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Section A :
Physique théorique.

The infinite volume limit of the φ_3^4 model

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ABSTRACT. — We construct a cluster expansion for the Euclidean φ_3^4 theory with large bare mass and small coupling constant. We show its convergence and prove the existence of an infinite volume theory, with a mass gap, and satisfying all the Osterwalder-Schrader axioms. We prove also that the Schwinger functions are infinitely differentiable with respect to the coupling constant and are the moments of a unique measure on $\mathcal{S}'_{\mathbb{R}}(\mathbb{R}^3)$.

I. THE MAIN RESULTS

I.0. Introduction

We prove the existence of the infinite volume limit for the φ_3^4 model in the weak coupling region. We proceed along the lines developped for the two dimensional $P(\varphi)_2$ model in [*1*]: i. e. to control the infinite volume limit we show there exists a convergent vacuum cluster expansion.

A way to perform this cluster expansion would have been to introduce a lattice of unit cubes and covariances with Dirichlet boundary conditions on subsets of this lattice, and then to estimate the basic quantities using the inductive expansion of J. Glimm and A. Jaffe [*3*] as modified by J. Feldman [*4*]. However a direct use of Dirichlet covariances generates two kinds of difficulties as remarked by Feldman in his thesis:

1) since Dirichlet covariances are generally not translation invariant

we lost the diagonal form of their Fourier transform and thus have to modify the inductive expansion of Glimm-Jaffe [3]. Also the use of momentum cutoff functions with compact support has for consequence that distant regions do not decouple exponentially,

2) because of the use of fields with momentum cutoff functions there is no longer strict localization at a point and we lost the decoupling at $s=0$ (see ref. [1]) except for sufficiently far away contours.

To overcome these difficulties in mimicking the two dimensional proof, we construct a new family of covariances possessing the main properties required from the Dirichlet covariances to perform the cluster expansion. These covariances are obtained by combination of two remarks:

1) one can construct covariances which behaves like locally constant Dirichlet covariances and which are bounded. Roughly speaking, they can be written as bilinear sums of averaged Dirichlet covariances, normalized with respect to the free covariance; the summation extends over the lattice,

2) the free covariance of mass M has kernel $e^{-M|x-y|}/|x-y|$ for x and y in $\mathbb{R}^3$. We write it as $\dfrac{e^{-(M-m)|x-y|}}{|x-y|}\, e^{-m|x-y|}$ for $M \geqslant m \geqslant 0$. The factor $e^{-m|x-y|}$ is expanded as a sum over the lattice and used to control the convergence of the sum of averaged Dirichlet covariances. It also gives an exponential decoupling for distant regions. The factor $\dfrac{e^{-m|x-y|}}{|x-y|}$ exhibits the local structure $\left(\dfrac{1}{|x|} \text{ behaviour at the origin}\right)$. The momentum cutoff is introduced in this factor (remark its Fourier transform is diagonal). This last step needs some comment. Generally approximate expressions (the approximation is related to the high momentum behaviour) are obtained by smoothing the random variable with momentum cutoff functions $\tilde{\eta}_\kappa$: a « field » $\varphi(x)$ of covariance $C(x, y)$ is replaced by $(\varphi * \tilde{\eta}_\kappa)(x) = \varphi_\kappa(x)$. Here we adopt another attitude. We obtain approximate expressions by using random variables associated to approximate covariances: the approximate field $\varphi_\kappa(x)$ has covariance $C_\kappa(x, y)$; it will be considered as the Gaussian random variable of covariance $C_\kappa(x, y)$ and mean zero. This second point of view is more general since one can construct many approximate covariances which cannot be associated with approximate fields of the form $\varphi * \tilde{\eta}_\kappa$.

This way of introducing the momentum cutoff is therefore compatible with the principle of the cluster expansion and produces only slight modifications of the inductive expansion.

We introduce the new covariances in chapter II and define the cluster expansion. The combinatoric and the estimates on graphs of the modified inductive expansion is given in chapter III. The main theorems are proven in chapter IV.

I.1. The main results

Let $\varphi(x)$ be the Gaussian random variable of mean zero and covariance

$$C_\kappa(x, y) = \frac{1}{(2\pi)^3} \int \frac{e^{ik(x-y)}}{k^2 + M^2} \eta_\kappa(k) d^3k$$

where

$$\eta_\kappa(k) = \eta\left(\frac{k}{\kappa}\right)$$

is a cutoff function.

Let $d\Phi_{C_\kappa}$ be the associated Gaussian measure, and $\Lambda(x)$ be the characteristic function of some compact subset Λ of $\mathbb{R}^3$. The interaction $V_\kappa(\Lambda, \mu)$ is defined by

$$V_\kappa(\Lambda, \mu) = \mu \int :\varphi^4:(x)\Lambda(x)dx + \frac{\mu^2}{2} \int \left(\int :\varphi^4:(x)\Lambda(x)dx \right)^2 d\Phi_{C_\kappa} - \frac{\mu^3}{6} \int \left(\int :\varphi^4:(x)\Lambda(x)dx \right)^3 d\Phi_{C_\kappa} - \frac{\mu^2}{2} \delta m^2 \int :\varphi^2:(x)\Lambda(x)dx$$

with

$$\delta m^2 = -4^2.6 \int C_\kappa(x, 0)^3 dx \qquad \text{and } \mu \geqslant 0 .$$

The φ_3^4 model is defined as the collection of the expectation values of the product of smeared fields with respect to the measure (Schwinger functions)

$$dq(\Lambda, \mu, \kappa) = \frac{e^{-V_\kappa(\Lambda,\mu)} d\Phi_{C_\kappa}}{\int e^{-V_\kappa(\Lambda,\mu)} d\Phi_{C_\kappa}}$$

Then we have

THEOREM I.1. — *Let μ be small and M be large enough, then the φ_3^4 model has an infinite volume limit which satisfies the axioms of Osterwalder-Schrader and exhibits an exponential decoupling.*

As an obvious consequence of this theorem we get

COROLLARY I.2. — *There exists a relativistic field theory satisfying the Wightman axioms and corresponding to the φ_3^4 model.*

We now state the central theorem

THEOREM I.3. — *Let μ be small and M large enough, then there exists a Schwartz space norm $|\ .\ |$ such that the Schwinger functions for any p, $p > 2$*

$$S_n(\mu; f_1, \ldots, f_n) = \lim_{\Lambda \to \infty} \lim_{\kappa \to \infty} \int \varphi(f_1) \ldots \varphi(f_n) dq(\Lambda, \mu, \kappa)$$

are bounded by $O(1)O(1)^n(n!)^{1-\frac{1}{p}} \prod_{i=1}^{n} |f_i|$.

Moreover as in two dimension they satisfy a strong decrease cluster property [2].

Theorem I.1 results from this last theorem since axiom E'_0 is obviously satisfied. All other axioms are evident.

We consider functions F, F′, ... of the fields defined as products of Wick monomials localized in unit cubes of a unit cover of $\mathbb{R}^3$ and bounded with respect to some norm $|||\,.\,|||_{\delta,\alpha}$ (closely related to the norm $||\,.\,||_{2,\delta,\alpha}$ of Feldman [4]). The support of such function is defined as in [1]. Let $n_F(\Delta)$ be the degree of the fields in F in Δ.

THEOREM I.4. — *Let μ be small and* M *large enough, then*

a)

$$\lim_{\Lambda\to\infty}\lim_{\kappa\to\infty}\int F dq(\Lambda,\mu,\kappa)$$

exists and is bounded uniformely in Λ, μ, κ, as $\Lambda \to \infty$ and $\kappa \to \infty$.

b) *These limits satisfy a strong decrease cluster property (see* [2]*). In particular, let* F *and* F′ *be two functions and let*

$$d = \text{dist}\,\{\,\text{suppt } F, \text{suppt } F'\,\} \neq 0\,,$$

then there exists a constant C *and a positive constant* $\bar{M}$ *independent of* F *and* F′ *such that*

$$\int FF'dq(\Lambda,\mu,\kappa) - \int Fdq(\Lambda,\mu,\kappa)\int F'dq(\Lambda,\mu,\kappa) \leqslant Ce^{-\bar{M}d}\,|||\,FF'\,|||_{\delta,\alpha} \times \prod_{\Delta}[O(1)n_F(\Delta)]^{n_F(\Delta)}[O(1)n_{F'}(\Delta)]^{n_{F'}(\Delta)}$$

uniformly in Λ, μ, κ.

The next theorem gives informations on the regularity of the infinite volume limit with respect to $\mu \geqslant 0$, for μ close to the origin,

THEOREM I.5. — *Under the conditions of theorem* I.4,

$$\lim_{\Lambda\to\infty}\lim_{\kappa\to\infty}\int F dq(\Lambda,\mu,\kappa)$$

is C^∞ in μ and

$$\left|\left(\frac{d}{d\mu}\right)^n \int Fdq(\Lambda,\mu,\kappa)\right| \leqslant C_1C_2^n(n!)^L\,|||\,F\,|||_{\delta,\alpha}\prod_{\Delta}[O(1)n_F(\Delta)]^{n_F(\Delta)}$$

for some constants C_1, C_2 and L, *and uniformly in Λ and κ as $\Lambda \to \infty$ and $\kappa \to \infty$.*

We have also:

THEOREM I.6. — *Under the same conditions, the infinite volume Schwinger functions are the moments of a unique measure on* $\mathcal{S}'(\mathbb{R}^3)$, *namely*

$$dq(\mu) = \lim_{\Lambda\to\infty}\lim_{\kappa\to\infty} dq(\Lambda, \mu, \kappa)$$

Finally the following theorem improves the bounds for the Schwinger functions in theorem I.3, using more explicitly the strength of the interaction.

THEOREM I.7. — *Under the same conditions, for* $\mu > 0$ *and for any* ρ, $\rho > 4/3$ *we have*

$$| S_n(\mu ; f_1, \ldots, f_n) | \leqslant O(1)O(1)^n(n!)^{1-1/\rho}\prod_{i=1}^{n}\frac{|f_i|}{\mu}$$

Note. — Similar results have been obtained by J. Feldman and K. Osterwalder [10] using another family of covariances.

II. THE CLUSTER EXPANSION

This chapter is divided in three parts. In the first one we define the new class of covariances and show they behave like Dirichlet covariances. In the second part we introduce the momentum cutoff functions. Finally the vacuum cluster expansion is defined in the last part following the lines of references [1] and [2]. A large part of the combinatoric relative to the cluster expansion is done and the convergence is reduced to a bound which will be proved in the next chapter.

II.1. The covariances

We divide $\mathbb{R}^3$ into a lattice of unit cubes, $\mathbb{R}^3 = \bigcup_{z\in Z^3}\Delta_0 + z$, where Δ_0 is the unit cube centered at the origin.

The set of all faces of unit cubes of this cover is $(Z^3)^*$. As in [1] we introduce the set of bonds $\mathbb{B}$ which can be either $(Z^3)^*$ or a subset of $(Z^3)^*$. For any subset $\Gamma \subset \mathbb{B}$, $\Gamma^c = \mathbb{B}\backslash\Gamma$ and we identify the subset Γ of $\mathbb{B}$ with the corresponding subset of $\mathbb{R}^3$. As in [1], we define an expansion labelled by the subsets Γ, and to each Γ is associated a covariance B^Γ. The set of the linear convex combinations of B^Γ is called $\mathscr{C}$. We require on the elements of $\mathscr{C}$ three conditions:

1) $C = (-\Delta + M^2)^{-1}$ is in $\mathscr{C}$.

2) Let $\mathbb{R}^3\backslash\Gamma^c = X_1 \cup \ldots \cup X_r$ where $X_i \cap X_j = \emptyset$ for $i \neq j$ and each X_i is connected. Then $B^\Gamma(x, y) = 0$ unless x and y belong to the same X_i.

3) Except for little change, the covariances B^{Γ} allow to perform the φ_3^4 inductive expansion of Glimm-Jaffe [3] as modified by Feldman [4].

Condition 2) ensures that B^{Γ} behaves essentially as a covariance with Dirichlet boundary conditions, for this reason we say that B^{Γ} is of the Dirichlet type.

II.1.1. DEFINITION OF B^{Γ}

Let $\mathbb{B} = (Z^3)^*$ and $\Gamma \subset \mathbb{B}$. Let M, m and m_1 be positive numbers with $M > m_1$ and x and y be in $\mathbb{R}^3$, then the covariance operator B^{Γ} is defined by

DÉFINITION II.1.1.1.

$$B^{\Gamma}(x, y) = m_1 \sum_{z \in Z^3} \frac{\bar{C}_m^{\Gamma}(x ; z]}{\bar{C}_m(x ; z]} C_{M-m_1}(x, y) D_z^{m_1}(x, y) \frac{\bar{C}_m^{\Gamma}(y ; z]}{\bar{C}_m(y ; z]} \quad \text{(II.1.1.1)}$$

We now explain each term of formula II.1.1.1

$$C_m(x, y) = \frac{1}{(2\pi)^3} \int \frac{e^{ik(x-y)}}{k^2+m^2} dk = (-\Delta+m^2)^{-1}(x, y) = \frac{1}{4\pi} \frac{e^{-m|x-y|}}{|x-y|} \quad \text{(II.1.1.2)}$$

$C_m^{\Gamma}(x, y)$ is, in three dimension what is noted $C_{m,\Gamma^c}(x, y)$ in reference [1], i. e.

$$C_m^{\Gamma}(x, y) = (-\Delta_{\Gamma^c} + m^2)^{-1}(x, y) = \int_0^{\infty} e^{-m^2 T} \int \prod_{b \in \Gamma^c} J_b^T(z) dz_{x,y}^T dT \quad \text{(II.1.1.3)}$$

where now $dz_{x,y}^T$ is the Wiener density for paths in $\mathbb{R}^3$ and

$$J_b^T(z) = \begin{cases} 0 & \text{if } z(\tau) \in b \quad 0 \leqslant \tau \leqslant T \\ 1 & \text{otherwise} \end{cases}$$

Then

$$\bar{C}_m^{\Gamma}(x ; z] = \sum_{z' \in Z^3} \chi_{z'}(x) \int \chi_{z'}(u) C_m^{\Gamma}(u, v) \chi_z(v) du dv$$

where $\chi_z(x) = \chi_{\Delta_0 + z}(x)$ is the characteristic function of the cube $\Delta_0 + z$.

Finally, let $f \in C_0^{\infty}(\mathbb{R}^3)$ such that $\int f(x) dx = 1$ and $f(x) \geqslant 0$ and define

$$\theta(x) = \int f(x + y) \chi_{\Delta_0}(y) dy$$

and

$$\theta_z(x) = \theta(x + z) \quad \text{for} \quad z \in Z^3$$

then

$$\sum_{z \in Z^3} \theta_z(x) = 1 \quad \text{(II.1.1.4)}$$

and we define

$$\mathrm{D}_z^{m_1}(x, y) = \frac{1}{(2\pi)^3\pi^2} \int \frac{e^{ik_1(x-z)}}{k_1^2 + m_1^2} \theta_z(z) \frac{e^{ik_2(z-y)}}{k_2^2 + m_1^2} dk_1 dk_2 dz$$

$$= 8\pi \int \mathrm{C}_{m_1}(x, z)\theta_z(z)\mathrm{C}_{m_1}(z, y)dz \quad \text{(II.1.1.5)}$$

Remark that $\mathrm{D}_z^{m_1}(x, y)$ is positive since C and θ are positive.

From (II.1.1.4) and (II.1.1.5) it follows that

$$\sum_{z' \in \mathrm{Z}^3} \mathrm{D}_{z'}^{m_1}(x, y) = \frac{1}{(2\pi)^3\pi^2} \int \frac{e^{ik_1(x-z)}}{k_1^2 + m_1^2} \frac{e^{ik_2(z-y)}}{k_2^2 + m_1^2} dk_1 dk_2 dz$$

$$= \frac{1}{\pi^2} \int \frac{e^{ik(x-y)}}{(k^2 + m_1^2)^2} dk = \frac{e^{-m_1|x-y|}}{m_1} \quad \text{(II.1.1.6)}$$

Thus if $\Gamma = (\mathrm{Z}^3)^*$

$$\mathrm{B}^{(\mathrm{Z}^3)^*}(x, y) = m_1 \sum_{z \in \mathrm{Z}^3} \mathrm{C}_{\mathrm{M}-m_1}(x, y)\mathrm{D}_z^{m_1}(x, y) = \mathrm{C}_{\mathrm{M}}(x, y)$$

Let us now comment on the definition of B^Γ. The terms $\bar{\mathrm{C}}_m^\Gamma(x\,;z]$ and $\bar{\mathrm{C}}_m^\Gamma(y\,;z]$ allow us to proceed to a cluster expansion as in [1], and since they are constant in unit cubes they do not interfere with the refinements of unit cubes of the φ_3^4 expansion of [1]. The division of each of the $\bar{\mathrm{C}}^\Gamma$ by $\bar{\mathrm{C}}$ is to normalize the Dirichlet contribution. In particular, from $0 \leqslant \mathrm{C}^\Gamma \leqslant \mathrm{C}$ follows that $0 \leqslant \bar{\mathrm{C}}^\Gamma \leqslant \bar{\mathrm{C}}$ and therefore

$$\frac{\bar{\mathrm{C}}^\Gamma}{\bar{\mathrm{C}}} \leqslant 1$$

Finally a factor $e^{-m_1|x-y|}$ is extracted from $\mathrm{C}_{\mathrm{M}}(x, y)$ and written as in (II.1.1.6) in order to insure the convergence of the sum over z. The mass m will be chosen large enough to control the combinatoric factors of the cluster expansion as in [1]. The mass m_1 will be chosen larger than m to control essentially the $(\bar{\mathrm{C}}_m)^{-1}$ factors, and M will be larger than m_1.

Another question is the introduction of the momentum cutoff. The followed procedure will be a justification for the choice of a central term of the form $\mathrm{C}_{\mathrm{M}-m_1}$ in formula (II.1.1.1). In fact a B_η^Γ, where η is a cutoff function, will be defined from formula (II.1.1.1) by replacing $\mathrm{C}_{\mathrm{M}-m_1}$ by $\mathrm{C}_{\mathrm{M}-m_1,\eta}$:

$$\mathrm{C}_{\mathrm{M}-m_1,\eta}(x, y) = \frac{1}{(2\pi)^3} \int \frac{e^{ik(x-y)}}{k^2 + (\mathrm{M} - m_1)^2} \eta(k)dk$$

We will enter the details in section 3.

II.1.2. PROPERTIES OF THE COVARIANCES

We first study the properties of B^Γ, then those of convex linear combinations, finally we give some estimates.

a) *Properties of* B^Γ.

We write $B^\Gamma = \sum_{z \in Z^3} B_z^\Gamma$ with

$$B_z^\Gamma(x, y) = \frac{\bar{C}_m^\Gamma(x ; z]}{\bar{C}_m(x ; z]} C_{M-m_1}(x, y) m_1 D_z^{m_1}(x, y) \frac{\bar{C}_m^\Gamma(y ; z]}{\bar{C}_m(y ; z]} \quad (II.1.2.1)$$

Now, we prove that B^Γ is of Dirichlet type:

According to condition 2) of section 1, suppose $\mathbb{R}^3 \backslash \Gamma^c = X_1 \cup \ldots \cup X_n$ with X_i connected and $X_i \cap X_j = \emptyset$, then consider a term B_z^Γ in B^Γ. From the definition of C^Γ, x and y should be in the same connected component, and thus, in $B_z^\Gamma(x, y)$, x, y and $\Delta_0 + z$ are in the same component otherwise $B_z^\Gamma(x, y) = 0$. This proves the lemma. Moreover, in

$$B^\Gamma(x, y) = \sum_{z \in Z^3} B_z^\Gamma(x, y)$$

the sum is restricted to the z's such that $\Delta_0 + z$ are in the same component that x and y.

Also it is easy to show that B_z^Γ is a positive continuous bilinear form on $\mathscr{S}(\mathbb{R}^3) \times \mathscr{S}(\mathbb{R}^3)$. The restriction of B_z^Γ to $L^2(X_i)$ define a non degenerate form on $\mathscr{S}(X_i) \times \mathscr{S}(X_i)$.

So we conclude as in [1] that B^Γ is a covariance.

Let now $\mathbb{B}$ be any subset of bonds. We define a multiparameter family of covariances by

$$C_{(s)}(x, y) = \sum_{\Gamma \subset \mathbb{B}} \prod_{b \in \Gamma} s_b \prod_{b \in \Gamma^c} (1 - s_b) B^\Gamma(x, y)$$

b) *Properties of* $C_{(s)}$.

Being a convex linear combination of B^Γ, $C_{(s)}$ is a covariance.

We now state a lemma which expresses the effect of derivatives with respect to (s).

LEMMA II.1.2.1

$$\partial^\gamma C_{(s)}(x, y) = \prod_{b \in \gamma} \frac{d}{ds_b} C_{(s)}(x, y) = \sum_{\Gamma \supset \gamma} \prod_{b \in \Gamma \backslash \gamma} s_b \prod_{b \in \Gamma^c} (1 - s_b)$$

$$\times \sum_{z \in Z^3} \left[\sum_{\substack{\gamma_1 \cup \gamma_2 = \gamma \\ \gamma_1 \cap \gamma_2 = \emptyset}} \frac{\overline{(\partial^{\gamma_1} C)}{}_m^{\Gamma \backslash \gamma_2}(x ; z]}{\bar{C}_m(x ; z]} C_{M-m_1}(x, y) m_1 D_z^{m_1}(x, y) \frac{\overline{(\partial^{\gamma_2} C)}{}_m^\Gamma(y ; z]}{\bar{C}_m(y ; z]} \right] \quad (II.1.2.2)$$

where

$$(\partial^\gamma C)_m^\Gamma(x, y) = \int e^{-m^2 T} \int \prod_{b \in \Gamma^c} J_b^T \prod_{b \in \gamma} (1 - J_b^T) dz_{x,y}^T dT \quad \text{(II.1.2.3)}$$

and $\overline{(\partial^\gamma C)_m^\Gamma}$ *is the average, as before, of* $(\partial^\gamma C)_m^\Gamma$.

By linearity it is enough to deal separately with each term of the sum over Z, and the proof of (II.1.2.3) follows by induction.

From the definition (II.1.2.4) one sees that

$$(\partial^\gamma C)_m^\Gamma(x, y) \leqslant \int e^{-m^2 T} \int \prod_{b \in \gamma} (1 - J_b^T) dz_{x,y}^T dT = \partial^\gamma C_m(x, y) \quad \text{(II.1.2.4)}$$

as defined in [1].

c) *Estimates.*

Let Δ and Δ' be unit cubes of the form $\Delta_0 + z$, $z \in Z^3$, we have

$$\int \chi_\Delta(x) \chi_{\Delta'}(y) C_m(x, y) dx dy \geqslant O(1) e^{-4m} \frac{e^{-m d_e(\Delta, \Delta')}}{d_e(\Delta, \Delta') + 1} \quad \text{(II.1.2.5)}$$

We want now some bounds on $(\partial^\gamma C)_m^\Gamma$.

First one has from (II.1.2.3), (II.1.2.4) and reference [1]

$$O \leqslant (\partial^\gamma C)_m^\Gamma(x, y) \leqslant \partial^\gamma C_m(x, y) \leqslant C_m(x, y)$$

When averaged these inequalities show that

$$O \leqslant \frac{\overline{(\partial^\gamma C)_m^\Gamma}(x\,;\,z]}{\bar{C}_m(x\,;\,z)} \leqslant 1 \quad \text{(II.1.2.6)}$$

We need for $(\partial^\gamma C)_m^\Gamma$ a strong decay property. We proceed as in [1] using the bound (II.1.2.4). We have

$$\partial^\gamma C_m(x, y) = \sum_{l \in L(\gamma)} [\partial^\gamma C_m(x, y)]_l = \sum_{l \in L(\gamma)} \int_0^\infty e^{-m^2 T} \int_{w(l)} dz_{x,y}^T dT \quad \text{(II.1.2.7)}$$

where $L(\gamma)$ is the set of all possible linear orderings of the faces $b \in \gamma$, and $l \in L(\gamma)$; $w(l)$ is the set of Wiener paths which cross all faces $b \in \gamma$ and whose order of first crossing is l.

Let $b_1, \ldots, b_{|\gamma|}$ be the elements of γ, as ordered by l. Let b'_2 be the first of the b's not touching $b_1 = b'_1$, let b'_3 be the first of the b's after b'_2 and not touching b'_2, etc. Suppose there are m such elements: $b'_1, \ldots, b'_m$; define

$$|l| = \sum_{j=1}^{m-1} \operatorname{dist}(b'_{j+1}, b'_j)$$

To bound $[\partial^\gamma C_m(x, y)]_l$ we do as Spencer [7] and get

$$[\partial^\gamma C_m(x, y)]_l \leqslant e^{-m\,\mathrm{dist}\,(x, b'_1) - m|l|} \sup_{\xi \in b'_m} C_m(\xi, y)$$

and hence if x is in some cube Δ and y in Δ'

$$\int [\partial^\gamma C_m(x, y)]_l \chi_\Delta(x)\chi_{\Delta'}(y)dxdy \leqslant e^{-md_e(\Delta, b'_1) - m|l| - md_e(b'_m, \Delta')}$$

According to formula (II.1.2.3) one can write

$$(\partial^\gamma C)^\Gamma_m = \sum_{l \in L(\gamma)} (\partial^\gamma C)^\Gamma_{m,l}$$

as in (II.1.2.7) and one gets

LEMMA II.1.2.2. — *Let x be in the unit cube Δ and let Δ' be the unit cube $\Delta_0 + z$ then*

$$O \leqslant \overline{(\partial^\gamma C)}^\Gamma_{m,l}(x\,;\,z] \leqslant e^{-m(d_e(\Delta, b'_1) + |l| + d_e(b'_m, \Delta'))} \tag{II.1.2.8}$$

We now introduce the momentum cutoff.

