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ANDRZEJ STARUSZKIEWICZ

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Lorentz Covariant Quantum Mechanics of Charged Particles in the Two-Dimensional Space-Time

by

Andrzej STARUSZKIEWICZ

Max-Planck-Institut für Physik und Astrophysik, München
(On leave of absence from the Jagellonian University, Cracow, Poland)

ABSTRACT. — Quantum-mechanical equation of motion for n charged particles in a two-dimensional space-time is derived. The equation describes relativistic effects up to order $(v/c)^2$ but is formally Lorentz covariant. Covariance is achieved by the use of advanced interactions.

INTRODUCTION

Hill and Rudd [*1*] and the author [*2*] [*3*] noted that some classical many-body problems in the special theory of relativity are solvable by means of ordinary differential (i. e. nonhereditary) equations. In particular, the exact Hamiltonian for two charged particles was calculated [*3*] and, for the case of two-dimensional space-time, a system of Lorentz covariant mechanics was constructed [*2*] [*4*]. The aim of this paper is to investigate the quantum-mechanical version of this system. First of all, however, we should like to discuss the physical meaning of our construction.

In the cases considered so far the goal of keeping a many-body system both covariant and nonhereditary has been achieved at the expense of introducing retarded and advanced interactions in a peculiar, non-symmetric way. Arguments have been presented [*5*] [*6*] that this very unphysical feature cannot really be avoided. Now, a theory of charged particles which is formally Lorentz covariant but uses a mixture of retarded and advanced interactions is physically valid only up to order $(v/c)^2$ because to this order there is still no difference between the retarded and advanced Lorentz forces. One immediately asks the question: what is the point in having a covariant theory whose physical validity does not extend beyond the $(v/c)^2$ effects? Does it make sense, for example, to keep kinetic energy terms in the form $m/\sqrt{1-(v/c)^2}$ if the potential is known only to order $(v/c)^2$? Our answer is: yes, it does make sense because, apart from an obvious esthetic appeal of a Lorentz covariant theory, it just so happens that most calculations in the formally covariant theory are much simpler than in the $(v/c)^2$ approximation. For example, our exact two-body Hamiltonian [3] is actually much simpler than Darwin's approximate one [7]; the retarded Coulomb potential, which is quite complicated in approximate calculations, is simply $1/r$ in the covariant approach.

So much about the physical meaning of our approach. In the following sections we construct the Lorentz covariant equations of motion. We hope that their physical meaning has been adequately explained in this introduction.

THE STRUCTURE OF NEWTONIAN DYNAMICS

The Newtonian dynamics has the following mathematical structure: there exists a relation between events, called simultaneity, which is reflexive (an event is simultaneous with itself), symmetric (if A is simultaneous with B then B is simultaneous with A), transitive (if A is simultaneous with B and B with C, then A is simultaneous with C) and invariant with respect to the Galilean group. By the familiar process of abstraction, the relation of simultaneity divides all events into disjoint equivalence classes. Two events are supposed to be in a mutual dynamical relationship if and only if they belong to the same equivalence class, i. e. if they are simultaneous.

Is there a relation between events which can replace simultaneity in the special theory of relativity? The usual Einstein (i. e. coordinate) simultaneity is reflexive, symmetric and transitive but not invariant. Another simple relation: « two events are separated by a null interval » is reflexive, symmetric and invariant but not transitive. However, in two-dimensional space-time the relation « two events are on the same null straight line » is reflexive, symmetric, transitive and invariant. On the basis of this simple observation one can construct a mechanics which is as simple and beautiful as the classical one.

THE TWO-DIMENSIONAL RELATIVISTIC MECHANICS

Let $c = 1$ and let $u = t - x$, $v = t + x$, where t and x denote respectively time and space coordinate in some inertial reference system. Two events are said to be v-simultaneous if they have the same v-coordinate. The relation of v-simultaneity is easily seen to be reflexive, symmetric, transitive and invariant i. e. to have all the properties of Newtonian simultaneity. Consequently, we shall treat u as the « coordinate » and v as « time ». We shall not bother about the inherent lack of time and space reflection symmetry in this treatment; later this shortcoming will be removed.

