delta + 1)(B_\beta) = (\alpha\hat{\delta} + 1)(B_\beta)$$

One then has

$$\| B_\beta \| \leqslant \| (\alpha\hat{\delta} + 1)(B_\beta) \| = \| A_\beta \| \leqslant \| A \|$$

Next note that $\mathfrak{M}_1$, the unit sphere of $\mathfrak{M}$, is σ-weakly compact and thus there exists a subnet B_γ of B_β which converges σ-weakly to some $B \in \mathfrak{M}$ such that $\| B \| \leqslant \| A \|$, i. e.

$$w\text{-}\lim_\gamma B_\gamma = B$$

$$w\text{-}\lim_\gamma (\alpha\delta + 1)(B_\gamma) = A$$

Since δ is σ-weakly closed one deduces that $B \in D(\delta)$ and $(\alpha\delta + 1)(B) = A$. Thus we have proved that

$$(\alpha\delta + 1)(D(\delta)) = \mathfrak{M} \quad , \quad \alpha \in \mathbb{R}\backslash\{ 0 \}$$

and

$$\| (\alpha\delta + 1)(B) \| \geqslant \| B \| \quad , \quad B \in D(\delta) .$$

It follows from Theorem 1 that δ is a generator.

3. SPATIAL DERIVATIONS

In this section we consider a special class* of the derivations of a von Neumann algebra, the spatial derivations.

A symmetric derivation δ of a von Neumann algebra $\mathfrak{M}$, on a Hilbert space $\mathfrak{H}$, is said to be spatial if there exists a symmetric operator H on $\mathfrak{H}$ with domain D(H) invariant under D(δ) and such that

$$\begin{aligned}\delta(\mathrm{A}) &= i[\mathrm{H}, \mathrm{A}]\\ &= i(\mathrm{HA} - \mathrm{AH})\end{aligned}$$

on D(H).

Since every symmetric operator H is closeable a straightforward argument reveals that every spatial derivation is σ-weakly closeable. We next give a criterium that characterizes the spatial derivations and generalizes the invariance condition used in Theorem 4 of [*1*] and Theorem 6 of [*2*]. The *-algebra $\mathfrak{D}$ used in the phrasing of the following theorem may be considered as the domain of a derivation of a C*-algebra or a von Neumann algebra.

THEOREM 3. — *Let δ be a symmetric derivation defined on a *-algebra $\mathfrak{D}$ which acts on a Hilbert space $\mathfrak{H}$. Let $\Omega \in \mathfrak{H}$ be a unit vector cyclic under $\mathfrak{D}$ and denote by ω the corresponding vector state, i. e.*

$$\omega(\mathrm{A}) = (\Omega, \mathrm{A}\Omega) \quad , \quad \mathrm{A} \in \mathfrak{D}$$

The following conditions are equivalent

1. *There exists a symmetric operator* H *on* $\mathfrak{H}$ *with the properties*

a) $$\mathrm{D}(\mathrm{H}) = \mathfrak{D}\Omega + \mathbb{C}\Omega$$

b) $$\delta(\mathrm{A})\psi = i[\mathrm{H}, \mathrm{A}]\psi$$

for all $\mathrm{A} \in \mathfrak{D}$ *and* $\psi \in \mathrm{D}(\mathrm{H})$.

2. *There exists a constant* $\mathrm{L} \geqslant 0$ *such that*

$$|\,\omega(\delta(\mathrm{A}))\,|^2 \leqslant \mathrm{L}(\omega(\mathrm{A}^*\mathrm{A}) + \omega(\mathrm{AA}^*))$$

for all $\mathrm{A} \in \mathfrak{D}$.

Furthermore, if 2 *is valid,* H *may be chosen such that*

$$\|\,\mathrm{H}\Omega\,\|^2 \leqslant \mathrm{L}/2$$

Proof. — 1 $\Rightarrow$ 2. For $\mathrm{A} \in \mathfrak{D}$ one has

$$|\,\omega(\delta(\mathrm{A}))\,| = |\,(\mathrm{H}\Omega, \mathrm{A}\Omega) - (\mathrm{A}^*\Omega, \mathrm{H}\Omega)\,| \leqslant \|\,\mathrm{H}\Omega\,\|\,(\|\,\mathrm{A}\Omega\,\| + \|\,\mathrm{A}^*\Omega\,\|)$$

(*) Non spatial derivations can be constructed, for example, for abelian von Neymann algebras.

Therefore

$$|\omega(\delta(A))|^2 \leqslant 2\,\|H\Omega\|^2(\|A\Omega\|^2 + \|A^*\Omega\|^2)$$
$$= 2\,\|H\Omega\|^2(\omega(A^*A) + \omega(AA^*))$$

$2 \Rightarrow 1$. Consider the Hilbert space

$$\mathfrak{H}_+ = \mathfrak{H} \oplus \bar{\mathfrak{H}}$$

where $\bar{\mathfrak{H}}$ is the conjugate space of $\mathfrak{H}$. Thus $\mathfrak{H}_+$ consists of all pairs $\{\varphi, \psi\}$, $\varphi, \psi \in \mathfrak{H}$ with addition, scalar multiplication, and inner product defined by

$$\{\varphi_1, \psi_1\} + \{\varphi_2, \psi_2\} = \{\varphi_1 + \varphi_2, \psi_1 + \psi_2\}$$
$$\lambda\{\varphi, \psi\} = \{\lambda\varphi, \bar{\lambda}\psi\}$$
$$(\{\varphi_1, \psi_1\}, \{\varphi_2, \psi_2\}) = (\varphi_1, \varphi_2) + (\psi_2, \psi_1)$$

Let $\hat{\mathfrak{H}}$ denote the subspace of $\mathfrak{H}_+$ spanned by vectors of the form $\{A\Omega, A^*\Omega\}$ with $A \in \mathfrak{D}$. Define a linear functional η on this subspace by

$$\eta(\{A\Omega, A^*\Omega\}) = i\omega(\delta(A))$$

From condition 2 one has

$$|\eta(\{A\Omega, A^*\Omega\})| \leqslant L^{\frac{1}{2}}\,\|\{A\Omega, A^*\Omega\}\|$$

Hence η is well defined and $\|\eta\| \leqslant L^{\frac{1}{2}}$. By the Riesz representation theorem there exists a vector $\{\varphi, \psi\}$ in the closure of $\hat{\mathfrak{H}}$ such that

$$\begin{aligned} i\omega(\delta(A)) &= \eta(\{A\Omega, A^*\Omega\}) \\ &= (\{\varphi, \psi\}, \{A\Omega, A^*\Omega\}) \\ &= (\varphi, A\Omega) + (A^*\Omega, \psi) \end{aligned}$$

Next using the symmetry of δ one finds

$$\begin{aligned} i\omega(\delta(A)) &= -\overline{i\omega(\delta(A^*))} \\ &= -[\overline{(\varphi, A^*\Omega)} + \overline{(A\Omega, \psi)}] \\ &= -[(\psi, A\Omega) + (A^*\Omega, \varphi)] \end{aligned}$$

Taking the mean of the two expressions for $\omega(\delta(A))$ one has

$$i^{-1}\omega(\delta(A)) = (\Omega_\delta, A\Omega) - (A^*\Omega, \Omega_\delta)$$

where

$$\Omega_\delta = \frac{1}{2}(\psi - \varphi)$$

Without loss of generality we assume $\mathbb{1} \in \mathfrak{D}$ and $\delta(\mathbb{1}) = 0$. Let us define the operator H by $D(H) = \mathfrak{D}\Omega$ and

$$HA\Omega = i^{-1}\delta(A)\Omega + A\Omega_\delta \quad , \quad A \in \mathfrak{D}.$$

One then computes that

$$\begin{aligned} &(HA\Omega, B\Omega) - (A\Omega, HB\Omega) \\ &= -\frac{1}{i}\omega(\delta(A^*)B) - \frac{1}{i}\omega(A^*\delta(B)) + (\Omega_\delta, A^*B\Omega) - (\Omega, A^*B\Omega_\delta) \\ &= -\frac{1}{i}\omega(\delta(A^*B)) + \frac{1}{i}\omega(\delta(A^*B)) \\ &= 0 \end{aligned}$$