II.2. The momentum cutoff

As in reference [3] we choose our momentum function to be $\eta(\rho) \in C^\infty_0(\mathbb{R})$ and we define for $k = (\{k^l\}, l = 0, 1, 2)$ and $\alpha \in (\mathbb{R}^+)^3$,

$$\eta_\alpha(k) = \prod_{l=0}^{2} \eta_{\alpha^l}(k^l) = \prod_{l=0}^{2} \eta\left(\frac{k^l}{\alpha^l}\right), \qquad \eta_{\alpha,\beta} = \eta_\alpha - \eta_\beta$$

The cutoffs α are chosen in the sequence

$$O, M_1, M_2, \ldots, M_{j+1} = M_j^{(1+\nu)} = M_1^{(1+\nu)^j}, \ldots$$

We introduce also

$$\eta^{j_l}(k^l) = \eta_{M_{j_l}}(k^l) - \eta_{M_{j_l-1}}(k^l) \geqslant 0$$

and for $j \in (\mathbb{N})^3$

$$\eta^j(k) = \prod_{l=0}^{2} \eta^{j_l}(k^l) \tag{II.2.1}$$

Let η_1 and η_2 be two momentum cutoff functions (i. e. η_α, $\eta_{\alpha,\beta}$, $\eta^j \ldots$), we define $B^\Gamma_{\eta_1,\eta_2}$ by replacing in formula (II.1.1.1) C_{M-m_1} by

$$C_{M-m_1,\eta_1,\eta_2}(x, y) = \frac{1}{(2\pi)^3} \int \frac{e^{ik(x-y)}}{k^2 + (M - m_1)^2} \eta_1(k)\eta_2(k)dk \tag{II.2.2}$$

When $\eta_1 = \eta_2$ or when we do not emphasize the role of the momentum cutoff we write B^Γ_η, $C_{M-m_1,\eta}$, $C_{(s),\eta}$ or $C_\kappa(s) := C_{(s),\eta_\kappa}$. The η's being positive,

$C_{M-m_1,\eta}$ is a positive bilinear form (a real but not positive valued function).

Lemma II.2.1 still applies to $C_{(s),\eta}$ which is also a positive continuous bilinear form; but due to the compact support of η it is a degenerate form. However one can still define Gaussian measures concentrated on subspaces of $\mathcal{S}'(\mathbb{R}^3)$ [9]. For these measures the usual functional calculus is left unchanged. We need only to be careful in considering the inverse operator of $C_{(s),\eta}$. So for each $C_{(s),\eta}$ we can define a Gaussian random variable of mean zero and covariance $C_{(s),\eta}$. Let us suppose now, for simplicity, that the starting cutoff is

$$\eta_{\bar{j}}(k) = \prod_{l=0}^{2} \eta\left(\frac{k^l}{M_{\bar{j}}}\right)$$

then

$$\eta_{\bar{j}}(k) = \sum_{j \leqslant \bar{j}} \eta^j(k) \tag{II.2.3}$$

where the sum extends over all $j \in \mathbb{N}^3$ such that $j_i \leqslant \bar{j}$, $i = 0, 1, 2$.

From the decomposition (II.2.3) one gets

$$C_{M-m_1,\eta_{\bar{j}},\eta_{\bar{j}}}(x, y) = \sum_{\substack{i,j \\ i \leqslant \bar{j}, j \leqslant \bar{j}}} C_{M-m_1,\eta^i,\eta^j}(x, y) =: \sum_{\substack{i,j \\ i \leqslant \bar{j}, j \leqslant \bar{j}}} C_{M-m_1}(x, i; y, j) \tag{II.2.4}$$

Replacing $C_{M-m_1,\eta}(x, y)$ by $C_{M-m_1}(x, i; y, j)$ one defines $B^\Gamma(x, i; y, j)$ and $C_{(s)}(x, i; y, j)$. Now $B^\Gamma(x, i; y, j)$ is of Dirichlet type.

Let us now define the space $\mathcal{B}(\mathbb{R}^3 \times \mathbb{N}^3)$ of functions $f(x, i)$ such that: $f(., i) \in \mathcal{S}(\mathbb{R}^3)$ for each i and $f(x, i)$ is different from zero only on a finite subset of $\mathbb{N}^3$. $C_{(s)}(x, i; y, j)$ is a positive continuous form, as easily checked, on $\mathcal{B} \times \mathcal{B}$. Let φ be the Gaussian random process over $\mathcal{B}$ of mean zero and covariance $C_{(s)}(x, i; y, j)$. It will play the role of the sequence of cutoff fields introduced during the inductive expansion of [3]. In particular if $\chi_{\alpha,\beta}(z)$, $z \in \mathbb{N}$ is the characteristic function of the set of integers between α and β, the Gaussian random process

$$f \to \sum_{i=(i_0,i_1,i_2)} \int \varphi(x, i) f(x) \prod_{l=0}^{2} \chi_{\alpha_l,\beta_l}(i_l) dx$$

can be identified with a field $\varphi_{\eta_{\alpha,\beta}}$ with « upper cutoff » β and « lower cutoff » α (see the remark above), and of covariance

$$\sum_{i,j} C_{(s)}(x, i; y, j) \prod_{l=0}^{2} \chi_{\alpha_l,\beta_l}(i_l) \prod_{l=0}^{2} \chi_{\alpha_l,\beta_l}(j_l) = C_{(s),\eta_{\alpha\beta},\eta_{\alpha\beta}}(x, y)$$

All formulas: change of measure, contraction formula, ... extend in an obvious way to φ.

An alternative point of view is to define a set $\{ \varphi_i(x) \}$ of jointly Gaussian random variables with covariance matrix $\{ C^{i,j}_{(s)}(x, y) \}$ where $\varphi_i(x) = \varphi(x, i)$ and $C^{i,j}_{(s)}(x, y) = C_{(s)}(x, i ; y, j)$, the field $\varphi_{\eta_{\alpha\beta}}$ being the sum of those random variables which are in the momentum range.

Let us now give more explicitely what are the change introduced by our definition of the field in the steps of the inductive expansion [3]. In the P – C expansion a local change in the momentum cutoff producing a lower upper cutoff is obtained here by a local change of characteristic function (those associated to the i-variable s). On the other hand the decomposition of the field φ in $\varphi_1 + \varphi_2$, starting point of the Wick construction, is obvious as is the restriction to some elementary intervals of the leg's momenta for the W-vertices. Finally

$$: \varphi^n_\eta : (x) = \int e^{ix(k_1 + \ldots + k_n)} : \varphi(k_1) \ldots \varphi(k_n) : \prod_{j=1}^{n} \eta_j(k_j) dk_j$$

is replaced by

$$\sum_{i_1 \ldots i_n} \prod_{j=1}^{n} \chi_j(i_j) : \varphi(x, i_1) \ldots \varphi(x, i_n) :$$

which we will generally write

$$\sum_i : \varphi^n : (x, i)$$

Remark. — To be more complete one should emphasized that the way we introduce the momentum cutoff function breaks the strict localization in momentum space (and therefore does not contradict the strict decoupling in position space). Let us look at an example.

Consider B^Γ_η for $\Gamma = (Z^3)^*$ given by

$$B^{(Z^3)^*}_\eta(x, y) = C_{M - m_1, \eta}(x, y) e^{-m_1 |x - y|}$$

as compared to $C_{M,\eta}(x, y)$ and take η to be with compact support. The covariance $C_{M,\eta}$ has Fourier transform $\eta(k)(k^2 + M^2)^{-1}$ but $B^{(Z^3)^*}_\eta$ has a Fourier transform which is the convolution product of $\eta(k)(k^2 + (M - m_1)^2)^{-1}$ by $(k^2 + m_1^2)^{-2}$ and one sees that this last covariance has no more compact support but decreases rapidly off the support of η. The same mechanism will generally apply with the effect of a little change in the way we get the estimates.

II.3. **The cluster expansion**

II.3.1. INTRODUCTION

The cluster expansion can be defined either as in [1] or as in [2]. In any case, the convergence results from the same basic estimate which will be proved subsequently. The notations will be mainly those of [1].

Let Λ be some finite subset of the lattice. Let $F = \prod_{i \in I} F_i$ be a finite product of Wick monomial F_i with support in $\Delta_i \subset \Lambda$, Δ_i unit cube of the lattice. Let $n_F(\Delta_i)$ be the number of legs of F localized in Δ_i and suppose that each F_i is bounded with respect to a norm $||| \, . \, |||_{\delta,\alpha}$, the norm of F being defined as

$$||| F |||_{\delta,\alpha} = \prod_{i \in I} ||| F_i |||_{\delta,\alpha}$$

Our basic object of study is

$$\mathscr{F}(s) = \int F e^{-V(\Lambda)} d\Phi_{C_{(s),\eta}} \tag{II.3.1.1}$$

where $d\Phi_{C_{(s),\eta}}$ is the Gaussian measure of mean zero and covariance $C_{(s),\eta}(x, i\,;\, y, j)$

$$V(\Lambda) = V_I(\Lambda) + V_C(\Lambda)$$

$$V_I(\Lambda) = \mu \int :\varphi^4:(x)\Lambda(x)dx \tag{II.3.1.2}$$

$$V_C(\Lambda) = \frac{1}{2}\langle V_I(\Lambda)^2 \rangle - \frac{1}{6}\langle V_I(\Lambda)^3 \rangle - \frac{\mu^2}{2}\sum_{\Delta \subset \Lambda} \delta m^2(\Delta) \int :\varphi^2:(x)\chi_\Delta(x)dx$$

the bracket $\langle \, . \, \rangle$ denotes the expectation value with respect to $d\Phi_{C_{(s),\eta}}$. The counterterm $\delta m^2(\Delta)$ will be defined later.

In what follows μ will be chosen equal to 1. In fact we fixe the interval of values of μ to be [0, 1] and all estimates are bounded by their values at $\mu = 1$. It is only when we normalize (II.3.1.1) that we will take care on the values of μ.

All Wick products are taken relatively to the covariance $C_{(s),\eta}$ and by $\varphi(x)$ we mean as announced in II.2 the sum of $\varphi(x, i)$ times the characteristic function of the momentum cutoff support.

Let Y be a big cube, union of lattice cubes, and containing Λ. Let $\mathbb{B}_Y$ be the set of all the lattice faces strictly contained in Y, $\mathbb{B}_Y$ will be our set of bonds. Let $\mathscr{F}_Y := \mathscr{F}(s)$ when $s_b = 1$ if $b \in B_Y$, $s_b = 0$ otherwise, then

$$\mathscr{F}_Y = \sum_{\Gamma \subset \mathbb{B}} \int_0^1 \partial^\Gamma \mathscr{F}(s(\Gamma)) \prod_{b \in \Gamma} ds_b$$

where $\partial^\Gamma = \prod_{b \in \Gamma} \frac{d}{ds_b}$ and

$$s_b(\Gamma) = \begin{cases} s_b & \text{if} \quad b \in \Gamma \\ 0 & \text{if} \quad b \notin \Gamma \end{cases}$$

The fact that $\mathscr{F}_Y$ converges as Y tends to infinity will be shown later. We first show the decoupling for $\mathscr{F}(s(\Gamma))$.

Suppose $\mathbb{R}^3\backslash\Gamma^c = X_1 \cup \ldots \cup X_r$, X_i being connected and $X_i \cap X_j = \emptyset$, then, since

$$C_{(s(\Gamma)),\eta}(x, y) = \sum_{\gamma\subset\Gamma} \prod_{b\in\gamma} s_b \prod_{b\in\Gamma\backslash\gamma} (1 - s_b) B_\eta^\gamma(x, y)$$

and according to the fact that B^Γ is of Dirichlet type, the measure $d\Phi_C$ factorizes. In the same way $V(\Lambda) = \sum_{i=1}^{r} V_i(\Lambda) = \sum_{i=1}^{r} V(X_i \cap \Lambda)$, thus, since F is of a factorized form one gets

LEMMA II.3.1.1. — *$\mathscr{F}(\Lambda, s(\Gamma)) = \mathscr{F}(s(\Gamma))$ decouples at $s = 0$.*

In the same way $\mathscr{F}(s)$ is smooth and regular at infinity and we have therefore the equivalent of proposition 3.2 of [*1*].

The convergence of the cluster expansion will result essentially from

PROPOSITION II.3.1.2. — *Let $\Gamma \subset \mathbb{B}_Y$ and X be one of the connected component generated by Γ. Let $F = \prod_i F_i$ with* suppt *$F_i \subset \Delta_i \subset X$, then there exists two constants C and K, C independent of M, m_1, m for m, m_1 and M large enough, and K as large as we want provided m is large enough, such that*

$$\lim_{\kappa\to\infty} \partial^\Gamma \int F e^{-V(\Lambda\cap X)} d\Phi_{C_\kappa(s)}$$

exists and is bounded uniformly in κ as $\kappa \to \infty$ by

$$\left(\prod_{\Delta_i\subset X} n_F(\Delta_i)\,!\right) C^{\sum_{\Delta_i\subset X} n_F(\Delta_i)} |||\,F\,|||_{\delta,\alpha} e^{-K|\Gamma|}$$

This proposition is proved in chapters III and IV.

Let us define by $C_\kappa^{\partial Y}$ the covariance $B_\kappa^{\partial Y}$ with Dirichlet conditions on ∂Y and cutoff η_κ. Then

COROLLARY II.3.1.3. — *There exists $\mu_0 > 0$, such that for κ large enough and $0 \leqslant \mu \leqslant \mu_0$*

$$\int e^{-V(\Delta_0,\mu)} d\Phi_{C_\kappa^{\partial\Delta_0}} \geqslant \frac{1}{2}$$

This corollary follows from the continuity of V as function of μ and from proposition II.3.1.2 applied for $\Gamma = \emptyset$, $\Lambda = \Lambda_0$ and $Y = \Delta_0$. See chapter IV.

Proposition 5.1 of ref. [1] generalizes easily to three dimension and one proves with the above proposition and corollary the convergence of the cluster expansion following the lines of [2].

This proves that the φ_3^4 model defined as the expectation values with respect to the measure

$$dq(\Lambda, \mathrm{Y}, \mu, \kappa) = \frac{e^{-\mathrm{V}_\kappa(\Lambda,\mu)} d\Phi_{\mathrm{C}_\kappa^{\partial \mathrm{Y}}}}{\int e^{-\mathrm{V}_\kappa(\Lambda,\mu)} d\Phi_{\mathrm{C}_\kappa^{\partial \mathrm{Y}}}}$$

has a limit as κ, Y and Λ tend to infinity and satisfies the equivalent of theorem I.4 and theorem I.5 for $n = 0$.

However, we want to prove these theorems for the approximate measure $dq(\Lambda, \mu, \kappa)$ defined by Glimm-Jaffe and Feldman. This result will be obtained in chapter IV by showing that, roughly speaking,

$$\lim_{\mathrm{Y}\to\infty} \lim_{\kappa\to\infty} dq(\Lambda, \mathrm{Y}, \mu, \kappa) = \lim_{\kappa\to\infty} dq(\Lambda, \mu, \kappa)$$

The necessity of taking the limit with respect to κ follows from the remark that

$$\lim_{\mathrm{Y}\to\infty} \mathrm{C}_\kappa^{\partial \mathrm{Y}}(x, y) = \mathrm{C}_{\mathrm{M}-m_1,\kappa}(x, y) e^{-m_1|x-y|}$$

is different from $\mathrm{C}_\kappa(x, y) := \mathrm{C}_{\mathrm{M},\kappa}(x, y)$.

II.3.2. The estimates of the cluster expansion

Let $\tilde{\Gamma} \subset \mathbb{B}_\mathrm{Y}$, X be one of the connected component generated by $\tilde{\Gamma}$ and Γ the restriction of $\tilde{\Gamma}$ to X.

Suppose that F has support in X and let us omit the momentum cutoff in our notation, then consider

$$\mathrm{A} = \partial^\Gamma \int \mathrm{F} e^{-\mathrm{V}(\mathrm{X})} d\Phi_{\mathrm{C}(s)}$$

As it will be explained in the next chapter, A is a sum of terms of the form

$$\sum_{\substack{\pi \in \mathscr{P}(\Gamma) \\ \pi = (\gamma_1, \ldots, \gamma_k)}} \sum_{\Delta_1, \ldots, \Delta_{2k}} \int \prod_{i=1}^{2k} \chi_{\Delta_i}(x_i) \partial^{\gamma_1} \mathrm{C}_{(s)}(x_1, x_2) \ldots \partial^{\gamma_k} \mathrm{C}_{(s)}(x_{2k-1}, x_{2k}) \int \mathrm{R} e^{-\mathrm{V}'} d\Phi_{\mathrm{C}_{(s)}} \prod_{i=1}^{2k} dx_i \quad (\mathrm{II}.3.2.1)$$

where $\mathscr{P}(\Gamma)$ is the set of all partitions $(\gamma_1, \ldots, \gamma_k)$ of Γ, the sum extends over all unit cubes of the lattice and R and V′ are functions of the field to be defined later.

We bound (II.3.2.1) using the results of section II.1 and the fact that

$$\sum_{\Gamma \supset \gamma} \prod_{b \in \Gamma \setminus \gamma} s_b \prod_{b \in \Gamma^c} (1 - s_b) = \prod_{b \in \mathbb{B}_Y \setminus \gamma} (s_b + (1 - s_b)) = 1$$

by

$$\sum_{\pi \in \mathscr{P}(\Gamma)} \sum_{\Delta_1, \dots, \Delta_{2k}} \Bigg(\prod_{i=1}^{k} \sum_{z_i \in Z^3} \sum_{\substack{\gamma_{i_1} \cup \gamma_{i_2} = \gamma_i \\ \gamma_{i_1} \cap \gamma_{i_2} = \emptyset}} \sum_{\substack{l_{i_1} \in L(\gamma_{i_1}) \\ l_{i_2} \in L(\gamma_{i_2})}}$$

$$\frac{\overline{(\partial^{\gamma_{i_1}} C)}_{m, l_{i_1}}(x_{2i-1}\,;\, z_i]}{\bar{C}_m(x_{2i-1}\,;\, z_i]} e^{-m(1+\tau_1)|r_{z_i, x_{2i-1}}|} e^{-m(1+\tau_1)|r_{z_i, x_{2i}}|} \frac{\overline{(\partial^{\gamma_{i_2}} C)}_{m, l_{i_2}}(x_{2i}\,;\, z_i]}{\bar{C}_m(x_{2i}\,;\, z_i]} \Bigg)$$

$$\sup_{z_1, \dots, z_k} \left| \int \prod_{i=1}^{k} \chi_{\Delta_{2i-1}}(x_{2i-1}) \chi_{\Delta_{2i}}(x_{2i}) C_{M-m_1}(x_{2i-1}, x_{2i}) m_1 \tilde{D}_{z_i}(x_{2i-1}, x_{2i}) \right.$$

$$\left. \int R e^{-V'} d\Phi_{C(s)} \prod_{i=1}^{2k} dx_i \right| \qquad \text{(II.3.2.2)}$$

In this formula we have used $O \leqslant J_b^T \leqslant 1$ and introduced

$$\tilde{D}_{z_i}^{m_1}(x_{2i-1}, x_{2i}) = e^{m(1+\tau_1)(|r_{z_i, x_{2i-1}}| + |r_{z_i, x_{2i}}|)} D_{z_i}^{m_1}(x_{2i-1}, x_{2i})$$

where $r_{z_i, x_{2i-1}}$ (resp. $r_{z_i, x_{2i}}$) is the vector translation from z_i to the center of the cube Δ_{2i-1} (resp. Δ_{2i}) containing x_{2i-1} (resp. x_{2i}), and $0 < \tau_1$, m_1 being larger than $m(1 + \tau_1)$.

We remember that to l_{i_1} corresponds a certain ordering of the bonds of γ_{i_1} labelled by $b_1^{i_1'}, \dots, b_{m_i}^{i_1'}$ where $b_1^{i_1'} = b_1^{i_1}$ is the first bond crossed by paths starting at x_{2i-1}. By symmetry, to l_{i_2} corresponds the ordering $b_1^{i_2'}, \dots, b_{m_{i_2}}^{i_2'}$ with $b_1^{i_2}$ the first bond crossed by paths starting at x_{2i}.

Let us remark that for an element of the sum over $\gamma_{i_1} \cup \gamma_{i_2}$ it may happen that either γ_{i_1} and γ_{i_2} are differents from zero or one of them is zero. If $\gamma_{i_\alpha} = \emptyset$ then the ratio $\dfrac{\overline{(\partial^{\gamma_{i_\alpha}} C)}}{\bar{C}}$ reduces to 1.

Since

$$O \leqslant r_{z_i, x_{2i-1}} - d_e(\Delta_0 + z_i, \Delta_{2i-1}) \leqslant \sqrt{3} \qquad \text{(II.3.2.3)}$$

the term

$$\frac{\overline{(\partial^{\gamma_{i_1}} C)}_{m, l_{i_1}}(x_{2i-1}\,;\, z_i]}{\bar{C}_m(x_{2i-1}\,;\, z_i]} e^{-m(1+\tau_1)(|r_{z_i, x_{2i-1}}| + |r_{z_i, x_{2i}}|)} \frac{\overline{(\partial^{\gamma_{i_2}} C)}_{m, l_{i_2}}(x_{2i}\,;\, z_i]}{\bar{C}_m(x_{2i}\,;\, z_i]}$$

is bounded by

$$O(1) e^{8m} (d_e(\Delta_{2i-1}, \Delta_0 + z_i) + 1) e^{-m[d_e(\Delta_{2i-1}, b_1^{i_1'}) + |li_1| + d_e(b_{m_{i_1}}^{i_1'}, \Delta_0 + z_i)]}$$
$$e^{m[d_e(\Delta_{2i-1}, \Delta_0 + z_i) - |r_{z_i, x_{2i-1}}|]} e^{-m\tau_1[|r_{z_i, x_{2i-1}}| + |r_{z_i, x_{2i}}|]}$$
$$e^{-m[d_e(\Delta_{2i}, b_1^{i_2'}) + |li_2| + d_e(b_{m_{i_2}}^{i_2'}, \Delta_0 + z_i)]} e^{m[d_e(\Delta_{2i}, \Delta_0 + z_i) - |r_{z_i, x_{2i}}|]}$$
$$(d_e(\Delta_{2i}, \Delta_0 + z_i) + 1) \qquad \text{(II.3.2.4)}$$

for $\gamma_{i_1} \neq \emptyset$, $\gamma_{i_2} \neq \emptyset$ and by

$$O(1)e^{4m}(d_e(\Delta_{2i-1}, \Delta_0 + z_i) + 1)e^{-m[d_e(\Delta_{2i-1}, b_1^{i'}) + |l_i| + d_e(b_{m_i}^{i'}, \Delta_0 + z_i)]}$$
$$e^{m[d_e(\Delta_{2i-1}, \Delta_0 + z_i) - |r_{z_i, x_{2i-1}}|]} e^{-m\tau_1 |r_{z_i, x_{2i-1}}|} e^{-m(1+\tau_1)|r_{z_i, x_{2i}}|} \quad \text{(II.3.2.5)}$$

if for example $\gamma_{i_1} = \gamma_i$, $\gamma_{i_2} = \emptyset$.

Now, using (II.3.2.3), there exists $K_1(m\tau_1)$ such that

$$(d_e(\Delta_{2i-1}, \Delta_0 + z_i) + 1)e^{-\frac{m\tau_1}{2}|r_{z_i, x_{2i-1}}|} \leqslant K_1(m\tau_1) \quad \text{(II.3.2.6)}$$

On the other hand, from the same inequality results

$$e^{m[d_e(\Delta_{2i}, \Delta_0 + z_i) - |r_{z_i, x_{2i}}|]} \leqslant 1$$

and a similar bound for $2i - 1$ instead of $2i$.