For a free particle

$$\begin{aligned} \text{action} &= -m\int\sqrt{(dt)^2-(dx)^2} = -m\int\sqrt{dudv} \\ &= -m\int\sqrt{\dot{u}}dv, \qquad \dot{u} = du/dv. \end{aligned}$$

Similarly for two particles

$$\text{action} = -m_1\int\sqrt{\dot{u}_1}dv - m_2\int\sqrt{\dot{u}_2}dv.$$

All this is trivial. The point is, however, that it is just as easy to introduce the electromagnetic interaction. The action

$$-m_1\int\sqrt{\dot{u}_1}dv - m_2\int\sqrt{\dot{u}_2}dv - e_1e_2\int\frac{\dot{u}_1+\dot{u}_2}{|u_1-u_2|}dv$$

is Lorentz invariant, as can be seen from the fact that the Lorentz transformation has the form

$$u' = e^{-\lambda}u, \qquad v' = e^{\lambda}v, \qquad \lambda = \text{hyperbolic angle},$$

and describes exactly the following physical situation: the first particle (or: the particle on the left-hand side) is acted upon by the retarded Lorentz force of the second particle while the second particle is acted upon by the advanced Lorentz force of the first particle. For n particles we have the action

$$-\int\sum_i m_i\sqrt{\dot{u}_i} + \sum_{i>k} e_ie_k\frac{\dot{u}_i+\dot{u}_k}{|u_i-u_k|}dv.$$

The Hamiltonian for one particle is $\mathrm{H} = -m^2/4p$, where

$$p = \partial\mathrm{L}/\partial\dot{u} = \partial(-m\sqrt{\dot{u}})/\partial\dot{u}$$

and $\mathrm{H} = \dot{u}(\partial\mathrm{L}/\partial\dot{u}) - \mathrm{L}$. The Hamiltonian is not equal to the total energy: $\mathrm{H} = (1/2)(\mathrm{E} - \mathrm{P})$, where E is the energy and P is the momentum. For two free particles $\mathrm{H} = -m_1^2/4p_1 - m_2^2/4p_2$, for two charged particles

$$\mathrm{H} = -\frac{m_1^2}{4\left(p_1+\frac{e_1e_2}{|u_1-u_2|}\right)} - \frac{m_2^2}{4\left(p_2+\frac{e_1e_2}{|u_1-u_2|}\right)}$$

and so on.

QUANTUM MECHANICS

The Schrödinger equation for one free particle has the form ($\hbar = 1$)

$$i\frac{\partial\psi}{\partial v} = \mathrm{H}\psi, \qquad \mathrm{H} = -\frac{m^2}{4p}, \qquad p = -i\frac{\partial}{\partial u}.$$

The inverse of momentum may be defined e. g. by means of the Fourier transform; it is, however, simpler to observe that the Schrödinger equation, if acted upon by the p-operator, becomes

$$\frac{\partial^2\psi}{\partial u \partial v} + \frac{m^2}{4}\psi = 0,$$

which is the Klein-Gordon equation. This is encouraging because from a somewhat unusual approach we have obtained a familiar result. Of course we also have obtained the well-known difficulties connected with the negative frequency solutions but, as we tried to make clear in the introduction, our mechanics, despite its formal covariance, is supposed to be physically valid only when potential and kinetic energies are small when compared to the rest masses. In this region the Klein-Gordon equation can be safely used.

In the same way, i. e. multiplying the Hamiltonian by the product of all momenta, we obtain for two free particles

$$\frac{\partial^3\psi}{\partial u_1 \partial u_2 \partial v} + \frac{m_1^2}{4}\frac{\partial\psi}{\partial u_2} + \frac{m_2^2}{4}\frac{\partial\psi}{\partial u_1} = 0,$$

and for n particles

$$\frac{\partial^{n+1}\psi}{\partial u_1 \partial u_2 \ldots \partial u_n \partial v} + \frac{m_1^2}{4}\frac{\partial^{n-1}\psi}{\partial u_2 \partial u_3 \ldots \partial u_n} + \frac{m_2^2}{4}\frac{\partial^{n-1}\psi}{\partial u_1 \partial u_3 \ldots \partial u_n} + \ldots + \frac{m_n^2}{4}\frac{\partial^{n-1}\psi}{\partial u_1 \partial u_2 \ldots \partial u_{n-1}} = 0.$$

All these equations are of course covariant.

For two charged particles there is an ordering problem because the operators

$$p_1 + \frac{e_1 e_2}{|u_1 - u_2|} \quad \text{and} \quad p_2 + \frac{e_1 e_2}{|u_1 - u_2|}$$

do not commute. We shall apply the simplest ordering, namely take the symmetrized product of all relevant operators. In this way we get for two charged particles

$$(1) \quad \left\{\left(\frac{\partial}{\partial u_1} + i\frac{e_1 e_2}{|u_1 - u_2|}\right)\left(\frac{\partial}{\partial u_2} + i\frac{e_1 e_2}{|u_1 - u_2|}\right)\right\}_{\mathrm{sym}}\frac{\partial\psi}{\partial v} + \frac{m_1^2}{4}\left(\frac{\partial}{\partial u_2} + i\frac{e_1 e_2}{|u_1 - u_2|}\right)\psi + \frac{m_2^2}{4}\left(\frac{\partial}{\partial u_1} + i\frac{e_1 e_2}{|u_1 - u_2|}\right)\psi = 0,$$

where the subscript sym denotes symmetrization. The generalization to n particles is obvious.