This calculation shows firstly that H is well defined, i. e. $A\Omega = 0$ implies $HA\Omega = 0$, and, secondly that H is symmetric. Finally, for $A, B \in \mathfrak{D}$, one has

$$\begin{aligned}\delta(A)B\Omega &= \delta(AB)\Omega - A\delta(B)\Omega \\ &= iHAB\Omega - AB\Omega_\delta - AiHB\Omega + AB\Omega_\delta \\ &= i[H, A]B\Omega\end{aligned}$$

The final remark of the theorem follows from the calculation

$$\begin{aligned}\| H\Omega \|^2 = \| \Omega_\delta \|^2 &\leqslant \frac{1}{4}(\| \varphi - \psi \|^2 + \| \varphi + \psi \|^2) \\ &= \frac{1}{2}(\| \varphi \|^2 + \| \psi \|^2) \\ &= \frac{1}{2}\| \eta \|^2 \leqslant L/2\end{aligned}$$

In general there are several inequivalent ways of defining a spatial derivation. The different possibilities occur because the commutators [H, A], with H unbounded, are not unambiguously defined and various conventions can be adopted to give meaningful definitions. Each of these conventions leads to a distinct definition of a spatial derivation. In the case of particular interest the spatial derivations are implemented by self-adjoint H and then there is no ambiguity, as the following result demonstrates

THEOREM 4. — *Let* H *be a self-adjoint operator on a Hilbert space* $\mathfrak{H}$ *and let*

$$\sigma_t(A) = e^{itH}Ae^{-itH} \quad , \quad A \in \mathfrak{L}(\mathfrak{H})$$

be the corresponding one-parameter group of automorphisms of $\mathfrak{L}(\mathfrak{H})$. *Denote by* δ *the infinitesimal generator of* σ.

For $A \in \mathfrak{L}(\mathfrak{H})$ *given, the following conditions are equivalent.*

1. $A \in D(\delta)$
2. *There exists a core* D *for* H *such that the sesquilinear form*

$$\psi, \varphi \in D \times D \rightarrow i(H\psi, A\varphi) - i(\psi, AH\varphi) \in \mathbb{C}$$

is bounded.

3. *There exists a core* D *for* H *such that* $AD \subseteq D$ *and the mapping*

$$\psi \in D \rightarrow i[H, A]\psi \in \mathfrak{H}$$

is bounded.

If condition 2 is valid the bounded operator associated with the sesquilinear form is $\delta(A)$. *Similarly the bounded mapping of condition 3 defines* $\delta(A)$.

Proof. — A closure argument shows that conditions 2 and 3 imply the corresponding conditions with $D = D(H)$, whilst the converse is trivial. Thus we may assume that $D = D(H)$ in the following.

$1 \Rightarrow 2$. Assume $A \in D(\delta)$ and $\psi, \varphi \in D(H)$ then

$$(\psi, \delta(A)\varphi) = \lim_{t\to 0} \frac{1}{t} \{ (\psi, e^{itH}Ae^{-itH}\varphi) - (\psi, A\varphi) \}$$
$$= \lim_{t\to 0} \left(\frac{1}{t}(e^{-itH} - 1)\psi, Ae^{-irH}\varphi \right)$$
$$+ \lim_{t\to 0} \left(\psi, A\frac{1}{t}(e^{-itH} - 1)\varphi \right)$$
$$= (-iH\psi, A\varphi) + (\psi, A(-iH)\varphi)$$

$2 \Rightarrow 3$ assume that there exists a bounded operator B such that

$$(H\psi, A\varphi) = (\psi, AH\varphi) - i(\psi, B\varphi)$$

for $\psi, \varphi \in D(H)$. This relation demonstrates that

$$\psi \rightarrow (H\psi, A\varphi)$$

is continuous for fixed $\varphi \in D(H)$. Hence $A\varphi \in D(H^*) = D(H)$ and, further $(H\psi, A\varphi) = (\psi, HA\varphi)$. Therefore one has

$$(\psi, B\varphi) = i(\psi, [H, A]\varphi)$$

The implication $3 \Rightarrow 2$ is trivial and the proof is completed by the following.

$2 \Rightarrow 1$. Assume that A satisfies condition 2 and that B is the corresponding bounded operator. Using the technique of the first part of the proof one verifies, for $\varphi, \psi \in D(H)$, that

$$\lim_{t\to 0} \frac{1}{t} (\psi_1(\sigma_{s+t}(A) - \sigma_s(A))\varphi) = (\psi, \sigma_s(B)\varphi)$$

Hence

$$(\psi(\sigma_s(A) - A)\varphi) = \left(\psi \int_0^s dt\sigma_t(B)\varphi \right)$$

Therefore one has

$$\sigma_s(A) = A + \int_0^s dt\sigma_t(B)$$

and from this relation it follows that $A \in D(\delta)$ and $\delta(A) = B$.

This theorem establishes that if δ is a spatial derivation of a von Neumann algebra $\mathfrak{M}$ on $\mathfrak{H}$ given by a self-adjoint operator H on a core for this operator then δ extends to a derivation of $\mathfrak{L}(\mathfrak{H})$ which is a generator. Using the characterizations of generators given in Theorems 1 and 2 one then deduces the following result.

THEOREM 5. — *Let δ be a symmetric derivation of a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$. Assume that there exists a self-adjoint operator* H *on $\mathfrak{H}$, such that $\delta(A) = i[H, A]$, for $A \in D(\delta)$, where the commutator $i[H, A]$ is defined by any of the equivalent conditions of Theorem* 4.

It follows that

1. *δ is σ-weakly closeable*

2 *a*) $\|(\alpha\delta + 1)(A)\| \geqslant \|A\|$, $A \in D(\delta)$, $\alpha \in \mathbb{R}$

b) $(\alpha\delta + 1)(A) \geqslant 0 \Rightarrow A \geqslant 0$, $A \in D(\delta)$, $\alpha \in \mathbb{R}$

Further the σ-weak closure of δ is a generator if, and only if, the subspaces $(\alpha\delta + 1)(D(\delta))$ are weakly dense in $\mathfrak{M}$ for all $\alpha \in \mathbb{R}\backslash\{0\}$ or, equivalently, for $\alpha = \pm 1$.

Alternatively if δ has a weakly dense set of analytic vectors then the σ-weak closure of δ is a generator.

Proof. — First consider the extension $\hat{\delta}$ of δ to $\mathfrak{L}(\mathfrak{H})$ defined by Theorem 4. $\hat{\delta}$ is a generator of a group of automorphisms of $\mathfrak{L}(\mathfrak{H})$ and hence conditions 2 *a* and 2 *b* follow from Theorem 1. It also follows from Corollary 1 that $(\alpha\hat{\delta} + 1)^{-1}$ is $\sigma(\mathfrak{L}(\mathfrak{H}), \mathfrak{L}(\mathfrak{H})_*)$ continuous. Thus if $(\alpha\delta + 1)(D(\delta))$ is $\sigma(\mathfrak{M}, \mathfrak{M}_*)$ dense in $\mathfrak{M}$ and $A \in \mathfrak{M}$ there exists a net $B_\beta \in D(\delta)$ such that

$$A = \lim_\beta (\alpha\delta + 1)(B_\beta)$$

But then

$$B_\beta = (\alpha\hat{\delta} + 1)^{-1}(\alpha\delta + 1)(B_\beta)$$

and hence B_β converges in the $\sigma(\mathfrak{L}(\mathfrak{H}), \mathfrak{L}(\mathfrak{H})_*)$ topology to $B \in \mathfrak{L}(\mathfrak{H})$. Finally as $\mathfrak{M}$ is closed one concludes that $B \in \mathfrak{M}$. Thence $A = (\alpha\bar{\delta} + 1)(B)$ where $\bar{\delta}$ denotes the σ-weak closure of δ. Therefore δ is a generator by Theorem 1.

To deduce the last statement it suffices to prove that $\bar{\delta}$ is the σ-weak closure of its restriction to the set of analytic elements because the result then follows from Theorem 2. The proof of this statement follows, however, from inequality 2 *a*, and the fact that $\mathfrak{M}_a \subseteq (\alpha\delta + 1)(\mathfrak{M}_a)$, by the same reasoning as given in the last part of the proof of Theorem 2.