Moreover one has for any $z \in Z^3$

$$d_e(\Delta_{2i}, b_{m_i}^{i'}) \leqslant |r_{z_i, x_{2i}}| + d_e(b_{m_i}^{i'}, \Delta_0 + z_i) + (1 + \sqrt{3})$$

from which follows that in (II.3.2.5) one can replace

$$e^{-md_e(b_{m_i}^{i'}, \Delta_0 + z_i)} e^{+m|r_{z_i, x_{2i}}|}$$

by

$$e^{-md_e(\Delta_{2i}, b_{m_i}^{i'})} e^{m(1+\sqrt{3})}$$

We now emphasize the choices of the $b_1^{i'_1}$'s or $b_{m_{i_1}}^{i'_1}$ by introducing the definitions

1) if $\gamma_{i_1} \neq \emptyset$, $\gamma_{i_2} \neq \emptyset$, l_{i_1} and l_{i_2} being given

$$b_1^{i'_1} =: b_{2i-1}$$
$$b_1^{i'_2} =: b_{2i}$$

2) if $\gamma_{i_1} = \gamma_i$, $l_{i_1} = l_i$ being given

$$b_1^{i'_1} =: b_{2i-1}$$
$$b_{m_{i_1}}^{i'_1} =: b_{2i}$$

3) if $\gamma_{i_2} = \gamma_i$, $l_{i_2} = l_i$ being given

$$b_{m_{i_2}}^{i'_2} =: b_{2i-1}$$
$$b_1^{i'_2} =: b_{2i}$$

Then (II.3.2.4) is bounded by

$$O(1)e^{10m}K_1^2 e^{-m[d_e(\Delta_{2i-1}, b_{2i-1}) + d_e(\Delta_{2i}, b_{2i})] - m(|l_{i_1}| + |l_{i_2}|)}$$
$$e^{-\frac{m\tau_1}{2}[|r_{z_i, x_{2i-1}}| + |r_{z_i, x_{2i}}|]} \quad \text{(II.3.2.4)'}$$

and (II.3.2.5) by

$$O(1)e^{7m}K_1 e^{-m[d_e(\Delta_{2i-1}, b_{2i-1}) + d_e(\Delta_{2i}, b_{2i})]} e^{-m|l_i|} e^{-\frac{m\tau_1}{2}[|r_{z_i, x_{2i-1}}| + |r_{z_i, x_{2i}}|]} \quad \text{(II.3.2.5)'}$$

We perform the sum over z_i using

$$\sum_{z_i} e^{-\frac{m\tau_1}{2}[|r_{z_i,x_{2i-1}}|+|r_{z_i,x_{2i}}|]} \leqslant \left[\sum_{z_i} e^{-\frac{m\tau_1}{2}|r_{z_i,x_{2i-1}}|}\right]\left[\sum_{z_i} e^{-\frac{m\tau_1}{2}|r_{z_i,x_{2i}}|}\right] \leqslant \mathrm{K}_2(m\tau_1) \quad \text{(II.3.2.7)}$$

Introducing the scaled distance d, which reduces here to

$$d = \sup(1, d_e) \leqslant 1 + d_e$$

one gets that (II.3.2.2) is bounded by

$$\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=(\gamma_1,\dots,\gamma_k)}} \sum_{\Delta_1,\dots,\Delta_{2k}} \sum_{\substack{\gamma_{i_1}\cup\gamma_{i_2}=\gamma_i\\ i=1,\dots,k}} \sum_{\substack{l_{i_1}\in\mathrm{L}(\gamma_{i_1})\\ l_{i_2}\in\mathrm{L}(\gamma_{i_2})\\ i=1,\dots,k}} [\mathrm{O}(1)e^{12m}\mathrm{K}_1^2\mathrm{K}_2]^k$$

$$\prod_{i=1}^{k}\left\{ e^{-m[d(\Delta_{2i-1},b_{2i-1})+d(\Delta_{2i},b_{2i})]}[d(\Delta_{2i-1}, b_{2i-1})]^{n_2}[d(\Delta_{2i}, b_{2i})]^{n_2}e^{-m[|l_{i_1}|+|l_{i_2}|}\right\}$$

$$\sup_{\substack{z_i\in\mathrm{Z}_3\\ i=1,\dots,k}} \sup_{\substack{b_{2i},b_{2i-1}\\ i=1,\dots,k}} \prod_{i=1}^{k}[d(\Delta_{2i-1}, b_{2i-1})]^{-n_2}[d(\Delta_{2i}, b_{2i}]^{-n_2}$$

$$\left|\int \mathrm{C}_{\mathrm{M}-m_1}(x_1, x_2).m_1\tilde{\mathrm{D}}_{z_1}^{m_1}(x_1, x_2)\ \dots\ \mathrm{C}_{\mathrm{M}-m_1}(x_{2k-1}, x_{2k})m_1\tilde{\mathrm{D}}_{z_k}^{m_1}(x_{2k-1}, x_{2k})\right.$$

$$\left.\int \mathrm{R}e^{-\mathrm{V}'}d\Phi_{\mathrm{C}}\prod_{i=1}^{2k}\chi_{\Delta_i}(x_i)dx_i\right| \quad \text{(II.3.2.8)}$$

where n_2 is some positive integer to be fixed later.

In the last expression the supremum extends over all choices of $2k$ *distinct* bonds $b_i \in \Gamma$.

Now, we notice that the sum $\sum_{\substack{\gamma_{i_1}\cup\gamma_{i_2}=\gamma_i\\ \gamma_{i_1}\cap\gamma_{i_2}=\emptyset}}$ is bounded by $2^{|\gamma_i|}$ thus a factor $2^{\sum_i|\gamma_i|} = 2^{|\Gamma|}$ bounds the partitions of γ_i into two pieces. A factor 2 for each γ_{i_j} allows us to decide if γ_{i_1} or γ_{i_2} are empty or not, and since there is at most $|\Gamma|$ elements, we get a factor $2^{|\Gamma|}$. This allows us to replace

$$\sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=(\gamma_1,\dots,\gamma_k)}} \sum_{\substack{\gamma_{i_1}\cup\gamma_{i_2}=\gamma_i\\ \gamma_{i_1}\cap\gamma_{i_2}=\emptyset\\ i=1,\dots,k}} \sum_{\substack{l_{i_1}\in\mathrm{L}(\gamma_{i_1})\\ l_{i_2}\in\mathrm{L}(\gamma_{i_2})\\ i=1,\dots,k}} \prod_{i=1}^{k} e^{-m(|l_{i_1}|+|l_{i_2}|)}\ \dots$$

by

$$\sum_{\substack{\pi \in \mathscr{P}(\Gamma) \\ \pi = (\gamma_1, \ldots, \gamma_j)}} \sum_{l_i \in L(\gamma_i)} e^{-m \sum_{i=1}^{p} |l_i|} \quad . \quad \sup_{\substack{\pi \in \mathscr{P}(\Gamma) \\ \pi = (\gamma_1, \ldots, \gamma_k) \\ \gamma_{i_1} \cup \gamma_{i_2} = \gamma_i, \gamma_{i_1} \cap \gamma_{i_2} = \emptyset \\ i = 1, \ldots, k}}$$

Now, we have a lemma corresponding to lemma 8.2 of [1].

LEMMA II.3.2.1

$$\sum_{\substack{\pi \in \mathscr{P}(\Gamma) \\ \pi = (\gamma_1, \ldots, \gamma_p)}} \sum_{l_i \in L(\gamma_i)} e^{-\frac{m}{2} \sum_i |l_i|} \leqslant e^{K_3 |\Gamma|}$$

with K_3 *independent of* m, *for* m *large enough.*

Proof. — As in [1].

We use the remaining $e^{-\frac{m}{2} \sum_i |l_i|}$ to get convergent factors. In fact

$$|l_i| \geqslant \frac{|\gamma_i|}{34} - 1$$

since each face has at most 32 faces which touch it. Thus

$$\Sigma |l_i| \geqslant \frac{|\Gamma|}{34} - j$$

but $j \leqslant 2k$, so

$$e^{-\frac{m}{2} \sum_i |l_i|} \leqslant e^{-\frac{m}{68} |\Gamma| + mk}$$

Then m will be chosen large enough $m > 68 K_3$, thus defining

$$K_4(m) = \frac{m}{68} - K_3$$

The factor e^{mk} will be inserted in formula (II.3.2.8) by replacing e^{12m} by e^{13m}.

Let $M(\Delta)$ be the number of Δ_i, $i = 1, \ldots, 2k$ equals to Δ, we then prove the following lemma corresponding to the main part of lemma 10.2 of [1].

LEMMA II.3.2.2. — *Let* n_1 *be some positive number, then for* m *larger than some* $m(n_1)$

$$\prod_{\Delta} M(\Delta)^{n_1 M(\Delta)} \leqslant e^{\frac{m}{4} \sum_{j=1}^{2k} d(\Delta_j, b_j)}$$

Proof. — For fixed Δ, the number of b_i such that $d(\Delta, b_i) \leqslant r$, $r \geqslant 1$ and $\Delta_i = \Delta$ is bounded by $O(1) r^3$. Thus the number M_j of b_i such that $d(\Delta, b_i) \leqslant d(\Delta, b_j)$ satisfies

$$M_j^{1/3} \leqslant O(1) d(\Delta, b_j)$$

Now

$$M(\Delta)^{4/3} \leqslant O(1) \sum_{j,\Delta_j=\Delta} M_j^{1/3} \leqslant O(1) \sum_{j,\Delta_j=\Delta} d(\Delta, b_j)$$

hence n_1 being some fixed number

$$M(\Delta)^{n_1 M(\Delta)} \leqslant e^{n_1 M(\Delta)\log M(\Delta)} \leqslant e^{2n_1 M(\Delta)^{4/3}} \leqslant e^{O(1) n_1 \sum_{j,\Delta_j=\Delta} d(\Delta, b_j)}$$

and the lemma follows by taking $m \geqslant 4n_1 O(1)$.

Using

$$[d(\Delta, b)]^{n_2} e^{-\frac{m}{2} d(\Delta, b)} \leqslant \frac{n_2 !}{m^{n_2}} \leqslant K_5(n_2)$$

for $m > 1$ one gets

PROPOSITION II.3.2.3. — *For m large enough,* (II.3.2.8) *is bounded by*

$$\sup_k [K_7(m, n_2)]^k K_6(m)^{-|\Gamma|} \sum_{\Delta_1, \dots, \Delta_{2k}} \prod_{\Delta} [M(\Delta)]^{-n_1 M(\Delta)} \sup_{\substack{z_i \\ i=1,\dots,k}} \sup_{\substack{b_i \\ i=1,\dots,2k}}$$

$$\prod_{i=1}^{2k} [d(\Delta_i, b_i)]^{-n_2} e^{-\frac{m}{4} d(\Delta_i, b_i)} \left| \int C_{M-m_1}(x_1, x_2) m_1 \tilde{D}_{z_1}^{m_1}(x_1, x_2) \dots \right.$$

$$\left. C_{M-m_1}(x_{2k-1}, x_{2k}) m_1 \tilde{D}_{z_k}^{m_1}(x_{2k-1}, x_{2k}) \int R e^{-V'} d\Phi_C \prod_{i=1}^{2k} \chi_{\Delta_i}(x_i) dx_i \right| \qquad \text{(II.3.2.9)}$$

with $K_7(m, n_2) = O(1) K_1^2 K_2 e^{13m} K_5^2$ *and* $\log K_6(m) = K_4(m) - 2 \log 2$.

We fixe $m\tau_1 = O(1)$, thus K_1 and K_2 are independent of m. Since K_4 is large if m is large, so it is for K_6.

Let us remark it remains a sum over $\Delta_1, \dots, \Delta_{2k}$. It will be treated with the help of combinatoric factors although it was possible to prove a lemma completely equivalent to lemma 10.2 of [*1*].

III. THE MODIFIED INDUCTIVE EXPANSION

We will show in this chapter how to bound $\partial^\Gamma \int F e^{-V} d\Phi_{C_{(s),\kappa}}$. In a first part we define the expansion and show in particular how we get terms of the form (II.3.2.1). In the second part we derive the combinatoric factors which bound the number of these terms and the number of graphs generated by the inductive expansion applied to each of these terms. An overall factor of the form $K^{-|\Gamma|}$ will then be exhibited ensuring the convergence of the cluster expansion.

We remenber that

$$V = \sum_i \int :\varphi^4:(x, i)\chi_{\Lambda\cap X}(x)dx + \frac{1}{2}\int\left(\sum_i \int :\varphi^4:(x, i)\chi_{\Lambda\cap X}(x)dx\right)^2 d\Phi_{C(s(\Gamma))} - \frac{1}{6}\int\left(\sum_i \int :\varphi^4:(x, i)\chi_{\Lambda\cap X}(x)dx\right)^3 d\Phi_{C(s(\Gamma))} - \sum_{\Delta\subset\Lambda\cap X} \frac{\delta m^2_{(s)}(\Delta)}{2}\int\sum_i :\varphi^2:(x, i)\chi_\Delta(x)dx \quad \text{(III.0.1)}$$

the $\sum_i$ extending from 0 to the starting cutoff, X being one of the connected component of $\mathbb{R}^3\backslash\Gamma^c$, the covariance $C_{(s)}(x, i; y, j)$ being restricted for the x-variables as a bilinear form on $\mathcal{S}'(X) \times \mathcal{S}'(X)$.

As before F is a product of Wick products, restricted tó $\Lambda \cap X$, and all Wick products are taken relatively to $C_{(s(\Gamma))}$.

To be complete we give the definition of $\delta m^2_{(s)}(\Delta)$. Let $u \in \Delta$, then

$$\delta m^2_{(s)}(\Delta) = -4^2.6\int_{\mathbb{R}^3}\left[\sum_\Gamma \prod_{b\in\Gamma} s_b \prod_{b\in\Gamma^c}(1 - s_b)\sum_{z\in Z^3}\left(\frac{\bar{C}^\Gamma_m(u ; z]}{\bar{C}_m(u ; z]}\right)^2 m_1 D_z^{m_1}(0, x)\sum_{i,j} C_{M-m_1}(0, i ; x, j)\right]^3 dx \quad \text{(III.0.2)}$$

Finally remenber that in (III.0.1) the $\sum_i$ can be replaced by $\sum_i \chi(i)$ where χ is the characteristic function of the range of allowed values of i.

Now we explain the inductive expansion.

III.1. The inductive expansion

To start

a) we perform the first P − C expansion as in [*3*] and [*4*]. The F − legs are treated as the G_2 − legs of [*4*], i. e.: they initiate no contractions and are localized in unit cubes.

As explained at the end of chapter II, the change of the momentum cutoff is obtained by a change of the characteristic function $\chi(i)$. To the s dependent momentum cutoff for one leg:

$$\eta(s, k) = s\eta(k) + (1 - s)\eta'(k)$$

corresponds the s-dependent characteristic function

$$\chi(s, i) = s\chi(i) + (1 - s)\chi'(i)$$

Then

b) we do the derivatives with respect to the s_b 's, $b \in \Gamma$. To a derivation on s_b is associated a derived propagator (many derivations can be associated to one derived propagator)

$$\frac{d}{ds_b} \sum_{i,j} \chi(i)\chi(j)C_{(s)}(x, i ; y, j)$$

As in [*1*] we localize x and y in unit cubes of the lattice.

Steps *a*) and *b*) replace the starting term by a sum of terms of the type II.3.2.1. However we should explicit what is the structure of $Re^{-V'}$ in formula II.3.2.1.

At the end of step *a*), we are left with a sum of terms of the form $\int G(s)e^{-V(s)}d\Phi$ (s is here the interpolating variable of the momentum cutoff). This $V(s)$ is our V'.

Let us see now what happens when we apply $\frac{d}{ds_b}$ on $\int Ge^{-V}d\Phi$:

$$\frac{d}{ds_b} \int Ge^{-V}d\Phi = \int \left(\frac{d}{ds_b}G\right)e^{-V}d\Phi - \int G\frac{dV}{ds_b}e^{-V}d\Phi + \frac{1}{2}\int \frac{d}{ds_b}C.\Delta Ge^{-V}d\Phi \quad \text{(III.1.1)}$$

where by $\frac{d}{ds_b}C.\Delta$ we mean:

$$\int dxdy \sum_{i,j} \frac{d}{ds_b}C(x, i ; y, j)\frac{\partial^2}{\partial\varphi(x, i)\partial\varphi(y, j)}$$

Now we remark that:

$$\frac{d}{ds_b} :\varphi(z, i_1) \dots \varphi(z, i_n): = -\frac{1}{2}\sum_{i,j} \int \frac{dC}{ds_b}(x, i ; y, j)\frac{\partial^2}{\partial\varphi(x, i)d\varphi(y, j)} :\varphi(z, i_1) \dots \varphi(z, i_n): dxdy = -\frac{1}{2}\frac{d}{ds_b}C.\Delta :\varphi(z, i_1) \dots \varphi(z, i_n):$$

This formula is proved easily by explicitation of the Wick products in terms of products of fields.

This remark proves that the derivation of the Wick products is compensated by the derivation of the measure, and we omit such terms.

Let us now exhibit another cancellation.

Consider

$$-\left(\frac{d}{ds}\mathrm{V}\right)e^{-\mathrm{V}}+\frac{1}{2}\left(\frac{d\mathrm{C}}{ds}.\Delta\right)e^{-\mathrm{V}}$$

$$=\left\{\frac{1}{2}\sum_{\Delta}\left(\frac{d}{ds}\delta m^2(\Delta)\right)\sum_{i}\int:\varphi^2:(x,i)\chi_\Delta(x)dx\right.$$

$$-\frac{1}{2}\frac{d}{ds}\int\left(\sum_i\int:\varphi^4:(x,i)dx\right)^2 d\Phi+\frac{1}{6}\frac{d}{ds}\int\left(\sum_i\int:\varphi^4:(x,i)dx\right)^3 d\Phi$$

$$+\frac{1}{2}\int\sum_{k,l=1}^{4}\sum_{i_k,j_l}\frac{d}{ds}\mathrm{C}(x,i_k;y,j_l):\varphi(x,i_1)\dots\hat{\varphi}(x,i_k)\dots\varphi(x,i_4):$$

$$:\varphi(y,j_1)\dots\hat{\varphi}(y,j_l)\dots\varphi(y,j_4):dxdy$$

$$-\sum_{\Delta}\frac{\delta m^2(\Delta)}{2}\int\sum_{k=1}^{4}\sum_{l=1}^{2}\sum_{i_k,j_l}\frac{d}{ds}\mathrm{C}(x,i_k;y,j_l)$$

$$:\varphi(x,i_1)\dots\hat{\varphi}(x,i_k)\dots\varphi(x,i_4):\varphi(y,j_{l'})\chi_\Delta(y)dxdy$$

$$+\frac{1}{8}\sum_{\Delta,\Delta'}\delta m^2(\Delta)\delta m^2(\Delta')\int\sum_{k,l=1}^{2}\frac{d}{ds}\mathrm{C}(x,i_k;y,j_l)$$

$$\left.\varphi(x,i_{k'})\varphi(y,j_{l'})\chi_\Delta(x)\chi_{\Delta'}(y)dxdy\right\}e^{-\mathrm{V}}=\{\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3+\mathrm{B}_1+\mathrm{B}_2+\mathrm{B}_3\}\,e^{-\mathrm{V}}$$

with $l\neq l'$, $k\neq k'$ and $\hat{\varphi}$ means that this term is omitted.

Then we introduce the following terminology: a vertex created by a s derivation is called a E-vertex. The number of E-vertices in a lattice cube Δ is $n_{\mathrm{E}}(\Delta)$. The contraction of E-legs to the exponent generates, as in [*3*], $\mathrm{C_E}$-vertices. Returning to formula (III.1.3) it is easy to show that the contractions of the E-legs in the B_1 term produce terms which exactly cancel A_2 and A_3. If we contract the E-legs in A_1, we produce terms that we can associate with those coming from the B_1 term by contraction of the E-legs and, in which two E-vertices form a mass subdiagram derivated with respect to s.

So our prescription for each derivation $\dfrac{d}{ds}\mathrm{C}(x,y)\dfrac{\partial^2}{\partial\varphi(x)\partial\varphi(y)}$ is

— first perform the two differentiations,

— then contract the E-legs in the terms B_1 and A_1. With these conventions, in counting the terms coming from a s-derivation we have to take account only of those terms coming from B_1, B_2 and B_3.

c) We proceed as in Feldman [*4*]: F-legs are treated as G_2-legs of Feldman. Each F-leg in Δ is divided by $n_{\mathrm{F}}(\Delta)$. Similarly each E-leg in Δ is divided by $[3n_{\mathrm{E}}(\Delta)]^{n_3}$. The integer n_3 bounds the number of $\mathrm{C_E}$-legs created by one E-leg. We will choose $n_3 \geqslant 3$. This factor will play for the E-legs the same role as played by $n_F(\Delta)^{-1}$ for the F-legs.

In the same way, for the squaring operation we treat the F, E and C_E-legs as G_2-legs in Feldman: the factors $[3n_E(\Delta)]^{n_3}$ by E-leg and $n_F(\Delta)^{-1}$ by F-leg are attached to their legs and so squared with them.

d) We do the low momentum expansion as in Glimm-Jaffe [3]. The Wick construction is performed through the modification of terminology introduced in II.2 where we have defined $\varphi(x, i)$. As in Feldman F, E and C_E-vertices initiate no action before the low momentum contraction and, at the low momentum contraction step they are treated as Feldman's G_2-vertices. We take the boundary cutoff at the r^{th} step to be M_{r-1}.

This modified inductive expansion yields an upper bound of

$$\left| \partial^\Gamma \int \mathrm{F} e^{-\mathrm{V}} d\Phi \right|$$

in the form of a sum of graphs times the factors we got at the end of chapter II. We now estimate the number of terms using the method of combinatoric factors.

III.2. **The combinatoric factors**

A given term of the sum bounding $\left| \partial^\Gamma \int \mathrm{F} e^{-\mathrm{V}} d\Phi \right|$ is a graph G with k derived propagators localized in unit cubes, multiplicated by the coefficients of formula (II.3.2.9), by $n_F(\Delta)$ by F-leg in Δ and $[3n_E(\Delta)]^{n_3}$ by E-leg in Δ ; remember that each F-leg (resp. E-leg) in the graph is divided by $n_F(\Delta)$ (resp. $[3n_E(\Delta)]^{n_3}$).

This gives for each graph G a factor A(G) which is bounded by the products of the following factors

— K_6^{-1} by bond $b \in \Gamma$,
— K_7 by derived propagator,
— $M(\Delta)^{-n_1 M(\Delta)}$ by unit cube Δ,
— $[3n_E(\Delta)]^{n_3}$ by E-leg in Δ,
— $n_F(\Delta)$ by F-leg in Δ,

$$— \prod_{i=1}^{2k} e^{-\frac{m}{4} d(\Delta_i, b_i)} \prod_{i=1}^{2k} [d(\Delta_i, b_i)]^{-n_2} \tag{III.2.1}$$

Here the bonds b_i are the bonds $b_1, \ldots, b_{2k}$ of chapter II.

We consider successively the different operations of the inductive expansion. For each operation we count the number c of terms created by this operation, and we attribute a combinatoric factor c to each term produced. So that each graph G has a factor $c(G)$ which is the product of the different factors c we have attributed to G, and we have $\sum_G c(G)^{-1} \leqslant 1$. As a consequence, this gives

$$\sum_G A(G)\,|\,G\,| \leqslant \sup_G c(G)A(G)\,|\,G\,|$$

Before we state the next proposition we introduce a convention in the notation. The maximum lower cutoff λ is defined as in [3] for all old vertices, however for E or C_E-vertices we define λ to be $M_{u(\Delta)}^{1/(1+\nu)} = M_{u(\Delta)-1}$ where $M_{u(\Delta)}$ is the upper cutoff in the unit cube Δ at the end of the first $P_1 - C_1$ expansion.

PROPOSITION III.2.1. — *The bounds on* $c(G)$ *given by Glimm-Jaffe apply equally well provided we include*

— $O(1)$ *by vertex,*
— $d^{O(1)}$ *by line non existing in* [3],
— $d(\Delta_i, b_i)^{O(1)}$ *for each derivation,*
— $O(1)\lambda^{O(1)\varepsilon}$ *by* E-*vertex,*
— $O(1)\lambda^{O(1)\varepsilon}$ *by* F-*leg.*

Proof. — These bounds being invariant by raising them to a finite power, the number of substeps below being finite, it suffices to verify that each one has combinatoric factors verifying the above bounds.

We proceed as in [3] examining each step. We only explicit the new cases.

We suppose the contraction occurs in the i^{th} inductive step. A leg in Δ is contracted to a leg in Δ'.

— Case P : creation of a P-vertex. It is unchanged.

— Case C_1 : division of each leg in two parts during the low momentum contraction. We consider the case of a E or C_E-vertex, the case of a F-vertex has been considered by Feldman [4]. A E or C_E-vertex with upper cutoff at M_j is fully contracted at the i^{th} step if $j \leqslant i - 1$. The maximum combinatoric factor for such a vertex is 2^{9j} (a E or C_E-vertex has at most 3 legs and each momentum has 3 components). Thus for M_1 large enough, the vertex being localized in Δ

$$2^{9j} \leqslant M_{u(\Delta)}^{5\varepsilon}$$

If the vertex is a C_E-vertex we transfer this factor to the generating E-vertex. Let Δ'' be the localization cube of the generating E-vertex. Suppose $\Delta'' \notin \mathscr{D}_C$, then we apply condition c) of [3] chapter 2 and

$$M_{u(\Delta)}^{\varepsilon} \leqslant cd(\Delta, \Delta'')^4 M_{u(\Delta'')}^{\varepsilon}$$

If $\Delta'' \in \mathscr{D}_C$, this means that $M_{u(\Delta'')+1}$ verifies condition c) otherwise the expansion would have been stopped at this value of the upper cutoff, so:

$$M^{\varepsilon}_{u(\Delta)} \leqslant cd(\Delta, \Delta'')^4 M_{u(\Delta'')+1} = cd(\Delta, \Delta'')^4 M^{\varepsilon(1+\nu)}_{u(\Delta'')}$$

Thus in any cases, we have for any two cubes in $\Lambda \cap X$

$$M^{\varepsilon}_{u(\Delta)} \leqslant cd(\Delta, \Delta'')^4 M^{2\varepsilon}_{u(\Delta'')} \tag{III.2.2}$$

and we can bound the factor $M^{5\varepsilon}_{u(\Delta)}$ for a C_E-vertex by giving a factor $c^5\lambda^{20\varepsilon}$ to the E-vertex associated with and d^{20} to the line (ν is chosen less than 1): i. e. $O(1)\lambda^{O(1)\varepsilon}$ to the E-vertex and $d^{O(1)}$ to the line.

