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Asymptotic perturbation expansion for the S-matrix and the definition of time ordered functions in relativistic quantum field models

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ABSTRACT. — In weakly coupled $P(\Phi)_2$ theories, perturbation theory in the coupling constant is asymptotic to the S-matrix elements and scattering is non-trivial. This is derived from regularity properties of the Schwinger functions and a new connection between Schwinger — and generalized time ordered functions.

PART I

SCATTERING IN WEAKLY COUPLED $P(\Phi)_2$ MODELS. PROPERTIES OF THE MODELS AND MAIN RESULTS

I.1. Introduction

This paper is devoted to the analysis of scattering in models of relativistic scalar quantum fields with a $\lambda P(\Phi)$ interaction in two space-time

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dimensions. Here $P(x) = \sum_{n=0}^{2N} a_n x^n$, $a_{2N} > 0$. We intend to show that perturbation theory in λ is asymptotic to the S-matrix. The main idea of our proof is to reduce this result to differentiability of the Schwinger functions which is known [D1]. The following is the basic existence theorem for the Schwinger functions.

THEOREM 1. — *Given P, there is an $\varepsilon > 0$ such that for $0 \leq \lambda/m_0^2 \leq \varepsilon$, $v_j \in \mathbb{Z}^+$ the limits*

$$S_{\Phi^{v_1} \dots \Phi^{v_n}}(x_1, \dots, x_n; m_0, \lambda) = \lim_{g \rightarrow 1} \frac{\int \prod_{i=1}^n : \Phi^{v_i} : (x_i) e^{-\lambda \int d^2x : P(\Phi) : (x) g(x)} d\mu_{m_0^2}}{\int e^{-\lambda \int d^2x : P(\Phi) : (x) g(x)} d\mu_{m_0^2}} \quad (\text{I.1})$$

exist [GJS1]. These limits are the restrictions to Euclidean points of the analytic continuation of Wightman functions satisfying all the Wightman axioms [G, OS]. The corresponding theories have a positive physical mass

$m(m_0, \lambda) \leq \sqrt{\frac{3}{2}} m_0$ and no other singularities below $\sqrt{2} m_0$ [GJS2, S].

We have used standard notation $: d\mu_{m_0^2}$ is the Gaussian measure on $\mathcal{S}'(\mathbb{R}^2)$ with mean zero and covariance $(-\Delta + m_0^2)^{-1}$, and $: \Phi^v :$ is Wick ordered with respect to this covariance. The constants λ and m_0 are the coupling constant and the bare mass respectively. We shall extend the notation $S_{\Phi^{v_1} \dots \Phi^{v_n}}$ by linearity to indices of the form $\sum a_j \Phi^j$.

Theorem 1 guarantees the existence of a Haag-Ruelle scattering theory. For the proof that perturbation theory in λ is asymptotic to the S-matrix elements we use the following two ingredients:

A) We prove that the sharp time Schwinger functions smeared in the space variables are bounded uniformly in the time variables and in $\lambda/m_0^2 \in [0, \varepsilon)$ (Theorem 2). This result and Theorem 1 permit us to show that the function \hat{H} , defined by

$$\begin{aligned} i^n \delta \left(\sum_{k=1}^n p_k \right) \hat{H}_{v_1 \dots v_n}((ip_1^0, \bar{p}_1), \dots, (ip_n^0, \bar{p}_n), m_0, \lambda) \\ = (2\pi)^{-n} \int d^2x_1 \dots d^2x_n S_{\Phi^{v_1} \dots \Phi^{v_n}}^T(x_1, \dots, x_n; m_0, \lambda) \exp i \sum_{j=1}^n p_j x_j \quad (\text{I.2}) \end{aligned}$$

is the restriction to Euclidean points of a function $H = H_{v_1 \dots v_n}$ analytic in the n point axiomatic domain (Theorem 3). Theorem 3 is an axiomatic

result based on regularity properties of the Schwinger functions (such as the ones proven in Theorem 2), exponential decay of their truncations and Osterwalder-Schrader bounds on their analytic continuations [OS]. Its proof is deferred to Part II.

B) We use the differentiability of the Schwinger functions with respect to λ [D1] (Lemma 6a) and m_0^2 (Lemma 6b) to derive differentiability of H in these parameters (Theorem 7). We prove differentiability of the bare mass with respect to the physical mass and the coupling constant (Theorem 8) and finally the differentiability of the S-matrix elements (Theorems 10, 12) as functions of λ for fixed physical mass. Our proof requires bounds on generalized time ordered and retarded functions which are uniform in $\lambda \in [0, \varepsilon m_0^2]$ (Theorem 5).

In Part II we give a definition of generalized time ordered functions in the framework of the Euclidean formulation of quantum field theory. We show how these functions are related to the Schwinger functions. The results of Part II are model independent and have some interest in their own right. They have applications to other models, in particular the massive sine-Gordon equation.

Remark. — Osterwalder and S enior have proven independently the non-triviality of the S-matrix in weakly coupled $P(\Phi)_2$ models [OS1], and Dimock has shown, by different methods, differentiability of the Green functions [D2] which will also be sufficient to show $S \neq I$.

We wish to thank V. Glaser for very helpful discussions.

1.2. Estimates on the Schwinger functions of $P(\Phi)_2$ and the main results

THEOREM 2. — Given P and $m_0 > 0$ there exists a Schwartz norm $\| \cdot \|$ depending only on $\bar{\nu} = \max \nu_j$ such that the functions

$$S_{\Phi^{\nu_1} \dots \Phi^{\nu_n}}(x_1^0, \dots, x_n^0; m_0, \lambda) \\ = \int \prod_{j=1}^n d\bar{x}_j f_j(\bar{x}_j) S_{\Phi^{\nu_1} \dots \Phi^{\nu_n}}((x_1^0, \bar{x}_1), \dots, (x_n^0, \bar{x}_n); m_0, \lambda)$$

are well defined and bounded by

$$\text{const. } n^{\bar{\nu}} (n!)^{\bar{\nu}} \prod_{j=1}^n \| f_j \|,$$

uniformly in $0 \leq \lambda < \varepsilon m_0^2$, for any $\varepsilon > 0$.

Proof. — ⁽¹⁾ Let $\langle \cdot \rangle_\lambda$ and $\langle \cdot \rangle_0$ denote the expectations with respect

⁽¹⁾ For the reader well acquainted with [GJS1] a shorter proof follows by directly substituting the test function $\delta \otimes f$ in equ. (9.5) and using the H older inequality only in the smeared variables.

to the measures $d\mu_{\lambda, m_0^2}$ and $d\mu_{m_0^2}$, where $d\mu_{\lambda, m_0^2}$ denotes the physical measure on $\mathcal{S}'(\mathbb{R}^2)$ whose moments are the Schwinger functions of Theorem 1. By definition,

$$S_{\Phi^{v_1} \dots \Phi^{v_n}}(x_1^0, \dots, x_n^0; m_0, \lambda) = \left\langle \prod_{j=1}^n : \Phi^{v_j} : (\delta_{x_j^0} \otimes f_j) \right\rangle_{\lambda}. \quad (\text{I.3})$$

We set $x_j = (x_j^0, \bar{x}_j)$. Wick's theorem gives

$$\prod_{j=1}^n : \Phi^{v_j} : (x_j) = \sum_{\alpha_j \leq v_j, \forall j} \left\langle \prod_{j=1}^n \binom{v_j}{\alpha_j} : \Phi^{v_j - \alpha_j} : (x_j) \right\rangle_0 : \prod_{j=1}^n \Phi^{\alpha_j}(x_j) :. \quad (\text{I.4})$$

Let $C(x-y)$ be the kernel of $(-\Delta + m_0^2)^{-1}$ and define $d_x(y) = C(x-y)$. Let π be any permutation of $\{ \underbrace{x_1, \dots, x_1}_{\alpha_1}, \underbrace{x_2, \dots, x_2}_{\alpha_2}, \dots, \underbrace{x_n, \dots, x_n}_{\alpha_n} \}$.

Using integration by parts on function space and the Leibniz formula we obtain the equation

$$\begin{aligned} & \left\langle : \prod_{j=1}^n \Phi^{\alpha_j}(x_j) : \right\rangle_{\lambda} \\ &= \sum_{\substack{\pi, M: \beta_1 + \dots + \beta_M = \Sigma \alpha_i \\ \beta_i \geq 1}} \frac{1}{M!} (-\lambda)^M \left\langle \prod_{i=1}^M \beta_i!^{-1} : P^{(\beta_i)} : (d_{x_{\pi(b_{i-1}+1)}} \dots d_{x_{\pi b_i}}) \right\rangle_{\lambda}, \end{aligned} \quad (\text{I.5})$$

where $P^{(\beta)}$ is the β -th derivative of P and $b_i = \sum_{l=1}^i \beta_l$. The Glimm-Jaffe $: \Phi^j :$ bounds give [F]

$$\left| \lambda^M \left\langle \prod_{i=1}^M : P^{(\beta_i)} : (d_{x_{\pi(b_{i-1}+1)}} \dots d_{x_{\pi b_i}}) \right\rangle_{\lambda} \right| \leq (\text{const.}')^{n\bar{v}} M!, \quad (\text{I.6})$$

uniformly in β_1, \dots, β_M with $\Sigma \beta_i \leq n\bar{v}$, $0 \leq \lambda < \varepsilon m_0^2$, and x_1, \dots, x_n ; (this follows from the fact that

$$d_{x_{\pi(b_{i-1}+1)}} \dots d_{x_{\pi b_i}} \in \bigcap_{1 \leq r < \infty} L^r(\mathbb{R}^2)$$

and from the Euclidean form of the $: \Phi^j :$ bounds). The proof now follows from

(1) The number of terms in the sum on the r. h. s. of (I.5) is bounded by $(\sum_1^n \alpha_i)!$ and $M \leq \sum_1^n \alpha_i$.

(2)

$$\left| \left\langle \prod_{j=1}^n : \Phi^{v_j - \alpha_j} : (\delta_{x_j^0} \otimes f_j) \right\rangle_0 \right| \leq \left(\sum_{j=1}^n (v_j - \alpha_j) \right)! \prod_{j=1}^n \| f_j \| ,$$

for some Schwartz norm $\| \cdot \|$. (This is a standard estimate.)

(3) The number of terms in the sum on the r. h. s. of (I.4) is bounded

$$\text{by } \prod_{i=1}^n v_i \leq \bar{v}^n .$$

These facts, combined with (I.3) and (I.6) yield

$$\begin{aligned} & | S_{\Phi^{v_1} \dots \Phi^{v_n}}(x_1^0, \dots, x_n^0; m_0, \lambda) | \\ & \leq \sum_{\alpha_j \leq v_j, \forall j} (\text{const.}')^{n\bar{v}} \left[\left(\sum_{i=1}^n \alpha_i \right)! \right] \left(\sum_{i=1}^n v_i - \alpha_i \right)! \prod_{j=1}^n \| f_j \| \\ & \leq \prod_{i=1}^n v_i (\text{const.}')^{n\bar{v}} \left[\left(\sum_{i=1}^n v_i \right)! \right] \prod_{j=1}^n \| f_j \| \\ & \leq \text{const.}^{n\bar{v}} (n!)^{\bar{v}} \prod_{j=1}^n \| f_j \| . \end{aligned}$$

Q. E. D.

THEOREM 3. — *Let $\{ S_{\Phi^{v_1} \dots \Phi^{v_n}} \}$ be the locally integrable Schwinger functions of a theory satisfying the conclusions of Theorem 1 and such that for some Schwartz norm $\| \cdot \|_{\bar{v}}$, and for $x_1^0 < x_2^0 < \dots < x_n^0$,*

$$\begin{aligned} & | S_{\Phi^{v_1} \dots \Phi^{v_n}}(x_1^0, \dots, x_n^0; m_0, \lambda) | \\ & \leq (\alpha_{\bar{v}} n)^{\beta_{\bar{v}} n} \| f_n \|_{\bar{v}} \prod_{i=1}^{n-1} \{ (1 + | x_{i+1}^0 - x_i^0 |)^{\gamma_{\bar{v}}} | x_{i+1}^0 - x_i^0 |^{-\delta_{\bar{v}}} \| f_i \|_{\bar{v}} \} , \quad (\text{I.7}) \end{aligned}$$

for some finite $\alpha_{\bar{v}}, \beta_{\bar{v}}, \gamma_{\bar{v}}$ and $\delta_{\bar{v}} < 1$. Then the function $\hat{H}_{v_1 \dots v_n}$ defined in (I.2) can be extended to a function $H_{v_1 \dots v_n}(k_1, \dots, k_n; m_0, \lambda)$ of n complex 2-vectors k_1, \dots, k_n with zero sum, defined and analytic in the axiomatic domain of the n -point function in momentum space, with single particle poles at $\{ k : k_j^2 = m^2(m_0, \lambda) \}$ and thresholds above $M = \sqrt{2}m_0 > \frac{2}{\sqrt{3}}m$.