In the following section spatial derivations which are implemented by self-adjoint operators will be analyzed in the context of algebras with cyclic and separating vectors.

4. SEPARATING STATES

If δ is a spatial derivation of a von Neumann algebra $\mathfrak{M}$ with a cyclic and separating vector Ω one would expect that the characterizations of generators given in the previous sections could be strengthened. For example the range condition $(1 \pm \delta)(D(\delta)) = \mathfrak{M}$ which characterizes a generator by Theorem 5 reduces to the condition $(1 \pm \delta)(D(\delta))\Omega = \mathfrak{M}\Omega$. If, further, the operator H which implements δ satisfies $H\Omega = 0$ then this latter condition is equivalent to

$$(1 \pm iH)D(\delta)\Omega \supseteq \mathfrak{M}\Omega$$

whilst essential self-adjointness of H on $D(\delta)\Omega$ is equivalent to

$$\overline{(1 \pm iH)D(\delta)\Omega} \supseteq \mathfrak{M}\Omega$$

This leads to the following conjecture.

CONJECTURE. — *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with cyclic and separating vector Ω. Further let δ be a spatial derivation of $\mathfrak{M}$ implemented by a self-adjoint* H *such that* $\Omega \in D(H)$, $H\Omega = 0$ *and* H *is essentially self-adjoint on* $D(\delta)\Omega$.
It is conjectured that

$$e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M} \quad , \quad t \in \mathbb{R}$$

Note that when this conjecture is valid then it follows by a simple application of modular theory that $\exp\{itH\}$ commutes with the modular operator Δ and the modular conjugation J.

A simple case in which this conjecture may be verified is when δ is bounded, and hence spatial [*12*]. In fact the bounded derivations verifying the assumptions of the conjecture are exactly those of the form

$$\delta(A) = i[H, A]$$

with

$$H = h - JhJ$$

for some $h = h^* \in \mathfrak{M}$ such that

$$(1 - J)h\Omega = 0$$

More complicated cases of the conjecture are verified in the following subsections.

a) General Theory

In this subsection we verify the above conjecture in the special case that the self-adjoint H which implements δ has a positivity preserving resolvent in the sense that

$$(1 \pm iH)^{-1}\mathfrak{M}_+\Omega \subseteq \overline{\mathfrak{M}_+\Omega}$$

where $\mathfrak{M}_+$ is the set of positive elements of $\mathfrak{M}$. This is a weaker version of the property

$$(1 \pm \delta)^{-1}(\mathfrak{M}_+) \subseteq \mathfrak{M}_+$$

which is necessary to ensure that δ is a generator.

THEOREM 6. — *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with cyclic and separating vector Ω. Further let δ be a spatial derivation of $\mathfrak{M}$ implemented by a self-adjoint operator* H *such that* $\Omega \in D(H)$ *and* $H\Omega = 0$. *Denote by* $D(\bar{\delta})$ *the set*

$$D(\bar{\delta}) = \{ A ; A \in \mathfrak{M}, \quad i[H, A] = \bar{\delta}(A) \in \mathfrak{M} \}$$

The following conditions are equivalent

1. $$e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M} \quad , \quad t \in \mathbb{R}$$
2. $$(1 \pm iH)^{-1}\mathfrak{M}\Omega \subseteq D(\bar{\delta})\Omega$$
3. $$(1 \pm iH)^{-1}\mathfrak{M}_+\Omega \subseteq D(\bar{\delta})_+\Omega$$

and each of the following conditions are also equivalent

4. $$(1 \pm iH)^{-1}\mathfrak{M}_+\Omega \subseteq \mathfrak{M}_+\Omega$$
5. $$(1 \pm iH)^{-1}\mathfrak{M}_+\Omega \subseteq \overline{\mathfrak{M}_+\Omega}$$
6. $$e^{itH}\mathfrak{M}_+\Omega \subseteq \mathfrak{M}_+\Omega \quad , \quad t \in \mathbb{R}$$
7. $$e^{itH}\mathfrak{M}_+\Omega \subseteq \overline{\mathfrak{M}_+\Omega} \quad , \quad t \in \mathbb{R}$$

where the bar denotes weak (strong) closure.

The first set of conditions implies the second and if $D(\bar{\delta})\Omega$ *is a core for* H *the second set of conditions implies the first.*

Proof. — 1 $\Leftrightarrow$ 2. From Theorems 1 and 5 one has that condition 1 is equivalent to $(1 \pm \bar{\delta})(D(\bar{\delta})) = \mathfrak{M}$. But as Ω is separating and $H\Omega = 0$ this is equivalent to

$$(1 \pm iH)D(\bar{\delta})\Omega = \mathfrak{M}\Omega$$

or,

$$(1 \pm iH)^{-1}\mathfrak{M}\Omega = D(\bar{\delta})\Omega$$

Conversely

$$\mathfrak{M}\Omega \subseteq (1 \pm iH)(D(\bar{\delta}))\Omega = (1 \pm \bar{\delta})(D(\bar{\delta}))\Omega \subseteq \mathfrak{M}\Omega$$

and hence $(1 \pm \bar{\delta})(D(\bar{\delta})) = \mathfrak{M}$.

2 $\Leftrightarrow$ 3. As each element of $\mathfrak{M}$ is a linear superposition of four positive elements condition 3 immediately implies condition 2. Conversely if $A \in \mathfrak{M}$ then there is a $B \in D(\bar{\delta})$ such that

$$(1 \pm iH)^{-1}A\Omega = B\Omega$$

or,

$$A\Omega = (1 + iH)B\Omega = (1 + \bar{\delta})(B)\Omega$$

As Ω is separating this last condition is equivalent to

$$A = (1 + \bar{\delta})(B)$$

But if $A \geqslant 0$ then $B \geqslant 0$ from Theorem 5.

The rest of the proof will be divided into several lemmas which actually establish more than we have stated above. The extra information will be summarized in Corollary form.

As a preliminary let us recall some facts concerning commutation of unbounded operators. Two unbounded self-adjoint operators A and B are defined to commute strongly if the spectral projectors of A and B commute or, equivalently, if $\exp\{itA\}$ and $\exp\{isB\}$ commute for all $t, s \in R$. A bounded operator T is defined to commute strongly with an unbounded self-adjoint operator A if T commutes with the spectral projectors of A or, equivalently, with $\exp\{itA\}$ for all $t \in \mathbb{R}$. Alternative criteria for this latter situation are given by the following.

LEMMA 1. — *Let* A *be an unbounded self-adjoint operator on a Hilbert space* $\mathfrak{H}$ *and* T *a bounded operator.*

The following conditions are equivalent

1. T *commutes strongly with* A.
2. *There is a core* D *for* A *such that* $TD \subseteq D$ *and*

$$TA\psi = AT\psi \quad , \quad \psi \in D$$

3. *The domain* D(A) *of* A *is invariant under* T, *i. e.* $TD(A) \subseteq D(A)$ *and*

$$TA\psi = AT\psi \quad , \quad \psi \in D(A)$$

Proof. — If condition 1 is valid then

$$t \in \mathbb{R} \;\to\; e^{itA}Te^{-itA} \in \mathfrak{L}(\mathfrak{H})$$

is constant. Hence conditions 2 and 3 follow from Theorem 4. Conversely if condition 3 is true then

$$\begin{aligned}(e^{itA}Te^{-itA} - T)\psi &= i\int_0^t ds e^{isA}[A, T]e^{-isA}\psi \\ &= 0\end{aligned}$$

for all $\psi \in D(A)$. Hence condition 1 is true.

Now we adopt the following notation. As Ω is cyclic and separating for $\mathfrak{M}$ there exists a modular operator Δ, and also a modular conjugation J, associated with the pair $\mathfrak{M}$, Ω by Tomita-Takesaki theory. For each $0 \leqslant \alpha \leqslant \frac{1}{2}$ we define convex cones

$$V_\alpha = \Delta^\alpha \mathfrak{M}_+ \Omega$$

and let $\overline{V_\alpha}$ denote their weak (strong) closure. The modular operator defines a one-parameter group of modular automorphisms σ by

$$\sigma_{it}(A) = \Delta^{it}A\Delta^{-it} \quad , \quad t \in \mathbb{R}, \quad A \in \mathfrak{M}$$

LEMMA 2. — *Let* $\mathfrak{M}$ *be a von Neumann algebra with cyclic and separating vector* Ω *and* T *a bounded operator on* $\mathfrak{H}$ *such that*

$$TV_\alpha \subseteq \overline{V_\alpha} \quad , \quad T^*V_\alpha \subseteq \overline{V_\alpha}$$

for some $0 \leqslant \alpha \leqslant \frac{1}{2}$, $\alpha \neq \frac{1}{4}$.