— Case C_2 : contraction to the exponential. We choose the cube in which we contract with $O(1)d(\Delta, \Delta')^4$ (because the leg we contract is in the same cover that the exponential), and also we have a factor 2×4 : 2 because there is two terms that one can contract in the exponential, and 4 because each term has at most 4 legs.

In case of the contraction of a E-leg we attribute the factor $2 \times 4 \times O(1)$ to the E-vertex and $d(\Delta, \Delta')^4$ to the line. A E-vertex can contract at most 3 legs.

— Case C_3 : contraction to a C-leg. It is unchanged.

— Case C'_3 : contraction to a C_E-leg. A C_E-leg is in a unit cube, so in the original cover. A $O(1)d(\Delta, \Delta')^4$ suffices to choose the cube of the C_E-leg. A $O(1)d(\Delta', \Delta'')^4$ is sufficient to choose the cube of the E-vertex that has generated the C_E-vertex.

A $2^i 3n_E(\Delta'')$ is sufficient to choose the E-leg that had generated the C_E-vertex. A $O(1)$ ($O(1) \geqslant 3$) is sufficient to choose the C_E-leg of the C_E-vertex. As in case C_1 we bound the factor 2^i by attributing $O(1)\lambda^{O(1)\varepsilon}$ to the generating E-vertex, $d(\Delta, \Delta')^4$ to the line of contraction and $d(\Delta', \Delta'')^{O(1)}$ to the line generating the C_E-vertex. The factor $3n_E(\Delta'')$ is given to the E-vertex. Finally a E-vertex can give rise to at most 9 C_E-legs.

— Case C_4 : contraction to a P-leg. It is unchanged.

— Case C'_4 : contraction to a E-leg.

A $O(1)d(\Delta, \Delta')^4$ is sufficient to choose the unit cube of the E-vertex and a $2^i 3n_E(\Delta')$ to choose the E-leg in Δ'. We proceed as before and give $O(1)\lambda^{O(1)\varepsilon}$ to the E-vertex, $d(\Delta, \Delta')^4$ to the line and $3n_E(\Delta')$ to the E-leg.

— Case C''_4 : contraction to a F-leg. A $O(1)d^4(\Delta, \Delta')$ is sufficient to choose the unit cube of the F-leg. A $2^i n_F(\Delta')$ is sufficient to choose the F-leg in Δ'. Again these factors are bounded by the attribution of $O(1)\lambda^{O(1)\varepsilon} n_F(\Delta')$ to each F-leg and $d^4(\Delta, \Delta')$ to the line.

— Cases C_5 and C_6 : they are unchanged.

— Case S : squaring. It is the use of the formula $|R| \leqslant 1/2(\delta^{-1} + \delta R^2)$. We take δ as in [*3*] and [*4*]. It is proportional to the product of the combinatoric factors attributed to the different elements of the graphs prior to the use of the squaring formula. In these factors we do not count the factors $n_F(\Delta)$ and $3n_E(\Delta)$ which could have been attributed. However the factors

$n_F(\Delta)^{-1}$ and $(3n_E(\Delta))^{-n_3}$ by F or E-legs are in R and are therefore squared with their legs.

— Cases W_1 and W_2 : they are unchanged.

— Case E : derivation with respect to s. We consider different subcases.

— Case E_1 : differentiation of existing propagators (created during the $P_1 - C_1$ expansion or during the contraction of E-legs in earlier derivations). We choose the propagators we derive using a factor 2 by propagator to decide if we derive it or not. We then attribute to each vertex a $\sqrt{2}$ by leg contracted during the $P_1 - C_1$ expansion and by E-leg or C_E-leg contracted during the cluster expansion. This gives at most a factor 2^2 by P_1 and C_1-vertices, a factor 2^2 by E-vertex, a factor 2^2 by C_E-vertex that we attribute to the E-generating vertex and a factor $\sqrt{2}$ by F-leg.

Then we consider the use of $\partial/\partial\varphi$ relatively to a face b. A $O(1)d^4(b, \Delta')$ is sufficient to choose the cube Δ' in which we derive (so we can choose $n_2 = 4$), we attribute the O(1) to b (a face gives at most 2 differentiations) and $d(b, \Delta')^4$ to the derivation.

All the following cases will be alike the cases above except for the factor $O(1)d(\Delta, \Delta')^4$ which has been replaced by $O(1)d(b, \Delta')^4$. This is so, because the combinatoric to choose the leg that one contracts in Δ', is the same as the combinatoric to choose the leg that one differentiates in Δ'. So we just mention the different case giving rise to the same combinatoric except for the choice of the cube. Thus, the combinatoric factors are those of the corresponding case except for the factor $O(1)d(\Delta, \Delta')^4$ replaced by $O(1)d(b, \Delta')^4$. We attribute the factors in the same way but when we say otherwise.

— Case E_2 : differentiation of the exponential. It is like case C_2. We attribute the factor 2×4 to the face b.

— Case E_3 : differentiation of a C_1-leg. It is as in case C_3.

— Case E_3' : differentiation of a C_E-leg. It is as in case C_3'.

— Case E_4 : differentiation of a P_1-leg. It is as in case C_4.

— Case E_4' : differentiation of a E-leg. It is as in case C_4'.

— Case E_4'' : differentiation of a F-leg. It is as in case C_4''.

Finally we remark that the factor $n_F(\Delta)$ and $3n_E(\Delta)$ that has been attributed give at most a factor $n_F(\Delta)$ by F-leg and a factor $(3n_E(\Delta))^{O(1)}$ by E-leg. We choose $n_3 = O(1)$ and then these factors disappear with the factors $n_F(\Delta)^{-1}$ and $(3n_E(\Delta))^{-n_3}$ that have been attributed by F or E-leg (these factors exist always because we have squared them with the legs in the squaring operation).

Now we prove two lemmas. In one we follow the analysis of the cluster expansion. In the other using the analysis of [3] we show that we can attribute a decreasing factor $M_{u(\Delta)}^{-1}$ by E-vertex in Δ. This factor makes all the E-vertices, « convergent » vertices and ensure the convergence of the expansion. We also prove that we can attribute a factor $M_{u(\Delta)}^{-1}$ by unit cube $\Lambda \cap X$ as done in [3]. This will allow us to show that derivatives of all orders exist in the coupling constant.

LEMMA III.2.2

$$\prod_{\Delta} [M(\Delta)]^{-n_1 M(\Delta)} (3n_E(\Delta))^{3n_3 n_E(\Delta)} \leqslant O(1)^k$$

Proof. — To create a E-vertex in Δ, we have to derive in Δ. Thus one Δ_i, $i = 1, \ldots, 2k$ must be equal to Δ; from this, it follows that $n_E(\Delta) \leqslant M(\Delta)$. Since $\sum_{\Delta} M(\Delta) = 2k$ the expression is bounded by

$$\prod_{\Delta} [M(\Delta)]^{-(n_1 - 3n_3)M(\Delta)} 3^{3n_3 M(\Delta)} \leqslant O(1)^k$$

for $n_1 \geqslant 3n_3$, which proves the lemma.

In ref. [3], theorem 5.1, it is proved that one has at each P_1-vertex a convergent factor $\lambda^{-\varepsilon_1}$. We modify this theorem by putting only a factor $\lambda^{-\varepsilon_1/2}$. It remains therefore a factor $\lambda^{-\varepsilon_1/2}$ at our disposal. In the same way we « attribute » a factor M_1^3 to each bond, from which follows that we have at our disposal a factor M_1^{-3} per bond.

LEMMA III.2.3. — *Let $m \geqslant 1$ and $\nu \leqslant 1$, then the factors*

— $M_{u(\Delta)}^{-\varepsilon_1/2(1+\nu)}$ *by P_1-vertex,*
— M_1^{-3} *by bond*

can be replaced by

— M_1 *for the whole graph*
— $M_{u(\Delta)}^{-1}$ *by E-vertex*
— $M_{u(\Delta)}^{-1}$ *by unit cube in $\Lambda \cap X$.*

Proof. — Consider a E-vertex in Δ and suppose it has been created by derivation with respect to s_b. Since Δ is the localization cube of the E-vertex, $\Delta \subset \Lambda \cap X$ (the face b is not necessarily in Λ), and in the support of the exponential. Using the analysis of the end of section 2 in Glimm-Jaffe [3] one can consider several cases.

1) $\Delta \in \mathscr{D}_a$, then $M_{u(\Delta)} = M_1$.

We can attribute a factor M_1^{-1} to each of the E-vertices in cubes of $\mathscr{D}_a$. In fact, since a bond gives rise to at most two E-vertices one has:

$$[M_1^{-2}]^{\text{number of bonds}} = [M_1^{-1}]^{2(\text{number of bonds})} \leqslant [M_1^{-1}]^{\text{number of E vertices}}$$

On the other hand, since the number of cubes in X is smaller than the number of bonds plus one

$$[M_1]^{-(\text{number of bonds})} \leqslant M_1 [M_1]^{-|\Lambda \cap X|}$$

2) $\Delta \in \mathscr{D}_b$. Then according to condition b) of [3] there is at least $M_{u(\Delta)}^{\varepsilon}$

P_1-vertices in Δ. We just modify the procedure of [3] distributing these P_1-vertices.

We assign $1/4M^{\varepsilon}_{u(\Delta)}$ of these vertices to the cube Δ and $1/4M^{\varepsilon}_{u(\Delta)}$ to the other cubes for the purposes of the expansion as in [3] (so in [3] just replace $1/2M^{\varepsilon}_{u(\Delta)}$ by $1/4M^{\varepsilon}_{u(\Delta)}$). Now with $\bar{c} = \sum_{b\in(Z^3)^*} d(\Delta, b)^{-4}$ we have

$$\sum_{b\in\Gamma} \frac{1}{8} M^{\varepsilon}_{u(\Delta)}(d(\Delta, b)^4\bar{c})^{-1} \leqslant \frac{1}{8} M^{\varepsilon}_{u(\Delta)}$$

So we assign $1/8M^{\varepsilon}_{u(\Delta)}(d(\Delta, b)^4\bar{c})^{-1}$ P_1-vertices of Δ to the bond b, and since each bond is the origin of at most 2 E-vertices, we can attribute $1/16M^{\varepsilon}_{u(\Delta)}(d(\Delta, b)^4\bar{c})^{-1}$ P_1-vertices to each E-vertex created possibly in Δ by b.

Now we use the fact that we can use a factor $M_{u(\Delta)}^{-\varepsilon_1/2(1+\nu)}$ by P_1-vertex in Δ. This gives a factor

$$M_{u(\Delta)}^{-\frac{\varepsilon_1}{2(1+\nu)}\frac{1}{16}M^{\varepsilon}_{u(\Delta)}(d(\Delta,b)^4\bar{c})^{-1}}$$

for each E-vertex in Δ created by b.

We now use the extra factor $e^{-\frac{m}{4}d(\Delta,b)}$ that we have get from the cluster expansion. We take $m \geqslant 1$ and $\nu \leqslant 1$. From the inequality

$$\sup_{d(\Delta,b)} M_{u(\Delta)}^{-\frac{\varepsilon_1}{4}\frac{1}{16}M^{\varepsilon}_{u(\Delta)}(d(\Delta,b)^4\bar{c})^{-1}} e^{-\frac{1}{4}d(\Delta,b)} \leqslant M_{u(\Delta)}^{-1}$$

valid for M_1 large enough depending on ε_1 and ε, we see that we can give a factor $M_{u(\Delta)}^{-1}$ to each E-vertex localized in cubes of $\mathscr{D}_b$. We use $1/8M^{\varepsilon}_{u(\Delta)}$ of the P_1-vertices in Δ to attribute a factor $M_{u(\Delta)}^{-1}$ to the cube Δ, since for $\nu \leqslant 1$ and M_1 large enough

$$M_{u(\Delta)}^{-\frac{\varepsilon_1}{2(1+\nu)}\frac{1}{8}M^{\varepsilon}_{u(\Delta)}} \leqslant M_{u(\Delta)}^{-1}$$

This gives the factor $M_{u(\Delta)}^{-1}$ by cube in $\mathscr{D}_b$.

It remains $1/4M^{\varepsilon}_{u(\Delta)}$ P_1-vertices in $\Delta \in \mathscr{D}_b$ that we have not attributed, we use them in the following case.

3) $\Delta \in \mathscr{D}_c$, $\Delta \notin \mathscr{D}_b$, $\Delta \notin \mathscr{D}_a$.

This means that there exists $\Delta' \in \mathscr{D}_b$ such that (see [3])

$$M^{\varepsilon}_{u(\Delta')}d(\Delta, \Delta')^{-4}c^{-1} > M^{\varepsilon}_{u(\Delta)} \qquad \text{(III.2.3)}$$

(because [3] says that $\Delta' \notin \mathscr{D}_c$ and because of (III.2.3) $M_{u(\Delta')} \geqslant M_2$ we have that $\Delta' \notin \mathscr{D}_{a_\varepsilon}$ so it must be in $\mathscr{D}_b$).

From the $1/4M_{u(\Delta')}$ P_1-vertices in Δ' that we still have, we attribute $1/8M^{\varepsilon}_{u(\Delta')}d(b, \Delta')^{-4}\bar{c}^{-1}$ P_1-vertices to b. Now since

$$d(b, \Delta') \leqslant O(1)d(b, \Delta)d(\Delta, \Delta')$$

one gets that the number of P_1-vertices attributed is greater than:

$$\frac{O(1)}{8} M^{\varepsilon}_{u(\Delta')} d(b, \Delta)^{-4} d(\Delta, \Delta')^{-4} \bar{c}^{-1} \geqslant \frac{O(1)}{8} M^{\varepsilon}_{u(\Delta)} d(b, \Delta)^{-4} \left(\frac{\bar{c}}{c}\right)^{-1}$$

because of (III.2.3). We assign $O(1)/16 M^{\varepsilon}_{u(\Delta)} d(\Delta, b)^{-4} (\bar{c}/c)^{-1}$ of the P_1-vertices of Δ' to each of the possible E-vertex created by b in Δ. Using the factor $M_{u(\Delta')}^{-\frac{\varepsilon_1}{2(1+\nu)}}$ that we have by P_1-vertex, because of (III.2.3) and the fact that $M_{u(\Delta')} \geqslant M_{u(\Delta)}$ we get a factor

$$M_{u(\Delta)}^{-\frac{\varepsilon_1}{2(1+\nu)} \frac{O(1)}{16} M^{\varepsilon}_{u(\Delta)} d(\Delta, b)^{-4} \left(\frac{\bar{c}}{c}\right)^{-1}}$$

by E-vertex in Δ created by b.

We finish as in case 2).

Moreover we use the $1/8 M^{\varepsilon}_{u(\Delta')}$ P_1-vertices in Δ' that remains by attributing $1/8 M^{\varepsilon}_{u(\Delta')} d(\Delta, \Delta')^{-4} c^{-1}$ of these P_1-vertices to each cube Δ. Now because of (III.2.3), this gives at least $1/8 M^{\varepsilon}_{u(\Delta)}$ P_1-vertices of Δ' to Δ. Each of these P_1-vertices has a decreasing factor $M_{u(\Delta')}^{-\frac{\varepsilon_1}{2(1+\nu)}} \leqslant M_{u(\Delta)}^{-\frac{\varepsilon_1}{2(1+\nu)}}$ as we have seen. This gives a factor

$$M_{u(\Delta)}^{-\frac{\varepsilon_1}{2(1+\nu)} \frac{1}{8} M^{\varepsilon}_{u(\Delta)}} \leqslant M^{-1}_{u(\Delta)}$$

by cube Δ if M_1 is large enough depending on ε and ε_1 ($\nu \leqslant 1$).

This finishes the proof of the lemma.

Define

$$K_8 = K_6 M_1^{-3} O(1)^{-1}$$
$$K_9 = K_7 O(1)$$

where the O(1) are taken from proposition III.2.1, then

PROPOSITION III.2.4. — *We have*

$$\sum_{G} A(G) \,|\, G \,| \leqslant M_1 \sup A(G) c(G) \,|\, G \,|$$

where the product $A(G)c(G)$ *is bounded by adding to the ombinatoric factors of* [3]

— K_8^{-1} *by bond* $b \in \Gamma$,

— K_9 *by derivated propagator,*

— $[O(1) n_F(\Delta)]^{n_F(\Delta)}$ *by unit cube* Δ,

— $M^{-1}_{u(\Delta)}$ *by unit cube* Δ,

— $O(1) \lambda^{O(1)\varepsilon} M^{-1}_{u(\Delta)}$ *by* E-*vertex in* Δ,

— $\lambda^{O(1)\varepsilon}$ *by* F-*leg,*

— $d^{O(1)}$ *by contraction that does not exist in* [3].

Now we bound $|G|$ itself. We proceed as in [3] in general and as in [4] for the F-legs.

First we show that we can compensate the exponential factors in the distance that we have put in $\tilde{D}_z^{m_1}$ (see chapter II), and also powers of the scaled distance.

Then we decompose the big graphs, and finally bound the small graphs.

III.3. **Estimates on graphs**

We prove in this part that the covariances introduced in chapter II lead to essentially the same bounds as in Glimm-Jaffe [3] and Feldman [4].

III.3.1. The localization factors

We treat here with the localization factors per line. The case of lines involving Wick-vertices will be separated from the others.

a) *All lines except those involving Wick-vertices.*

The variables being localized in unit or smaller cubes, we use the bound $\bar{C}^{\Gamma}/\bar{C} \leqslant 1$ since these factors are locally constant on unit cubes. The sum over z is treated as in chapter II part 2.

A propagator from a vertex localized in a cube Δ_1 to a vertex in Δ_2 is then of the form

$$O(1)\chi_{\Delta_1}(x)\frac{e^{ik_1(x-z)}}{k_1^2+m_1^2}e^{m(1+\tau_1)|r_1|}\frac{e^{ik(x-y)}}{k^2+(M-m_1)^2}m_1\theta_z(z)$$
$$\eta(k)e^{m(1+\tau_1)|r_2|}\frac{e^{ik_2(z-y)}}{k_2^2+m_1^2}\chi_{\Delta_2}(y)dkdk_1dk_2dz \quad (III.3.1.1)$$

here r_1 (resp. r_2) is the vector translation from z the center of the cube Δ_1 (resp. Δ_2) and, as in chapter II part 2 we have extract convergent factors $e^{-m(1+\tau_1)(|r_1|+|r_2|)}$ replacing $D_z^{m_1}$ by $\tilde{D}_z^{m_1}$ (see formula II.2.3). Those factors are used for the same purposes as in this part (to insure the convergence of the sum over z). The cutoff function $\eta(k)$ represents here a sum of products of cutoffs since for the contraction we proceed as in Glimm-Jaffe [3], contracting not a single $\varphi(x, i)$ but a sum $\sum_i \varphi(x, i)$ going from the lower cutoff to the upper cutoff of the leg. The decomposition into « elementary » fields $\varphi(x, i)$ is only done for the W-legs as in [3].

We translate all cubes to the origin and do the integration over x and y remenbering that other propagators can be attached to the same vertex. We get

$$e^{m(1+\tau_1)|r_1|}e^{m(1+\tau_1)|r_2|}\int e^{ik_1r_1}e^{-ik_2r_2}e^{ikr_{12}}\tilde{\chi}_1(K_1+k+k_1)\frac{1}{k_1^2+m_1^2}$$
$$\eta(k)\frac{1}{k^2+(M-m_1)^2}m_1\tilde{\theta}(k_1-k_2)\frac{1}{k_2^2+m_1^2}\tilde{\chi}_2(K_2+k+k_2)dk_1dk_2dk \quad (III.3.1.2)$$

where r_{12} is the vector distance from the center of Δ_1 to the center of Δ_2, and $\tilde{\chi}_1$ (resp. $\tilde{\chi}_2$) is the Fourier transform of the characteristic function of the cube Δ_1 (resp. Δ_2) translated to the origin.

We want, as in [3], to obtain the localization factors d^{-4n}, thus we will show that (III.3.1.2) multiplied by $|r_{12}|^{4n}$ for $r_{12} \neq 0$ has the required bound. As in [3] we replace $|r_{12}|^{4n}$ by $(-\Delta)^{2n}$ acting on $e^{ikr_{12}}$ and use integration by parts. In the same way

$$e^{m(1+\tau_1)|r_1|} = \sum_0^\infty \frac{[m(1+\tau_1)]^n}{n!}|r_1|^n = \sum_0^\infty \frac{[m(1+\tau_1)]^{2n}}{(2n)!}|r_1|^{2n}$$
$$+ \sum_0^\infty \frac{[m(1+\tau_1)]^{2n+1}}{(2n+1)!}|r_1|^{2n+1} = \sum_0^\infty \frac{[m(1+\tau_1)]^{2n}}{(2n)!}(r_1^2)^n$$
$$+ \frac{1}{m(1+\tau_1)|r_1|}\sum_1^\infty \frac{[m(1+\tau_1)]^{2n}}{(2n)!}(r_1^2)^n 2n \qquad \text{(III.3.1.3)}$$

Then for, let us say, $m(1+\tau_1)|r_1| \geqslant 1$, we use (III.3.1.3) replacing r_1^{2n} (resp. r_2^{2n}) by $(-\Delta_1)^n$ (resp. $(-\Delta_2)^n$) acting on $e^{ik_1r_1}$ (resp. $e^{ik_2r_2}$). By the way of partial integration we get a sum of products of derivatives with respect to k_1, k_2 and k. If $m(1+\tau_1)|r_1| \leqslant 1$, we bound $e^{m(1+\tau_1)|r_1|}$ by e.

We need estimates on these derivatives.

LEMMA III.3.1.1. — *Let*

$$D^n = \prod_{l=0}^{2}\left(\frac{\partial}{\partial k^l}\right)^{n_l}, \qquad \Sigma n_l = |n|,$$

then

1) $$\left|D^n \frac{1}{k^2+m^2}\right| \leqslant O(1)\frac{|n|!}{m^{|n|}}\frac{1}{k^2+m^2}$$

2) $$|D^n\tilde{\theta}(k)| \leqslant C(R, N)\frac{|n|!}{R^{|n|}}F(k)^N$$

for any $R > 0$ *and integer* $N \geqslant 0$.

Proof

1) is a consequence of the Cauchy representation for $\dfrac{1}{k^2+m^2}$.

2) follows from the theorem of Paley-Wiener which says that the

Fourier-Laplace transform of a C_0^∞-function $\tilde{\theta}(k)$ is bounded, for any $N>0$, by

$$C_N e^{A|\mathrm{Im}\, k|}\left(\frac{1}{1+|k|}\right)^N \qquad k \in \mathbb{C}^3$$

for some A related to the support of $\theta(x)$. Then one uses the Cauchy bounds to estimate $|D^n\tilde{\theta}(k)|$ using the fact that $\tilde{\theta}(k)$ is entire and therefore analytic in any polydisc centered at $k \in \mathbb{R}^3$.

The other bounds we need are those of Glimm-Jaffe [3] given in formulas 5.2.8, 5.2.9, 5.2.10 and 5.2.11.

We then replace in (III.3.1.2) $e^{m(1+\tau_1)|r_1|}$ and $e^{m(1+\tau_1)|r_2|}$ by (III.3.1.3) and we show that the sum over n converges. Using the factorization of the various bounds it is equivalent to show that

$$\sum_0^\infty \frac{[m(1+\tau_1)]^{2n}}{(2n)!} C^{2n}(-\Delta)^n\left(\tilde{\chi}_\Delta(k)\frac{1}{k^2+m_1^2}\tilde{\theta}(k)\right) \qquad \text{(III.3.1.4)}$$

is bounded. But (III.3.1.4) is bounded by

$$\sum_0^\infty \frac{[m(1+\tau_1)]^{2n}}{(2n)!} C^{2n}3^{3n} \sum_{|n_1+n_2+n_3|=2n} |D^{n_1}\tilde{\chi}(k)|\left|D^{n_2}\frac{1}{k^2+m_1^2}\right||D^{n_3}\tilde{\theta}(k)|$$

$$\leqslant O(1)\left(\sum_0^\infty \frac{[m(1+\tau_1)]^{2n}}{(2n)!} C'^{3n} \sum_{|n_1+n_2+n_3|=2n} |\Delta|^{|n_1|/3}\frac{|n_2|!}{m_1^{|n_2|}}\frac{|n_3|!}{R^{|n_3|}}\right)|\Delta| F_\Delta(k)$$

$$\frac{1}{k^2+m_1^2}F(k)^N$$

with C′ obviously defined and the sum in the bracket is bounded by

$$\left(\sum_0^\infty \frac{(m(1+\tau_1)C')^n}{n!}\right)\left(\sum_0^\infty \frac{(m(1+\tau_1)C')^n}{m_1^n}\right)\left(\sum_0^\infty \frac{(m(1+\tau_1)C')^n}{R^n}\right)$$

with $|\Delta|$ bounded by 1. Thus if we choose m_1 and R large enough we prove the convergence.

Finally proceeding as in Glimm-Jaffe [3], part 5.2 (i. e. refining the cubes) we have shown that for a graph G involving any vertices except Wick vertices

$$|G| \leqslant \left(\prod_{\mathscr{L}} \alpha_2 d_{\max}^{-4n}\right)|\bar{G}|$$

where $\mathscr{L}$ is the set of vertices such that $d_{\max} > 1$ and for a given vertex $d_{\max}$

is the maximum of the scaled distances for lines contracting to the vertex.