We defer the proof to Part II. Note that (I.7), with $\beta_{\bar{v}} = \bar{v}, \gamma_{\bar{v}} = \delta_{\bar{v}} = 0$, is a consequence of Theorem 2.

The basic content of Theorem 3 is that the momentum space analytic function $H_{v_1 \dots v_n}$ is, at Euclidean points, equal to the Fourier transform

of $S_{\Phi^{v_1 \dots v_n}}^T$. In general this need not be true (depending on the behaviour of $S_{\Phi^{v_1 \dots v_n}}^T$ at coinciding arguments).

THEOREM 4. — *The intersection of the axiomatic domain with the manifold $\{k : k_{r+1} = \dots = k_n = 0\}$ (with $1 \leq r < n$) is identical with the corresponding domain of an r point function (in the variables k_1, \dots, k_r), with the same thresholds. The restriction of $H_{v_1 \dots v_n}$ to this manifold has all the linear properties of an r point function (including tempered boundary values) with the exception that the singularities at $\{k : k_j^2 = m^2(m_0, \lambda)\}$, $1 \leq j \leq r$, may now be multiple poles.*

Theorem 4 is a standard fact (see e. g. [EG]).

Remark. — The multiple poles at $\{k : k_j^2 = m^2\}$, (where $1 \leq j \leq r$) arise from the poles of the form $[(k_j + k_l)^2 - m^2]^{-1}$, (where $k_l \equiv \sum_{s \in I} k_s$, and I is a subset of $\{r+1, \dots, n\}$) when k_l is set equal to zero (as a consequence of k_{r+1}, \dots, k_n being set equal to zero).

THEOREM 5. — *Given P and $m_0 > 0$, there exists an $\varepsilon > 0$ such that all bounds on the boundary values of*

$$H_{v_1 \dots v_n}(k_1, \dots, k_r, k_{r+1}, \dots, k_n; m_0, \lambda) |_{k_{r+1} = \dots = k_n = 0},$$

from each Ruelle tube $[R]$ (in the sense of tempered distributions) are uniform in $0 \leq \lambda < \varepsilon m_0^2$.

Proof. — This follows from the uniform bounds on the generalized time ordered and retarded functions established in Part II (Proposition 1 and Theorem 2) and from the positivity of the physical mass $m(m_0, \lambda)$, Theorem 1, by general axiomatic results [EG, BEGS].

Next we consider the dependence of various functions on λ and m_0 . The proof of differentiability can be traced back to the fact that the various derivatives of Schwinger functions can be identified with sums of integrals of other Schwinger functions. Let the superscript T denote truncation (see e. g. [D1]). The above identification is summarized for $P(\Phi)_2$ in Lemma 6a and Lemma 6b.

LEMMA 6a. — [D1] *There is an $\varepsilon > 0$ (depending on P) such that the Schwinger functions are C^∞ in λ for $0 \leq \lambda/m_0^2 \leq \varepsilon$, and the derivatives are given by*

$$\begin{aligned} \frac{d}{d\lambda} S_{\Phi^{v_1 \dots v_n}}^T(x_1, \dots, x_n; m_0, \lambda) \\ = - \int d^2 y S_{\Phi^{v_1 \dots v_n, P(\Phi)}}^T(x_1, \dots, x_n, y; m_0, \lambda). \quad (\text{I.8}) \end{aligned}$$

LEMMA 6b. — *There is an $\varepsilon > 0$ (depending on P) such that for $0 \leq \lambda/m_0^2 \leq \varepsilon$ the Schwinger functions are C^∞ in m_0^2 , and the derivatives are given by*

$$\begin{aligned} \frac{d}{dm_0^2} S_{\Phi^{v_1} \dots \Phi^{v_n}}^T(x_1, \dots, x_n; m_0, \lambda) \\ = \sum_{k=1}^n c(m_0) v_k (v_k - 1) S_{\Phi^{v_1} \dots \Phi^{v_{k-2}} \dots \Phi^{v_n}}^T(x_1, \dots, x_n; m_0, \lambda) \\ - c(m_0) \lambda \int d^2 y S_{\Phi^{v_1} \dots \Phi^{v_n}, \Phi''(\Phi)}^T(x_1, \dots, x_n, y; m_0, \lambda) \\ - \frac{1}{2} \int d^2 y S_{\Phi^{v_1} \dots \Phi^{v_n}, \Phi^2}^T(x_1, \dots, x_n, y; m_0, \lambda), \end{aligned} \quad (I.9)$$

where

$$c(m_0) = \frac{1}{2} \int d^2 k (k^2 + m_0^2)^{-2}.$$

Proof. — The proof of the formal relation (I.9) is a consequence of equ. (I.1) and properties of Gaussian measures, see Lemma A1 and A2 for details. Both sides of equ. (I.9) are limits of corresponding expressions with cutoffs. The right hand side has a limit, when the cutoffs are removed because the truncated functions cluster and hence the integrals over y converge (this is the argument used by Dimock). The differentiability of S can be concluded from the following:

PRINCIPLE 1. — If $f_n(\mu)$ is a sequence of continuous functions of μ , with continuous derivatives $\partial_\mu f_n$, and if f_n and $\partial_\mu f_n$ converge uniformly in μ to f and f_1 respectively then f has a continuous derivative $\partial_\mu f = f_1$.

THEOREM 7. — *The function $H_{v_1 \dots v_n}(k; m_0, \lambda)$ is C^∞ in m_0^2 and λ in the interval $0 \leq \lambda/m_0^2 < \varepsilon$, and analytic in k in the n point axiomatic domain, with single particle poles at $k_j^2 = m^2(m_0, \lambda)$ and thresholds above $2m_0^2$.*

Proof. — By Theorem 3 and Lemma 6a,

$$\partial_\lambda [H_{v_1 \dots v_n}(k; m_0, \lambda) \Big|_{\substack{\text{Euclidean} \\ \text{points}}}] = - H_{v_1 \dots v_n, P}(k, q; m_0, \lambda) \Big|_{q=0} \Big|_{\substack{\text{Euclidean} \\ \text{points}}}$$

i. e., if k is any Euclidean point,

$$\begin{aligned} H_{v_1 \dots v_n}(k; m_0, \lambda) = H_{v_1 \dots v_n}(k; m_0, \lambda_1) \\ + \int_{\lambda_1}^{\lambda} d\lambda' H_{v_1 \dots v_n, P}(k, q = 0; m_0, \lambda'). \end{aligned} \quad (I.10)$$

If we now consider a point \hat{k} in the axiomatic domain of analyticity of $H_{v_1 \dots v_n}(k; m_0, \lambda)$, we can find λ_1 so close to λ that (by Theorems 1, 3 and 5) \hat{k} remains in the axiomatic domain of analyticity of $H_{v_1 \dots v_n}(k; m_0, \lambda')$ for all λ' in the interval $[\lambda_1, \lambda]$ (in particular the poles, which depend on λ' ,

will move sufficiently little to stay away from \hat{k}). As a consequence, and because H is uniformly bounded in λ , see Theorem 5, the formula (I.10) can be analytically continued to \hat{k} and this proves that

$$\partial_\lambda H_{v_1 \dots v_n}(k; m_0, \lambda) = -H_{v_1 \dots v_n, p}(k, q; m_0, \lambda)|_{q=0},$$

for all k in the axiomatic analyticity domain of the n -point function. By iterating this procedure, the existence of derivatives of all orders in λ is immediately obtained. The derivatives with respect to m_0^2 , and mixed derivatives with respect to λ and m_0^2 , are constructed in the same way, with the help of Lemma 6b.

THEOREM 8. — *There is an $\varepsilon > 0$ and for each $a > 0$ a C^∞ function $\lambda \rightarrow m_0(a, \lambda)$ on $0 \leq \lambda < \varepsilon a^2$ such that for $0 \leq \lambda < \varepsilon a^2$ one has*

$$m(m_0(a, \lambda), \lambda) = a.$$

Proof. — Consider the two point function

$$F(p^2; m_0, \lambda) = H_{11}(p, -p; m_0, \lambda).$$

Since $S_{\Phi\Phi}$ is the Schwinger function of a Wightman theory, and by the analysis of [GJS2, S],

$$[p^2 - m^2(m_0, \lambda)]F(p^2; m_0, \lambda)$$

extends to a holomorphic function in the cut plane $\mathbb{C} - (M^2 + \mathbb{R}^+)$, $M^2 \geq 2m_0^2$, $0 < m^2(m_0, \lambda) \leq \frac{3}{2}m_0^2$ (cf. also Theorem 1), i. e. F has only a pole at $m^2(m_0, \lambda)$ in this cut plane. Take the fixed contour $\Gamma = \{z \in \mathbb{C} \mid |z| = b\}$, with $\frac{7}{4}m_0^2 < b < 2m_0^2$ for all values of m_0^2 in a small interval. Γ encloses $m^2(m_0, \lambda)$ for $0 \leq \lambda \leq m_0^2 \varepsilon$, some $\varepsilon > 0$. By the Cauchy formula, we have

$$\begin{aligned} Z(m_0, \lambda)^2 &:= [p^2 - m^2(m_0, \lambda)]F(p^2; m_0, \lambda)|_{p^2=m^2(m_0, \lambda)} \\ &= \frac{1}{2\pi i} \oint_{\Gamma} F(z; m_0, \lambda) dz, \quad (\text{I.11}) \end{aligned}$$

and

$$\begin{aligned} m^2(m_0, \lambda)[p^2 - m^2(m_0, \lambda)]F(p^2; m_0, \lambda)|_{p^2=m^2(m_0, \lambda)} \\ = \frac{1}{2\pi i} \oint_{\Gamma} zF(z; m_0, \lambda) dz. \quad (\text{I.12}) \end{aligned}$$

The function $Z(m_0, \lambda)$ is the field strength renormalization (which is finite in two dimensional space-time). By Theorem 7, $F(z; m_0, \lambda)$ is C^∞ in m_0 and λ . Since $Z^2(m_0, 0) = 1$, we have $Z^2(m_0, \lambda) \neq 0$ for λ/m_0^2 sufficiently small so that we can divide equ. (I.12) by (I.11). We have thus shown ⁽²⁾.

⁽²⁾ Related results have been proved by Glimm-Jaffe and by Spencer.

LEMMA 9. — *There is an $\varepsilon > 0$ (depending on \mathbf{P}) such that the physical mass $m(m_0, \lambda)$ and the field strength $Z(m_0, \lambda)$ are C^∞ functions in λ and m_0 for $0 \leq \lambda/m_0^2 \leq \varepsilon$.*

The assertion of Theorem 8 can be reduced by Lemma 9 and the implicit function theorem to the claim: For $a > 0$,

$$\frac{\partial}{\partial m_0^2} m^2(m_0, \lambda) |_{m_0=a} \neq 0, \quad (\text{I.13})$$

for sufficiently small λ , $0 \leq \lambda < \eta \cdot a^2$, $\eta > 0$. To prove (I.13), observe that from scaling one has (in two dimensional space-time)

$$m^2(m_0, \lambda) = m_0^2 u(\lambda/m_0^2).$$

By Lemma 9, $u(t)$ is C^∞ in $0 \leq t < \varepsilon$, so

$$\partial_{m_0^2} m^2(m_0, \lambda) |_{m_0=a} = u(\lambda/a^2) - \lambda/a^2 u'(\lambda/a^2).$$

But this is different from zero for small λ/a^2 because $u(0) = 1$ and u' is bounded, hence (I.13) follows and this completes the proof of Theorem 8.