It follows that T *strongly commutes with* Δ *and* T *commutes with J.*

Remark. — If $\alpha = \frac{1}{4}$ then this result is in general false. It is easy to establish in this case that $[T, J] = 0$ but T does not necessarily commute with Δ. For example each operator T of the form $T = AJAJ$, $A \in \mathfrak{M}$, maps $V_{\frac{1}{4}}$ into $\overline{V_{\frac{1}{4}}}$ and if Δ commutes with all of these operators then from Theorem 4 of [*13*] or lemma 2.9 of [*14*] one finds

$$\Delta\psi = \psi$$

for all $\psi \in \overline{V_{\frac{1}{4}}}$, i. e. Δ is the identity. Even if one adds the assumption that T is unitary and $T\Omega = \Omega$ it is impossible to draw the conclusion. For example if $\mathfrak{N} = \mathfrak{M} \oplus \mathfrak{M}'$ on $\mathfrak{H} \oplus \mathfrak{H}$ and $\Omega_{\mathfrak{N}} = \Omega \oplus \Omega$ then

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is such that $T\mathfrak{N}T = \mathfrak{N}'$, $T\Omega_{\mathfrak{N}} = \Omega_{\mathfrak{N}}$ and hence $T\overline{V_{\frac{1}{4}}(\mathfrak{N})} \subseteq \overline{V_{\frac{1}{4}}(\mathfrak{N})}$. But nevertheless $T\Delta_{\mathfrak{N}} = \Delta_{\mathfrak{N}}^{-1}T$.

Proof. — First we will prove that T and T* commute strongly with $J\Delta^{\frac{1}{2}-2\alpha}$.

Choose $A \in \mathfrak{M}$ and $B \in \mathfrak{M}'$ such that A and B are entire with respect to the modular automorphism group σ. One has, with $S = J\Delta^{\frac{1}{2}}$,

$$\begin{aligned}(T^*B\Omega, S\sigma_{-\alpha}(A)\Omega) &= (T^*B\Omega, \Delta^\alpha A^*\Omega) \\ &= (B\Omega, T\Delta^\alpha A^*\Omega) \\ &= \lim_\beta (B\Omega, \Delta^\alpha C_\beta\Omega)\end{aligned}$$

for some directed set $C_\beta \in \mathfrak{M}$. Therefore

$$\begin{aligned}(T^*B\Omega, S\sigma_{-\alpha}(A)\Omega) &= \lim_\beta (\sigma_\alpha(B)\Omega, SC_\beta^*\Omega) \\ &= \lim_\beta (C_\beta^*\Omega, \sigma_{-\alpha}(B^*)\Omega) \\ &= \lim_\beta (\Delta^\alpha C_\beta^*\Omega, \sigma_{-2\alpha}(B^*)\Omega) \\ &= (T\Delta^\alpha A\Omega, S^*\sigma_{2\alpha}(B)\Omega)\end{aligned}$$

where we have now used the fact that T maps the cone V_α into $\overline{V_\alpha}$ to deduce that

$$T\Delta^\alpha A^*\Omega = \lim_\beta \Delta^\alpha C_\beta\Omega$$

implies that

$$T\Delta^\alpha A\Omega = \lim_\beta \Delta^\alpha C_\beta^*\Omega$$

Next one finds

$$(B\Omega, TS\sigma_{-\alpha}(A)\Omega) = (T\Delta^\alpha A\Omega, S^*\Delta^{2\alpha}B\Omega)$$

and as the left hand side is continuous in $B\Omega$ one deduces that $T\Delta^\alpha A\Omega \in D(\Delta^{2\alpha}S)$ and

$$\begin{aligned}TS\Delta^{-\alpha}A\Omega &= \Delta^{2\alpha}ST\Delta^\alpha A\Omega \\ &= S\Delta^{-2\alpha}T\Delta^\alpha A\Omega\end{aligned}$$

Replacing A by $\sigma_{-\alpha}(A)$ one has

$$TS\Delta^{-2\alpha}A\Omega = S\Delta^{-2\alpha}TA\Omega$$

or

$$TJ\Delta^\beta A\Omega = J\Delta^\beta TA\Omega$$

with $\beta = \dfrac{1}{2} - 2\alpha$. But the set of vectors $A\Omega$ with A σ-entire is a core for Δ^β for all β. Hence by Lemma 1

$$TD(\Delta^\beta) \subseteq D(\Delta^\beta)$$

and

$$TJ\Delta^{\beta}\psi = J\Delta^{\beta}T\psi \quad , \quad \psi \in D(\Delta^{\beta})$$

Similarly

$$T^*J\Delta^{\beta}\psi = J\Delta^{\beta}T^*\psi \quad , \quad \psi \in D(\Delta^{\beta})$$

Therefore, by transposition, using $JD(\Delta^{-\beta}) \subseteq D(\Delta^{\beta})$ one finds that $TD(\Delta^{-\beta}) \subseteq D(\Delta^{-\beta})$ and

$$\Delta^{\beta}JT\psi = T\Delta^{\beta}J\psi \quad , \quad \psi \in D(\Delta^{-\beta})$$

Now one has

$$\begin{aligned}(T\Delta^{\beta}J)(J\Delta^{\beta}\psi) &= (\Delta^{\beta}JT)(J\Delta^{\beta}\psi) \\ &= (\Delta^{\beta}J)(J\Delta^{\beta}T\psi) \quad , \quad \psi \in D(\Delta^{2\beta})\end{aligned}$$

i. e.

$$T\Delta^{2\beta}\psi = \Delta^{2\beta}T\psi \quad , \quad \psi \in D(\Delta^{2\beta})$$

Hence T commutes strongly with $\Delta^{2\beta}$ by Lemma 1. But this is equivalent to T commuting strongly with Δ.

Finally

$$TJ\Delta^{\beta}\psi = J\Delta^{\beta}T\psi = JT\Delta^{\beta}\psi \quad , \quad \psi \in D(\Delta^{\beta})$$

But as Δ is invertible this implies

$$TJ = JT$$

LEMMA 3. — *Let $\mathfrak{M}$ be a von Neumann algebra with cyclic and separating vector Ω and* T *a bounded operator on $\mathfrak{H}$ such that*

$$TV_{\alpha} \subseteq \overline{V_{\alpha}} \quad , \quad T^*V_{\alpha} \subseteq \overline{V_{\alpha}}$$

for some $0 \leqslant \alpha \leqslant 1/2$, $\alpha \neq 1/4$.

It follows that

$$TV_{\beta} \subseteq \overline{V_{\beta}} \quad , \quad T^*V_{\beta} \subseteq \overline{V_{\beta}}$$

for all $0 \leqslant \beta \leqslant 1/2$.