The propagators in $\bar{G}$ are

$$O(1)m_1 \mid \Delta_1 \mid F_{\Delta_1}(K_1 + k + k_1)\frac{1}{k_1^2 + m_1^2}\eta(k)\frac{1}{k^2 + (M - m_1)^2}F(k_1 - k_2)^N \times \frac{1}{k_2^2 + m_1^2} \mid \Delta_2 \mid F_{\Delta_2}(K_2 + k + k_2) \quad (III.3.1.5)$$

b) *The Wick vertices.*

For graph involving Wick vertices we will proceed as in Glimm-Jaffe [3] section 6.2 and we will show in part 3 of this chapter that we get the equivalent of estimate 6.2.16 of [3].

Finally we have that

$$|G| \leqslant \Big(\prod_{\text{lines}} d^{-n}\Big)\Big(\prod_{\text{F-legs}} \alpha_2\Big)\Big(\prod_{\text{E-vertices}} \alpha_2^4\Big)\Big(\prod_{\substack{\text{all vertices}\\ \text{but F, E, } C_E \text{ vertices}}} \alpha_2\Big)|\bar{G}|$$

where we have attributed to the generating E-vertex the factors coming from C_E-vertices.

III.3.2. The decomposition of large graphs

We proceed similarly as in [3] and [4]. We decompose a graph in big subgraphs consisting in:

1) a single F_i-graph,
2) a E-subgraph, i. e. a E-vertex and the C_E-vertices it generates,
3) a single W-vertex,
4) a P-vertex and the C-vertices it generates.

The only differences is then to bound the new type of subgraph: a E-vertex and its C_E-vertices. We decompose its Hilbert-Schmidt norm in the same manner as in [3] for the decomposition of P-subgraphs (note that a E-subgraph is simpler since one does not contract the legs of the C_E-vertices).

Now we ensure the convergence of the norm of each C_E-vertex. By proposition (III.2.4) each E-vertex has a convergent factor:

$$\lambda^{O(1)\varepsilon}M_{u(\Delta)}^{-1} \leqslant \lambda^{-(1+O(1)\varepsilon)}$$

We keep a factor $\lambda^{-(1+O(1)\varepsilon)/4}$ for the E-vertex and attribute the remaining to its C_E-vertices (a E-vertex generates at most 3 C_E-vertices). Using the transfert inequality III.2.2, we can attribute to each C_E-vertex a factor $\lambda^{-(1+O(1)\varepsilon)/8}$ provided we attribute a factor O(1) to the generating E-vertex and a factor $d^{O(1)}$ to the generating line.

PROPOSITION III.3.2.1.

$$|\bar{G}| \leqslant \prod_{\substack{\text{lines generating}\\ C_E \text{ vertices}}} d^{O(1)} \prod_{E\text{ vertices}} O(1)\lambda^{\frac{3}{4}(1-O(1)\varepsilon)} \prod_{C_E\text{ vertices}} \lambda^{\frac{1}{8}(-1+O(1)\varepsilon)}$$

$\times$ *product of the norms of the vertices*

See [3].

For the F-legs we proceed as in Feldman [4]. So our norm for each F_i : $|||F_i|||_{\delta,\alpha}$ is the norm $||\,.\,||_{2,\delta,\alpha}$ of Feldman except the fact that we have modified the propagators. Because we have decided to take each F_i as a subgraph we shall obtain a bound in

$$\prod_{\Delta_i \in X} |||F_i|||_{\delta,\alpha}$$

Finally to ensure the convergence of the cluster expansion we want to obtain a convergent factor by derivated propagator.

III.3.3. BOUNDS ON SMALL GRAPHS

As in [3], we have

LEMMA III.3.3.1. — *If we choose* $M \geqslant 2m_1$ *then*

a) $||$  $||_{3,1}$ $\leqslant O(1)\lambda^{\varepsilon_5}m_1^{-\varepsilon_6}$ $\qquad \varepsilon_5 > 4\varepsilon_6$

a') $\leqslant O(|\Delta|^{\varepsilon_3})(\log\lambda)^{1/2}$

b) $||$  $||_{2,2}$ $\leqslant O(1)\lambda^{-(\varepsilon_3-\varepsilon_5)}m_1^{-\varepsilon_6}$

b') $\leqslant O(|\Delta|^{\varepsilon_3})\lambda^{-\varepsilon_3}$

c) $||$  $||_{\text{H.S.}}$ $\leqslant O(1)\lambda^{-(\varepsilon_3-\varepsilon_5)}m_1^{-\varepsilon_6}$

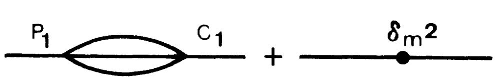
c') $\leqslant O(|\Delta|^{\varepsilon_3})\lambda^{-\varepsilon_3}$

d) $||W||_{\text{H.S.}} \leqslant O(|\Delta|^{\varepsilon_3})\prod_{\text{legs}} b^{-3(1+\nu)/2}$

e) $|P\equiv W| \leqslant |P\equiv P|\,|W\equiv W| \leqslant O(|\Delta|^{\varepsilon_3}|\Delta'|^{\varepsilon_3})\lambda^{-\varepsilon_3}\prod_{\text{legs}} b^{-3(1+\nu)/2}$

where ε_5 *can be taken as small as we want.*

LEMMA III.3.3.2.

$||$ 

$||_{\text{H.S.}} \leqslant O(1)d^{-n}\lambda^{-\varepsilon_3}$

where d *is the scaled distance between* Δ *and* Δ'.

This proposition proves that in the case of the mass subdiagram plus its counterterm the procedure of localization factors is still valid. This bound is the same as in Glimm-Jaffe [3].

We now give a proof of the bound of proposition II.3.1.2. The E or C_E-vertices are bounded by lemma III.3.3.1. *a*), *b*) and *c*), which bounds also δm^2.

In a first choice ε is chosen such that $1 - O(1)\varepsilon > 0$. Then we choose ε_5 such that $\varepsilon_5 < 1/8(1 - O(1)\varepsilon)$ and $4\varepsilon_6 < \varepsilon_5$, this last choice fixes the O(1) in lemma III.3.3.1. The $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3, \nu$ are then chosen as in [3], remenbering that we have replaced ε_1 by $\varepsilon_{1/2}$, and M_1 is chosen large enough depending on these choices, in particular through lemma III.2.3. Finally we choose n in proposition III.3.1 large enough to compensate all the factors $d^{O(1)}$ introduced by lines.

Thus, in a graph, a C_E-vertex is bounded by

$$O(1)\lambda^{-(1-O(1)\varepsilon)/8}\lambda^{\varepsilon_5} \leqslant O(1)$$

We give this O(1) to the generating E-vertex and bound the C_E-vertices by 1. Now we bound all factors O(1) attributed to the E-vertices by taking m_1 large enough such that $m_1^{-\varepsilon_6/2}O(1) \leqslant 1$.

On the other hand we bound the factor K_9 by derivated propagator by taking m_1 large enough so that $K_9 m_1^{-\varepsilon_6/4} \leqslant 1$, since the number of derivated propagators is less than twice the number of E-vertices. We then obtain

$$\left| \partial^{\Gamma} \int F e^{-V(\Lambda \cap X)} d\Phi \right| \leqslant C^{\underline{t}} K_8^{-|\Gamma|} \, ||| F |||_{\delta,\alpha} \prod_{\Delta} [O(1) n_F(\Delta)]^{n_F(\Delta)} O(1)^{|\Lambda \cap X|}$$

the constant depends on $M_1, \varepsilon, \ldots$ but not on m and m_1 big enough. Given any $K > 0$ and M_1 being fixed, one chooses m large enough such that $K_8^{-|\Gamma|} O(1)^{|\Lambda \cap X|} \leqslant e^{-K|\Gamma|} O(1)$ (if X is one of the connected component of $\mathbb{R}^3 \backslash \Gamma^c$, then $|X| < |\Gamma| + 1$). Remark that here we have bounded $M_{u(\Delta)}^{-1}$ by 1. We have thus terminated the proof of the last part of proposition II.3.2.1 for a general F.

The case of a product of fields will be treated later.

The proof of lemma III.3.3.1 and also of the equivalent of proposition 6.3.1 of [3] are obtained using the two following lemmas.

LEMMA III.3.3.3. — *Let $0 \leqslant \lambda \leqslant 1$ and n large enough, then*

$$\int \frac{1}{(k+u)^2 + m_1^2} F(u)^n F_{\Delta}(\lambda u + k) d^3u \leqslant \frac{O(1)}{k^2 + m_1^2} F_{\Delta}(k)$$

Proof. — With the notation of [3]:

$$F_{\Delta}(u) = \prod_{i=0}^{2} \mu_{|\Delta|^{1/3}}(u_i)^{-1}$$

so

$$\int \frac{1}{(k+u)^2 + m_1^2} \mathrm{F}(u)^n \mathrm{F}_\Delta(\lambda u + k) d^3 u$$
$$\leqslant \int \frac{1}{(k+u)^2 + m_1^2} \prod_{i=0}^{2} \mu(u_i)^{-n} \mu_{\lambda|\Delta|^{1/3}}\left(u_i + \frac{k_i}{\lambda}\right)^{-1} du_i .$$

Now we define $b^2 = m_1^2 + (k_2 + u_2)^2 + (k_1 + u_1)^2$ and consider

$$\frac{1}{b^2} \int \mu_{\frac{1}{b}}(k_0 + u_0)^{-2} \mu(u_0)^{-n} \mu_{\lambda|\Delta|^{1/3}}\left(u_0 + \frac{k_0}{\lambda}\right)^{-1} du_0 \quad \text{(III.3.3.1)}$$

which is bounded by

$$\frac{1}{b^2}\left(\int \mu_{\frac{1}{b}}(k_0 + u_0)^{-4} \mu(u_0)^{-n} du_0 \right)^{1/2}$$
$$\left(\int \mu(u_0)^{-n} \mu_{\lambda|\Delta|^{1/3}}\left(u_0 + \frac{k_0}{\lambda}\right)^{-2} du_0 \right)^{1/2} \quad \text{(III.3.3.2)}$$

First we consider

$$\int \mu_{\frac{1}{b}}(k_0 + u_0)^{-4} \mu(u_0)^{-n} du_0 = \int \mu_{\frac{1}{b}}(k_0 + u_0)^{-4} \mu_{\frac{1}{b}}(u_0)^{-4} \mu_{\frac{1}{b}}(u_0)^4 \mu(u_0)^{-n} du_0$$

Then using $\mu_{\frac{1}{b}}(u_0) \leqslant \mu(u_0)$ since $b \geqslant 1$ $(m_1 \geqslant 1)$ and

$$\mu_{\frac{1}{b}}^{-1}(k_0 + u_0) \mu_{\frac{1}{b}}^{-1}(u_0) \leqslant \mathrm{O}(1) \mu_{\frac{1}{b}}^{-1}(k_0) \qquad \mathrm{O}(1) \geqslant \frac{2}{\sqrt{3}}$$

the first bracket in (III.3.3.2) is bounded by

$$\left(\mu_{\frac{1}{b}}^{-4}(k_0) \mathrm{O}(1) \int \mu(u_0)^{-n+4} du_0 \right)^{1/2} \leqslant \mathrm{O}(1) \mu_{\frac{1}{b}}^{-2}(k_0)$$

In the same way, the second bracket is bounded by

$$\mathrm{O}(1) \mu_{\lambda|\Delta|^{1/3}}^{-1}\left(\frac{k_0}{\lambda}\right) \leqslant \mathrm{O}(1) \mu_{|\Delta|^{1/3}}^{-1}(k_0)$$

So finally (III.3.3.1) is bounded by

$$\frac{\mathrm{O}(1)}{b^2 + k_0^2} \mu_{|\Delta|^{1/3}}^{-1}(k_0)$$

It suffices to repeat the operation for the other variables to get the lemma.

LEMMA III.3.3.4. — *Let* $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \lambda' \leqslant 1$ *and* $M \geqslant 2m_1$ *then*

$$\int F_\Delta(K + \lambda(k+q)) \frac{m_1}{[k^2 + (M - m_1)^2][q^2 + m_1^2]^2} F_{\Delta'}(K' + \lambda'(k+q)) d^3k d^3q \leqslant O(1) \int \frac{F_\Delta(K + \lambda l) F_{\Delta'}(K' + \lambda' l)}{l^2 + m_1^2} dl$$

Proof. — Define $l = k + q$; one has to prove that

$$\int \frac{m_1}{(l-q)^2 + (M - m_1)^2} \frac{1}{(q^2 + m_1^2)^2} d^3q \leqslant \frac{O(1)}{l^2 + m_1^2}$$

Since $M \geqslant 2m_1$ the left hand side is bounded by

$$m_1 \int \frac{1}{(l-q)^2 + m_1^2} \frac{1}{(q^2 + m_1^2)^2} d^3q \leqslant \frac{1}{m_1^2} \int \frac{1}{\left(\frac{l}{m_1} - q\right)^2 + 1} \frac{1}{q^2 + 1} d^3q \leqslant \frac{O(1)}{l^2 + m_1^2}$$

This finishes the proof.

Thus for all small graphs but the W-vertices and the subdiagrams of mass we can proceed exactly as in [3], chapter 6, after using lemma III.3.3.3 and lemma III.3.3.4 to reduce the propagators of the form (III.3.1.5) to the usual form.

But in [3] we can replace each propagator by:

$$\frac{1}{k^2 + m_1^2} \leqslant \frac{1}{(k^2 + m_1^2)^{1 - \varepsilon_6/2}} m_1^{-\varepsilon_6} \leqslant \frac{1}{(k^2 + 1)^{1 - \varepsilon_6/2}} m_1^{-\varepsilon_6}$$

Now it is obvious to see that the estimates of chapter 6, section 2 of [3] with this new propagator gives lemma III.3.3.1 cases *a*), *b*), and *c*) with $\varepsilon_5 > 4\varepsilon_6$ but as close to $4\varepsilon_6$ as we want.

Case of W-*vertices.*

The discussion of part 6.2 of [3] applies equally well to our theory. Instead of going in the details we show on an example how is the connection. Thus consider a propagator between a $\delta\varphi$-leg and a φ-leg. One gets

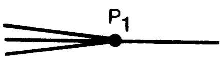
$$\tilde{\chi}_{\Delta_1}(K_1 + \lambda(k + k_1)) \frac{\tilde{\chi}_{\Delta_1}((1-\lambda)(k + k_1))}{\tilde{\chi}_{\Delta_1}(0)} \eta(k) \frac{1}{k_1^2 + m_1^2} \frac{1}{k^2 + (M - m_1)^2}$$
$$\tilde{\theta}(k_1 - k_2) \frac{1}{k_2^2 + m_1^2} \tilde{\chi}_{\Delta_2}(-K_2 - k - k_2)|_0^1 \quad \text{(III.3.3.4)}$$

where we have omitted the distance factors. We then express (III.3.3.4) as an integral of derivatives with respect to λ. We apply on it the analysis of section III.3.1 of this chapter, we use the lemmas III.3.3.3 and III.3.3.4 to finally terminate along the lines of part 6.2 of [3]. In particular this gives the equivalent of formula 6.2.16.

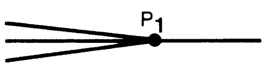
Case of the subdiagrams of mass.

It is the proof of lemma III.3.3.2.

We have only to consider the case of a P_1-vertex giving rise to a mass term. Moreover we look only on the cases where there is none of the external legs of the mass subdiagram with lower cutoff at $\lambda \geqslant M_{u(\Delta)-1}$, because in these cases we can use the better version of lemma III.3.3.1 given in [3] (which we obtain using lemmas III.3.3.3 and III.3.3.4 and following [3]) which says that

$\|$  $\mu^{1/4-\varepsilon}\|_{3,1} \leqslant O(1) \log \lambda$

and since $\mu^{-(1/4-\varepsilon)} \leqslant \lambda^{-(1/4-\varepsilon)}$ we obtain

$\|$  $\|_{3,1} \leqslant O(1)\lambda^{-\varepsilon_3} \qquad \varepsilon_3 < 1/4 - \varepsilon$

So it gives a convergent P_1-diagram. For the mass counterterm it is similar

$|$  $|^{1/2} \leqslant O(1)\lambda^{-\varepsilon_3}$

We do not look at the case of mass terms which are not localized in unit cubes because in this case we can bound them with lemma (III.3.3.1, a')).

We therefore restrict ourselves to unit cubes and look at the sum of a mass subdiagram  and of its counterterm. We have a counterterm only if the two vertices of the mass subdiagram are in the same cube Δ. We remark also that due to the form of the mass counterterm the factors $\bar{C}^{\Gamma}/\bar{C}$ are the same for the diagram and its counterterm and thus factor out. We get (apart these factors)

$$\int \left\{ \tilde{\chi}_\Delta\left(k_1 + k_1' + \sum_{i=2}^{4}(k_i + k_i')\right)\tilde{\chi}_{\Delta'}\left(-k_5 - k_5' - \sum_{i=2}^{4}(k_i + k_i'')\right) \frac{m_1^3 \prod_{i=2}^{4} \tilde{\theta}(k_i'' - k_i')\eta(k_i)}{\prod_{i=2}^{4}(k_i^2 + (M - m_1)^2)(k_i'^2 + m_1^2)(k_i''^2 + m_1^2)} - \tilde{\chi}_\Delta(k_1 + k_1' - k_5 - k_5'))\delta_{\Delta\Delta'} \frac{m_1^3\delta\left(\sum_{i=2}^{4}(k_i + k_i')\right)\prod_{i=2}^{4}\tilde{\theta}(k_i'' - k_i')\eta(k_i)}{\prod_{i=2}^{4}(k_i^2 + (M - m_1)^2)(k_i'^2 + m_1^2)(k_i''^2 + m_1^2)} \right\} \prod_{i=2}^{4} dk_i dk_i' dk_i'' \qquad \text{(III.3.3.5)}$$

where we have omitted the propagators in k_1, k'_1 and k_5, k'_5 and

$$\delta_{\Delta\Delta'} = \begin{cases} 0 & \text{if} \quad \Delta \neq \Delta' \\ 1 & \text{if} \quad \Delta = \Delta' \end{cases}$$

Now we write the term in the bracket as, setting $\sum_{i=2}^{4}(k_i + k'_i) = \mathrm{P}$

$$\tilde{\chi}_\Delta(k_1 + k'_1 + \mathrm{P})\left[\tilde{\chi}_{\Delta'}\left(-k_5 - k'_5 - \mathrm{P} + \sum_{i=2}^{4}(k'_i - k''_i)\right) - \tilde{\chi}_{\Delta'}(-k_5 - k'_5 - \mathrm{P})\right]$$

$$\times \frac{m_1^3\left[\prod_{i=2}^{4}\tilde{\theta}(k''_i - k'_i)\right]\eta\left(\mathrm{P} - k'_2 - \sum_{i=3}^{4}(k_i + k'_i)\right)\eta(k_3)\eta(k_4)}{\left[\left(\mathrm{P} - k'_2 - \sum_{i=3}^{4}(k_i + k'_i)\right)^2 + (\mathrm{M} - m_1)^2\right](k'^2_2 + m_1^2)(k''^2_2 + m_1^2) \times \times \prod_{i=3}^{4}(k_i^2 + (\mathrm{M} - m_1)^2)(k'^2_i + m_1^2)(k''^2_i + m_1^2)} +$$

$$+ \tilde{\chi}_\Delta(k_1 + k'_1 + \mathrm{P})\tilde{\chi}_{\Delta'}(-k_5 - k'_5 - \mathrm{P})m_1^3\prod_{i=2}^{4}\tilde{\theta}(k''_i - k'_i)$$

$$\left\{ \frac{\eta\left(\mathrm{P} - k'_2 - \sum_{i=3}^{4}(k_i + k'_i)\right)\eta(k_3)\eta(k_4)}{\left[\left(\mathrm{P} - k'_2 - \sum_{i=3}^{4}(k_i + k'_i)\right)^2 + (\mathrm{M} - m_1)^2\right](k'^2_2 + m_1^2)(k''^2_2 + m_1^2) \times \times \prod_{i=3}^{4}(k_i^2 + (\mathrm{M} - m_1)^2)(k'^2_i + m_1^2)(k''^2_i + m_1^2)} - \right.$$

$$\left. - \frac{\eta\left(-k'_2 - \sum_{i=3}^{4}(k_i + k'_i)\right)\eta(k_3)\eta(k_4)}{\left[\left(-k'_2 - \sum_{i=3}^{4}(k_i + k'_i)\right)^2 + (\mathrm{M} - m_1)^2\right](k'^2_2 + m_1^2)(k''^2_2 + m_1^2) \prod_{i=3}^{4}(k_i^2 + (\mathrm{M} - m_1)^2)(k'^2_i + m_1^2)(k''^2_i + m_1^2)} \right\} \tag{III.3.3.6}$$

A) We consider first the second difference. We proceed as in [3], part 6.3, decomposing this difference in two parts.

$$\frac{\eta(\mathrm{P} - \mathrm{K}) - \eta(-\mathrm{K})}{[(\mathrm{P} - \mathrm{K})^2 + (\mathrm{M} - m_1^2)]} \qquad \mathrm{M} \geqslant 2m_1$$

Using $|\eta(\mathrm{P} - \mathrm{K}) - \eta(-\mathrm{K})| \leqslant \mathrm{O}(1)|\mathrm{P}|^{\delta}\lambda^{-\delta}$ as in [3], we get

$$\int \frac{m_1^3 dk_2' dk_3' dk_4'}{\left[\left(\mathrm{P} - k_2' - \sum_{i=3}^{4}(k_i + k_i')\right)^2 + m_1^2\right](k_2'^2 + m_1^2)^2(k_3'^2 + m_1^2)^2(k_4'^2 + m_1^2)^2}$$

Once we have done the integration on $k_i'' - k_i'$ with lemma III.3.3.3. Now using lemma III.3.3.4, one gets:

$$\frac{\mathrm{O}(1)}{[(\mathrm{P} - k_3 - k_4)^2 + m_1^2]} \leqslant \frac{\mathrm{O}(1)}{[(\mathrm{P} - k_3 - k_4)^2 + 1]}$$

and now we have the same expression as in [3] part 6.3, so we finish in the same way and obtain the same bound

2) $$[(\mathrm{P} - \mathrm{K})^2 + (\mathrm{M} - m_1)^2]^{-1} - [\mathrm{K}^2 + (\mathrm{M} - m_1)^2]^{-1}$$

From [3], p. 373, we get the inequality: $0 \leqslant \varepsilon_7 \leqslant 1/3$

$$|\mu(k)^{-2} - \mu(l - k)^{-2}| \leqslant \mathrm{O}(1)\mu(l)^{3\varepsilon_7}[\mu(k)^{-2-\varepsilon_7} + \mu(l - k)^{-2-\varepsilon_7}]$$

Thus for $|l| \geqslant 1$ we have

$$|\mu(k)^{-2} - \mu(l - k)^{-2}| \leqslant \mathrm{O}(1)|l|^{3\varepsilon_7}[\mu(k)^{-2-\varepsilon_7} + \mu(l - k)^{-2-\varepsilon_7}]$$

For $|l| \leqslant 1$ consider the difference

$$|[(l - k)^2 + 1]^{-1} - [k^2 + 1]^{-1}| \\ \leqslant |l^2 - 2l - k|\{[(l - k)^2 + 1]^{-2} + [k^2 + 1]^{-2}\}$$

Then for any $0 \leqslant \varepsilon_7 \leqslant 1/3$ one has that:

$$\frac{|l^2 - 2l.k|}{[l^2 - 2l.k + k^2 + 1]^{1-\varepsilon_7/2}} \leqslant \mathrm{O}(1)|l| \leqslant \mathrm{O}(1)|l|^{3\varepsilon_7}$$

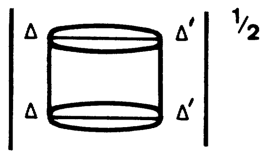
$$\frac{|l^2 - 2l.k|}{[k^2 + 1]^{1-\varepsilon_7/2}} \leqslant \mathrm{O}(1)|l| \leqslant \mathrm{O}(1)|l|^{3\varepsilon_7}$$

So finally proceeding by scaling, using $\mathrm{M} - m_1 \geqslant m_1$ we obtain

$$|[(\mathrm{P} - \mathrm{K})^2 + (\mathrm{M} - m_1)^2]^{-1} - [\mathrm{K}^2 + (\mathrm{M} - m_1)^2]^{-1}| \\ \leqslant \mathrm{O}(1)|\mathrm{P}|^{3\varepsilon_7}\{[(\mathrm{P} - \mathrm{K})^2 + m_1^2]^{-1-\varepsilon_7/2} + [\mathrm{K}^2 + m_1^2]^{-1-\varepsilon_7/2}\}$$

Now we do the integration on $k_i'' - k_i'$ with lemma III.3.3.3 and on $k_i + k_i'$ using lemma III.3.3.4. We are then in the same situation as in [3] and we obtain the same bound.

B) Now we look at the first difference in (III.3.3.6). We bound it by its Hilbert-Schmidt norm, introducing the propagators for k_1 and k_5.