THEOREM 10. — (i) *For a $\mathbf{P}(\Phi)_2$ theory with physical mass $m > 0$ and a bare coupling constant λ satisfying $0 \leq \lambda < \varepsilon m^2$ ($\varepsilon > 0$ depending on \mathbf{P}), the function*

$$G(k_1, \dots, k_n; m, \lambda) = \left\{ \prod_{j=1}^n (k_j^2 - m^2) \right\} H_{1\dots 1}(k_1, \dots, k_n; m_0(m, \lambda), \lambda)$$

is C^∞ in λ and holomorphic in k_1, \dots, k_n (with $\sum_j k_j = 0$), in the « axiomatic domain » of the n -point function with thresholds above $(2/\sqrt{3})m$ and no single particle poles (at $k_j^2 = m^2$).

(ii) *The Taylor expansion of $G(k; m, \lambda)$ in λ at $\lambda = 0$, for k taken in the axiomatic domain (as described above), is given by standard perturbation theory.*

Proof. — (i) G has already been shown to be holomorphic in k in the above mentioned axiomatic domain (without one-particle-poles), and C^∞ in λ for $k_j^2 \neq m^2$. From this we conclude, by Principle 2 (below) that it is also C^∞ in λ at the points of the domain where $k_j^2 = m^2$ for some or all j . The assertion about the thresholds follows from Theorem 3.

PRINCIPLE 2. — Consider a function $f(z, \lambda)$ of $z \in \mathbb{C}$ and λ in some real interval I , holomorphic in z for $|z| < r$ and all $\lambda \in I$, jointly C^1 in z and λ

for $\frac{r}{8} < |z| < r$ and $\lambda \in I$, such that $\frac{\partial}{\partial \lambda} f(z, \lambda)$ is holomorphic in z and jointly continuous in z and λ for $\frac{r}{8} < |z| < r$, and $\lambda \in I$. The formula

$$f(z, \lambda) = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta f\left(z + \frac{r}{2} e^{i\vartheta}, \lambda\right),$$

valid for $|z| < \frac{r}{4}$, $\lambda \in I$ shows that $f(z, \lambda)$ is jointly continuous in z and λ whenever $|z| < r$, $\lambda \in I$. Moreover define

$$f_1(z, \lambda) = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \partial_\lambda f\left(z + \frac{r}{2} e^{i\vartheta}, \lambda\right),$$

for $|z| < \frac{r}{4}$, $\lambda \in I$. This function is analytic in z and jointly continuous in z and λ , and, for all $|z| < \frac{r}{4}$,

$$\int_{\lambda_0}^{\lambda} f_1(z, \lambda') d\lambda' = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \left[f\left(z + \frac{r}{2} e^{i\vartheta}, \lambda\right) - f\left(z + \frac{r}{2} e^{i\vartheta}, \lambda_0\right) \right].$$

This shows that f is jointly C^1 in z and λ , and holomorphic in z , in $|z| < r$, $\lambda \in I$, and $\frac{\partial f}{\partial \lambda}$ is holomorphic in z there.

Applying now Principle 2 successively to the variables k_j^2 , keeping k_{j+1}, \dots, k_n away from the poles, and iterating the procedure, we find that $G(k; m, \lambda)$ is C^∞ in λ and holomorphic in k in the axiomatic domain (without poles at $k_j^2 = m^2$) and the same is true of its successive derivatives in λ .

(ii) The r^{th} derivative of $G(\hat{k}; m, \lambda)$ with respect to λ (when \hat{k} is in the axiomatic domain) is the analytic continuation to \hat{k} of the restriction to Euclidean k of $\left(\frac{\partial}{\partial \lambda}\right)^r G(k; m, \lambda)$. The latter is, for $\lambda = 0$, the r^{th} order term of standard perturbation theory (evaluated at Euclidean points) by [D1]. The assertion (ii) now follows since perturbation theory has the axiomatic analyticity properties.

Theorem 10 is sufficient to prove that the analytic continuation of S-matrix elements to the complex points of the mass-shell which belong to the axiomatic domain, is C^∞ in λ and has a Taylor series at $\lambda = 0$ given by standard perturbation theory. Indeed this quantity is simply the restriction to the complex mass-shell $\{k : k_j^2 = m^2, \forall j\}$ of $Z(m_0(m, \lambda), \lambda)^{-n} G(k; \lambda)$.

We note that this is satisfactory for the case of the four-point function; in this case every physical point of the real mass-shell is on the boundary of the intersection of the mass-shell with the « axiomatic domain ». For

the n -point function, only a subset of the physical points enjoy this property (see e. g. [BEG]). This is still sufficient to prove that the S-matrix is non trivial in $P(\Phi)_2$ theories of degree ≥ 4 ⁽³⁾.

To strengthen Theorem 10, we turn our attention to the boundary values of G at real Minkowskian points. All the successive derivatives of G with respect to λ have boundary values, in the sense of tempered distributions, from each Ruelle tube, at the real points (this is inherited from the same property for functions of the form

$$H_{v_1 \dots v_n}(k_1, \dots, k_r, k_{r+1}, \dots, k_n) |_{k_{r+1} = \dots = k_n = 0}$$

as already noted above). In order to show that taking derivatives in λ and taking boundary values are commuting operations we use:

PRINCIPLE 3. — Let $\{F_n(\lambda)\}$ be a sequence of tempered distributions over \mathbb{R}^N depending differentiably (as elements of $\mathcal{S}'(\mathbb{R}^N)$) on a real parameter λ in some interval. Suppose that, for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and every integer r with $0 \leq r \leq s$, the limit

$$\lim_{n \rightarrow \infty} \left\langle \left(\frac{\partial}{\partial \lambda} \right)^r F_n(\lambda), \varphi \right\rangle = L_r(\varphi; \lambda) \quad (\text{I. 14})$$

exists and is bounded in λ . It then defines a tempered distribution depending on λ . For $0 \leq r < s$, the identity

$$\left\langle \left[\left(\frac{\partial}{\partial \lambda} \right)^r F_n(\lambda) - \left(\frac{\partial}{\partial \lambda} \right)^r F_n(\lambda_0) \right], \varphi \right\rangle = \int_{\lambda_0}^{\lambda} d\lambda' \left\langle \left(\frac{\partial}{\partial \lambda} \right)^{r+1} F_n(\lambda'), \varphi \right\rangle,$$

and the fact that $L_{r+1}(\varphi; \lambda')$ is bounded in λ' , show that $L_r(\varphi; \lambda)$ depends continuously on λ , and that the convergence in (I.14) is uniform in λ .

By Principle 1, it follows that $L_r(\varphi; \lambda) = \frac{\partial}{\partial \lambda} L_{r-1}(\varphi; \lambda)$ for $1 \leq r < s$.

We apply this to the case

$$\langle F_n(\lambda), \varphi \rangle = \int G\left(p + \frac{i}{n}q; m, \lambda\right) \varphi(p) dp$$

where q is chosen in the cone which is the base of one of the Ruelle tubes, and p runs over real Minkowski momentum space. The required boundedness in λ is supplied by Theorem 5. As a consequence, the boundary value (from any Ruelle tube) of $G(k; m, \lambda)$ is (as a tempered distribution depending on the parameter λ), infinitely differentiable in λ . Its r^{th} deri-

⁽³⁾ The statement depends on the definition of the word « non trivial ». We understand it to mean:

- 1) The cross-section is not zero.
- 2) The cross-section depends on the energy.

To check points 1) and 2), it suffices to look e. g. in Φ_2^4 at the two-two or three-three particle cross-section.

vative with respect to λ is the (tempered-distribution) boundary value of $\left(\frac{\partial}{\partial\lambda}\right)^r G(k; m, \lambda)$.

This proves :

LEMMA 11. — *The boundary values of $G(k; m, \lambda)$ from Ruelle tubes at Minkowskian momenta are tempered distributions depending in a C^∞ manner on λ ; their Taylor expansion at $\lambda = 0$ is given by standard perturbation theory. These properties are shared by the Fourier transform of the amputated truncated chronological function.*

The last assertion is due to the well-known fact [R] that the Fourier transform of the amputated truncated chronological function is obtained by patching together (by means of a partition of the unit independent of λ) the boundary values of G from the various Ruelle tubes.

THEOREM 12. — *At « non overlapping points » of the real mass-shell, the S-matrix elements of a $P(\Phi)_2$ theory, with fixed physical mass $m > 0$ and bare coupling constant λ , are C^∞ in λ as tempered distributions in the momenta for $0 \leq \lambda \leq \varepsilon m^2$ (where $\varepsilon > 0$ depends on P). Their Taylor expansion at $\lambda = 0$ is given by standard perturbation theory.*

Proof. — S-matrix elements are given by the restriction to the real mass-shell of $Z(m_0(m, \lambda), \lambda)^{-n} \tilde{\tau}_{\text{amp}}^T(p_1, \dots, p_n; m, \lambda)$ where $\tilde{\tau}_{\text{amp}}^T(p; m, \lambda)$ is a suitable boundary value of $G(k; m, \lambda)$. In view of the independence of the mass-shell on λ , and of the simple dependence on λ of $Z(m_0(m, \lambda), \lambda)^{-n}$, it suffices to prove the differentiability assertion for the restriction to the mass-shell of $\tilde{\tau}_{\text{amp}}^T$. The latter is, as a tempered distribution, restrictible to the mass-shell, at non-overlapping points, as a consequence of the linear properties of the n -point function possessed by $G(k; m, \lambda)$ (a well-known theorem of Hepp [H]). But these properties are also enjoyed by all the derivatives $\left(\frac{\partial}{\partial\lambda}\right)^r \tilde{\tau}_{\text{amp}}^T$. (Note in particular the absence of one particle poles proved in Theorem 10.) Therefore the $\left(\frac{\partial}{\partial\lambda}\right)^r \tilde{\tau}_{\text{amp}}^T$ are restrictible to the mass shell at non overlapping momenta. Theorem 5 shows that each of these restrictions (as a tempered distribution in the mass-shell variables) is locally bounded in λ . By applying Principle 3 once more, we find that

$$\left(\frac{\partial}{\partial\lambda}\right) \left[\left(\frac{\partial}{\partial\lambda}\right)^r \tilde{\tau}_{\text{amp}}^T \Big|_{p_i^2 = m^2} \right] = \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{r+1} \tilde{\tau}_{\text{amp}}^T \right\} \Big|_{p_i^2 = m^2} .$$

This and Lemma 11 yield Theorem 12.

PART II

**AXIOMATIC RESULTS ;
GENERALIZED TIME-ORDERED FUNCTIONS
IN THE EUCLIDEAN FORMULATION
OF QUANTUM FIELD THEORY**

II.1. Definitions and a reformulation of Theorem 3

For the convenience of the reader, we first recapitulate some well-known definitions and then reformulate Theorem 3 of Part I. We show that the conclusions of Theorem 3 are equivalent to the commutativity of a certain diagram (Diagram 1) which asserts connections between Wightman distributions, Schwinger functions, the momentum space analytic functions and generalized time-ordered and retarded functions. We note that (a large part of) the methods developed in Part II can be applied to (non-relativistic scattering theory and) other quantum field models for which Theorems 1 and 2 (or the bounds (I. 7)) are available. For the case of the massive sine-Gordon equation see [FS].