Proof. — From [*15*] Theorem 2.5

$$\overline{V_{\frac{1}{4}}} = \overline{\Delta^{\frac{1}{4}-\alpha}\overline{V}_{\alpha}}$$

Therefore

$$\begin{aligned}TV_{\frac{1}{4}} &= \Delta^{\frac{1}{4}-\alpha}TV_{\alpha} \\ &\subseteq \Delta^{\frac{1}{4}-\alpha}\overline{V_{\alpha}} \\ &\subseteq \overline{V_{\frac{1}{4}}}\end{aligned}$$

Next from [*15*] Theorem 2.8

$$\overline{V_{\beta}} = \Delta^{\beta-\frac{1}{4}}\left\{ V_{\frac{1}{4}} \cap D(\Delta^{\beta-\frac{1}{4}}) \right\}$$

Hence

$$\begin{aligned}TV_{\beta} &\subseteq T\overline{V_{\beta}} \\ &= \Delta^{\beta-\frac{1}{4}}T\left\{ \overline{V_{\frac{1}{4}}} \cap D(\Delta^{\beta-\frac{1}{4}}) \right\}\end{aligned}$$

But $TD(\Delta^{\beta-\frac{1}{4}}) \subseteq D(\Delta^{\beta-\frac{1}{4}})$ and as T is bounded $T\overline{V_{\frac{1}{4}}} \subseteq \overline{V_{\frac{1}{4}}}$ from which one concludes

$$TV_{\beta} \subseteq \overline{V_{\beta}}$$

LEMMA 4. — *Let $\mathfrak{M}$ be a von Neumann algebra with cyclic and separating vector Ω and* T *a bounded operator on $\mathfrak{H}$ such that*

$$TV_\alpha \subseteq \overline{V_\alpha} \quad , \quad T^*V_\alpha \subseteq \overline{V_\alpha}$$

and

$$T\Omega = \lambda\Omega = T^*\Omega$$

where $\lambda \in \mathbb{R}$ and $0 \leqslant \alpha \leqslant 1/2$, $\alpha \neq 1/4$.

It follows that

$$TV_\alpha \subseteq V_\alpha \quad , \quad T^*V_\alpha \subseteq V_\alpha$$

Proof. — From Lemma 3, it follows that

$$TV_0 \subseteq \overline{V_0} \quad , \quad T^*V_0 \subseteq \overline{V_0}$$

We will first show that T and T* actually map V_0 into V_0. First we show

$$T\mathfrak{M}'_+\Omega \subseteq \mathfrak{M}'_+\Omega \quad , \quad T^*\mathfrak{M}'_+\Omega \subseteq \mathfrak{M}'_+\Omega$$

Assume $A \in \mathfrak{M}'_+$ and $B \in \mathfrak{M}_+$. We then have

$$(TA\Omega, B\Omega) = (A\Omega, T^*B\Omega) \geqslant 0$$

where we have used $T^*V_0 \subseteq \bar{V}_0$. Choose $B_\alpha \in \mathfrak{M}_+$ such that

$$B_\alpha\Omega \to T^*B\Omega$$

and then one finds

$$\begin{aligned}(TA\Omega, B\Omega) &= \lim_\alpha (A\Omega, B_\alpha\Omega) \\ &= \lim_\alpha (B_\alpha^{\frac{1}{2}}\Omega, AB_\alpha^{\frac{1}{2}}\Omega) \\ &\leqslant \overline{\lim_\alpha} \, \| A \| (\Omega, B_\alpha\Omega) \\ &= \| A \| (T\Omega, B\Omega) \\ &= \lambda \| A \| (\Omega, B\Omega)\end{aligned}$$

By assumption $\lambda \geqslant 0$ and hence by a theorem of Segal (see, for example, [9], Theorem 1.21-10) there exists a $C \in \mathfrak{M}'_+$ such that $\| C \| \leqslant \lambda \| A \|$ and

$$(TA\Omega, B\Omega) = (C\Omega, B\Omega) \quad , \quad B \in \mathfrak{M}_+$$

Therefore

$$TA\Omega = C\Omega$$

or, alternatively stated,

$$T\mathfrak{M}'_+\Omega \subseteq \mathfrak{M}'_+\Omega \subseteq \overline{\mathfrak{M}'_+\Omega}$$

Similarly

$$T^*\mathfrak{M}'_+\Omega \subseteq \mathfrak{M}'_+\Omega \subseteq \overline{\mathfrak{M}'_+\Omega}$$

Applying the same argument but with $\mathfrak{M}$ and $\mathfrak{M}'$ interchanged one concludes that

$$TV_0 \subseteq V_0 \quad , \quad T^*V_0 \subseteq V_0$$

Finally

$$TV_\alpha = T\Delta^\alpha V_0 = \Delta^\alpha TV_0 \subseteq \Delta^\alpha V_0 = V_\alpha$$

and a similar result for T*.

COROLLARY 1. — *Under the conditions of the Lemma, if* $A, B \in \mathfrak{M}_+$ *are such that*

$$T\Delta^\alpha A\Omega = \Delta^\alpha B\Omega$$

then it follows that

$$\| B \| \leqslant \lambda \| A \|$$

This result was actually established in the proof of the Lemma.

LEMMA 5. — *Let* $\mathfrak{M}$ *be a von Neumann algebra with cyclic and separating vector* Ω *and let* δ *be a spatial derivation of* $\mathfrak{M}$ *determined by a self-adjoint operator* H *such that* $\Omega \in D(H)$ *and*

$$H\Omega = 0$$

Assume further that $D(\delta)\Omega$ *is a core for* H *and*

$$(iH + 1)^{-1}\mathfrak{M}_+\Omega \subseteq \mathfrak{M}_+\Omega$$

It follows that

$$(iH + 1)^{-1}\mathfrak{M}_+\Omega \subseteq D(\bar{\delta})_+\Omega$$

where

$$D(\bar{\delta}) = \{ A \in \mathfrak{M} ; [iH, A] \in \mathfrak{M} \}$$

Proof. — First note that by decomposing a general $A \in \mathfrak{M}$ as a superposition of positive elements one deduces that

$$(iH + 1)^{-1}A\Omega = B\Omega$$

implies that

$$(iH + 1)^{-1}A^*\Omega = B^*\Omega$$

Next choose $A \in \mathfrak{M}_+$ and a self-adjoint $C \in D(\delta)$. Introduce B by

$$(iH + 1)^{-1}A\Omega = B\Omega$$

It then follows that

$$\delta(C)B\Omega + C(iH + 1)B\Omega = (iH + 1)CB\Omega$$

Thus $CB\Omega \in D(H)$ and

$$(iH + 1)^{-1}F\Omega = CB\Omega$$

where

$$F = \delta(C)B + CA$$

Hence it follows that

$$(iH + 1)^{-1}F^*\Omega = BC\Omega$$

i. e. $BC\Omega \in D(H)$ and

$$(iH + 1)BC\Omega = AC\Omega + B\delta(C)\Omega$$

This last relation may be rewritten as

$$i(HB - BH)C\Omega = (A - B)C\Omega$$

This last equality extends easily to non self-adjoint $C \in D(\delta)$ and then it follows from Theorem 4 that $B \in D(\bar{\delta})$.

Let us now return to the proof of the theorem. Conditions 4 and 5 are equivalent by Lemma 4 with the choice $T = (1 + iH)^{-1}$ and $\alpha = 0$. Similarly conditions 6 and 7 are equivalent by the same Lemma with $T = \exp\{ iHt \}$. Next note that conditions 5 and 7 are equivalent as a result of the algorithms

$$e^{itH} = \lim_n \left(1 - i\frac{t}{n}H\right)^{-n}$$

$$(1 + iH)^{-1} = \int_{-\infty}^{0} dte^{t}e^{itH}$$

(To pass from condition 5 to condition 7 it is first necessary to use the Neumann series for the resolvent to establish that

$$(1 + \alpha iH)^{-1}\mathfrak{M}_{+}\Omega \subseteq \overline{\mathfrak{M}_{+}\Omega}$$

for all $\alpha \in \mathbb{R}$).

The final statement of the theorem follows from Lemma 5.

COROLLARY 2. — *Adopt the assumptions and notation of Theorem* 6 *including the assumption that* $D(\bar{\delta})\Omega$ *is a core for* H.

The following conditions are equivalent

1. $$e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M} \quad , \quad t \in \mathbb{R}$$
2. *For some* $0 \leqslant \alpha \leqslant 1/2$, $\alpha \neq 1/4$ *one has*
$$e^{itH}V_{\alpha} \subseteq \overline{V_{\alpha}} \quad , \quad t \in \mathbb{R}\,.$$
3. H *commutes strongly with the modular operator* Δ *and*
$$e^{itH}V_{1/4} \subseteq \overline{V_{1/4}}$$

The implications $1 \Leftrightarrow 2 \Rightarrow 3$ follow from Theorem 6 and Lemma 3. The final implication $3 \Rightarrow 2$ follows from the proof of Lemma 3.

COROLLARY 3. — *Adopt the assumptions and notation of Theorem 6 including the assumption that* $D(\bar{\delta})\Omega$ *is a core for* H. *Let* $\mathfrak{M}_s$ *denote the set of self-adjoint elements of* $\mathfrak{M}$.