The square of the Hilbert-Schmidt (H. S.) norm has the form of a product of two differences of characteristic functions. We will exhibit a cancellation for one of the difference bounding the second one by the sum of the characteristic functions. We write this difference:

$$\tilde{\chi}_{\Delta'}\left(-k_5 - k'_5 - \mathrm{P} + \sum_{i=2}^{4}(k'_i - k''_i)\right) - \tilde{\chi}_{\Delta'}(-k_5 - k'_5 - \mathrm{P})$$
$$= \int_0^1 d\lambda\left[\frac{d}{d\lambda}\tilde{\chi}_{\Delta'}\left(-k_5 - k'_5 - \mathrm{P} + \lambda\sum_{i=2}^{4}(k'_i - k''_i)\right)\right]$$

We divide Δ and Δ' in small cubes of the same size Δ_i and Δ'_i such that $|\Delta_i|$, $|\Delta'_i| \leqslant \mathrm{M}_{u(\Delta)}^{-3}$ (Δ is the cube of the P-vertex). We write then the H. S. norm as a sum over i and i' of graphs which vertices (for the first difference) are in Δ_i and Δ'_i.

We have that

$$\left|\frac{d}{d\lambda}\tilde{\chi}_{\Delta'_i}\right| \leqslant |\Delta_i|^{1/3}\mathrm{O}(1)\,|\Delta_i|\,\mathrm{F}_{\Delta_i}\mu\left(\sum_{i=2}^{4}(k'_i - k''_i)\right)$$

We then proceed as in [3] obtaining localization factors which will ensure the convergence of the sum over Δ_i and Δ'_i using:

$$\sum_{\Delta_i,\Delta'_i} d^{-4}(\Delta_i, \Delta'_i) \leqslant \mathrm{O}(1)\,|\Delta_i|^{-1}$$

We perform the integration on $k''_i - k'_i$ and on $k_i + k'_i$ using lemmas III.3.3.3 and III.3.3.4. It remains:

$$\mathrm{O}(1)\,|\Delta_i|^{1/3}\,|\Delta_i|^{-1}\,|\Delta_i|^2 \int \frac{\mathrm{F}_{\Delta_i}(k_1 + k_2 + k_3 + k_4)\mathrm{F}_{\Delta'_i}(k_5 + k_2 + k_3 + k_4)}{(k_2^2 + 1)(k_3^2 + 1)(k_4^2 + 1)}$$
$$\times \frac{1}{(k_1^2+1)(k_5^2+1)}\,\frac{\mathrm{F}(k_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4)\mathrm{F}(k_5 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4)}{(\bar{k}_2^2 + 1)(\bar{k}_3^2 + 1)(\bar{k}_4^2 + 1)}\, dk_1 dk_5 \prod_{i=2}^{4} dk_i d\bar{k}_i$$
$$\text{(III.3.3.7)}$$

where we have noted $|\Delta| F_\Delta = F$ for $|\Delta| = 1$ and used:

$$\frac{1}{k^2 + m_1^2} \leqslant \frac{1}{k^2 + 1}$$

From

$$\int \frac{dk_2 dk_3}{(k_2^2 + 1)(k_3^2 + 1)[(P - k_2 - k_3)^2 + 1]} \leqslant O(1) \log \lambda$$

and

$$|\Delta_i| \int F_{\Delta_i}(k_1 + P)F_{\Delta_i}(k_5 + P)dP \leqslant O(1)F_{\Delta_i}(k_1 - k_5)^{1-\varepsilon}$$

we get that (III.3.3.7) is bounded by:

$$\begin{aligned} O(1)|\Delta_i|^{1/3}(O(1)\log\lambda)^2 &\int \frac{F_{\Delta_i}(k_1 - k_5)^{1-\varepsilon}F(k_1 - k_5)^{1-\varepsilon}}{(k_1^2 + 1)(k_5^2 + 1)} dk_1 dk_5 \\ &\leqslant O(1)M_{u(\Delta)}^{-1}(O(1)\log\lambda)^2 \int \frac{F(k_1 - k_5)^{1-\varepsilon}}{(k_1^2 + 1)(k_5^2 + 1)} dk_1 dk_5 \\ &\leqslant O(1)M_{u(\Delta)}^{-1+\varepsilon} \leqslant O(1)\lambda^{-2\varepsilon_3} \end{aligned}$$

Remark that there is no problem to get the localization factors for the mass subdiagram: we deal with unit cubes so we can replace $d(\Delta, \Delta')$ by dist (Δ, Δ') = euclidean distance between Δ and Δ'. When dist $(\Delta, \Delta') \leqslant 1$ we do not need localization factors. When dist $(\Delta, \Delta') > 1$, we bound the subdiagram using lemma III.3.3.1 a') and get localization factors. Then we use $d^{-O(1)}$ to show the convergence $d^{-O(1)} \leqslant \text{dist}(\Delta, \Delta')^{-O(1)}\lambda^{-O(1)} \leqslant \lambda^{-O(1)}$.

$$d^{-O(1)} \leqslant \text{dist}(\Delta, \Delta')^{-O(1)}\lambda^{-O(1)} \leqslant \lambda^{-O(1)}.$$

Thus multiplying the bound of lemma III.3.3.1 by $\lambda^{-O(1)}$ we get the result.

This finishes the proof of lemma III.3.3.2.

IV. PROOF OF THE MAIN RESULTS

In this chapter we prove the main results, however in order to simplify the presentation, we omit to repeat in the statements the conditions imposed to the various parameters of the theory which are not directly under consideration. In particular this remark applies for the masses M, m_1 and m which are fixed by the considerations of chapter III and for the coupling constant.

IV.1. Existence of the limits

Let Y be an union of unit cubes, $Y \supset \Lambda$ and let $C_\kappa^{\partial Y}$ and C_κ the covariances introduced in section II.3. We prove that the cutoff unnormalized Schwinger functions with boundary conditions on ∂Y converge as Y and κ tend to infinity to the finite volume unnormalized Schwinger functions of J. Feldman [4]. This results from the following proposition.

PROPOSITION IV.1.1. — *Given any* $\varepsilon > 0$, *there exists* $\kappa_0 > 0$ *and* $Y_0(\Lambda)$, $Y_0 \supset \Lambda$ *such that for* $\kappa > \kappa_0$ *and* $Y \supset Y_0$

$$\left| \int F e^{-V(\Lambda)} d\Phi_{C_\kappa^{\partial Y}} - \int F e^{-V(\Lambda)} d\Phi_{C_\kappa} \right| < \varepsilon$$

Proof. — Define $C_t = tC_\kappa + (1-t)C_\kappa^{\partial Y}$ and write

$$\int F e^{-V(\Lambda)} d\Phi_{C_\kappa} - \int F e^{-V(\Lambda)} d\Phi_{C_\kappa^{\partial Y}} = \int_0^1 dt \frac{d}{dt} \int F e^{-V(\Lambda)} d\Phi_{C_t}$$

To bound the difference, we bound

$$\frac{d}{dt} \int F e^{-V(\Lambda)} d\Phi_{C_t}.$$

We then proceed as before. We first perform the $P_1 - C_1$ expansion for $\int F e^{-V(\Lambda)} d\Phi_{C_t}$, then we derive with respect to t (this operation is similar to the derivation with respect to s) and finally we apply the other steps of the inductive expansion.

The small graphs are bounded using the decomposition of C_t as $tC_\kappa + (1-t)C_\kappa^{\partial Y}$ and the estimates of chapter III.3 and of ref. [3] chapter 6. To insure the convergence we use the factor $M_{u(\Delta)}^{-1}$ by cubes of lemma III.2.3 in order to compensate the logarithmic divergences coming from vertices created by the t-derivation. The propagators being of the same type as in chapter III, we obtain the same bound as in Feldman but with the norm $|||\cdot|||_{\delta,\alpha}$. However each term of the expansion has a derived propagator $C_\kappa - C_\kappa^{\partial Y}$. We write it as

$$\begin{aligned} C_\kappa^{\partial Y}(x,y) - C_\kappa(x,y) = C_\kappa^{\partial Y}(x,y) &- m_1 \sum_{z \in Z^3} C_{M-m_1,\kappa}(x,y) D_z^{m_1}(x,y) \\ &+ m_1 \sum_{z \in Z^3} C_{M-m_1,\kappa}(x,y) D_z^{m_1}(x,y) - C_\kappa(x,y) \end{aligned}$$

We first look at terms with a propagator given by the first difference

$$\begin{aligned} C_\kappa^{\partial Y}(x,y) &- m_1 \sum_{z \in Z^3} C_{M-m_1,\kappa}(x,y) D_z^{m_1}(x,y) \\ &= m_1 \sum_{z \in Z^3} \left(\frac{\bar{C}_m^{\partial Y}(x\,;z]}{\bar{C}_m(x\,;z]} \frac{\bar{C}_m^{\partial Y}(y\,;z]}{\bar{C}_m(y\,;z]} - 1 \right) C_{M-m_1,\kappa}(x,y) D_z^{m_1}(x,y) \end{aligned}$$

We treat the sum over z as in chapter II and use

$$\bar{C}_m(x\,;\,z]e^{-m(1+\varepsilon)\operatorname{dist}(x,z)} \leqslant O(1)$$

where the exponential factor comes from the transformation of D in $\tilde{D}$ (see chapters II and III). So

$$\left| 1 - \frac{\bar{C}_m^{\partial Y}(x\,;\,z]}{\bar{C}_m(x\,;\,z]} \frac{\bar{C}_m^{\partial Y}(y\,;\,z]}{\bar{C}_m(y\,;\,z]} \right| e^{-m(1+\varepsilon)[\operatorname{dist}(x,z)+\operatorname{dist}(y,z)]}$$
$$\leqslant O(1)\{\,|\,\bar{C}_m(x\,;\,z] - \bar{C}_m^{\partial Y}(x\,;\,z]\,| + |\,\bar{C}_m(y\,;\,z] - \bar{C}_m^{\partial Y}(y\,;\,z]\,|\,\}$$

where we have used $C_m^{\partial Y}/C_m \leqslant 1$.

The path interpretation of C shows immediatly that

$$|\,\bar{C}_m(x\,;\,z] - \bar{C}_m^{\partial Y}(x\,;\,z]\,| \leqslant O(1)e^{-m\operatorname{dist}(x,\partial Y)}$$

Because $x \in \Lambda$ we see that if Y is big enough, independently of κ, the corresponding part of the difference is as small as we want in absolute value.

Now we look at terms with a propagator given by the second difference. We use

$$m_1 \sum_{z \in Z^3} D_z^{m_1}(x, y) = \frac{m_1}{\pi^2} \int \frac{e^{il(x-y)}}{(l^2 + m_1^2)^2} dl$$

We have

$$m_1 \sum_{z \in Z^3} C_{M-m_1,\kappa}(x, y) D_z^{m_1}(x, y) = \frac{m_1}{(2\pi)^3\pi^2} \int \frac{e^{ik(x-y)}}{k^2 + (M-m_1)^2} \eta_\kappa(k) \frac{e^{il(x-y)}}{(l^2+m_1^2)^2} dk dl$$

and

$$C_{M,\kappa}(x, y) = \frac{1}{(2\pi)^3} \int \frac{e^{ik(x-y)}}{k^2 + M^2} \eta_\kappa(k) dk$$
$$= \frac{m_1}{(2\pi)^3\pi^2} \int \frac{e^{ik(x-y)}}{k^2 + (M - m_1)^2} \eta_\kappa(k + l) \frac{e^{il(x-y)}}{(l^2 + m_1^2)^2} dk dl$$

So the propagator of the second difference is of the same type as in chapter III but with cutoff: $\eta_\kappa(k) - \eta_\kappa(k + l)$.

We use that for ε as small as we want ($\varepsilon \leqslant 1$)

$$|\,\eta_\kappa(k + l) - \eta_\kappa(k)\,| \leqslant O(1)\frac{|\,l\,|^\varepsilon}{\kappa^\varepsilon}$$

Now

$$\int \frac{m_1\,|\,l\,|^\varepsilon dl}{((k - l)^2 + (M - m_1)^2)(l^2 + m_1^2)^2} \leqslant \int \frac{dl}{((k - l)^2 + (M - m_1)^2)(l^2 + m_1^2)^{\frac{3-\varepsilon}{2}}}$$
$$\leqslant \frac{1}{(k^2 + m_1^2)^{1-\varepsilon/2}} \quad \text{if} \quad M \geqslant 2m_1$$

Thus we can get the same estimates as in chapter III and obtain in this way that the second part of the difference is as small as we want provided we take κ big enough, independently of Y.

We now prove a proposition which shows the limit, as the momentum cutoff tends to infinity, of the cluster expansion.

PROPOSITION IV.1.2. — *With the notation of chapter II*

$$\partial^{\Gamma} \int F e^{-V(\Lambda \cap X)} d\Phi_{C_{\kappa}(s)}$$

has a limit as $\kappa \to \infty$.

Proof. — Define the covariances $C_j(s)$ with upper cutoff $\kappa = M_j$ and $C_{j'}(x)$ with upper cutoff $M_{j'}$. Consider

$$I = \partial^{\Gamma} \int F e^{-V(\Lambda \cap X)} d\Phi_{C_j} - \partial^{\Gamma} \int F e^{-V(\Lambda \cap X)} d\Phi_{C_{j'}}$$

$$= \sum_{i=j'}^{j-1} \partial^{\Gamma} \int F e^{-V(\Lambda \cap X)} d\Phi_{C_{i+1}} - \partial^{\Gamma} \int F e^{-V(\Lambda \cap X)} d\Phi_{C_i} = \sum_{i=j'}^{j-1} D_i$$

To bound we bound each difference.

Define $C_i(s, t) = tC_{i+1} + (1 - t)C_i$ and

$$D_i = \int_0^1 dt \frac{d}{dt} \int F e^{-V(\Lambda \cap X)} d\Phi_{C_i(s,t)}$$

Now in $\int F e^{-V(\Lambda \cap X)} d\Phi_{C_i(s,t)}$ we can do the $P_1 - C_1$ expansion. The parameter t, from the point of view of this step, being considered as a first s-parameter of the $P - C$ expansion.

Then we differentiate with respect to t and perform afterwards the differentiation with respect to the s_b. As pointed out before, the differentiation with respect to t has the same form that the differentiation with respect to the s parameters and therefore generates the same combinatoric factors (up to change on the O(1) factors). As in proposition IV.1.1 in the estimates, we ensure the convergence using $M_{u(\Delta)}^{-1/2}$ from the $M_{u(\Delta)}^{-1}$ factor of lemma III.2.3. Thus we obtain (with a little change in the O(1) factors) the usual bounds, except that in each term of the expansion one of the propagators was replaced by $\frac{d}{dt} C_i(s, t)$. This derivative has a lower cutoff at M_i. From $M_{u(\Delta)} > M_i$, we see that the remaining $M_{u(\Delta)}^{-1/2}$ factor gives to this term a decreasing overall factor $M_i^{-1/2}$.

Now summing over i, one has that I is bounded by something proportional to

$$\sum_{i=j'}^{j-1} M_i^{-1/2} \leqslant O(1) M_{j'}^{-1/2}$$

which goes to zero as $j' \to \infty$. We have therefore shown the existence of a Cauchy sequence and thus the existence of the limit.

This proposition IV.1.2 finishes the proof of proposition II.3.1.2.

We prove corollary II.3.1.3.

As will be shown in the next section $\int e^{-V(\Delta_0,\mu)} d\Phi_{C_\kappa^{\partial\Delta_0}}$ is a smooth function of μ whatever is κ.

On the other hand, from proposition IV.1.2, this expression has a limit as $\kappa \to \infty$ (uniformly in μ, $0 \leqslant \mu \leqslant 1$), thus from

$$\lim_{\mu\to 0} \int e^{-V(\Delta_0,\mu)} d\Phi_{C_\kappa^{\partial\Delta_0}} = 1$$

follows that there exists $\bar{\mu}_0$ and κ_0, such that

$$\frac{1}{2} \leqslant \int e^{-V(\Delta_0,\mu)} d\Phi_{C_\kappa^{\partial\Delta_0}} \leqslant 2$$

for $0 \leqslant \mu \leqslant \bar{\mu}_0$, $\kappa \geqslant \kappa_0$. We choose from now on $0 \leqslant \mu \leqslant \mu_0 = \inf(1, \bar{\mu}_0)$ and precise in this way the meaning of « μ small » in chapter I.

We now prove a proposition showing that the infinite volume limit we have defined in chapter II, part 3 is also the infinite volume limit of the theory defined by Feldman [4].

PROPOSITION IV.1.3. — *Given any* $\varepsilon > 0$, *there exists* K_0 and $Y_0(\Lambda)$, $Y_0 \supset \Lambda$ *such that for* $\kappa \geqslant \kappa_0$, $Y \supset Y_0$

$$\left| \int F dq(\Lambda, Y, \mu, \kappa) - \int F dq(\Lambda, \mu, \kappa) \right| < \varepsilon$$

This proposition is the equivalent of proposition IV.1.1 for the normalized expression.

Proof. — Proposition IV.1.2 and corollary II.3.1.3 show through the expansion of [2] that $\int F dq(\Lambda, Y, \mu, \kappa)$ has a limit as κ, Y and $\Lambda \subset Y$ go to infinity.

Proposition IV.1.1 shows that

$$\int F e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa^{\partial Y}} \underset{\substack{\kappa\to\infty \\ Y\to\infty}}{\longrightarrow} \int F e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa}$$

To prove the proposition it suffices then to show that the denominators are bounded from below independently of Y and κ.

As a consequence of the Kirkwood-Salsburg argument, see [1] we have

$$\int e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa^{\partial Y}} \geqslant O(1) e^{-|\Lambda|} \left| \int e^{-V(\Delta_0,\mu)} d\Phi_{C_\kappa^{\partial\Delta_0}} \right|^{|\Lambda|}$$

Then for $0 \leqslant \mu \leqslant \mu_0$

$$\int e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa^{\partial Y}} \geqslant O(1)2^{-2|\Lambda|}$$

and by continuity for Y and κ large enough

$$\int e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa}$$

is bounded from below by a term proportional to $O(1)2^{-2|\Lambda|}$. This finishes the proof.

This proposition and the results of chapter II prove theorem I.4.

IV.2. **Existence of derivatives with respect to the coupling constant**

We prove for $0 \leqslant \mu \leqslant \mu_0$

PROPOSITION IV.2.1. — *Let* $Y \supset \Lambda$ *and* $n \in \mathbb{N}$, *then*

$$\lim_{\Lambda\to\infty} \lim_{Y\to\infty} \lim_{\kappa\to\infty} \frac{d^n}{d\mu^n} \int F dq(\Lambda, Y, \mu, \kappa)$$

exists and is bounded by

$$O(1)O(1)^n(n!)^{O(1)} \prod_\Delta (O(1)n_F(\Delta))^{n_F(\Delta)} \, ||| F |||_{\delta,\alpha}$$

uniformly in κ, Y *and* Λ.

At $\mu = 0$ the derivatives have to be understood as right derivatives.

Proof. — We first localize the derivation with respect to μ in the following way. Let $\bigcup_z \Delta_0 + z$ be a unit cover of $\mathbb{R}^3$, then

$$\frac{d}{d\mu} = \sum_z \frac{d}{d\mu_z}$$

where $\frac{d}{d\mu_z}$ is defined by

1) $$\frac{d}{d\mu_z}\left(\mu^k \int :\varphi^n:(x)\Lambda(x)dx\right) = k\mu^{k-1} \int :\varphi^n:(x)\Lambda(x)\chi_{\Delta_0+z}(x)dx$$

2) $$\frac{d}{d\mu_z} \int \mu^n\left(\int :\varphi^4:(x)\Lambda(x)dx\right)^n d\Phi$$
$$= n\mu^{n-1} \int \left(\int :\varphi^4:(x)\Lambda(x)\chi_{\Delta_0+z}(x)dx\right)\left(\int :\varphi^4:(x)\Lambda(x)dx\right)^{n-1} d\Phi \tag{IV.2.1}$$

Now

$$\frac{d^n}{d\mu^n}\int F dq(\Lambda, Y, \mu, \kappa) = \sum_{z_1,\ldots,z_n} \frac{d}{d\mu_{z_1}} \cdots \frac{d}{d\mu_{z_n}} \int F dq(\Lambda, Y, \mu, \kappa)$$

Remark. — As one can see easily from their definition, the localized derivations are compatible with the decoupling at $s = 0$.

We define by $\langle\ \rangle_{n,T}$ the truncated function of n arguments. Then

$$\frac{d}{d\mu_{z_1}} \cdots \frac{d}{d\mu_{z_n}} \int F dq(\Lambda, Y, \mu, \kappa) = \left\langle F, \frac{d}{d\mu_{z_1}}, \ldots, \frac{d}{d\mu_{z_n}} \right\rangle_{n+1,T}$$

where by convention

$$\left\langle G \frac{d}{d\mu_{z_{i_1}}} \cdots \frac{d}{d\mu_{z_{i_\kappa}}} \right\rangle = \left(\frac{d}{d\mu_{z_{i_1}}} \cdots \frac{d}{d\mu_{z_{i_\kappa}}} \int G e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa^{\partial Y}}\right)\left(\int e^{-V(\Lambda,\mu)} d\Phi_{C_\kappa^{\partial Y}}\right)^{-1}$$

Here $G = 1$ or F.

This form is similar to the expression given by Dimock [*6*]. We now expand each term as in [*2*] taking $\kappa \geqslant \kappa_0$ in order corollary II.3 applies. With

LEMMA IV.2.2. — *Let X be some connected region of $\mathbb{R}^3$, let Γ be a subset of bonds such that $\mathbb{R}^3 \backslash \Gamma^c = X$, let F be with support in $\Lambda \cap X$ and $z_1, \ldots, z_k \in Z^3$ such that $\Delta_0 + z_i \subset X$ $i = 1, \ldots, k$ then*

$$\lim_{\kappa\to\infty} \frac{d}{d\mu_{z_1}} \cdots \frac{d}{d\mu_{z_k}} \partial^\Gamma \int F e^{-V(\Lambda\cap X,\mu)} d\Phi_{C_\kappa(s)}$$

exists and is bounded by

$$O(1) \prod_\Delta (O(1) n_F(\Delta))^{n_F(\Delta)} O(1)^k (k!)^{O(1)} \||| F \|||_{\delta,\alpha} e^{-K|\Gamma|}$$

uniformly in κ as $\kappa \to \infty$.

The constant K can be taken as large as we want provided m is large enough.

and the above remark one can as in [*2*], resum the cluster expansion for each term of the sum IV.2.1 and get a strong cluster decrease. Performing the sum over the unit cubes (as in [*2*]) and applying the analogue of proposition IV.1.3 we have proved theorem I.5.

Proof of lemma IV.2.2. — To bound

$$\frac{d}{d\mu_{z_1}} \cdots \frac{d}{d\mu_{z_k}} \partial^\Gamma \int F e^{-V(\Lambda\cap X,\mu)} d\Phi_{C_{(s),\kappa}}$$

we proceed as follows.

First we do the $P_1 - C_1$ expansion, then the cluster expansion as in chapter III.

Then we differentiate with respect to μ: each time a derivation in μ creates a new vertex, say a D-vertex

A) we contract its legs, giving rise possibly to C_D-vertices,

B) we contract the legs of the C_D-vertices.

The exceptions to B) are as in [3], chapter II:

1) a D-vertex forms with a C_D-vertex a mass subdiagram, then we contract only the remaining legs of the C_D-vertex,

2) a D-vertex is a mass counterterm, then we do not perform B).

In this way we obtain the cancellations of the other counterterms (as in [3]).

In bounding the graphs, we will consider a new type of subgraph: a D-vertex and its C_D-vertices. We bound it decomposing such a graph like a P-subdiagram, extracting localization factors and using proposition III.3.3.1. We obtain then that D and C_D-vertices have at most logarithmic divergences.

Below, we will obtain combinatoric factors and bounds or the same type as in chapter III. The factors by C_D-vertices are given to the generating D-vertex and we use the factor $M_{u(\Delta)}^{-1}$ by cube Δ obtained in lemma III.2.3 to compensate the factors attributed to D-vertices.

Let $n_D(\Delta)$ be the number of D-vertices in the unit cube Δ. Then $\sum_{\Delta} n_D(\Delta) \leqslant k$.

We look now at the combinatoric factors. They are the same as before but for the fact that we have two cases more:

— the derivation with respect to μ,

— the contraction to D or C_D-legs.

We have

LEMMA IV.2.3. — *The combinatoric factors are bounded as in chapter III provided we add*

— $O(1)(n_D(\Delta))^{O(1)} (\log \lambda)^{O(1)\varepsilon}$ *by* D-*vertex in* Δ,

— $d^{O(1)}$ *to the new lines.*

In fact the two new cases modify in an obvious way the O(1) factors which enters in the bound of the old combinatoric factors.

We now prove the lemma looking only at the new cases:

1) derivation with respect to μ.

We suppose it is localized in Δ. Now, or we derive the μ's of an old vertex (coming from the $P_1 - C_1$ expansion or from the cluster expansion) or we derive D or C_D-vertices (a mass counterterm which is in μ^2 gives a factor 2 that we can attribute to the vertex or to the generating vertex). We examine the different cases.

— Derivation of a P_1-vertex. There is at most $M^{\varepsilon}_{u(\Delta)}$ P_1-vertices in Δ. So we attribute the factor $M^{\varepsilon}_{u(\Delta)}$ to the P_1-vertex we derive.

— Derivation of a C_1-vertex. By a $O(1)d^4(\Delta, \Delta')$ we choose the cube Δ' of the generating P_1-vertex (or with a $O(1)d^4(\Delta, \Delta')d^4(\Delta', \Delta'')$ if the C_1-vertex is an outer C_1-vertex) and by $M^{\varepsilon}_{u(\Delta')}$ we choose the P_1-vertex. We attribute the $O(1)M^{\varepsilon}_{u(\Delta')}$ to the P_1-vertex and the d^4 to the lines.