In most of Part II (except § II. 2. 2), whenever we speak of a Schwinger or Wightman function, we consider only its dependence on the time-variables, i. e. we may consider that it has been smeared out in the space components with suitable fixed test-functions (the dependence on which we suppress in our notation). We denote $w_\pi(x_1, \dots, x_n)$ the permuted Wightman function, w_π^T the truncated permuted Wightman function (x_1, \dots, x_n are, as announced, the time-components of the variables, $x_j \equiv x_j^0$; we shall, in general, omit the superscript 0). The corresponding analytic Wightman function $w(z_1, \dots, z_n)$ and its truncated version $w^T(z_1, \dots, z_n)$ are analytic in the union of the tubes \mathcal{T}_π , where π is any permutation of $(1, \dots, n)$ and

$$\mathcal{T}_\pi = \{ z = (z_1, \dots, z_n) \in \mathbb{C}^n ; z_j = x_j + iy_j \text{ and } y_{\pi 1} < y_{\pi 2} < \dots < y_{\pi n} \}.$$

We also use the variables

$$\zeta_j^\pi = \xi_j^\pi + i\eta_j^\pi = z_{\pi j} - z_{\pi(j+1)}, \quad 1 \leq j \leq n.$$

In the sense of tempered distributions,

$$w_\pi(x) = \lim_{\substack{y \rightarrow 0 \\ x + iy \in \mathcal{T}_\pi}} w(x + iy).$$

Similar properties hold for the truncated functions. These distributions depend only on the differences of the variables, and

$$\tilde{w}_\pi^T(p_1, \dots, p_n) \delta\left(\sum_{j=1}^n p_j\right) = (2\pi)^{-n} \int e^{i\sum_j p_j x_j} w_\pi^T(x) dx_1 \dots dx_n$$

defines a tempered distribution $\tilde{w}^T(p)$ with support in

$$\{p : P_j^\pi \geq \mu \text{ for all } j = 1, \dots, n-1\},$$

with $P_j^\pi = \sum_{1 \leq k \leq j} p_{\pi k}$. Here $\mu \geq 0$ and, in the case of a mass-gap, $\mu > 0$.

It may happen that the following other objects can be defined by some unspecified means:

1) Truncated Schwinger functions

By this we mean a set of symmetrical tempered distributions over \mathbb{R}^n , denoted $S^T(y_1, \dots, y_n)$ depending only on the differences $(y_j - y_k)$, and coinciding with $w^T(-iy_1, -iy_2, \dots, -iy_n)$ in case $y_{\pi 1} > y_{\pi 2} > \dots > y_{\pi n}$ for some π .

2) Truncated generalized time-ordered functions

By this we mean a system of tempered distributions denoted $t_{X_1, \dots, X_v}^T(x_1, \dots, x_n)$, corresponding heuristically to

$$\left(\Omega, T\left(\prod_{j \in X_1} \Phi(x_j)\right) \dots T\left(\prod_{k \in X_v} \Phi(x_k)\right)\Omega\right)^T,$$

and having all the corresponding linear properties (X_1, \dots, X_v form a partition of $\{1, \dots, n\}$). These t_{X_1, \dots, X_v}^T determine a unique system of generalized retarded functions (g. r. f.) denoted $r^\mathcal{S}(x_1, \dots, x_n)$. The Fourier transform of any g. r. f., $\tilde{r}^\mathcal{S}$, with

$$\tilde{r}^\mathcal{S}(p) \delta\left(\sum_{j=1}^n p_j\right) = (2\pi)^{-n} \int e^{i\sum_{j=1}^n p_j x_j} r^\mathcal{S}(x) dx_1 \dots dx_n,$$

is the boundary value, from a tube $\mathcal{T}^{\mathcal{L}}$, of the momentum space analytic function $H(k_1, \dots, k_n)$, holomorphic in a certain domain of

$$\left\{ k = (k_1, \dots, k_n) : \sum_j k_j = 0 \right\}.$$

[Note : according to the announced point of view, here $H(k_1, \dots, k_n)$ denotes the momentum space analytic function integrated over (real) space-variables with test-functions, i. e.

$$\int H((k_1, \bar{p}_1), \dots, (k_n, \bar{p}_n)) \delta\left(\sum_{j=1}^n \bar{p}_j\right) \prod_j f_j(\bar{p}_j) d\bar{p}_j.$$

Accordingly the domain of analyticity to be described is contained in the intersection of the « axiomatic » domain with $\{(k, \vec{k}) : \text{Im } \vec{k} = 0\}$.

$H(k)$ is holomorphic in

$$\left\{ k = (k_1, \dots, k_n) : \sum_j k_j = 0, \text{ and, } \forall J \in \mathcal{P}_*(X), k_J^2 \notin \mu^2 + \mathbb{R}^+ \right\}.$$

Here $\mathcal{P}_*(X)$ is the set of proper subsets of $X = \{1, \dots, n\}$ and $k_J = \sum_{j \in J} k_j$. A tube $\mathcal{T}^{\mathcal{L}}$ is any connected component of

$$\{k = (k_1, \dots, k_n) : \sum k_j = 0 \text{ and, } \forall J \in \mathcal{P}_*(X), \text{Im } k_J \neq 0\}.$$

We shall call the $\mathcal{T}^{\mathcal{L}}$ « Ruelle tubes » for short. For all these results, see [R].

The function H is polynomially bounded at infinity (in directions interior to its domain of analyticity) and, in particular, if there is a mass gap (i. e. $\mu > 0$), $H(iq_1, \dots, iq_n)$ is a real-analytic function of q_1, \dots, q_n real (with $\sum q_j = 0$). The Fourier transform of this function,

$$\hat{S}^T(y_1, \dots, y_n) = \int i^{n-1} e^{i \sum_{j=1}^n y_j q_j} \delta\left(\sum_j q_j\right) H(iq_1, \dots, iq_n) dq_1 \dots dq_n,$$

is a symmetric tempered distribution which, by a well-known argument (« Wick rotations ») which we do not reproduce (see e. g. [BEGS]) coincides with $w^T(-iy)$ whenever $y_j - y_k \neq 0$ for all $j \neq k$.

We wish to find sufficient conditions on S^T and on the generalized time ordered functions, for \hat{S}^T to coincide *everywhere* with S^T in the sense of distributions, or, in other words, for the following diagram to be commutative.

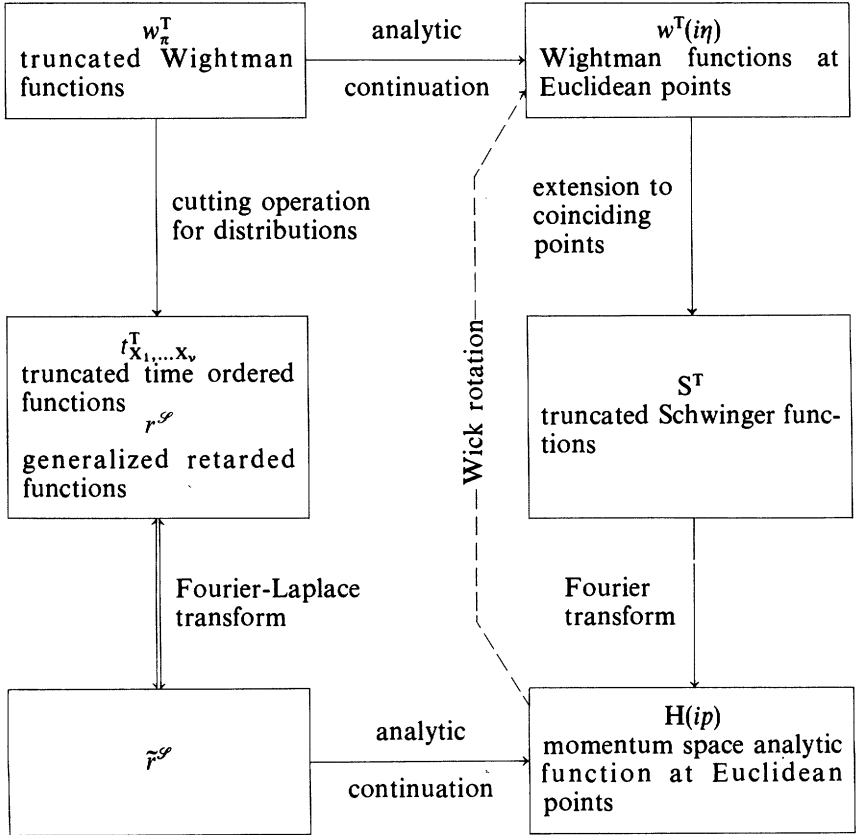


Diagram 1.

In the remainder of Part II, we show that local integrability of the Schwinger functions and Osterwalder-Schrader bounds [OS] of the form (I.7), together with the existence of a mass-gap, suffice to establish the commutativity of Diagram 1 (see Theorem 2, § II.2). This yields Theorem 3, § I.2. In the course of this proof, we derive bounds on the generalized time-ordered functions, which yield Theorem 5, § I.2.

Remark. — Another sufficient condition for the commutativity of Diagram 1 is that $S^T(y_1, \dots, y_n)$ and $r^\mathcal{S}(x_1, \dots, x_n)$ (recall that they are smeared out in the space-components) be locally L^2 . Then, in the presence of a mass-gap, $H(iq)$ can be shown to be L^2 as well as $\hat{S}^T = S^T$. (This observation is useful in the case of a theory which is the limit of theories whose n -point functions satisfy the L^2 -property. The commutativity of Diagram 1 is stable under taking limits.)

II.2. Construction of time-ordered functions and commutativity of Diagram 1

In this subsection we use the Euclidean formulation of quantum field theory and show how one may define generalized time-ordered functions in such a way that Diagram 1 is commutative, given a sequence of Schwinger functions satisfying the Osterwalder-Schrader axioms [OS] and the hypotheses of Theorem 3, § I.2.

II.2.1. A basic bound

We use a slight generalization of the « analytic continuation of bounds » provided by [OS]. In the following a Schwinger function $S(\eta_1, \dots, \eta_n)$ is supposed to be smeared out in the spatial variables with some fixed test functions. Let J and K be complementary subsets of $\{1, \dots, n\}$, and define

$$\Xi_J = \{ \zeta = \xi + i\eta \mid \eta_k < 0 \quad \forall k \text{ and } \zeta_k = 0 \text{ for } k \in J \}. \quad (\text{II.1})$$

PROPOSITION 1. — Let $\{S(y_1, \dots, y_{n+1})\}_{n=0}^{\infty}$ be a sequence of Schwinger functions satisfying the Osterwalder-Schrader axioms [OS] with, for $\eta_k < 0$,

$$|S(y_1, \dots, y_{n+1})| \leq (\alpha(n+1))^{\beta(n+1)} \prod_{j=1}^n (1 + |\eta_j|)^{\gamma} |\eta_j|^{-\delta}, \quad (\text{II.2})$$

for some finite $\alpha, \beta, \gamma, \delta$, and all n . Then there exist positive constants C, L, N independent of J and K (but depending on n) such that the restriction of the analytic continuation $w(z_1, \dots, z_{n+1})$ of $S(y_1, \dots, y_{n+1})$ to Ξ_J is bounded in modulus by

$$C \prod_{k \in K} (1 + |\zeta_k|)^L (-\eta_k)^{-N} \prod_{j \in J} (1 + |\zeta_j|)^{\gamma} (-\eta_j)^{-\delta}. \quad (\text{II.3})$$

Proof. — For $K = \emptyset$ (II.3) is trivial, and for $J = \emptyset$ the proposition is proved in [OS].

We consider the following functions of $B = |J_1| + |K_1|$ variables:

$$F_{(n)_1}(\{\eta\}_{K_1}) = \prod_{j \in J_1} (1 + |\eta_j|)^{-\gamma} |\eta_j|^{\delta} S(y_1, \dots, y_{B+1}). \quad (\text{II.4})$$

This function has an analytic extension in $\{\eta\}_{K_1}$ to all of Ξ_{J_1} because it

is the restriction of a function analytic in ζ_k on $\{\zeta : \eta \equiv \text{Im } \zeta < 0\}$, for all $k \in K_1$. This is seen by inserting T_{ζ_k} (where $\{T_t\}$ is the time-translation group on the physical Hilbert space obtained by Osterwalder-Schrader reconstruction $[G, OS]$) between times x_k^0 and x_{k+1}^0 ; see $[G, OS]$ for details. Therefore we may repeat the inductive analytic continuation of Schwinger functions derived in $[G, OS]$ for the case of the functions $F_{(m)J_1}(\{\zeta\}_{K_1})$. Let $\{\zeta\} \in \Xi_{J_1}$.

For $\{\zeta\}$ to be reached after the I^{th} inductive step in the inductive analytic continuation of $F_{(m)J_1}$ it is sufficient that $\{\zeta\}_{K_1}$ be reached after the I^{th} inductive step in the analytic continuation of $S(\{\zeta\}_{K_1})$ (which means that I as a function of $\{\zeta\} \in \Xi_{J_1}$ is independent of J_1).