The following are equivalent

1. $$e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M} \quad , \quad t \in \mathbb{R}\,.$$
2. $$(1 + iH)^{-1}\mathfrak{M}_s\Omega \subseteq \mathfrak{M}_s\Omega$$
3. *The resolvents* $(1 \pm iH)^{-1}$ *commute strongly with* J *and* Δ *and*
$$(1 \pm iH)^{-1}\mathfrak{M}\Omega \subseteq \mathfrak{M}\Omega$$

The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are straightforward. For the converse note that condition 2 implies

$$(1 \pm iH)^{-1}\mathfrak{M}_s\Omega \subseteq D(\bar{\delta})_s\Omega$$

by the reasoning used to prove Lemma 4. But this implies

$$(1 \pm iH)^{-1}\mathfrak{M}\Omega \subseteq D(\bar{\delta})\Omega$$

and then $2 \Rightarrow 1$ follows from Theorem 6.

Condition 3 implies condition 2 because

$$J\Delta^{\frac{1}{2}}A\Omega = A^*\Omega$$

In Theorem 6 the core condition for H is essential if one wishes to establish that the second set of conditions imply the automorphism property. This is seen by the following.

EXAMPLE. — *There exists a von Neumann algebra $\mathfrak{M}$ with cyclic and separating vector Ω and a spatial derivation δ of $\mathfrak{M}$, given by a self-adjoint operator* H *with the property that* $H\Omega = 0$, *such that*

$$e^{itH}\mathfrak{M}_+\Omega = \mathfrak{M}_+\Omega \quad , \quad t \in \mathbb{R}$$

but

$$e^{itH}\mathfrak{M}e^{-itH} \neq \mathfrak{M}$$

for some $t \in \mathbb{R}$.

In this example, whose construction we will indicate below, the one-parameter group α_t of mappings of $\mathfrak{M}$ onto $\mathfrak{M}$ defined by the requirement

$$\alpha_t(A)\Omega = e^{itH}A\Omega \quad , \quad A \in \mathfrak{M}$$

is a one-parameter group of Jordan *-automorphisms of $\mathfrak{M}$ which are not *-automorphisms for all t. This contrasts with the result that a continuous group of Jordan *-automorphisms of a von Neumann algebra $\mathfrak{M}$ which has discrete centre is a group of *-automorphisms [*16*]; a similar result is trivially true for abelian $\mathfrak{M}$.

Let M_2 be the algebra of 2×2 complex matrices and τ the normalized trace on M_2. The algebra can be realized on a four-dimensional Hilbert space $\mathfrak{H}_0$ such that

$$\tau(A) = (\Omega_0, A\Omega_0) \quad , \quad A \in M_2$$

where $\Omega_0 \in \mathfrak{H}_0$ is cyclic and separating. A calculation establishes that

$$(*) \qquad M_{2+}\Omega_0 = M'_{2+}\Omega_0$$

Let π be the circle group with normalized Haar measure and define a von Neumann algebra $\mathfrak{N}$ on

$$\mathfrak{H} = \mathfrak{H}_0 \otimes L^2(\pi)$$

by

$$\mathfrak{N} = M_2 \otimes L^\infty(\pi)$$

The vector

$$\Omega = \Omega_0 \otimes \mathbb{1}$$

is cyclic and separating for $\mathfrak{N}$.

Next let $E_1 \in L^\infty(\pi)$ be the projection defined by the characteristic function of the upper half of the circle and set

$$E = \mathbb{1} \otimes E_1$$

It follows that E is a projection in the centre of $\mathfrak{N}$. Now define $\mathfrak{M}$ by

$$\mathfrak{M} = \mathfrak{N}E + \mathfrak{N}'(\mathbb{1} - E)$$

It follows from (*) that

$$\mathfrak{M}_+\Omega = \mathfrak{N}_+\Omega$$

Let $U_1(t)$ be the unitary group of $L^2(\pi)$ defined by rotation through angle t and define U by

$$U(t) = \mathbb{1} \otimes U_1(t) \quad , \quad t \in \mathbb{R} .$$

Clearly

$$U(t)\mathfrak{N}U(t)^* = \mathfrak{N} \quad , \quad U(t)\Omega = \Omega$$

for all $t \in \mathbb{R}$ and hence

$$U(t)\mathfrak{M}_+\Omega = U(t)\mathfrak{N}_+\Omega = \mathfrak{N}_+\Omega = \mathfrak{M}_+\Omega \quad , \quad t \in \mathbb{R}$$

But clearly

$$U(t)\mathfrak{M}U(t)^* \neq \mathfrak{M}$$

unless $t \in 2\pi\mathbb{Z}$.

Note that this family of mappings arise from a derivation in the following sense. One may view $\mathfrak{M}$ as functions over the circle with values in M_2 on the upper half of the circle and M_2' on the lower half. If A is such a function which is differentiable and vanishes in open neighbourhoods around ± 1 then $U(t)AU(t)^* \in \mathfrak{M}$ for t sufficiently small and $t \to U(t)AU(t)^*$ is differentiable at the origin. As the set of such A is weakly dense the differential defines a derivation.

b) **Analytic Vectors**

The conjecture at the beginning of the section is valid if $D(\delta)$ contains a dense set of analytic elements for δ by Theorem 5. In this subsection we establish a result of this nature based on a weaker assumption involving the analytic vectors of H.

THEOREM 7. — *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with cyclic and separating vector Ω. Further let δ be a spatial derivation of $\mathfrak{M}$ implemented by an operator* H *such that* $H\Omega = 0$. *Further assume there exists a self-adjoint subspace $\mathfrak{B} \subseteq D(\delta)$ with the properties*

a) $\mathfrak{B}$ *is weakly dense in* $\mathfrak{M}$

b) $\delta(\mathfrak{B}) \subseteq \mathfrak{B}$

c) *For each* $B \in \mathfrak{B}$

$$\sum_{n\geqslant 0} \frac{|z|^n}{n!} \| \delta^n(B)\Omega \| < +\infty \quad , \quad 0 < |z| \leqslant t_B$$

It follows that

$$e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M} \quad , \quad t \in \mathbb{R}$$

Proof. — We aim at showing that if $A \in \mathfrak{B}$

$$(JB, \Omega, [e^{itH}Ae^{-itH}, C]JB_2\Omega) = 0 \quad , \quad t \in \mathbb{R}$$

for all $B_1, B_2 \in \mathfrak{B}$ and $C \in \mathfrak{M}'$. This is sufficient to establish that

$$e^{it\mathrm{H}}\mathrm{A}e^{-it\mathrm{H}} \in \mathfrak{M}$$

and then the density of $\mathfrak{B}$ in $\mathfrak{M}$ allows the deduction of the stated result.

Now if $\mathrm{A_N}$ is an arbitrary element of $\mathfrak{M}$

$$\begin{aligned} &|(\mathrm{JB}, \Omega, [e^{i\mathrm{H}t}\mathrm{A}e^{-i\mathrm{H}t}, \mathrm{C}]\mathrm{JB}_2\Omega)| \\ &\qquad \leqslant \|\mathrm{B}_1\|\|\mathrm{C}\|\|\psi_{\mathrm{N}}(\mathrm{A}, \mathrm{B}_2)\| + \|\mathrm{B}_2\|\|\mathrm{C}\|\|\psi_{\mathrm{N}}(\mathrm{A}^*, \mathrm{B}_1)\| \end{aligned}$$

where

$$\psi_{\mathrm{N}}(\mathrm{A}, \mathrm{B}) = (e^{i\mathrm{H}t}\mathrm{A}e^{-i\mathrm{H}t} - \mathrm{A_N})\mathrm{JB}\Omega$$

Next we consider the special choice

$$\mathrm{A_N} = \sum_{n_1 \leqslant \mathrm{N}_1} \cdots \sum_{n_m \leqslant \mathrm{N}_m} \frac{t_0^{n_1}}{n_1!} \cdots \frac{t_0^{n_m}}{n_m!} \delta^{n_1+n_2+\ldots+n_m}(\mathrm{A})$$

where m and t_0 are chosen such that $t = mt_0$ and $|t_0| < t_{\mathrm{A}}$. It then follows by the usual form of iteration argument associated with analytic vectors that

$$\mathrm{A_N JB}\Omega = \mathrm{JBJA_N}\Omega \xrightarrow[\mathrm{N}_1\ldots\mathrm{N}_m=\infty]{} \mathrm{JBJ}e^{it\mathrm{H}}\mathrm{A}\Omega$$

Therefore for each possible partition of the above type

$$\psi_{\mathrm{N}}(\mathrm{A}, \mathrm{B}) \xrightarrow[\mathrm{N}_1\ldots\mathrm{N}_m=\infty]{} \psi(\mathrm{A}, \mathrm{B})$$

where

$$\psi(\mathrm{A}, \mathrm{B}) = e^{it\mathrm{H}}\mathrm{A}e^{-it\mathrm{H}}\mathrm{JB}\Omega - \mathrm{JBJ}e^{it\mathrm{H}}\mathrm{A}\Omega$$

Next we examine the commutation properties of H and the modular operator with the aim of establishing that $\psi(\mathrm{A}, \mathrm{B}) = 0$.