— Derivation of a E-vertex. The number of E-vertices in Δ is $n_E(\Delta)$. So we attribute a $n_E(\Delta)$ to the E-vertex.

— Derivation of a C_E-vertex. We choose by a $O(1)d^4(\Delta, \Delta')$ the cube of the generating E-vertex and with a $n_E(\Delta')$ the E-vertex. We attribute $O(1)n_E(\Delta')$ to the E-vertex and d^4 to the generating line.

— Derivation of a D-vertex. The number of D-vertices in Δ is $n_D(\Delta)$. So we attribute $n_D(\Delta)$ to the derived D-vertex.

— Derivation of a C_D-vertex. There is at most $O(1)k$ C_D-vertices. So we attribute a factor $O(1)k$ to the generating D-vertex.

Now a P, E or D-vertex generates a finite number of C, C_E or C_D-vertices.

— Derivation of the exponential. There is no combinatoric factor because as we have seen all the counterterms disappear or are associated with terms coming from a φ^4 D-vertex.

— Contraction to a D or C_D-leg. There is at most $O(1)k2^i$ D or C_D-legs at the i^{th} step. We attribute then the $O(1)k$ to the D-vertex or to the generating D-vertex, and 2^i is attributed to the vertex of the leg. Because.

$$2^{4i} \leqslant (\log M_i)^{O(1)} \qquad O(1) \text{ depending on } v$$

this gives a factor $(\log \lambda)^{O(1)}$ at most by D or C_D-vertex.

Now we have with $O(1) \geqslant 1$

$$\lambda_\Delta^{\varepsilon} \leqslant O(1)d^4(\Delta, \Delta')\lambda_{\Delta'}^{2\varepsilon}$$

so

$$\varepsilon \log \lambda_\Delta \leqslant 4 \log O(1)d(\Delta, \Delta') + 2\varepsilon \log \lambda_{\Delta'}$$

Now

$$4 \log O(1)d(\Delta, \Delta') \leqslant 4O(1)d(\Delta, \Delta')$$

Take M_1 so large that $\varepsilon \log M_1 \geqslant 2$, then because $\lambda_{\Delta'} \geqslant M_1$ and $d(\Delta, \Delta') \geqslant 1$ we have

$$\log \lambda_\Delta^{\varepsilon} \leqslant 2.4.O(1)d(\Delta, \Delta') \log \lambda_{\Delta'}^{\varepsilon}$$

So, we can attribute, with a factor d to the line and $O(1)$ to the D-vertex, all the divergent logarithmic factors of the C_D-vertices to the D-vertices.

This finishes the proof of lemma IV.2.3.

The localization factors $d^{O(1)}$ are treated as in chapter III. Concerning the logarithmic localization factors by D-vertex we show now how to compensate them.

From chapter III we have a factor $M_{u(\Delta)}^{-1}$ by unit cube Δ. We use a part of it. There is at most $n_D(\Delta)$ D-vertices by cube and

$$M_{u(\Delta)}^{-1/O(1)} (\log M_{u(\Delta)})^{O(1)n_D(\Delta)} \leqslant O(1)^{n_D(\Delta)}(n_D(\Delta)!)^{O(1)}$$

$$\prod_\Delta O(1)^{n_D(\Delta)}(n_D(\Delta)!)^{O(1)} \leqslant O(1)^k(k!)^{O(1)}$$

So we obtain

$$\left| \partial^\Gamma \frac{d}{d\mu_{z_1}} \cdots \frac{d}{d\mu_{z_k}} \int F e^{-V(\Lambda\cap X,\mu)} d\Phi_{C(s)} \right|$$
$$\leqslant O(1)O(1)^k(k!)^{O(1)} e^{-K|\Gamma|} \prod_\Delta (O(1)n_F(\Delta))^{n_F(\Delta)} |||F|||_{\delta,\alpha}$$

This bound gives the proof of the proposition and also of the existence of the infinite volume limit. The analogue of proposition IV.1 gives the convergence as $\kappa \to \infty$. So finally we have proved theorem I.5.

IV.3. Analytic properties of the characteristic function

Let $f \in \mathscr{S}(\mathbb{R}^3)$ and $\alpha \in \mathbb{C}$. We prove in this part the following two results concerning

$$I(\alpha, \mu, f) = \lim_{\Lambda\to\infty} \lim_{\kappa\to\infty} \int e^{\alpha\varphi(f)} dq(\Lambda, \mu, \kappa)$$

First for μ small enough but different from zero $I(\alpha, \mu, f)$ is an entire function in α of order less or equal to ρ, $\rho > 4/3$. Secondly, for μ small enough (including possibly zero), it is an entire function in α of order less or equal to ρ, $\rho > 2$.

The first of these results is strongly related to the nature of the interaction (roughly speaking, $\alpha X + X^4 \geqslant -O(1)|\alpha|^{4/3}$) and is a kind of improvement of the φ-bounds in a weaker form: we deal with expectations and not with operators. The second one is valid whatever is the interaction and gives a weaker result than what we would have got from a φ-bound (as in [*8*]), since the order is strictly larger than 2. These results finish the proof of theorem I.3, and according to Fröhlich [*8*] and Feldman [*4*] imply theorems I.6 and I.7.

PROPOSITION IV.3.1. — *There exists a Schwartz norm* $|\,.\,|$ *such that for* $0 < \mu \leqslant \mu_0$ *and any* ρ, $\rho > 4/3$, *we have*

$$\lim_{\Lambda\to\infty} \lim_{\kappa\to\infty} \int F e^{\alpha\varphi(f)} dq(\Lambda, \mu, \kappa)$$

exists, is entire analytic in α of order ρ and is bounded by

$$O(1)\prod_{\Delta}[O(1)n_F(\Delta)]^{n_F(\Delta)}\,|||\,F\,|||_{\delta,\alpha}e^{O(1)|\alpha|^\rho\frac{|f|^\rho}{\mu^\rho}}$$

As a consequence the Schwinger functions satisfy

$$|\,S_n(\mu\,; f_1, \ldots, f_n)\,| \leqslant O(1)O(1)^n(n!)^{1-1/\rho}\prod_{i=1}^{n}\frac{|\,f_i\,|}{\mu}$$

Remark. — The factor $1/\mu$ which multiplies the norm reflects the fact that this bound depends on the nature of the interaction.

Proof of proposition IV.3.1. — Let $f \in C_0^\infty(\mathbb{R}^3)$, take Λ large enough such that supp $f \subset \Lambda$, and consider

$$\int F e^{-V(\Lambda,\mu)+\alpha\varphi(f)}d\Phi_{C_\kappa^{\partial Y}(s)}$$

To prove that the above expression possesses a limit and the required bound for Y, Λ and κ going to infinity, we consider $\alpha\varphi(f)$ as a part of the interaction and do the cluster expansion. Therefore it is enough to show that, for X and Γ as in proposition II.3.1.2, supp F and supp f contained in $\Lambda \cap X$, one has

$$\left|\,\partial^\Gamma\int F e^{-V(\Lambda\cap X)}e^{\alpha\varphi(f)}d\Phi_{C_\kappa(s)}\right| \leqslant O(1)\prod_{\Delta}(O(1)n_F(\Delta))^{n_F(\Delta)}$$
$$|||\,F\,|||_{\delta,\alpha}e^{-K|\Gamma|}e^{O(1)|\alpha|^\rho\frac{|f|^\rho}{\mu^\rho}} \quad \text{(IV.3.1)}$$

The analyticity will result from the analyticity at finite cutoffs and from Vitali's theorem.

Considering $\alpha\varphi(f)$ as a part of the interaction, we do the expansion of chapter III with this new interaction. This introduces the following modifications:

1) there is a new type of P, C, E or C_E-vertices: the φ-vertices,

2) when we apply the squaring operation in the first inductive step (after the derivations relatively to the $s_b's$), we first bound each integrand by its absolute value. Each term is of the following type:

$$\int \alpha^n R(s)e^{\alpha\varphi}e^{-V'}d\Phi_C$$

where R is of degree n in the φ-vertices. We bound it by

$$\int |\,\alpha\,|^n\,|\,R(s)\,|\,e^{\mathrm{Re}\,\alpha\varphi}e^{-V'}d\Phi_C$$

Then we use the squaring inequality:

$$|R(s)| \leqslant \frac{1}{2}[\delta^{-1} + \delta R(s)^2]$$

as in chapter III.

Thus after the first squaring operation it remains only real quantities.

3) for the Wick construction we have (see [3], chapter 3):

$$\delta\left(V - \operatorname{Re}\alpha \int_{\Delta'} \varphi(x)f(x)dx\right) \geqslant \delta V - \mu\frac{|\operatorname{Re}\alpha|}{\mu}\left|\int_{\Delta'} \varphi_1(x)f(x)dx\right|$$

Now

$$\frac{|\operatorname{Re}\alpha|}{\mu}\left|\int_{\Delta'} \varphi_1(x)f(x)dx\right| \leqslant \left(\int_{\Delta'} \varphi_1(x)^4 dx\right)^{1/4}\left(\int_{\Delta'} dx\right)^{3/4}\frac{|\operatorname{Re}\alpha|}{\mu}\sup_{x\in\Delta'}|f(x)|$$
$$= A\frac{|\operatorname{Re}\alpha|}{\mu}|\Delta'|^{3/4}\sup_{x\in\Delta'}|f(x)| = A\tilde{a}_{\Delta'},$$

with

$$\tilde{a}_{\Delta'} = \frac{|\operatorname{Re}\alpha|}{\mu}|\Delta'|^{3/4}\sup_{x\in\Delta'}|f(x)|$$

We obtain

$$\delta\left(V - \operatorname{Re}\alpha\int_{\Delta'}\varphi(x)f(x)dx\right) \geqslant \frac{1}{2}\mu\left\{A^4 - \sum_{i=0}^{3} A^i\frac{a_i}{\mu} - 2A\tilde{a}_{\Delta'}\right)$$

If $\sup|\alpha_i(\Delta')| \leqslant 1$ and if $\tilde{a}_{\Delta'} \leqslant 1$ we have (each a_i is a sum of α_j multiplied by μ^k, $k = 1, 2, 3$, $0 < \mu \leqslant 1$):

$$\frac{1}{2}\left\{\mu A^4 - \sum_{i=0}^{3} A^i a_i - 2\mu A\tilde{a}_{\Delta'}\right\} \geqslant -K_1$$

with a new definition of K_1.

We use then

$$e^{-(V-\operatorname{Re}\alpha\varphi)(\zeta)} \leqslant K_1 e^{-(V-\operatorname{Re}\alpha\varphi)(\zeta')} + (1-\chi)e^{-(V-\operatorname{Re}\alpha\varphi)(\zeta)} \quad (IV.3.2)$$

with $(1-\chi) \leqslant P(\Delta') + (\tilde{a}_{\Delta'})^l$.

So we have a new type of W-vertex with no legs. It is bounded by

$$\tilde{a}_{\Delta'} \leqslant |\Delta'|^{\varepsilon_3}|\Delta'|^{3/4-\varepsilon_3}\frac{|\operatorname{Re}\alpha|}{\mu}\sup_{x\in\Delta'}|f(x)| = |\Delta'|^{\varepsilon_3}\tilde{\tilde{a}}_{\Delta'}$$

with

$$\tilde{\tilde{a}}_{\Delta'} = \frac{|\operatorname{Re}\alpha|}{\mu}\sup_{x\in\Delta'}|f(x)|\,|\Delta'|^{3/4-\varepsilon_3}$$

So up to the factor $\tilde{\tilde{a}}_{\Delta'}$, this new vertex has the same bound as the other W-vertices (there is no $b^{-3(1+\nu)/2}$ localization factors because there is no leg, and therefore no contractions to this W-vertex). All the remaining of the expansion is then as in chapter III. Thus, the combinatoric factors except for a change of the O(1) factors due to the fact that we have one more term in the exponential and one more W-vertex, are as in chapter III.

The bounds on graphs are also of the same type, except that we have a new type of P, C, E or C_E-vertices with only one leg. We compute a bound of the H. S. norm of the φ-vertex:

$$\left\| \overset{\Delta'}{\circ\!\!-\!\!-\!\!-} \right\|_{\text{H.S.}} = \left| \overset{\Delta' \quad \Delta'}{\circ\!\!-\!\!-\!\!-\!\!\circ} \right|^{1/2}$$

Remenber now that the vertex function is not $\tilde{\chi}_{\Delta'}$ but $(f(x)\chi_{\Delta'}(x))^\sim = \tilde{f}_{\Delta'}$

$$\begin{aligned}
\tilde{f}_{\Delta'}(k) &= \int e^{ikx} f(x)\chi_{\Delta'}(x)dx \\
&= \int e^{ikx} f(x)\chi_{\Delta'_0}(x - r)dx = \int e^{ikr}e^{ikx}f(x + r)\chi_{\Delta'_0}(x)dx \\
&= e^{ikr}\int e^{ikx} f_r(x)\chi_{\Delta'_0}(x)dx = e^{ikr}\int \tilde{f}_r(k')\tilde{\chi}_{\Delta'_0}(k - k')dk'
\end{aligned}$$

where Δ'_0 is the translated to the origin of Δ', and r is the vector translation between Δ' and Δ'_0, and $f_r(x) = f(x + r)$. The localization factors that are obtained by derivations on k, act only on $\tilde{\chi}_{\Delta'_0}$ as in chapter III. Thus we obtain:

$$\begin{aligned}
\left| \overset{\Delta' \quad \Delta'}{\circ\!\!-\!\!-\!\!-\!\!\circ} \right| &\leqslant |\alpha|^2 \, [\sup_{k'} |(1 + k'^2)^2 \tilde{f}_r(k')|]^2 \, |\Delta'|^2 \\
&\times \int \frac{F_{\Delta'}(k + k_1 - k'_1)m_1 F(k_1 - k_2)^n \eta(k) F_{\Delta'}(k + k_2 - k'_2)}{(1 + k_1'^2)^2(k^2 + (M - m_1)^2)(k_1^2 + m_1^2)(k_2^2 + m_1^2)(1 + k_2'^2)^2} dk dk_1 dk_2 dk'_1 dk'_2
\end{aligned}$$

We do the integration on $k_1 - k_2$ using lemma III.3.3.3. The integral is bounded by

$$O(1)\int \frac{F_{\Delta'}(u - k'_4)m_1\eta(k)F_{\Delta'}(u - k'_2)}{(1 + \kappa_1'^2)^2(k^2 + (M - m_1)^2)((u - k)^2 + m_1^2)^2(1 + k_2'^2)^2} dk du dk'_1 dk'_2$$

We do the integration on k:

$$\begin{aligned}
&\int \frac{m_1\eta(k)dk}{(k^2 + (M - m_1)^2)((u - k)^2 + m_1^2)^2} \\
&\qquad \leqslant \int \frac{m_1^{1-2\varepsilon_3}dk}{(k^2 + m_1^2)^{1-\varepsilon_3}((u - k)^2 + m_1^2)^2} \sup_k \frac{\eta(k)}{(k^2 + m_1^2)^{\varepsilon_3}} m_1^{2\varepsilon_3} \\
&\qquad \leqslant \frac{1}{(u^2 + m_1^2)^{1-\varepsilon_3-\varepsilon_7}} m_1^{-2\varepsilon_7}\left(\frac{\lambda}{m_1}\right)^{-2\varepsilon_3} \qquad \text{for} \quad \varepsilon_3 \leqslant 1/2 \\
&\qquad \leqslant \frac{1}{(u^2 + m_1^2)^{1-\varepsilon_3-\varepsilon_7}} m_1^{-2\varepsilon_6}\lambda^{-2\varepsilon_3} \qquad \text{with} \quad \varepsilon_7 - \varepsilon_3 = \varepsilon_6 > 0
\end{aligned}$$

Then using

$$|\Delta'|^{1/4}(F_{\Delta'}(u - k_1'))^{1/4} \leqslant F(u - k_1')^{1/4}$$
$$|\Delta'|^{1/4}(F_{\Delta'}(u - k_2'))^{1/4} \leqslant F(u - k_2')^{1/4}$$

we bound the integral by

$$O(1)\,|\Delta'|^{-1/2} \int \frac{F(u - k_1')^{1/4} F(u - k_2')^{1/4}}{(1 + k_1'^2)^2 (1 + k_2'^2)^2 (1 + u^2)^{1-\varepsilon_3-\varepsilon_7}} du dk_1' dk_2' m_1^{-2\varepsilon_6} \lambda^{-2\varepsilon_3}$$

We use $\mu^{-1}(u - k) \leqslant O(1)\mu^{-1}(u)\mu(k)$ and get that the integral is bounded, if $\varepsilon_3 + \varepsilon_7 < 1/4$, by

$$O(1)\,|\Delta'|^{-1/2} \int \frac{F(u)^{1/2}}{(1 + u^2)^{1-\varepsilon_3-\varepsilon_7}} du \left(\int \frac{F(k')^{-1/4}}{(1 + k'^2)^2} dk' \right)^2 m_1^{-2\varepsilon_6} \lambda^{-2\varepsilon_3}$$
$$\leqslant O(1)\,|\Delta'|^{-1/2} m_1^{-2\varepsilon_6} \lambda^{-2\varepsilon_3}$$

Finally

$$\left\| \overset{\Delta'}{\circ\!\!-\!\!-} \right\|_{H.S.} \leqslant O(1)\,|\Delta'|^{\varepsilon_3} \lambda^{-\varepsilon_3} m_1^{-\varepsilon_6} \left\{ |\Delta'|^{3/4-\varepsilon_3} |\alpha| \sup_k |(1 + k^2)^2 \tilde{f}_r(k)| \right\}$$

Then the P or C-vertices of type φ in Δ' have the same bounds as the other vertices up to the factor

$$|\Delta'|^{3/4-\varepsilon_3} \sup_k |(1 + k^2)^2 \tilde{f}_r(k)|$$

which we bound by

$$O(1) \sum_{\substack{|n| \leqslant 4 \\ n = (n_0, n_1, n_2)}} \sup_k \left| \int e^{ik.x} D^n f(x + r) dx \right|$$
$$\leqslant O(1) \sum_{|n| \leqslant O(1)} \sup_{y \in \Delta''} \sup_x |(1 + x^2)^2 D^n f(x + y)| = |f|'_{\Delta''}$$

where Δ'' is the unit cube containing Δ'.

We have

$$\left\| \overset{\Delta'}{\circ\!\!-\!\!-} \right\|_{H.S.} \leqslant O(1)\,|\Delta'|^{\varepsilon_3} \lambda^{-\varepsilon_3} m_1^{-\varepsilon_6} |\Delta'|^{3/4-\varepsilon_3} |\alpha|\, |f|'_{\Delta''}$$
$$\leqslant O(1)\,|\Delta'|^{\varepsilon_3} \lambda^{-\varepsilon_3} m_1^{-\varepsilon_6} a_{\Delta'}$$

with

$$a_{\Delta'} = |\Delta'|^{3/4-\varepsilon_3} \frac{|\alpha|}{\mu} |f|'_{\Delta''} \qquad (0 < \mu \leqslant 1)$$

Note also that $\tilde{\tilde{a}}_{\Delta'} \leqslant a_{\Delta'}$. From now on, we replace in each bound, $\tilde{\tilde{a}}_{\Delta'}$ by $a_{\Delta'}$. Thus up to the factor $a_{\Delta'}$, we obtain for the new φ-vertices the same bounds as we got in chapter III for the corresponding old vertices. To bound this new factor we use once more a part of the converging factors that we have by P, E and W-vertices, leaving the other part for the purpose of the expansion, as in chapter III. So we have to choose all the parameters according to the new converging factors. We obtain up to modifications of the numerical factors, the bounds of chapter III multiplied by a factor that we now compute.

Consider that we are in the r^{th} inductive step.

i) First we consider the P − C expansion. Let C_r be the cover in the exponential, and $\Delta' \in C_r$. Let $n(\Delta')$ be the number of P_r-vertices of type φ created in Δ',

$$n(\Delta') \leqslant \lambda^{2\varepsilon} \quad \text{if} \quad r = 1, \qquad n(\Delta') \leqslant |\Delta'|^{-\varepsilon} \quad \text{if} \quad r > 1$$

The extra factors by P_r-vertices of type φ in Δ' give:

$$(a_{\Delta'})^{n(\Delta')} = \frac{(a^{\rho}_{\Delta'})^{n(\Delta')/\rho}}{(n(\Delta')/\rho)^{n(\Delta')/\rho}}(n(\Delta')/\rho)^{n(\Delta')/\rho} \leqslant e^{a^{\rho}_{\Delta'}} \qquad (n(\Delta')/\rho)^{n(\Delta')/\rho}$$

Each P_r-vertex has a converging factor $\lambda^{-\varepsilon_1}$ or $|\Delta'|^{\varepsilon_3}$. We keep $\lambda^{-\varepsilon_1/2}$ or $|\Delta'|^{\varepsilon_3/2}$ for all the purposes of the expansion of chapter III and IV part 1. Now we use the remaining part. For M_1 large enough we have

$$\begin{aligned} (n(\Delta')/\rho)^{n(\Delta')/\rho} &\leqslant (\lambda^{2\varepsilon}/\rho)^{n(\Delta')/\rho} \leqslant (\lambda^{2\varepsilon/\rho})^{n(\Delta')} && \text{if} \quad r = 1 \\ &\leqslant (|\Delta'|^{-\varepsilon}/\rho)^{n(\Delta')/\rho} \leqslant (|\Delta'|^{-\varepsilon/\rho})^{n(\Delta')} && \text{if} \quad r > 1 \end{aligned}$$

These factors are compensated by giving a factor $\lambda^{-\frac{2\varepsilon}{\rho}}$ ($|\Delta'|^{\frac{\varepsilon}{\rho}}$) by P_r-vertex of type φ.

Let $n(\Delta', \Delta)$ be the number of C-vertices of type φ generated in Δ' by P_r-vertices in Δ. By such C-vertex we can extract a factor

$$d^{-3}(\Delta, \Delta') \leqslant d^{-4/\rho}(\Delta, \Delta')$$

in modifying consequently the constants in chapter III. So, the contribution of these C-vertices is:

$$\frac{[(a_{\Delta'}d^{-4/\rho}(\Delta', \Delta))^{\rho}]^{n(\Delta',\Delta)/\rho}}{[n(\Delta', \Delta)/\rho]^{n(\Delta',\Delta)/\rho}}(n(\Delta', \Delta)/\rho)^{n(\Delta',\Delta)/\rho} \leqslant e^{a^{\rho}_{\Delta'}d^{-4}(\Delta,\Delta')}(n(\Delta', \Delta)/\rho)^{n(\Delta',\Delta)/\rho}$$

A P-vertex can generate at most 12 C-vertices of type φ. Thus

$$n(\Delta', \Delta) \leqslant O(1)\lambda^{2\varepsilon} \quad \text{if} \quad r = 1, \qquad n(\Delta', \Delta) \leqslant O(1)|\Delta'|^{-\varepsilon} \quad \text{if} \quad r > 1$$

This give that

$$\begin{aligned} (n(\Delta', \Delta)/\rho)^{n(\Delta',\Delta)/\rho} &\leqslant \left(\frac{O(1)}{\rho}\lambda^{2\varepsilon}\right)^{n(\Delta',\Delta)/\rho} \leqslant (O(1)\lambda^{\frac{2\varepsilon}{\rho}})^{n(\Delta',\Delta)} && \text{if} \quad r = 1 \\ &\leqslant \left(\frac{O(1)}{\rho}|\Delta'|^{-\varepsilon}\right)^{n(\Delta',\Delta)/\rho} \leqslant (O(1)|\Delta'|^{-\frac{\varepsilon}{\rho}})^{n(\Delta',\Delta)} && \text{if} \quad r > 1 \end{aligned}$$

A factor $O(1)\lambda^{-\frac{24}{\rho}\varepsilon}(O(1)|\Delta'|^{\frac{12}{\rho}\varepsilon})$ given to a P_r-vertex in Δ by C-vertex of type φ that it generates in Δ' is sufficient to compensate the factor

$$(n(\Delta', \Delta)/\rho)^{n(\Delta',\Delta)/\rho}$$

The P_r or C_r-vertices of type φ in Δ' give a total contribution

$$e^{a^\rho_{\Delta'}+\sum\limits_{\Delta\in C_r} d^{-4}(\Delta,\Delta')a^\rho_{\Delta'}} \leqslant e^{O(1)a^\rho_{\Delta'}}$$

So all the P_r and C_r-vertices of type φ give a factor $e^{O(1)\sum\limits_{\Delta'\in C_r} a^\rho_{\Delta'}}$ provided that we take ε such that

$$\lambda^{-\frac{\varepsilon_1}{2}}\lambda^{\frac{\varepsilon}{\rho}(\varepsilon+O(1)\varepsilon)} \leqslant 1, \qquad |\Delta'|^{\frac{\varepsilon_3}{2}}|\Delta'|^{-\frac{1}{\rho}(\varepsilon+O(1)\varepsilon)} \leqslant 1$$

This realized if $\varepsilon \leqslant O(1)^{-1}\inf(\varepsilon_1, \varepsilon_3)$ with $O(1) \geqslant 435$.

ii) Consider now the Wick expansion. W-vertices are created in the cubes of $\mathcal{D}'_r$, $\mathcal{D}'_r$ being formed only of disjoint cubes, see [3], chapter 3. Let $\Delta' \in \mathcal{D}'_r$. If W-vertices of the new type are created in Δ' their number is $l = 2(\log_2|\Delta'|^{-1})^3$. The extra factors give a contribution:

$$(a_{\Delta'})^l = \frac{(a^\rho_{\Delta'})^{l/\rho}}{(l/\rho)^{l/\rho}}(l/\rho)^{l/\rho} \leqslant e^{a^\rho_{\Delta'}}(l/\rho)^{l/\rho}$$

Now

$$(l/\rho)^{l/\rho} \leqslant (|\Delta'|^{-\varepsilon})^l$$

if M_1 is large enough. So the factor $(l/\rho)^{l/\rho}$ is compensated by a factor $|\Delta'|^{-\varepsilon}$ by W-vertex of the new type. Again from $|\Delta|^{-\varepsilon}|\Delta|^{\varepsilon_3/2} \leqslant 1$ we have that the contribution of the W-vertices in Δ' is bounded by $e^{a^\rho_{\Delta'}}$ and get for the W-vertices in $\mathcal{D}'_r$:

$$e^{\sum\limits_{\Delta'\in\mathcal{D}'_r} a^\rho_{\Delta'}}$$

iii) Consider finally the contribution of the cluster expansion. Let $n_d(\Delta)$ be the number of E-vertices of type φ in Δ ($|\Delta| = 1$) created by bonds such that $d(\Delta, b) \leqslant d$. We have $n_d(\Delta) \leqslant O(1)d^3$, this means that $n_d(\Delta)^{1/3} \leqslant O(1)d$. Then we proceed as in [1], see also chapter II, to obtain that

$$n(\Delta)^{1/3} := n_\infty(\Delta)^{1/3} \leqslant O(1)\sum_b d(\Delta, b)$$

where the sum is on the bonds which creates E-vertices in Δ.