We derive inductively bounds on $|F_{(m)J_1}(\{\zeta\}_{K_1})|$, for $\{\zeta\}_{K_1} \in C_{K_1}^{(I)}$, where $C_{K_1}^{(I)}$ is the set in $\mathbb{O}^{|K_1|}$ reached after I inductive steps. Let J_1 and K_1 be always complementary sets of integers. Let k_1 and k_2 be two consecutive elements of K_1 , (i. e. $\{k_1 + 1, k_1 + 2, \dots, k_2 - 1\} \subset J_1$, or $k_1 = 0$, $k_2 =$ smallest element of K_1 , or $k_1 =$ largest element of K_1 and $k_2 = |J_1| + |K_1| + 1$). We let $\rho(J_1, K_1)$ be the smallest integer such that $k_2 - k_1 \leq \rho(J_1, K_1)$, for all such choices of k_1, k_2 . For J and K as in the hypothesis of the proposition $\rho(J, K) \leq n$.

Let ε be some strictly positive number. We set

$$G_{(m)J_1}^\varepsilon(\{\zeta\}_{K_1}) = \prod_{k \in K_1} \{(1 - \zeta_k + \varepsilon)^{-\gamma} \varepsilon^\delta\} F_{(m)J_1}(\{\zeta - \varepsilon\}_{K_1}).$$

Induction hypothesis For all $M < I$, arbitrary J_1 and K_1 with $\rho(J_1, K_1) \leq n$ and for all $\{\zeta\}_{K_1} \in C_{|K_1|}^{(M)}$,

$$|G_{(m)J_1}^\varepsilon(\{\zeta\}_{K_1})| \leq (n\alpha(|K_1| + 1))^{n\beta(|K_1| + 1) + 2n\beta M(|K_1| + 1)} \quad (\text{II.5})$$

For $M = 0$ the induction hypothesis is a direct consequence of (II.1) and $\rho(J_1, K_1) \leq n$.

We now want to prove (II.5) for $M = I$. From the definition of $G_{(m)J_1}^\varepsilon$, the maximum principle for holomorphic functions and the Schwarz inequality with respect to the Osterwalder-Schrader inner product $[OS]$ we obtain

for all $\{\zeta\}_{K_1} \in C_{|K_1|}^{(I)}$,

$$|G_{(m)J_1}^\varepsilon(\{\zeta\}_{K_1})| \leq \max_{\substack{\{\zeta\}_{\bar{K}_l} \in C_{|\bar{K}_l}^{(I-1)} \\ l=1,2}} |G_{(m)\bar{J}_1}^\varepsilon(\{\zeta\}_{\bar{K}_1}) G_{(m)\bar{J}_2}^\varepsilon(\{\zeta\}_{\bar{K}_2})|^{1/2}, \quad (\text{II.6})$$

where $|\bar{J}_1| + |\bar{J}_2| = 2|J_1|$, $|\bar{K}_1| + |\bar{K}_2| = 2|K_1|$ and $\rho(\bar{J}_1, \bar{K}_1) \leq n$, $\rho(\bar{J}_2, \bar{K}_2) \leq n$, for all $\{\zeta\}_{K_1} \in C_{|K_1|}^{(I)}$; this follows easily from $[OS]$.

Applying now the induction hypothesis to the r. h. s. of (II. 6) we conclude that it is bounded by

$$\max_{k+l=2|K_1|} (n\alpha(k+1))^{n\beta(k+1)/2} (n\alpha(l+1))^{n\beta(l+1)/2} 2^{n\beta(|K_1|+1)(l-1)} \leq (n\alpha(|K_1|+1))^{n\beta(|K_1|+1)} 2^{n\beta(|K_1|+1)l},$$

which is (II. 5), for all $\{\zeta\}_{K_1} \in C_{K_1}^{(l)}$.

Thus the bound (II. 5) is true for all I and all J_1, K_1 with $\rho(J_1, K_1) \leq n$; ($K_1 \neq \emptyset$). We now set $2\epsilon = \min(-\eta_k), k \in K_1$ (as in [OS]).

Osterwalder and Schrader compute I as a function of the point $\{\zeta\}_{K_1}$ [OS], and this yields with (II. 5).

$$|F_{(n)J_1}(\{\zeta\}_{K_1})| \leq C \prod_{k \in K_1} (1 + |\zeta_k|)^L \sup_{k \in K_1} (-\eta_k)^{-N}, \tag{II. 7}$$

where C, L and N only depend on $\alpha, \beta, \gamma, \delta$ and n . If we now set $J_1 = J, K_1 = K$ and use definition (II. 4) of the function $F_{(n)J}(\{\eta\}_K)$ we obtain (II. 3) as a consequence of (II. 7), for all $\{\zeta\} \in \Xi_J$. Q. E. D.

In the following the conclusions of Proposition 1, in particular the bounds (II. 3) for some $\delta < 1$, and the local integrability of the Schwinger functions are our basic assumptions. All subsequent results follow from these assumptions, the positivity of the physical mass and the Wightman axioms.

We note that the hypotheses of Theorem 3, Part I, in particular inequality (I. 7), yield hypothesis (II. 1) of Proposition 1 for some $\delta < 1$. Theorem 2, Part I, proves (II. 1) with $\gamma = \delta = 0$ and $\beta = \bar{v}$. Thus all results of Part II apply to the $P(\Phi)_2$ models, under the assumptions of Theorem 1, or more generally to any models satisfying the hypotheses of Theorem 3, and yield proofs of Theorem 3 and 5.

II.2.2. Construction of time ordered functions

In this section we state the main result of Part II. Define

$$\Xi_J^\pi = \{z = x + iy \mid \eta_k^\pi < 0, \quad k = 1, \dots, n-1 \quad \text{and} \quad \xi_j^\pi = 0 \quad \text{for} \quad j \in J\}.$$

THEOREM 2. — Assume that there exist constants $C \geq 0, L \geq 0, N \geq 0, \delta \geq 0, L$ and N integer, $\delta < 1$, such that, for any two complementary subsets J, K of $\{1, \dots, n-1\}$, the restriction of w to the set Ξ_J^π is bounded in modulus by

$$C \left(1 + \sum_{r=1}^{n-1} |\zeta_r^\pi| \right)^L \prod_{j \in J} (-\eta_j^\pi)^{-\delta} \sup_{k \in K} (-\eta_k^\pi)^{-N} \tag{II. 8}$$

(for all permutations π). Then :

(i) It is possible to define in a natural manner

$$\prod_{j \in P} \theta(\zeta_j^\pi) \prod_{k \in Q} \theta(-\zeta_k^\pi) w_\pi(x),$$

(where P and Q are disjoint subsets of $\{1, \dots, n-1\}$), in particular as the limit, in the sense of tempered distributions, of

$$\prod_{j \in P} \theta(\zeta_j^\pi)(1 - \alpha(\zeta_j^\pi/\varepsilon)) \prod_{k \in Q} \theta(-\zeta_k^\pi)(1 - \alpha(\zeta_k^\pi/\varepsilon)) w_\pi(x),$$

(where α is a suitably chosen analytic function) as $\varepsilon \downarrow 0$ (see Lemma 4). These distributions yield a set of time ordered and generalized retarded functions which are all contained in a bounded subset of \mathcal{S}' depending only on C, L, N, n, δ .

(ii) If the w_π are the space smeared Wightman functions of a local theory, the so constructed generalized retarded functions, time ordered functions, etc. all have the usual support or causal properties [R].

(iii) If the Schwinger functions $S(y_1, \dots, y_n)$ are locally integrable extensions of $w(-iy_1, \dots, -iy_n)$ and if the \tilde{w}_π^T (Fourier transforms of truncated Wightman functions) have non-zero thresholds (i. e. if there is a mass gap) then Diagram 1 is commutative.

Proof. — The proof occupies the remainder of Part II. Standard arguments show:

LEMMA 3. — Let K and J be complementary subsets of $\{1, \dots, n-1\}$. For every $\varphi \in \mathcal{S}(\mathbb{R}^{K|})$ the limit

$$W_\pi^J(\varphi, \{\zeta^\pi\}_J) = \lim_{\eta_j^\pi \uparrow 0, k \in K} \int w(z) \varphi(\{\zeta^\pi\}_K) d\{\zeta^\pi\}_K \quad (\text{II.9})$$

exists and defines a function of $\{\zeta^\pi\}_J$ holomorphic in the tube

$$\{\zeta_j^\pi : \eta_j^\pi < 0, \forall j \in J\}.$$

If $J = P \cup Q$, $P \cap Q = \emptyset$ and $\text{Re} \{\zeta\}_P \equiv \{\zeta\}_P = 0$, then

$$\begin{aligned} & |W_\pi^J(\varphi, \{\zeta^\pi\}_J)| \\ & \leq C' \left(1 + \sum_{j \in J} |\zeta_j^\pi|\right)^L \prod_{s \in P} (-\eta_s^\pi)^{-\delta} \sup_{r \in Q} (-\eta_r^\pi)^{-N} \|\varphi\|_{n, n+L+N+3, N+1}, \end{aligned} \quad (\text{II.10})$$

with

$$\|\varphi\|_{n, L, N} = \sup_{|\alpha| \leq N} \sup_{(\zeta^\pi)_K} \left(n + \sum_{k \in K} |\zeta_k^\pi|\right)^L |D^\alpha \varphi(\{\zeta^\pi\}_K)|.$$

The constant C' depends only on C, N, L, n and δ occurring in (II. 8).

We shall also denote

$$W_{\pi}^J(\varphi, \{\zeta^{\pi}\}_J) = \int W_{\pi}^J(\{\xi^{\pi}\}_K, \{\zeta^{\pi}\}_J) \varphi(\{\xi^{\pi}\}_K) d\{\xi^{\pi}\}_K.$$

The inequality $\delta < 1$ will allow us to define in a natural manner tempered distributions in $\{\xi^{\pi}\}_K$, depending holomorphically on $\{\zeta^{\pi}\}_J$, formally given by

$$\left[\prod_{k \in K} \theta(\zeta_k^{\pi}) \right] W_{\pi}^J(\{\xi^{\pi}\}_K, \{\zeta^{\pi}\}_J),$$

and obeying a bound of the type of (II. 10). It will suffice to do this in the case $\pi = 1$ and we temporarily write ξ_j, η_j, ζ_j instead of $\xi_j^{\pi=1}, \eta_j^1, \zeta_j^1$, respectively. The case $K = \emptyset$ is trivial. For $K \neq \emptyset$ and any two disjoint subsets P and Q of K we define a tempered distribution in $\{\xi\}_K$ denoted

$$(P^+, Q^-)W^J(\{\xi\}_K, \{\zeta\}_J),$$

such that, for all $\varphi \in \mathcal{S}(\mathbb{R}^{|K|})$,

$$(P^+, Q^-)W^J(\varphi, \{\zeta\}_J) := \int (P^+, Q^-)W^J(\{\xi\}_K, \{\zeta\}_J) \varphi(\{\xi\}_K) d\{\xi\}_K$$

is holomorphic in $\{\zeta\}_J$ (for $\eta_j < 0$, all $j \in J$). The definition is the following:

Definition of $(P^+, Q^-)W^J(\varphi, \{\zeta\}_J)$:

1) Choose once and for all a function u of one complex variable such that:

- u is holomorphic in the half plane $\{\zeta \in \mathbb{C} : \text{Im } \zeta < 1\}$.
- $u^{(r)}(\zeta) \equiv \left(\frac{d}{d\zeta}\right)^r u(\zeta)$ vanishes for $\zeta = 0$ whenever $1 \leq r \leq N + 1$ and $u(0) = 1$.
- for all $\zeta \in \mathbb{C}$ with $\text{Im } \zeta < 1$, and all $r \leq N + 1$,

$$|u^{(r)}(\zeta)| < \text{Const. } (1 + |\zeta|)^{-A}$$

with $A \geq L + 2N + n + 6$.

[Example:

$$u(\zeta) = \sum_{2 \leq l \leq N+4} c_l (il - \zeta)^{-A}.$$

The c_l are determined so that $u^{(r)}(0) = \delta_{r,0}$ by a Vandermonde linear system].

For each $k \in K$, denote

$$(\mathcal{X}_{u,k}\varphi)(\{\xi\}_K) = u(\xi_k) \sum_{r=0}^{N+1} \frac{\xi_k^r}{r!} \left[\left(\frac{\partial}{\partial \xi_k} \right)^r \varphi(\{\xi\}_K) \Big|_{\xi_k=0} \right].$$

This extends to a function holomorphic in ξ_k in $\{\zeta_k = \xi_k + i\eta_k : \eta_k < 1\}$.