LEMMA 6. — *Adopt the assumptions and notation of Theorem 7. Let* Δ *and* J *denote the modular operator and modular conjugation associated with* Ω. *It follows that* iH *strongly commutes with* Δ *and* J.

Proof. — For $\mathrm{A} \in \mathfrak{B}$ introduce $\mathrm{A_N}$ as above. One then has

$$\|\mathrm{A_N}\Omega - e^{it\mathrm{H}}\mathrm{A}\Omega\| \xrightarrow[\mathrm{N}_1\ldots\mathrm{N}_m=\infty]{} 0$$

$$\|\mathrm{A_N^*}\Omega - e^{it\mathrm{H}}\mathrm{A}^*\Omega\| \xrightarrow[\mathrm{N}_1\ldots\mathrm{N}_m=\infty]{} 0$$

But the modular operator satisfies

$$\|\Delta^{\frac{1}{2}}(\mathrm{A_N} - \mathrm{A_M})\Omega\| = \|(\mathrm{A_N^*} - \mathrm{A_M^*})\Omega\|$$

and hence $e^{it\mathrm{H}}\mathrm{A}\Omega \in \mathrm{D}(\Delta^{\frac{1}{2}})$, whenever $\mathrm{A} \in \mathfrak{B}$, because $\Delta^{\frac{1}{2}}$ is closed. Further

$$(*) \qquad \|\Delta^{\frac{1}{2}}e^{it\mathrm{H}}\mathrm{A}\Omega\| = \|\mathrm{A}^*\Omega\| = \|\Delta^{\frac{1}{2}}\mathrm{A}\Omega\|$$

Next remark that $\mathfrak{B}$ is strongly dense in $\mathfrak{M}$ and hence each $\mathrm{B} \in \mathfrak{M}$ may be strongly approximated by a net $A_\alpha \in \mathfrak{B}$ such that A_α^* strongly approximates A^*. This last statement is a consequence of Corollary 1.8.11 in [8]. But we then have

$$\begin{aligned} \| \Delta^{\frac{1}{2}} e^{itH}(A_\alpha - A_\beta)\Omega \| &= \|(A_\alpha^* - A_\beta^*)\Omega \| \\ &= \| \Delta^{\frac{1}{2}}(A_\alpha - A_\beta)\Omega \| \end{aligned}$$

and by the same argument as above the relation (*) extends to $\mathfrak{B}$. As $\mathfrak{M}\Omega$ is a core for $\Delta^{\frac{1}{2}}$ one then deduces that $e^{iHt}D(\Delta^{\frac{1}{2}}) \subseteq D(\Delta^{\frac{1}{2}})$ and

$$(\Delta^{\frac{1}{2}} e^{itH}\psi, \Delta^{\frac{1}{2}} e^{itH}\psi) = (\Delta^{\frac{1}{2}}\psi, \Delta^{\frac{1}{2}}\psi)$$

for all $\psi \in D(\Delta^{\frac{1}{2}})$. By polarization one then deduces that

$$(\Delta^{\frac{1}{2}} e^{itH}\varphi, \Delta^{\frac{1}{2}} e^{itH}\psi) = (\Delta^{\frac{1}{2}}\varphi, \Delta^{\frac{1}{2}}\psi)$$

for all $\varphi, \psi \in D(\Delta^{\frac{1}{2}})$. But if $\psi \in D(\Delta)$ the right hand side is continuous in φ and hence the left hand side is continuous in $e^{iHt}\varphi$. Therefore $e^{iHt}\psi \in D(\Delta)$ and we have

$$e^{-itH}\Delta e^{itH}\psi = \Delta\psi$$

Hence Δ commutes strongly with iH by Lemma 1.

To derive the properties of J note that

$$J\Delta^{\frac{1}{2}}A\Omega = A^*\Omega \quad , \quad A \in \mathfrak{M}$$

and, by using the approximants A_N,

$$\begin{aligned} J\Delta^{\frac{1}{2}} e^{iHt}A\Omega &= e^{iHt}A^*\Omega \\ &= e^{iHt}J\Delta^{\frac{1}{2}}A\Omega \quad , \quad A \in \mathfrak{M} \end{aligned}$$

Using the commutation properties of $\Delta^{\frac{1}{2}}$ one then has

$$[J, e^{itH}]\Delta^{\frac{1}{2}}A\Omega = 0$$

But $\mathfrak{M}\Omega$ is a core for $\Delta^{\frac{1}{2}}$ and $\Delta^{\frac{1}{2}}$ is invertible. Thus

$$[J, e^{itH}] = 0$$

Let us now return to the proof of the theorem. It remains to show that $\psi(A, B) = 0$. But if $C \in \mathfrak{B}$ one has

$$\begin{aligned} (JC\Omega, \psi(A, B)) &= (JC\Omega, e^{itH}Ae^{-itH}JB\Omega) - \lim_{N_1 \dots N_m \to \infty} (JC\Omega, A_N JB\Omega) \\ &= (JC\Omega, e^{itH}Ae^{-itH}JB\Omega) \\ &\quad - \lim_{N_1 \dots N_m \to \infty} \sum_{\substack{n_1 \leq N_1 \\ p_1 \leq N_1}} \cdots \sum_{\substack{n_m \leq N_m \\ p_m \leq N_m}} \frac{t_0^{n_1 + p_1}}{n_1 ! \, p_1 !} \cdots \frac{t_0^{n_m + p_m}}{n_m ! \, p_m !} \\ &\qquad \times (J(-iH)^{n_1 + \dots + n_m}C\Omega, AJ(-iH)^{p_1 + \dots + p_m}B\Omega) \\ &= (JC\Omega, e^{iHt}Ae^{-iHt}JB\Omega) - (Je^{-iHt}C\Omega, AJe^{-iHt}B\Omega) \\ &= 0 . \end{aligned}$$

In the second equality we have used $\delta(A) = i[H, A]$ and $HJ + JH = 0$, and in the third we have used B, C $\in \mathfrak{B}$. The final conclusion follows from $[J, \exp\{iHt\}] = 0$.

Finally we note that $J\mathfrak{B}\Omega$ is a dense set of vectors, $\psi(A, B)$ is orthogonal to this set, and hence this vector is zero. This completes the proof.

c) Abelian Algebras

It is well known that if M is a differentiable manifold, $\mathfrak{A}$ the $*$-algebra of C^∞ functions on M, δ a derivation defined on $\mathfrak{A}$, and $x_1, \ldots, x_n$ a system of local coordinates, then δ is given locally by a first order differential operator

$$(\delta f)(x_1, \ldots, x_n) = \sum_{i=1}^{n} g_i(x_1, \ldots, x_n)\frac{\partial f}{\partial xi}(x_1, \ldots, x_n)$$

where the g_i's are C^∞-functions on the coordinate patch (*) [*17*]. Thus derivations of abelian C*- or W*-algebras may be viewed as generalized first order linear differential operators. The study of whether a derivation generates a group of $*$-automorphisms of an abelian C*-algebra, or its weak closure, is thus an extension of the global existence theory for first order linear partial differential equations.