The contribution of the extra factors for the E-vertices in Δ is

$$(a_\Delta)^{n(\Delta)} = \frac{(a^\rho_\Delta)^{n(\Delta)/\rho}}{(n(\Delta)/\rho)^{n(\Delta)/\rho}}(n(\Delta)/\rho)^{n(\Delta)/\rho} \leqslant e^{a^\rho_\Delta}(n(\Delta)/\rho)^{n(\Delta)/\rho}$$

Now

$$(n(\Delta)/\rho)^{n(\Delta)/\rho} \leqslant n(\Delta)^{n(\Delta)} \leqslant e^{O(1)n(\Delta)^{4/3}} \leqslant e^{O(1)\sum\limits_b d(\Delta,b)} \qquad (n(\Delta) \geqslant 1)$$

Thus we get a factor $e^{O(1)d(\Delta,b)}$ by bond creating E-vertices in Δ. Now, consider the C_E-vertices in Δ' created by E-vertices in Δ and let their number be $n(\Delta, \Delta')$. Then the contribution of the extra factors for these vertices is (with each C_E-vertex of type φ we associate a localization factor $d^{-4/\rho}(\Delta, \Delta')$ that we can get as in chapter III)

$$(a_{\Delta'} d^{-4/\rho}(\Delta, \Delta'))^{n(\Delta,\Delta')} = \frac{(a_{\Delta'}^{\rho} d^{-4}(\Delta, \Delta'))^{n(\Delta,\Delta')/\rho}}{(n(\Delta, \Delta')/\rho)^{n(\Delta,\Delta')/\rho}} (n(\Delta, \Delta')/\rho)^{n(\Delta,\Delta')/\rho} \leqslant e^{a_{\Delta'}^{\rho} d^{-4}(\Delta,\Delta')}(n(\Delta, \Delta')/\rho)^{n(\Delta,\Delta')/\rho}$$

A E-vertex can generate at most 3 C_E-vertices. So, as before, we can compensate $(n(\Delta, \Delta')/\rho)^{n(\Delta,\Delta')/\rho}$ $(n(\Delta, \Delta') \leqslant 3n(\Delta))$ giving to each E-vertex in Δ (created by a derivation with respect to s_b) a factor $e^{O(1)d(\Delta,b)}$ by C_E-vertex generated in Δ'. Finally with an attribution of $e^{8O(1)d(\Delta,b)}$ by bond b which derives in Δ (that we can compensate by taking m large enough, so that in chapter II we get an extra factor $e^{-4O(1)d(\Delta,b)}$ by derivation in Δ due to b) we get for the E and C_E-vertices of type φ a factor:

$$\prod_{|\Delta'|=1} e^{a_{\Delta'}^{\rho} + O(1) a_{\Delta'}^{\rho} \sum_{\Delta'} d^{-4}(\Delta,\Delta')} \leqslant e^{O(1)\sum_{\Delta'} a_{\Delta'}^{\rho}}$$

Then we have obtained for the left hand side of (IV.3.1) the bound

$$O(1)\prod_{\Delta}(O(1)n_F(\Delta))^{n_F(\Delta)} ||| F |||_{\delta,\alpha} e^{-K|\Gamma|}$$

multiplied by

$$B = e^{O(1)\left\{\sum_{\Delta,|\Delta|=1} a_{\Delta}^{\rho} + \sum_r\left(\sum_{\Delta\in C_r} a_{\Delta}^{\rho} + \sum_{\Delta\in\mathcal{D}_r'} a_{\Delta}^{\rho}\right)\right\}}$$

We put this expression in a better form. Let

$$\Delta_r \in \mathscr{C}_r \quad \text{then} \quad \sum_r |\Delta_r|^{\varepsilon_8} \leqslant O(1)$$

$$\Delta_r \in \mathscr{D}_r' \quad \text{then} \quad \sum_r |\Delta_r|^{\varepsilon_8} \leqslant O(1)$$

for $\varepsilon_8 > 0$. This comes from the fact that if $|\Delta_r| \leqslant 1$ then $|\Delta_r| \leqslant M_{3r}^{-2}$ for any $\Delta_r \in \mathscr{C}_r$, $\Delta_r \in \mathscr{D}_r'$.

So we can perform the $\sum_r$ provided we give to each $a_{\Delta'}^{\rho}$ a factor $|\Delta'|^{-\varepsilon_8}$.

We choose ε_3 and ε_8 such that $\varepsilon_3\rho + \varepsilon_8 \leqslant \frac{3}{4}\rho - 1$ $(\rho > 4/3)$ then

$$\sum_{\substack{\Delta'\subset\Delta''\\ \Delta'\in C_r}} |\Delta'|^{-\varepsilon_8} a_{\Delta'}^{\rho} \leqslant \frac{|\alpha|^{\rho}}{\mu^{\rho}}(|f|_{\Delta''}')^{\rho} \qquad \left(\sum_{\substack{\Delta'\subset\Delta''\\ \Delta'\in C_r}} |\Delta'| = 1\right)$$

and

$$\sum_{\Delta'\subset\Delta'',\Delta'\in\mathscr{D}'_r} |\Delta'|^{-\varepsilon} a^{\rho}_{\Delta'} \leqslant \frac{|\alpha|^{\rho}}{\mu^{\rho}}(|f|'_{\Delta''})^{\rho} \qquad \Big(\sum_{\substack{\Delta'\subset\Delta''\\ \Delta'\in\mathscr{D}'_r}} |\Delta'| = 1\Big)$$

We obtain in this way that

$$B \leqslant e^{O(1)\frac{|\alpha|^{\rho}}{\mu^{\rho}}\sum_{\Delta}(|f|'_{\Delta})^{\rho}}$$

where the sum is over the unit cubes.

Now

$$\sum_{\Delta}(|f|'_{\Delta})^{\rho} \leqslant O(1)\sup_{\Delta} d_e^4(\Delta, \Delta_0)(|f|'_{\Delta})^{\rho}$$
$$\leqslant O(1)\sup_{\Delta}\Big[\sum_{|n|\leqslant O(1)} \sup_{y\in\Delta}\sup_{x} |(1+x^2)^2(1+y^2)^2 D^n f(x+y)|\Big]^{\rho}$$

because $d_e(\Delta, \Delta_0) \leqslant O(1)(1+y^2)^{1/2}$, $y\in\Delta$ and $d_e^4(\Delta, \Delta_0) \leqslant O(1)(1+y^2)^{2\rho}$. Notice that $\sup_{\Delta}\sup_{y\in\Delta} \equiv \sup_{y}$ and that if $z = x + y$

$$(1+x^2)(1+y^2) \leqslant (1+(x+y)^2)^2 = (1+z^2)^2$$

We obtain

$$\sum_{\Delta}(|f|'_{\Delta})^{\rho} \leqslant O(1)\Big[\sum_{|n|\leqslant O(1)} \sup_{z}(1+z^2)^4|D^n f(z)|\Big]^{\rho} = O(1)|f|^{\rho}$$

which defines the Schwartz norm $|\,.\,|$.

So finally

$$B \leqslant e^{O(1)|\alpha|^{\rho}\frac{|f|^{\rho}}{\mu^{\rho}}}$$

from which follows the bound on the Schwinger functions. The result extends by continuity to functions $f\in\mathscr{S}(\mathbb{R}^3)$.

We prove in a similar way:

PROPOSITION IV.3.2. — *There exists $\mu''_0 > 0$ such that for $0 \leqslant \mu \leqslant \mu''_0$ and $\rho > 2$*

$$\lim_{\Lambda\to\infty}\lim_{\kappa\to\infty}\int e^{\alpha\varphi(f)}dq(\Lambda, \mu, \kappa)$$

exists, is entire analytic in α of order $\leqslant \rho$ and is bounded by

$$e^{O(1)|\alpha|^{\rho}|f|^{\rho}}$$

for the same Schwartz norm as in Proposition IV.3.1. As a consequence

$$|S_n(\mu; f_1, \ldots, f_n)| \leqslant O(1)O(1)^n(n!)^{1-\frac{1}{\rho}}\prod_{i=1}^{n}|f_i|$$

Proof. — The method consists in extracting from the free interaction a small quadratic term, hence diminishing the bare mass, in order to bound from below $\alpha\varphi(f)$ independently of the interaction.

We consider the following covariances:

$$\tilde{C}_\kappa(x, y) = \frac{1}{(2\pi)^3} \int \frac{e^{ik.(x-y)}\eta_\kappa(k)}{k^2 + M^2 - \xi^2(1 - \eta_\kappa(k))} dk \qquad \tilde{\tilde{C}}_\kappa = C_{\sqrt{M^2 - \xi^2},\kappa}$$

for $0 < \xi^2 < \inf(1, M^2)$.

The φ^4 theory with covariance $\tilde{C}_\kappa$ tends as κ tends to infinity to the theory with covariance $C_{M,\kappa}$ since

$$\lim_{\kappa\to\infty} \frac{1}{k^2 + M^2}\eta_\kappa(k) = \lim_{\kappa\to\infty} \frac{\eta_\kappa(k)}{k^2 + M^2 - \xi^2(1 - \eta_\kappa(k))}$$

So we can prove in the same way as in proposition IV.1.1 that

$$\lim_{\kappa\to\infty} \int Fe^{-V}d\varphi_{C_{M,\kappa}} = \lim_{\kappa\to\infty} \int Fe^{-V}d\varphi_{\tilde{C}_\kappa}$$

Now we want to change the integration measure, going from $d\varphi_{\tilde{C}_\kappa}$ to $d\varphi_{\tilde{\tilde{C}}_\kappa}$.

To this purpose we want to apply lemma 2, chapter 2 of ref. [2]. However we need a slight modification in the proof of this lemma since our cutoff covariances are degenerate. More precisely one can check that the identity (9) of chapter II, ref. [2] is still valid if we take C′ equal to

$$C' = C + C*h*C + C*h*C*h*C + \dots \tag{IV.3.3}$$

provided the sum converges (the $*$ product is defined as in [2]). Here $C' = \tilde{\tilde{C}}_\kappa$ and we just show that taking the limit value of h:

$$h(x, y) = \xi^2\delta(x - y), \qquad \xi^2 < M^2$$

give $C' = \tilde{C}_\kappa$. In fact, the Fourier transform of the right hand side of (IV.3.3) is

$$\frac{\eta_\kappa(k)}{k^2 + M^2 - \xi^2(1 - \eta_\kappa(k))} \sum_{n=0}^{\infty} \left[\frac{\xi^2\eta_\kappa(k)}{k^2 + M^2 - \xi^2(1 - \eta_\kappa(k))}\right]^n$$

but the sum converges for $2\xi^2 < M^2$ remenbering that $0 \leqslant \eta_\kappa \leqslant 1$ and we get that the Fourier transform of C′ is

$$\frac{\eta_\kappa(k)}{k^2 + M^2 - \xi^2}$$

Thus if $\tilde{P}(\varphi)_{\tilde{C}_\kappa} = \tilde{\tilde{P}}(\varphi)_{\tilde{\tilde{C}}_\kappa}$ where $\tilde{P}$ and $\tilde{\tilde{P}}$ are two polynomials of Wick products relatively to respectively $\tilde{C}_\kappa$ and $\tilde{\tilde{C}}_\kappa$, we have that

$$\int \tilde{P}(\varphi)_{\tilde{C}_\kappa} d\Phi_{\tilde{C}_\kappa} = \lim_{\Lambda'\to\infty} \frac{\int \tilde{\tilde{P}}(\varphi)_{\tilde{\tilde{C}}_\kappa} e^{-\xi^2:\varphi^2:(\Lambda')} d\Phi_{\tilde{\tilde{C}}_\kappa}}{\int e^{-\xi^2:\varphi^2:(\Lambda')} d\Phi_{\tilde{\tilde{C}}_\kappa}}$$

We have with

$$g = \lim_{x\to 0} \tilde{C}_\kappa(x, 0) - \tilde{\tilde{C}}_\kappa(x, 0) = -\frac{\xi^2}{(2\pi)^3}\int \frac{\eta_\kappa(k)^2 dk}{(k^2+M^2-\xi^2)(k^2+M^2-\xi^2(1-\eta_\kappa(k)))}$$

that

1) $|g| \leqslant O(1)$
2) $:\varphi^2:_{\tilde{C}_\kappa}(x) = :\varphi^2:_{\tilde{\tilde{C}}_\kappa}(x) - g$
   $:\varphi^4:_{\tilde{C}_\kappa}(x) = :\varphi^4:_{\tilde{\tilde{C}}_\kappa}(x) - 6g:\varphi^2:_{\tilde{\tilde{C}}_\kappa}(x) + 3g^2$
3) $\delta m^2_{\tilde{\tilde{C}}_\kappa} = -4^2.6\int(\tilde{C}_\kappa(x, 0))^3 dx = \delta m^2_{\tilde{C}_\kappa} + l(\kappa, \xi^2), \qquad |l(\kappa, \xi^2)| \leqslant O(1)$

since

$$\int(\tilde{C}_\kappa(x, 0))^i dx \leqslant O(1) \quad \text{and} \quad \int(\tilde{\tilde{C}}_\kappa(x, 0))^i dx \leqslant O(1), \qquad i = 1, 2$$

Finally a theory $\varphi^4 + a\varphi^2$ has no new primitive divergent diagrams.
We get that:

$$\int e^{\alpha\varphi(f)} e^{-V(\Lambda,\mu)} d\Phi_{\tilde{C}_\kappa} = \lim_{\Lambda'\to\infty} K(\Lambda, \Lambda', \mu, \kappa) \int e^{-\tilde{V}(\Lambda,\mu)+\alpha\varphi(f)-\xi^2:\varphi^2:(\Lambda)} d\Phi_{\tilde{\tilde{C}}_\kappa}$$

where

$$\tilde{V}(\Lambda, \mu) = \mu:\varphi^4:(\Lambda) + \frac{\mu^2}{2}\langle(:\varphi^4:(\Lambda))^2\rangle_{\tilde{\tilde{C}}_\kappa} - \frac{\mu^3}{6}\langle(:\varphi^4:(\Lambda))^3\rangle_{\tilde{\tilde{C}}_\kappa}$$
$$- \frac{\mu^2}{2}\delta m^2_{\tilde{\tilde{C}}_\kappa}:\varphi^2:(\Lambda) - 6\mu g:\varphi^2:(\Lambda) - \frac{\mu^2}{2} l(\kappa, \xi^2):\varphi^2:(\Lambda)$$

and $K(\Lambda, \Lambda', \mu, \kappa)$ is the normalization factor:

$$\left(\int e^{-\xi^2:\varphi^2:(\Lambda')} d\Phi_{\tilde{\tilde{C}}_\kappa}\right)^{-1} e^{-3\mu g^2|\Lambda| - \frac{\mu^2}{2}\delta m^2_{\tilde{\tilde{C}}_\kappa} g|\Lambda| - \frac{\mu^2}{2} g l|\Lambda|}$$
$$\times\, e^{\frac{\mu^2}{2}(\langle(:\varphi^4:(\Lambda))^2\rangle_{\tilde{\tilde{C}}_\kappa} - \langle(:\varphi^4:(\Lambda))^2\rangle_{\tilde{C}_\kappa}) - \frac{\mu^3}{6}(\langle(:\varphi^4:(\Lambda))^3\rangle_{\tilde{\tilde{C}}_\kappa} - \langle(:\varphi^4:(\Lambda))^3\rangle_{\tilde{C}_\kappa})}$$

So we have that

$$\lim_{\Lambda'\to\infty} \frac{\int e^{-\tilde{V}(\Lambda,\mu)+\alpha\varphi(f)-\xi^2:\varphi^2:(\Lambda')} d\Phi_{\tilde{\tilde{C}}_\kappa}}{\int e^{-\tilde{V}(\Lambda,\mu)-\xi^2:\varphi^2:(\Lambda')} d\Phi_{\tilde{\tilde{C}}_\kappa}} = \frac{\int e^{-V(\Lambda,\mu)+\alpha\varphi(f)} d\Phi_{\tilde{C}_\kappa}}{\int e^{-V(\Lambda,\mu)} d\Phi_{\tilde{C}_\kappa}}$$

Now if $\mu \leqslant \mu_0 < 1$ and if κ is big enough $\int e^{-V(\Lambda,\mu)} d\Phi_{C_{M,\kappa}}$ is bounded from below independently of μ and κ, then it is also true from $\int e^{-V(\Lambda,\mu)} d\Phi_{\tilde{C}_\kappa}$ and we have that

$$\lim_{\kappa\to\infty} \frac{\int e^{-V(\Lambda,\mu)+\alpha\varphi(f)} d\Phi_{\tilde{C}_\kappa}}{\int e^{-V(\Lambda,\mu)} d\Phi_{\tilde{C}_\kappa}} = \lim_{\kappa\to\infty} \frac{\int e^{-V(\Lambda,\mu)+\alpha\varphi(f)} d\Phi_{C_{M,\kappa}}}{\int e^{-V(\Lambda,\mu)} d\Phi_{C_{M,\kappa}}}$$

because these limits exist as can be shown following Feldman [4]. Then using Vitali's theorem we have only to consider the Schwinger functions with $d\Phi_{\tilde{\tilde{C}}_\kappa}$ and to proceed as in the proof of proposition IV.3.1.

We take $\Lambda' \supset \Lambda$. We start with the covariance $\tilde{\tilde{C}}_\kappa = C_{\sqrt{M^2-\xi^2},\kappa}$ and introduce as in chapter II interpolating covariances $\tilde{\tilde{C}}(s)$.

Again as in proposition IV.3.1 we have just to prove that

$$\partial^\Gamma \int e^{-\tilde{\tilde{V}}} d\Phi_{\tilde{\tilde{C}}_\kappa(s)} \leqslant O(1) e^{-K|\Gamma|} e^{O(1)|\alpha|^p |f|^p}, \qquad p > 2 \qquad \text{(IV.3.4)}$$

with

$$\tilde{\tilde{V}} = \tilde{V}(\Lambda \cap X) + \xi^2 :\varphi^2: (\Lambda \cap X) - \alpha\varphi(f)$$

and supp $f \subset \Lambda' \cap X$, for μ small enough. More precisely fixing ξ^2 small, using the continuity and Jensen inequality, for κ sufficiently large, there exists μ_0' such that for $0 \leqslant \mu \leqslant \mu_0'$

$$\int e^{-\xi^2:\varphi^2:(\Delta) - \tilde{V}(\Delta,\mu)} d\Phi_{\tilde{\tilde{C}}_\kappa^{\partial\Delta}} \geqslant 1/2$$

and we take $\mu \leqslant \mu_0'' = \inf(\mu_0, \mu_0')$.

Now to prove (IV.3.4) we proceed as in the proof of (IV.3.1) with the following modifications in the Wick expansion (see [3], chapter III)

$$\delta\tilde{\tilde{V}} = \delta\tilde{V} + \delta\left(\xi^2 \int :\varphi^2: - \operatorname{Re}\alpha \int_{\Delta'} \varphi(x) f(x) dx\right) = \delta V - \left(6\mu g + \frac{\mu^2}{2} l\right)$$
$$\times \left(\int :\varphi_1^2: + 2\int :\varphi_1\varphi_2:\right) + \xi^2 \int :\varphi_1^2: + 2\xi^2 \int :\varphi_1\varphi_2:$$
$$- \operatorname{Re}\alpha \int_{\Delta'} \varphi_1(x) f(x) dx$$

We use

$$\left| 6\mu g + \frac{\mu^2}{2} l(\kappa, \xi^2) \right| \leqslant \mu O(1)$$
$$-\left| \int :\varphi_1^2: \right| \geqslant -A^2 |\Delta'|^{1/2} - O(1) |\Delta'|^{1/2}$$
$$-\left| 2\int :\varphi_1\varphi_2: \right| \geqslant -2A |\Delta'|^{-1/4} \left| \int \varphi_2 \right| - 2\int \varphi_1\varphi_2 - O(|\Delta'|^{1/2})$$

according to [3].

Now let $T = \left(\int_{\Delta'} \varphi_1^2(x)dx \right)^{1/2}$, then $|\overline{\varphi_1}| \leqslant |\Delta'|^{-1/2}T$ and

$$\int \varphi_1(x)f(x)dx \leqslant T\,|\Delta'|^{1/2}\,|f|'_{\Delta'}$$

Define $\tilde{t}_{\Delta'} = \frac{|\alpha|}{\xi^2}|\Delta'|^{1/2}\,|f|'_{\Delta'}$.

We get

$$\delta\tilde{\tilde{V}} \geqslant \frac{1}{2}\left[\mu A^4 - \sum_{i=0}^{3} A^i a_i - \mu A^2 O(1)|\Delta'|^{1/2} - \mu A O(1)|\Delta'|^{-1/4}\left|\int\varphi_2\right| \right.$$
$$\left. - \mu O(1)\left(\left|\int\delta\varphi_1\varphi_2\right| + O(|\Delta'|^{1/2})\right)\right] + \xi^2\left[T^2 - \sum_{i=0} T^i t_i - T\tilde{t}_{\Delta'}\right]$$

with

$$t_1 = 2|\Delta'|^{-1/2}\left|\int\varphi_2\right|$$
$$t_0 = 2\left|\int\delta\varphi_1\varphi_2\right| + O(|\Delta'|^{1/2})$$

If $|\tilde{t}_{\Delta'}| \leqslant 1$ and $\sup \alpha_j(\Delta') \leqslant 1$ we have that all the coefficients of A and T are bounded above by O(1) independently of Δ'. Thus, under this condition

$$\delta\tilde{\tilde{V}} \geqslant -K_2$$

for some $K_2 = -\log K_1$.

Then we replace (IV.3.2) by

$$e^{-\tilde{\tilde{V}}(\zeta)} \leqslant K_1 e^{-\tilde{\tilde{V}}(\zeta')} + (1-\chi)e^{-\tilde{\tilde{V}}(\zeta)} \tag{IV.3.5}$$

where

$$1 - \chi \leqslant P(\Delta') + (\tilde{t}_{\Delta'})^l$$

We use (IV.3.5) as formula (IV.3.2) before.

The bound of a $:\varphi^2:$-vertex is of the same type as the bound of the convergent P-vertices. Therefore we proceed as in the proof of (IV.3.1) up to modifications of the O(1) factors. We obtain the bounds of chapter III multiplied by the contribution of the P, C, E, C_E, and W φ-vertices. We obtain then for $\rho > 2$ the bound

$$\left|\partial^\Gamma \int e^{-\tilde{V}(\Lambda\cap X,\mu) - \xi^2:\varphi^2:(\Lambda'\cap X) + \alpha\varphi(f)} d\Phi_{\tilde{\tilde{C}}_\kappa(s)}\right| \leqslant O(1)e^{-K|\Gamma|}$$
$$\times\, e^{O(1)\left(\frac{|\alpha|}{\xi^2}\right)^\rho \left\{ \sum_{\Delta'',\text{unit cubes}} (|f|'_{\Delta''})^\rho + \sum_{\Delta''}\sum_r \left(\sum_{\substack{\Delta'\in C_r \\ \Delta'\subset\Delta''}} (|\Delta'|^{1/2-\varepsilon_3}|f|'_{\Delta''})^\rho + \sum_{\substack{\Delta'\in\mathcal{D}'_r \\ \Delta'\subset\Delta''}} (|\Delta'|^{1/2-\varepsilon_3}|f|'_{\Delta''})^\rho \right) \right\}}$$

Choosing ε_3 and ε such that:

$$|\Delta'|^{\rho\left(\frac{1}{2}-\varepsilon_3\right)-\varepsilon_8} \leqslant |\Delta'| \qquad \text{i. e. } \frac{\rho}{2} - \rho\varepsilon_3 - \varepsilon \geqslant 1$$

we can proceed as in IV.3.1 getting in this way the required bound.

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