2) For every $\varphi \in \mathcal{S}(\mathbb{R}^{|K|})$,

$$P \subset K, \quad Q \subset K, \quad P \cap Q = \emptyset, \quad K \setminus (P \cup Q) = K',$$

define ⁽⁴⁾,

$$\begin{aligned}
 & (P^+, Q^-)W^J(\varphi, \{\zeta\}_J) \\
 &= \sum_{\bar{P} \subset P, \bar{Q} \subset Q} \int d\{\xi\}_{\bar{P} \cup \bar{Q} \cup K'} \prod_{k \in P \setminus \bar{P}} \int_0^{-\infty} id\eta_k \prod_{q \in Q \setminus \bar{Q}} \int_0^{-\infty} -id\eta_q \\
 & \times W^{J \cup (P \setminus \bar{P}) \cup (Q \setminus \bar{Q})}(\{\xi\}_{\bar{P} \cup \bar{Q} \cup K'}, [\{i\eta\}_{(P \setminus \bar{P}) \cup (Q \setminus \bar{Q})}, \{\zeta\}_J]) \\
 & \times \prod_{k \in \bar{P}} \theta(\xi_k)(1 - \mathcal{H}_{u,k}) \prod_{q \in \bar{Q}} \theta(-\xi_q)(1 - \mathcal{H}_{u,q}) \prod_{s \in (P \setminus \bar{P}) \cup (Q \setminus \bar{Q})} \mathcal{H}_{u,s} \\
 & \times \varphi(\{\xi\}_{\bar{P} \cup \bar{Q} \cup K'}, \{i\eta\}_{(P \setminus \bar{P}) \cup (Q \setminus \bar{Q})}). \tag{II.11}
 \end{aligned}$$

This definition makes sense because (e. g.)

$$\prod_{k \in \bar{P}} \theta(\xi_k)(1 - \mathcal{H}_{u,k}) \prod_{q \in \bar{Q}} \theta(-\xi_q)(1 - \mathcal{H}_{u,q}) \varphi(\{\xi\}_K)$$

has continuous derivatives of orders $\leq N + 1$ and is an admissible test function for $W^J(\{\xi\}_K, \{\zeta\}_J)$. The integrals over the η_j variables in the last term of the definition converge absolutely because of the bound of the type $\prod_s (-\eta_s)^{-\delta}$ satisfied by W^J . One can easily verify that a bound of the form ^s

$$\begin{aligned}
 |(P^+, Q^-)W^J(\varphi, \{\zeta\}_J)| &\leq C'' \left(1 + \sum_{j \in J} |\zeta_j|\right)^L \prod_{s \in S} (-\eta_s)^{-\delta} \sup_{r \in \mathbb{R}} (-\eta_r)^{-N} \\
 &\times \|\varphi\|_{n, n+L+N+3, n(N+1)}, \tag{II.12}
 \end{aligned}$$

holds (here $R \cup S = J$, $R \cap S = \emptyset$, and $\xi_s = 0$ for all $s \in S$).

The constant C'' depends only on the constants appearing in (II.8) and on the choice of u . This completes the proof of the first half of part (i) of Theorem 2.

We need the following two spaces of holomorphic functions of one complex variable:

$$\mathcal{H}_+(B, N + 1) \quad (\text{resp. } \mathcal{H}_-(B, N + 1))$$

is the space of functions χ of one complex variable, holomorphic in the quarter plane $\{\zeta \in \mathbb{C} : \text{Re } \zeta > -1, \text{Im } \zeta < 1\}$ (resp. holomorphic in $\{\zeta \in \mathbb{C} : \text{Re } \zeta < 1, \text{Im } \zeta < 1\}$), and bounded there by:

$$\left| \left(\frac{d}{d\zeta}\right)^r \chi(\zeta) \right| \leq C(\chi)(1 + |\zeta|)^{-B}$$

for all $r \leq N + 1$.

⁽⁴⁾ We write $1 = (1 - \mathcal{H}_{u,s}) + \mathcal{H}_{u,s}$ and expand the product over $s \in P \cup Q$. Then we turn integration contours in the variables for which the $\mathcal{H}_{u,s}$ factor has been chosen.

The following lemma completes the proof of Theorem 2 (i).

LEMMA 4. — *The tempered distributions $(P^+, Q^-)W^J(\varphi, \{\zeta\}_J)$ defined by the equation (II.11) have the following properties*

- (i) *They verify the bound (II.12) which depends only on the bound (II.8).*
 (ii) *If $\varphi(\{\xi\}_K)$ is of the form*

$$\varphi(\{\xi\}_K) = \left[\prod_{k \in P_1 \cup Q_1} \chi_k(\xi_k) \right] \psi(\{\xi\}_{K \setminus (P_1 \cup Q_1)}),$$

where $P_1 \subset P$, $Q_1 \subset Q$, $P \setminus P_1 = P_2$, $Q \setminus Q_1 = Q_2$, $K \setminus (P_1 \cup Q_1) = K_2$, and $\chi_k \in \mathcal{H}_+(\mathbf{B}, N+1)$ for all $k \in P_1$, $\chi_k \in \mathcal{H}_-(\mathbf{B}, N+1)$ for all $k \in Q_1$, with $\mathbf{B} \geq L + 2N + n + 6$ then

$$(P^+, Q^-)W^J(\varphi, \{\zeta\}_J) = \left[\prod_{k \in P_1} \int_0^{-\infty} id\eta_k \right] \left[\prod_{k \in Q_1} \int_0^{-\infty} -id\eta_k \right] \left[\prod_{k \in P_1 \cup Q_1} \chi_k(i\eta_k) \right] (P_2^+, Q_2^-)W^{J \cup P_1 \cup Q_1}(\psi, \{\zeta\}_{J \cup P_1 \cup Q_1}) \Big|_{\substack{\zeta_k = i\eta_k \\ k \in P_1 \cup Q_1}} \quad (\text{II.13})$$

(iii) *If α is a function of one complex variable, $\alpha^{(r)}(\zeta)$ holomorphic in $\{\zeta \in \mathbb{C} : \text{Im } \zeta < 1\}$, bounded there by $\text{const.} \times (1 + |\zeta|)^{-A}$ together with its derivatives of order $\leq N+1$, ($A \geq L + 2N + n + 6$), and such that $\alpha^{(r)}(0) = \delta_{r0}$ for all $0 \leq r \leq N+1$, then, for all $\varphi \in \mathcal{S}(\mathbb{R}^{|\mathbf{K}|})$,*

$$(P^+, Q^-)W^J(\varphi, \{\zeta\}_J) = \lim_{\varepsilon_p \downarrow 0, \varepsilon_q \in P \cup Q} \int W^J(\{\xi\}_K, \{\zeta\}_J) \prod_{p \in P} \theta(\xi_p)(1 - \alpha(\xi_p/\varepsilon_p)) \prod_{q \in Q} \theta(-\xi_q)(1 - \alpha(\xi_q/\varepsilon_q)) \varphi(\{\xi\}_K) d\{\xi\}_K. \quad (\text{II.14})$$

The permuted objects $(P^+, Q^-)W_\pi^J$ have, mutatis mutandis, identical properties.

Proof. — Part (i) follows from the original definition.

It immediately follows from the definition that, if $a \in K \setminus (P \cup Q)$, and $\varphi \in \mathcal{S}(\mathbb{R}^{|\mathbf{K}|})$,

$$\begin{aligned} & ((P \cup \{a\})^+, Q^-)W^J(\varphi, \{\zeta\}_J) \\ &= \int (P^+, Q^-)W^J(\{\xi\}_K, \{\zeta\}_J) \theta(\xi_a)(1 - \mathcal{X}_{u,a}) \varphi(\{\xi\}_K) d\{\xi\}_K \\ &+ \int_0^{-\infty} id\eta_a \int (P^+, Q^-)W^{J \cup \{a\}}(\{\xi\}_{K \setminus \{a\}}, (i\eta_a, \{\zeta\}_J)) \\ &\times (\mathcal{X}_{u,a} \varphi)(\{\xi\}_{K \setminus \{a\}}, i\eta_a) d\{\xi\}_{K \setminus \{a\}}, \end{aligned} \quad (\text{II.15})$$

and a similar expression for $(P^+, (Q \cup \{a\})^-)W^j(\varphi, \{\zeta\}_j)$. As a consequence (ii) can be proved by induction on $|P|$ and $|Q|$, so that it suffices to treat a one dimensional example.

Let g be a function of one complex variable $\zeta = \xi + i\eta$, holomorphic in $\{\zeta \in \mathbb{C} : \eta < 0\}$ and satisfying:

$$\begin{aligned} |g(\zeta)| &< (1 + |\zeta|)^L (-\eta)^{-N}, \\ |g(i\eta)| &< (1 + |\eta|)^L (-\eta)^{-\delta}, \quad \delta < 1. \end{aligned}$$

Let g_b denote its boundary value in the sense of $\mathcal{S}'(\mathbb{R})$. It can be simply obtained by writing:

$$\begin{aligned} g(\zeta) &= \left(\frac{\partial}{\partial \zeta}\right)^{N+1} h_{N+1}(\zeta), \\ h_r(\zeta) &= \int_{\zeta_0}^{\zeta} h_{r-1}(\zeta') d\zeta', \quad h_0 = g, \quad \zeta_0 \text{ fixed with } \text{Im } \zeta_0 < 0. \end{aligned}$$

Then h_{N+1} is continuous in the closed lower half plane, and

$$\int g_b(\xi)\varphi(\xi)d\xi \equiv \int_{-\infty}^{+\infty} (-1)^{N+1} h_{N+1}(\xi) \left(\frac{\partial}{\partial \xi}\right)^{N+1} \varphi(\xi) d\xi \quad (\text{II. 16})$$

Define g_b^\pm as a tempered distribution as follows: $\forall \varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \int g_b^\pm(\xi)\varphi(\xi)d\xi &= \int g_b(\xi)[\theta(\pm \xi)(1 - \mathcal{X}_u)\varphi(\xi)]d\xi \\ &\quad \pm \int_0^{-\infty} id\eta g(i\eta)\mathcal{X}_u\varphi(i\eta). \end{aligned}$$

Note that $\mathcal{X}_u\varphi(\zeta)$ is holomorphic in $\{\zeta = \xi + i\eta\}$ and given by

$$\mathcal{X}_u\varphi(\zeta) = u(\zeta) \sum_{r=0}^{N+1} \frac{\zeta^r}{r!} \varphi^{(r)}(0),$$

with $\varphi^{(r)}(\zeta) = \left(\frac{d}{d\zeta}\right)^r \varphi(\zeta)$. The other term in the definition makes sense because $\theta(\pm \xi)(1 - \mathcal{X}_u)\varphi(\xi)$ has continuous derivatives of orders $\leq N+1$ and the definition (II. 16) can be used. If $\varphi^{(r)}$ vanishes at 0 for all $r \leq N+1$, the definition coincides with

$$\begin{aligned} \int g_b(\xi)\theta(\pm \xi)\varphi(\xi)d\xi &= \int h_{N+1}(\xi)(-1)^{N+1} \left(\frac{d}{d\xi}\right)^{N+1} [\theta(\pm \xi)\varphi(\xi)]d\xi \\ &= \int h_{N+1}(\xi)(-1)^{N+1} \theta(\pm \xi)\varphi^{(N+1)}(\xi)d\xi, \end{aligned}$$

and therefore g_b^\pm has support in \mathbb{R}^\pm . Moreover, for any $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle g^+ + g^-, \varphi \rangle = \int g_b(\xi)(1 - \mathcal{X}_u)\varphi(\xi)d\xi,$$

$$\int g_b(\xi)(\mathcal{X}_u\varphi)(\xi)d\xi = \int_{-\infty}^{+\infty} h_{N+1}(\xi)(-1)^{N+1}\left(\frac{\partial}{\partial\xi}\right)^{N+1}(\mathcal{X}_u\varphi)(\xi)d\xi.$$

Since $\left(\frac{\partial}{\partial\xi}\right)^{N+1}\mathcal{X}_u\varphi(\xi)$ is holomorphic in the lower half plane with sufficient decrease at infinity, we can close the contour and find 0. Therefore $g_b^+ + g_b^- = g_b$.