The author finds it remarkable that such a simple and covariant equation of motion for n charged particles can be obtained by application of such an elementary method which involves no arbitrariness other then the choice of ordering for the noncommuting operators.

TIME AND SPACE REFLECTION SYMMETRY

Eq. (1) is neither time nor space reflection invariant. Lack of such an invariance is inherent in our construction. A similar problem exists in the notion of relativistic spinor; the solution by Dirac is well-known: he introduced two spinors which are interchanged under reflection.

Let us write the equation which arises from (1) when u is replaced by v and v by u and call the new wave function χ:

$$(2)\quad \left\{\left(\frac{\partial}{\partial v_1} + i\frac{e_1 e_2}{|v_1 - v_2|}\right)\left(\frac{\partial}{\partial v_2} + i\frac{e_1 e_2}{|v_1 - v_2|}\right)\right\}_{\text{sym}}\frac{\partial \chi}{\partial u} + \frac{m_1^2}{4}\left(\frac{\partial}{\partial v_2} + i\frac{e_1 e_2}{|v_1 - v_2|}\right)\chi + \frac{m_2^2}{4}\left(\frac{\partial}{\partial v_1} + i\frac{e_1 e_2}{|v_1 - v_2|}\right)\chi = 0.$$

It is easy to see that, under time reflection, $u' = -v$, $v' = -u$. This is a symmetry operation for the system of Eq. (1) and (2), if it is accompanied by the transformation $\psi' = \bar{\chi}$, $\chi' = \bar{\psi}$. Similarly, the space reflection $u' = v$, $v' = u$ has to be accompanied by the transformation $\psi' = \chi$, $\chi' = \psi$ and the total reflection $u' = -u$, $v' = -v$ by the transformation $\psi' = \bar{\psi}$, $\chi' = \bar{\chi}$. Finally, let us observe that the function ψ is determined on a hyperplane $v_1 = v_2 = \ldots = v_n$ while the function χ is determined on the hyperplane $u_1 = u_2 = \ldots = u_n$. These two hyperplanes intersect on a two-dimensional plane $u_1 = u_2 = \ldots = u_n$, $v_1 = v_2 = \ldots = v_n$. *We assume that the functions ψ and χ are not independent but form together one wave function whose support consist of two pieces.* It follows from this interpretation that a solution ψ of Eq. (1) and χ of Eq. (2) form together an acceptable wave function if and only if $\psi = \chi$ for $u_1 = u_2 = \ldots = u_n$ and $v_1 = v_2 = \ldots = v_n$. In general this boundary condition is obviously nontrivial. However, in the case of two charged particles the condition is trivially satisfied because of singular nature of the Coulomb force: the wave equations (1) and (2) have a singular point for $u_1 = u_2$ and $v_1 = v_2$ respectively and we have to impose the usual regularity condition. It turns out that both functions ψ and χ have to vanish at the singularity which means that the condition $\psi = \chi$ for $u_1 = u_2$ and $v_1 = v_2$ is trivially satisfied in the form $\psi = 0 = \chi$ for $u_1 = u_2$ and $v_1 = v_2$. In general however, in particular for nonsingular forces, the condition will not be trivially satisfied and will constitute a dynamical connection between the two parts of the wave function.

THE TWO-BODY PROBLEM

To see if our equations of motion give physically plausible results we shall consider the two-body problem (1).

Because the potential depends on $u_1 - u_2$ only, it is convenient to introduce new variables $u_1 - u_2 = u$, $\alpha u_1 + \beta u_2 = \mathrm{U}$, $\alpha + \beta = 1$. For the same reason the Hamiltonian H and $p_1 + p_2$ are constants of motion:

$$\mathrm{H} = \frac{1}{2}(\mathrm{E} - \mathrm{P}), \qquad p_1 + p_2 = -\frac{1}{2}(\mathrm{E} + \mathrm{P}),$$

where E and P are the total energy and momentum. Therefore, we put in (1)

$$i\frac{\partial\psi}{\partial v} = \frac{1}{2}(\mathrm{E} - \mathrm{P})\psi, \qquad i\left(\frac{\partial\psi}{\partial u_1} + \frac{\partial\psi}{\partial u_2}\right) = \frac{1}{2}(\mathrm{E} + \mathrm{P})\psi.$$