The following is an abstract version of a recent theorem of Gallavotti and Pulvirenti [*18*]. The proof given here is a version of that part of the original proof which is relevant in the present case.

THEOREM. — *Let $\mathfrak{M}$ be an abelian von Neumann algebra on a Hilbert space $\mathfrak{H}$ with cyclic vector Ω. Further let δ be a spatial derivation of $\mathfrak{M}$ implemented by a self-adjoint operator* H *such that* $\Omega \in D(H)$ *and* $H\Omega = 0$ *and, further,* $D(\delta)\Omega$ *is a core for* H.

It follows that

$$e^{itH}\mathfrak{M}e^{-itH} = \mathfrak{M} \quad , \quad t \in \mathbb{R}$$

Proof. — As $\mathfrak{M}$ is abelian the modular operator Δ is the identity and the modular conjugation J satisfies

$$JA\Omega = A^*\Omega \quad , \quad A \in \mathfrak{M}$$

Therefore if $A \in D(\delta)$

$$\begin{aligned} iHJA\Omega &= \delta(A^*)\Omega \\ &= \delta(A)^*\Omega \\ &= JiHA\Omega \end{aligned}$$

As $D(\delta)\Omega$ is a core for H it follows from Lemma 1 that J commutes strongly with iH and hence commutes with $(1 \pm iH)^{-1}$.

(*) A C*-algebraic version of this theorem was derived in [*1*]. If δ is a closed derivation of an abelian C*-algebra $\mathfrak{A}$, $A = A^* \in D(\delta)$, and f is a continuously differentiable function then $f(A) \in D(\delta)$ and

$$\delta(f(A)) = \delta(A)f'(A)$$

In order to continue the proof we realize the abelian algebra $\mathfrak{M}$ as an algebra of multiplication operators. It follows from the Gelfand representation theory that $\mathfrak{H}$ may be represented as a Hilbert space of square integrable functions $L^2(X; d\mu)$ such that $\mathfrak{M}$ is identifiable with the algebra $L^\infty(X; d\mu)$ acting by multiplication, and the vector state $\omega(A) = (\Omega, A\Omega)$, $A \in \mathfrak{M}$ associated with Ω is given by the probability measure μ on X. Thus Ω is represented by the unit function and if $A \in L^\infty(X; d\mu) = \mathfrak{M}$ then

$$\omega(A) = \int_X d\mu(x)A(x)$$

If $A \in D(\delta)$ it can be identified both as an L^∞ and as an L^2 function and one has

$$\delta(A) = \delta(A)\mathbb{1} = \delta(A)\Omega = iHA\Omega = iHA$$

Theorem 8 now follows by combining the following Lemma with Theorem 6. Note that the method used in the Lemma also shows directly that $(1 + i\alpha H)^{-1}L^\infty \subseteq L^\infty$ hence making Corollary 3 also applicable.

LEMMA 7. — *If* $\alpha \in \mathbb{R}$ *then*

$$(1 + i\alpha H)^{-1}L^\infty_+(X; d\mu) \subseteq L^2_+(X; d\mu)$$

Proof. — Let $A \in L^\infty$ be positive. If $B = (1 + i\alpha H)^{-1}A$ then B is real because iH commutes strongly with the conjugation J. It remains to show that B is a positive L^2-function.

As $D(\delta)\Omega$ is a core for H one can choose a real sequence $B_n \in D(\delta)$ such that

$$(1 + i\alpha H)B_n\Omega \rightarrow A\Omega$$

and hence $B_n \rightarrow B$ in $L^2(X; d\mu)$. Clearly one also has $i\alpha HB_n \rightarrow A - B$ in $L^2(X; d\mu)$.

Next note that if f is a continuously differentiable function on $\mathbb{R}$ then

$$0 = \omega(\delta(f(B_n))) = \omega(f'(B_n)\delta(B_n))$$

by [1]. This relation may be reexpressed in Hilbert space terms as

$$(f'(B_n)^*\Omega, iHB_n\Omega) = 0$$

If we further assume that the derivative f', of f, is uniformly bounded then by L^2 convergence one immediately has

$$(f'(B)^*\Omega, (A - B)\Omega) = 0$$

Finally if g is a continuous, positive function with compact support then there is an f, satisfying the previous conditions, such that $f' = g$. Therefore

$$(g(B)\Omega, B\Omega) = (g(B)\Omega, A\Omega) \geqslant 0$$

or, reexpressed in measure space language,

$$\int_x d\mu(x) g(\mathrm{B}(x))\mathrm{B}(x) \geqslant 0$$

Hence $\mathrm{B}(x) \geqslant 0$ μ-almost everywhere. This completes the proof of the theorem.

Remark. — If the definition of a derivation δ of a von Neumann algebra $\mathfrak{M}$ is weakened to be a mapping from a weakly dense $*$-subalgebra $\mathrm{D}(\delta)$ into a space $\mathrm{R}(\delta)$ of elements affiliated with $\mathfrak{M}$ such that the symmetry property and the basic derivation property are still valid the Theorem 8 is still true [*18*] (one must of course assume that $\Omega \in \mathrm{D}(\delta(\mathrm{A}))$ for all $\mathrm{A} \in \mathrm{D}(\delta)$). The proof is unchanged.

REFERENCES

[*1*] O. BRATTELI and D. W. ROBINSON, Unbounded Derivations of C*-Algebras, *Commun. math. Phys.*, t. **42**, 1975, p. 253-268.
[*2*] O. BRATTELI and D. W. ROBINSON, Unbounded Derivations of C*-Algebras II, *Commun. math. Phys.*, t. **46**, 1976, p. 11-30.
[*3*] D. W. ROBINSON, Statistical Mechanics of Quantum Spin Systems II, *Commun. math. Phys.*, t. **7**, 1968, p. 337-348.
[*4*] M. B. RUSKAI, Time Development of Quantum Lattice Systems, *Commun. math. Phys.*, t. **20**, 1971, p. 193-204.
[*5*] M. PULVIRENTI and B. TIROZZI, Time Evolution of a Quantum Lattice System, *Commun. math. Phys.*, t. **30**, 1973, p. 83-98.
[*6*] D. A. DUBIN and G. SEVELL, Time Translations in the Algebraic Formulation of Statistical Mechanics, *Jour. Math. Phys.*, t. **11**, 1970, p. 2990-2998.
[*7*] O. E. LANFORD and D. W. ROBINSON, Approach to Equilibrium of Free Quantum Systems, *Commun. math. Phys.*, t. **24**, 1972, p. 193-210.
[*8*] S. SAKAI, *C*-Algebras and W*-Algebras*, Springer Verlag, Berlin, 1971.
[*9*] W. ARVESON, On Groups of Automorphisms of Operator Algebras, *Jour. Func. Anal.*, t. **15**, 1974, p. 217-243.
[*10*] K. YOSIDA, *Functional Analysis*, 4th Edition, Springer Verlag, Berlin, 1975.
[*11*] A. IKUNISHI and Y. NAKAGAMI, *On invariants $G(\sigma)$ and $\Gamma(\sigma)$ for an automorphism group of a von Neumann Algebra*, Tokyo Kogyo Daigaku Preprint, 1974.
[*12*] R. V. KADISON, Derivation of Operator Algebras, *Ann. Math.*, t. **83**, 1966, p. 280-293.
[*13*] H. ARAKI, Some Properties of Modular Conjugation Operator of von Neumann Algebras and a Non-Commutative Radon-Nikodym Theorem with a Chain Rule. *Pac. Jour. Math.*, t. **50**, 1974, p. 309-354.
[*14*] A. CONNES, Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, *Ann. Inst. Fourier*, Grenoble, 24.4, 1974, p. 121-155.
[*15*] H. ARAKI, Radon-Nikodym Theorems, Relative Hamiltonian and Applications, *Proc. Int. School of Physics*, Varenna, 1973.
[*16*] R. V. KADISON, Transformations of States in Operator Theory and Dynamics, *Topology*, t. **3**, 1965, p. 177-198.
[*17*] S. HELGASON, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.
[*18*] G. GALLAVOTTI and M. PULVIRENTI, Classical KMS Condition and Tomita-Takesaki Theory, *Commun. math. Phys.*, t. **46**, 1976, p. 1-9.

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