If $\chi \in \mathcal{H}_+(A, N+1)$ for sufficiently large A , we wish to prove that

$$\int g_b^+(\xi)\chi(\xi)d\xi = \int_0^{-\infty} id\eta g(i\eta)\chi(i\eta), \quad (\text{II.17})$$

by proving it for $\mathcal{X}_u\chi$ and $(1 - \mathcal{X}_u)\chi$. But, for $\mathcal{X}_u\chi$, this follows immediately from the definition, and it suffices to prove (II.17) for $(1 - \mathcal{X}_u)\chi$, i. e. it suffices to prove (II.17) when $\chi^{(r)}(0) = 0$ for all $r \leq N+1$. For such a χ ,

$$\begin{aligned} \int g_b^+(\xi)\chi(\xi)d\xi &= \int g_b(\xi)\theta(\xi)\chi(\xi)d\xi \\ &= (-1)^{N+1} \int_{-\infty}^{\infty} h_{N+1}(\xi)\left(\frac{\partial}{d\xi}\right)^{N+1}[\theta(\xi)\chi(\xi)]d\xi \\ &= (-1)^{N+1} \int_0^{\infty} h_{N+1}(\xi)\chi^{(N+1)}(\xi)d\xi. \end{aligned}$$

We can now rotate the contour of integration and obtain

$$\begin{aligned} (-1)^{N+1} \int_0^{-\infty} h_{N+1}(i\eta)\chi^{(N+1)}(i\eta)id\eta \\ = i \int_0^{-\infty} g(i\eta)\chi(i\eta)d\eta - \sum_{r=1}^{N+1} \lim_{\varepsilon \downarrow 0} (-1)^r h_r(-i\varepsilon)\chi^{(r-1)}(-i\varepsilon), \end{aligned}$$

and the last terms tend to 0. In the same manner we verify that, if $\chi \in \mathcal{H}_-(A, N+1)$, A sufficiently large,

$$\int g_b^-(\xi)\chi(\xi)d\xi = - \int_0^{-\infty} id\eta g(i\eta)\chi(i\eta). \quad (\text{II.18})$$

Finally consider a function α of one complex variable having the properties postulated for u (e. g. α could be u itself). We wish to show that, for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\int g_b^+(\xi)\varphi(\xi)d\xi = \lim_{\varepsilon \downarrow 0} \int g_b(\xi)\theta(\xi)(1 - \alpha(\xi/\varepsilon))\varphi(\xi)d\xi. \quad (\text{II.19})$$

Consider first a C^∞ function ψ on \mathbb{R} with derivatives $\psi^{(r)}$ such that $\psi^{(r)}(0) = 0$ for $0 \leq r \leq N + 1$ (and sufficiently decreasing at infinity). Then, for $r \leq N + 1$,

$$\psi^{(r)}(\xi) = \xi^{N+2-r} \varphi_r(\xi),$$

with continuous φ_r , and $(\theta\psi)^{(r)}(\xi) = \xi^{N+2-r} \theta(\xi) \varphi_r(\xi)$. Assume

$$|\varphi_r(\xi)| \leq \text{const.} (1 + |\xi|)^{-L-3-N}.$$

From (II.16) we know that, for any $f \in \mathcal{S}(\mathbb{R})$,

$$\left| \int g_b(\xi) f(\xi) d\xi \right| \leq \text{const.} \sup_{\xi, \alpha \leq N+1} (1 + |\xi|)^{L+N+1} |f^{(\alpha)}(\xi)|.$$

In particular, if

$$M(\varepsilon) = \int g_b(\xi) \alpha(\xi/\varepsilon) \theta(\xi) \psi(\xi) d\xi,$$

$|M(\varepsilon)|$ is bounded by a sum of terms of the form

$$\text{const.} \sup_{\xi} (1 + |\xi|)^{L+N+1} \varepsilon^{-r} |\alpha^{(r)}(\xi/\varepsilon)| |\xi|^{N+2-t} (1 + |\xi|)^{-L-N-3},$$

(with $r + t \leq N + 1$). This is bounded by

$$\text{const.} \sup_{\xi} \varepsilon^{-r} \frac{|\xi|^{N+2-t}}{(1 + |\xi/\varepsilon|)^A} \leq \text{const.} \varepsilon^{N+2-r-t} \sup_{\xi} \frac{|\xi|^{N+2-t}}{(1 + |\xi|)^A}.$$

Hence this is bounded by $\text{const.} \varepsilon$, if $A \geq N + 2$, and $M(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

On the other hand, if $\chi \in \mathcal{H}_+(A, N + 1)$, we have

$$\int g(\xi) \theta(\xi) (1 - \alpha(\xi/\varepsilon)) \chi(\xi) d\xi = \int_0^{-\infty} g(i\eta) (1 - \alpha(i\eta/\varepsilon)) \chi(i\eta) i d\eta,$$

by the preceding verification (case of an analytic test function vanishing at 0 with its $N + 1$ first derivatives). Also,

$$\begin{aligned} & \left| \int_0^{\infty} g(i\eta) \alpha(\varepsilon^{-1} i\eta) \chi(i\eta) i d\eta \right| \\ & \leq \int_{-\infty}^0 |g(i\eta) \chi(i\eta)| B(1 + \varepsilon^{-1} |\eta|)^A d\eta \\ & \leq \varepsilon^{1-\delta} \text{const.} \int_{-\infty}^0 d\eta (1 + |\eta|)^{-A} |\eta|^{-\delta} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \end{aligned}$$

This proves our assertion. The same holds for g_b^- (with the same α).

Applying the same technique in several independent variables we obtain part (iii) of the lemma.

As a special case of this lemma, the tempered distributions

$$\prod_{j \in P} \theta(\zeta_j^\pi) \prod_{k \in Q} \theta(-\zeta_k^\pi) w_\pi(x)$$

are well defined, with $P \cup Q = \{1, \dots, n-1\}$, $P \cap Q = \emptyset$, and, if $\chi_j \in \mathcal{H}_+(A, N+1)$ for all $j \in P$, $\chi_k \in \mathcal{H}_-(A, N+1)$ for all $k \in Q$, with $A \geq L + 2N + n + 6$, we have:

$$\begin{aligned} & \int \prod_{j \in P} \theta(\zeta_j^\pi) \chi_j(\zeta_j^\pi) \prod_{k \in Q} \theta(-\zeta_k^\pi) \chi_k(\zeta_k^\pi) w_\pi(x) d\zeta_1^\pi \dots d\zeta_{n-1}^\pi \\ &= (-1)^{|Q|} \int_0^{-\infty} id\eta_1^\pi \chi_1(i\eta_1^\pi) \dots \int_0^{-\infty} id\eta_{n-1}^\pi \chi_{n-1}(i\eta_{n-1}^\pi) w(i\eta). \quad (\text{II.20}) \end{aligned}$$

This completes the proof of Lemma 4, hence of Theorem 2 (i).

II.2.3. Verification of locality

(Proof of Theorem 2 (ii).)

We now turn to the aspect of the definition which concerns space-like components when they occur. They were ignored in the preceding discussion, which is valid e. g. in (not necessarily relativistic) theories where space-like components have been integrated with test-functions. For this paragraph, we reinstate the space-like components, denoted \bar{x}_j , with the corresponding $\bar{\zeta}_j^\pi = \bar{x}_{\pi j} - \bar{x}_{\pi(j+1)}$. They will remain real and essentially play the role of spectators. The notations x_j , ζ_j^π , ζ_j^π will be retained for the time components, and a four-component variable will be denoted e. g. (x_j, \bar{x}_j) or $(\zeta_j^\pi, \bar{\zeta}_j^\pi)$.

To verify that our definition of $\prod_{j \in I} \theta(\zeta_j^\pi) w_\pi(x, \bar{x})$ has the locality suggested by this heuristic notation (under the assumption that the original Wightman functions $w_\pi(x, \bar{x})$ have the correct locality property) we shall mainly rely on the limiting procedure described in (II.14) (Lemma 4 (iii)). The prototype of the locality property to be checked is as follows: let π be a permutation and K, L two disjoint subsequences of the form

$$\begin{aligned} K &= \pi(r), \pi(r+1), \dots, \pi(t+1), \\ L &= \pi(s), \pi(s+1), \dots, \pi(v+1), \quad \text{with} \quad t+1 < s. \end{aligned}$$

Let σ be the permutation obtained by interchanging the two subsequences K

and L, all other relative orders being preserved. Then, denoting $J = \{ \pi(r), \dots, \pi(t), \pi(s), \dots, \pi(v) \}$, we must have:

$$\prod_{j \in J} \theta(\xi_j^\pi) w_\pi(x, \bar{x}) = \prod_{j \in J} \theta(\xi_j^\pi) w_\sigma(x, \bar{x}).$$

in the region where the variables labelled by K are space-like separated from those labelled by L and by $\pi(t+2), \dots, \pi(s-1)$, and the latter are space-like separated from those labelled by L. Note that

$$\prod_{j \in J} \theta(\xi_j^\pi) = \prod_{k \in R} \theta(\xi_k^\sigma), \quad \text{where } R = \sigma \circ \pi^{-1}(J).$$

We can thus apply the limiting procedure (II.14) with the same α function simultaneously in all the variables $\{ \xi_j^\pi \}_{j \in J}$ (which are also the variables $\{ \xi_k^\sigma \}_{k \in R}$). Before the limit is taken, the identity holds because of the locality of the w_π , and it remains true in the limit.

As a consequence, the generalized retarded functions and time-ordered functions obtained by this construction have all the geometrical and algebraic properties usually postulated in general field theory ([R]). The momentum space analytic function is well defined and holomorphic in the usual « axiomatic domain » (see e. g. [BEGS]). To investigate the commutativity of Diagram 1, we can again restrict our attention to time-components.

II.2.4. The momentum space analytic function in the time-components

(Proof of Theorem 2 (iii)).

From now on, space-like components of the variables will disappear again, and we shall deal with the truncated permuted Wightman functions denoted w_π^T (and to which the preceding theory applies unchanged). Denote

$$h_\pi(x) = \prod_{j=1}^{n-1} \theta(\xi_j^\pi) w_\pi^T(x),$$

$$\delta \left(\sum_{j=1}^n p_j \right) \tilde{h}_\pi(p) = (2\pi)^{-n} \int dx_1 \dots dx_n h_\pi(x) \exp i \sum_{j=1}^n p_j x_j, \quad (\text{II.21})$$

$$\delta \left(\sum_{j=1}^n p_j \right) \tilde{w}_\pi^T(p) = (2\pi)^{-n} \int dx_1 \dots dx_n w_\pi^T(x) \exp i \sum_{j=1}^n p_j x_j. \quad (\text{II.22})$$

For a fixed π , it is convenient to use the variables $P_j^\pi = \sum_{r=1}^j p_{\pi r}$, since $\sum p_k x_k = \sum_{j=1}^{n-1} P_j^\pi \xi_j^\pi$. This gives

$$\tilde{w}_\pi^T(p) = (2\pi)^{-(n-1)} \int w_\pi^T(x) \exp i \sum_{j=1}^{n-1} P_j^\pi \xi_j^\pi d\xi_1^\pi \dots d\xi_{n-1}^\pi, \quad (\text{II. 23})$$

$$\tilde{h}_\pi(p) = (2\pi)^{-(n-1)} \int \prod_{j=1}^{n-1} \theta(\xi_j^\pi) w_\pi^T(x) \exp i \sum_{j=1}^{n-1} P_j^\pi \xi_j^\pi d\xi_1^\pi \dots d\xi_{n-1}^\pi. \quad (\text{II. 24})$$

We assume the thresholds of the theory are bounded below by $\mu > 0$ so that the support of \tilde{w}_π^T is contained in

$$\{ p : P_j^\pi \geq \mu, j = 1, \dots, n-1 \}.$$

Since \tilde{w}_π^T is tempered we can write

$$\tilde{w}_\pi^T(p) = \left(\prod_{j=1}^{n-1} \frac{\partial}{\partial P_j^\pi} \right)^V F_\pi(P_1^\pi, \dots, P_{n-1}^\pi),$$

where F_π is continuous and polynomially bounded and has the same support property. As a consequence, if $\eta_j^\pi < 0$ for all $j = 1, \dots, n-1$, we have (for some $V \geq 0$ and some $Z \geq 0$)

$$|w_\pi^T(iy)| \leq \text{const.} \left[\prod_{j=1}^{n-1} (-\eta_j^\pi)^V \exp(\mu \eta_j^\pi) \right] \left[\prod_k (-\eta_k^\pi) \right]^{-Z}.$$