In this way Eq. (1) is reduced to the form

$$\left\{\left[\frac{\alpha}{2i} + \frac{1}{\mathrm{E}+\mathrm{P}}\frac{d}{du} + \frac{ie_1e_2}{(\mathrm{E}+\mathrm{P})\,|\,u\,|}\right]\left[\frac{\beta}{2i} - \frac{1}{\mathrm{E}+\mathrm{P}}\frac{d}{du} + \frac{ie_1e_2}{(\mathrm{E}+\mathrm{P})\,|\,u\,|}\right]\right\}_{\mathrm{sym}} \frac{\mathrm{M}^2}{i}\psi$$
$$+ \frac{m_1^2}{2}\left[\frac{\beta}{2i} - \frac{1}{\mathrm{E}+\mathrm{P}}\frac{d}{du} + \frac{ie_1e_2}{(\mathrm{E}+\mathrm{P})\,|\,u\,|}\right]\psi$$
$$+ \frac{m_2^2}{2}\left[\frac{\alpha}{2i} + \frac{1}{\mathrm{E}+\mathrm{P}}\frac{d}{du} + \frac{ie_1e_2}{(\mathrm{E}+\mathrm{P})\,|\,u\,|}\right]\psi = 0,$$

where $\mathrm{M}^2 = \mathrm{E}^2 - \mathrm{P}^2$. Let us choose α and β so that the term proportional to $d\psi/du$ vanishes; to this end we put

$$\alpha = \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{\mathrm{M}^2}\right), \qquad \beta = \frac{1}{2}\left(1 + \frac{m_2^2 - m_1^2}{\mathrm{M}^2}\right).$$

For small velocities $\mathrm{M} = m_1 + m_2$ and we obtain the usual nonrelativistic expression for α and β.

To simplify the equation further we introduce the invariant variable

$$\zeta = \frac{(\mathrm{E}+\mathrm{P})(u_1 - u_2)}{2\mathrm{M}} = \frac{(\mathrm{E}+\mathrm{P})u}{2\mathrm{M}}$$

and the equation of internal motion takes on the form

$$(3) \quad -\frac{d^2\psi}{d\zeta^2} + \left[\frac{\mathrm{M}^2 - m_1^2 - m_2^2}{\mathrm{M}}\frac{e_1e_2}{|\,\zeta\,|} - \frac{(e_1e_2)^2}{\zeta^2}\right]\psi$$
$$= \frac{1}{4\mathrm{M}^2}[\mathrm{M}^2 - (m_1 + m_2)^2][\mathrm{M}^2 - (m_1 - m_2)^2]\psi.$$

The derivation of Eq. (3) admittedly involves a certain arbitrariness in the ordering of noncommuting operators: we simply took the symmetrized product. Nevertheless, the result is in excellent agreement with our exact classical two-body Hamiltonian [3]. The physical content of the last equation and of the exact classical result may be described as follows: if the kinetic energy term in the electromagnetic two-body problem is conventionally put equal to p^2, then the Coulomb potential is $(M^2 - m_1^2 - m_2^2)/M$ times the ordinary Coulomb potential while the numerical value of the Hamiltonian is

$$[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]/4M^2 .$$

Let us check that the non-relativistic limit is correct. We say that a two-body system is non-relativistic if its mass can be written in the form

$$M = m_1 + m_2 + \varepsilon ,$$

where ε is so small a number that its square can be neglected. Introducing ε into Eq. (3) and neglecting the term proportional to $(e_1 e_2)^2$ we find

$$-\frac{1}{2\mu}\frac{d^2\psi}{d\zeta^2} + \frac{e_1 e_2}{|\zeta|}\psi = \varepsilon\psi ,$$

where

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} ,$$

which means that the non-relativistic limit is indeed correct.

The one-particle limit $m_2 \to \infty$ is also correct. In this case $M = m_2 + \varepsilon$, where m_2 is very large. Putting this into Eq. (3) and taking the limit $m_2 \to \infty$ we find

$$-\frac{d^2\psi}{d\zeta^2} + 2\varepsilon\frac{e_1 e_2}{|\zeta|}\psi - \frac{(e_1 e_2)^2}{\zeta^2}\psi = (\varepsilon^2 - m_1^2)\psi .$$

This is to be compared with the Klein-Gordon equation for particle 1 in the external Coulomb field generated by particle 2 i. e. with the equation

$$(p_\mu - e_1 A_\mu)(p^\mu - e_1 A^\mu)\psi = m_1^2\psi ,$$

where

$$A_0 = \frac{e_2}{|x^1|}, \qquad A_1 = 0, \qquad p_\mu = i\partial_\mu, \qquad \mu = 0, 1 .$$

A simple calculation shows that these two equations are, in fact, identical.

CONCLUSIONS

We have constructed a quantum mechanics of charged particles which describes correctly relativistic effects up to order $(v/c)^2$ and is formally Lorentz covariant. Covariance is achieved by the use of an advanced interaction. If a similar construction is possible in four-dimensional space, it would be obviously interesting to find it. If such a construction is impossible, the two-dimensional model probably has little relevance for four-dimensional physics.

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