From this it follows that for $\eta_j < 0, \forall j$,

$$\left| w^T(i\eta) \exp \sum_{k=1}^{n-1} (-i P_k^\pi \eta_k^\pi) \right| \leq B_1(\eta),$$

$$B_1(\eta) = \text{const.} \left[\prod_k (-\eta_k^\pi) \right]^{-Z} \exp \left(\frac{\mu}{2} \sum_{j=1}^{n-1} \eta_j^\pi \right).$$

On the other hand, we may assume without loss of generality that $Z > 3/2$ and choose $\bar{\delta}, \delta < \bar{\delta} < 1$. Then by (II.8), if $|P_j^\pi| < (1/2)\mu(1 - \bar{\delta})/(4(Z - 1))$ $j = 1, \dots, n-1$, we also have

$$\left| w^T(i\eta) \exp \sum_{k=1}^{n-1} (-i P_k^\pi \eta_k^\pi) \right| \leq B_2(\eta),$$

$$B_2(\eta) = \text{const.} \prod_{k=1}^{n-1} (-\eta_k^\pi)^{-\delta} \exp\left(\frac{\mu(1-\delta)}{4(Z-1)} \sum_{j=1}^{n-1} -\eta_j^\pi\right).$$

By combining the two bounds, we get a bound

$$\begin{aligned} \left| w^T(i\eta) \exp \sum_{k=1}^{n-1} (-iP_k^\pi \eta_k^\pi) \right| &\leq B_1(\eta)^{\frac{1-\delta}{Z-\delta}} B_2(\eta)^{\frac{Z-1}{Z-\delta}} \\ &\leq \text{const.} \prod_k (-\eta_k^\pi)^{-1 + \frac{(\delta-\delta)(Z-1)}{Z-\delta}} \exp\left(\frac{\mu(1-\delta)}{4(Z-1)} \sum_{j=1}^{n-1} \eta_j^\pi\right), \end{aligned}$$

which is L^1 for the measure $\prod_{j=1}^{n-1} d\eta_j^\pi \theta(-\eta_j^\pi)$.

Denote temporarily

$$\rho_\pi(ip) = (2\pi)^{-(n-1)} \int w^T(iy) \prod_{1 \leq k \leq n-1} [\theta(-\eta_k^\pi) \exp(-iP_k^\pi \eta_k^\pi) (-id\eta_k^\pi)]. \quad (\text{II.25})$$

The above estimate shows that $\rho_\pi(ip)$ is (for real p) the restriction of a function, denoted $\rho_\pi(k)$, holomorphic in

$$\left\{ k = (k_1, \dots, k_n) : \sum k_j = 0 \quad \text{and} \quad |\text{Re } K_j^\pi| < \frac{\mu(1-\delta)}{8(Z-1)}, \forall j \right\}. \quad (\text{II.26})$$

Here $K_j^\pi \equiv \sum_{r=1}^j k_{\pi r}$.

We now return to $\tilde{h}_\pi(p)$. It is formally given by

$$\tilde{h}_\pi(p) = (-2\pi i)^{1-n} \int \tilde{w}_\pi^T(p') \prod_{j=1}^{n-1} \frac{dP_j^\pi}{P_j^\pi - (P_j^\pi + i0)}.$$

We wish to show that, in accordance with this heuristic formula,

$$\tilde{h}_\pi(p) = \lim_{\text{Im } Q_j^\pi \downarrow 0, \forall j} f_\pi(p + iq), \quad (\text{II.27})$$

where $f_\pi(k)$ is holomorphic in

$$\Delta_\pi = \left\{ k = (k_1, \dots, k_n) : \sum_j k_j = 0 \quad \text{and} \quad K_j^\pi \in \mathbb{C} \setminus (\mu + \mathbb{R}^+), \forall j \right\}.$$

To verify that this is the case we consider the two Laplace transforms

$$(2\pi)^{1-n} \int \left[\prod_{\substack{1 \leq j \leq n-1 \\ j \neq a}} \varepsilon_j \theta(\varepsilon_j \zeta_j^\pi) d\zeta_j^\pi \right] \theta(\zeta_a^\pi) d\zeta_a^\pi \left[\exp i \sum_{j=1}^{n-1} K_j^\pi \zeta_j^\pi \right] w_\pi^T(x),$$

and

$$- (2\pi)^{1-n} \int \left[\prod_{\substack{1 \leq j \leq n-1 \\ j \neq a}} \varepsilon_j \theta(\varepsilon_j \zeta_j^\pi) d\zeta_j^\pi \right] \theta(-\zeta_a^\pi) d\zeta_a^\pi \left[\exp i \sum_{j=1}^{n-1} K_j^\pi \zeta_j^\pi \right] w_\pi^T(x),$$

which are respectively analytic in

$$\{ k : \operatorname{Im} K_a^\pi > 0, \quad \operatorname{Im} K_j^\pi \varepsilon_j > 0, \quad \forall j \neq a \}$$

and

$$\{ k : \operatorname{Im} K_a^\pi < 0, \quad \operatorname{Im} K_j^\pi \varepsilon_j > 0, \quad \forall j \neq a \}.$$

Here $\varepsilon_j = \pm 1$.

The difference of their boundary values is the Fourier transform of

$$\prod_{j \neq a} \varepsilon_j \theta(\varepsilon_j \zeta_j^\pi) w_\pi^T(x)$$

and (because the procedure (II. 14) can be used) has support in $\{ p : P_a^\pi \geq \mu \}$. This proves our contention.

In particular $f_\pi(k)$ is holomorphic in the same domain (II. 26) as $\rho_\pi(k)$. We shall prove that $f_\pi(ip)$ coincides with $\rho_\pi(ip)$ on the set

$$\{ ip : P_j^\pi > 0 \quad \forall j \}, \quad (\text{II. 28})$$

and that, as a consequence, f_π and ρ_π coincide at all pure imaginary points. At points in (II. 28), f_π coincides with the Laplace transform

$$f_\pi(k) = (2\pi)^{1-n} \int \prod_{j=1}^{n-1} \theta(\zeta_j^\pi) d\zeta_j^\pi w_\pi^T(x) \exp \Sigma i K_j^\pi \zeta_j^\pi,$$

($\operatorname{Im} K_j^\pi > 0$). The test function $\prod_{1 \leq j \leq n-1} \exp i K_j^\pi \zeta_j^\pi$ is an admissible one

for $\prod_{j=1}^{n-1} \theta(\zeta_j^\pi) w_\pi^T(x)$ because it decreases when $\zeta_j^\pi \rightarrow \infty$. Moreover, if $K_j^\pi = iQ_j^\pi$, $Q_j^\pi > 0$, it is the limit, as $B \rightarrow \infty$, $B > 0$, of

$$\prod_{j=1}^{n-1} \left(\frac{B}{B + i\zeta_j^\pi} \right)^A \exp(-Q_j^\pi \zeta_j^\pi),$$

which is in $\mathcal{H}_+(A, N + 1)$, for large A . Hence

$$f_\pi(iq) = \lim_{B \rightarrow \infty} (2\pi)^{1-n} \int w^\Gamma(iy) \prod_{j=1}^{n-1} (-id\eta_j^\pi) \left[\frac{B}{B - \eta_j^\pi} \right]^A \theta(-\eta_j^\pi) \exp - iQ_j^\pi \eta_j^\pi.$$

Since $w^\Gamma(iy)$ is L^1 , the limit exists and coincides with the Fourier transform (II.25) of $w^\Gamma(iy) \prod_{j=1}^{n-1} \theta(-\eta_j^\pi)$, i. e. $f_\pi(iq) = \rho_\pi(iq)$.

Denote $G(k) = \sum_\pi f_\pi(k)$. Then

$$i^{n-1} \delta \left(\sum_{j=1}^n q_j \right) G(iq) = (2\pi)^{-n} \int w^\Gamma(iy) e^{-i \sum_{j=1}^n q_j y_j} dy_1 \dots dy_n.$$

To verify that the function $G(k)$ is none other than the « momentum space analytic function » $H(k)$, we note that the latter is characterized by the following properties:

(a) $H(k)$ is analytic in

$$\{ k : \forall J \in \mathcal{P}_*(X), \operatorname{Im} k_J \neq 0 \}.$$

(b)
$$\lim_{\substack{\theta \rightarrow 0 \\ 0 < \theta < \pi/2}} H(e^{i\theta} p) = \tilde{r}^\Gamma(p),$$

for all p such that $\forall J \in \mathcal{P}_*(X), p_J \neq 0$.

The definition of $G(k)$ shows that it possesses the property (a), and that, if $p_J \neq 0 \forall J \in \mathcal{P}_*(X)$,

$$\lim_{\substack{\theta \rightarrow 0 \\ 0 < \theta < \pi/2}} G(e^{i\theta} p) = \sum_\pi \tilde{h}_\pi(p).$$

But the r. h. s. is the Fourier transform of $\sum_\pi \prod_j \theta(\xi_j^\pi) w_\pi^\Gamma(\xi) = t^\Gamma(\xi)$.

This completes the proof of Theorem 2.

Remark. — Theorem 2, § II.2, and the result of this subsection combined with Theorem 2, § I.2, and results of [BEGS] yield a precise version of Theorem 3 and 5, section I.2.

APPENDIX

TWO COMPUTATIONS

LEMMA A1. —

$$\frac{d}{da} : \Phi^n :_a(x) = \frac{1}{2} n(n-1) \int d^2k (k^2 + a)^{-2} : \Phi^{n-2} :_a(x).$$

Proof. — By Wick reordering, for $\kappa < \infty$

$$\begin{aligned} \partial_a : \Phi_\kappa^n :_a(x) &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \{ : \Phi_\kappa^n :_{a+\delta}(x) - : \Phi_\kappa^n :_a(x) \} \\ &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \sum_{l=1}^{[n/2]} : \Phi_\kappa^{n-2l} :_a \frac{n}{2^l l! (n-2l)!} \left(\int_{|k| \leq \kappa} d^2k \frac{1}{k^2 + a} - \frac{1}{k^2 + a + \delta} \right)^l \right\}. \end{aligned}$$

But

$$\lim_{\kappa \rightarrow \infty} \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(\int_{|k| \leq \kappa} d^2k \frac{1}{k^2 + a} - \frac{1}{k^2 + a + \delta} \right)^l = \begin{cases} \int d^2k (k^2 + a)^{-2} & \text{if } l = 1 \\ 0 & \text{if } l \geq 2 \end{cases},$$

so the assertion follows.

COROLLARY.

$$\frac{d}{da} : P(\Phi) :_a = \frac{1}{2} : P''(\Phi) :_a \int d^2k (k^2 + a)^{-2}.$$

LEMMA A2. —

$$\begin{aligned} &\frac{d}{da} \frac{\int F(\Phi) d\mu_a}{\int G(\Phi) d\mu_a} \\ &= -\frac{1}{2} \frac{\int F(\Phi) \int d^2y : \Phi^2 :_a(y) d\mu_a \int F(\Phi) d\mu_a \int G(\Phi) \int d^2y : \Phi^2 :_a(y) d\mu_a}{\int G(\Phi) d\mu_a \left(\int G(\Phi) d\mu_a \right)^2} \end{aligned}$$

Proof. — Let $b, c > 0$. We use the representation

$$\int F(\Phi) d\mu_a = \lim_{\Lambda \rightarrow \mathbb{R}^2} \frac{\int F(\Phi) \exp \left(- \left(\frac{a}{2} - \frac{b}{2} \right) \int_\Lambda d^2y : \Phi^2 :_c(y) \right) d\mu_b}{\int G(\Phi) \exp \left(- \left(\frac{a}{2} - \frac{b}{2} \right) \int_\Lambda d^2y : \Phi^2 :_c(y) \right) d\mu_b}$$

The derivative exists (Principle 1, p. 7) and is independent of b, c , hence the result.

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