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Section A :
Physique théorique.

Scattering theory with highly singular oscillating potentials

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ABSTRACT. — Continuing recent investigations of Pearson [*1*] [*2*], and Amrein and Georgescu [*7*], on asymptotic completeness of wave operators for singular attractive or repulsive potentials, we study the case of a spherically symmetric potential which is singular and strongly oscillating near the origin. We consider all potentials which are $L_1(a, \infty)$ for each $a > 0$, and satisfy the following conditions: let $W(r) = -\int_r^\infty V(t)dt$. We assume *a*) $W \in L_1(0, 1)$; *b*) $\lim rW(r) = 0$ as $r \to 0$. Under these hypotheses, we show that

1) The spectrum of the Hamiltonian $H_l = -\dfrac{d^2}{dr^2} + \dfrac{l(l+1)}{r^2} + V(r)$, $l = 0, 1, \ldots$, is simple;

2) The essential spectrum is $[0, \infty)$;

3) H_l has no singularly continuous spectrum;

4) The bound states are finite in number, and non-degenerate. It is known that, under our assumptions, there are no positive eigenvalues.

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5) The Møller wave operators exist as strong limits, and are asymptotically complete.

These results generalize those of the previous authors.

6) Moreover, if we assume that $rV(r) \in L_1(1, \infty)$, we can define the Jost function exactly as in the regular case, and it has exactly the same analytic and asymptotic properties. In particular, it contains complete information about the spectrum, scattering, and bound states, and the inverse scattering formalisms of Gel'fand-Levitan-Jost-Kohn, and Marchenko, apply without any modifications.

1. INTRODUCTION

We consider in this paper scattering of a particle by a spherically symmetric potential which is short range and highly singular at the origin. *The potential is assumed to be locally absolutely integrable for* $r > 0$. By short range, we mean that $V(r) = 0(r^{-2-\varepsilon})$ as $r \to \infty$ $(\varepsilon > 0)$, or more generally,

$$\int_1^\infty r \, | V(r) | \, dr < \infty \qquad (^1) \tag{1}$$

By singular at the origin, we mean, as usual, that

$$\int_0^1 r \, | V(r) | \, dr = \infty \tag{2}$$

If we split the potential into its positive and negative parts, we have two cases, according to whether rV_- is integrable or not at the origin. In the first case, there is no problem for formulating scattering theory, no matter how singular V_+ is [*1*], and the usual conventional formalism applies. In this paper we consider the case where

$$\int_0^1 r \, | V_-(r) | \, dr = \infty \tag{3}$$

We also assume that

$$\int_0^1 rV_+(r) dr = \infty \tag{4}$$

since, if rV_+ is integrable at the origin, V_+ is regular, which means that the potential is equivalent to a strongly singular attractive potential. This case is also rather well understood, and for a very large class of potentials of this kind (very singular and attractive) the asymptotic completeness has recently been proved by Pearson [*1*], and Amrein and Georgescu [*7*].

(1) There is no special meaning attached to the lower limit 1. It is chosen merely for convenience.

According to Pearson's analysis, a potential which violates the asymptotic completeness must be very singular and violently oscillating near the origin, and should satisfy both (3) and (4). Such an example has indeed been found by Pearson [2]. In this example, the absolutely continuous positive spectrum is doubly degenerate on a finite part of the positive axis. Moreover, there are no positive-energy eigenvalues, and the negative eigenvalues are finite in number. Finally, the Hamiltonian is essentially self-adjoint and bounded from below. The scattering operator is of course non-unitary, so that this example leads, with a real potential, to the absorption of part of the incoming particles, which remain bound to the scattering center.

In this example one can verify that

$$\int_0^1 r(V_+ + V_-)dr = \infty \tag{5}$$

In other words, there are no compensations between V_+ and V_-. However, it is not clear in Pearson's paper [2] whether condition (5) is necessary for the breakdown of asymptotic completeness. Our purpose is to show that in some sense (5) is necessary. Indeed, we show in the next section that if

$$\left| \int_0^1 rV(r)dr \right| < \infty \tag{6}$$

where V is locally absolutaly integrable for $r > 0$, and if

$$W(r) = -\int_r^1 V(t)dt \tag{7a}$$

is such that $W \in L_1(0, 1)$ and

$$\lim_{r \to 0} rW(r) = 0 \tag{7b}$$

the S-wave Schrodinger equation (S-wave for simplicity !)

$$\varphi'' + E\varphi = V\varphi \tag{8a}$$

has a regular solution satisfying

$$\varphi(E, 0) = 0, \qquad \varphi'(E, 0) = 1 \tag{8b}$$

An integration by parts show that (6) and (7*b*) lead to

$$\left| \int_0^1 W(r)dr \right| < \infty \tag{9}$$

The reason for asking the absolute integrability of W will become clear in the next section. We shall show there that everything is then exactly as for the usual regular potentials satisfying

$$\int_0^\infty r \,|\, V \,|\, dr < \infty \tag{10}$$

This means that the Hamiltonian is self-adjoint and bounded from below, that the absolutely continuous part of the spectrum is the whole positive axis and is simple, that there are no bound states with positive energies, and, finally that the singular spectrum is void. Needless to say that asymptotic completeness holds, and that the S-matrix is unitary.

We shall in fact see that as for regular potentials, we can formulate everything in terms of the Jost function, which is known to be by far the most convenient tool in dealing with central potentials [3] [4].

We deduce from the above results the usual completeness relation for the wave-functions $\{ \varphi(E, r) \}$, where E runs through the whole spectrum. Similarly, we deduce the Gel'fand-Levitan and Marchenko representations for the regular solution and the Jost solution, respectively, in section 3. It follows from these results that the inverse scattering methods of Gel'fand-Levitan, Jost-Kohn, and Marchenko apply to our case without any modification. We end up this section with some illuminating examples.

This enlargement of the class of admissible potentials to which the usual Jost function formalism applies is far from trivial since, as can be easily seen, wildly oscillating potentials associated with

$$W_1 = \frac{\sin 1/r}{r^\alpha}\theta(1 - r), \qquad \alpha < 1 \tag{11a}$$

$$W_2 = \frac{\sin e^{1/r}}{r^\alpha}\theta(1 - r), \qquad \alpha < 1 \tag{11b}$$

and in fact

$$W_3 = \frac{B(r)}{r^\alpha (\log 1/r)^\beta}\theta\left(\frac{1}{2} - r\right) \tag{11c}$$

where B is any bounded function on [0, 1] which is differentiable for $r > 0$, and $\alpha < 1$, or $\alpha = 1$, $\beta > 1$, ... etc., are allowed.

Singular oscillating potentials at infinity have recently been subjecte to extensive studies [5] [6]. One of their features, which we do not have here, is the occurence of bound states with positive binding energies. Here, since we assume the potential to be of short range, and for positive energies, it is known that the asymptotic behaviour for large r of all the solutions of the Schrödinger equation is given by a linear combination of e^{ikr} and e^{-ikr} [3] [4]. It is then obvious that no positive-energy solution can be square-integrable at infinity, a well-known fact. Our analysis is somewhat the analogue of the work done for oscillating potential at infinity.

As a final remark, let us notice that, if we define W by (24), which differs from (7a) by a constant, it follows from (1) and $W \in L_1(0, 1)$, that in fact $W \in L_1(0, \infty)$. As will become clear in the next section, in order to prove the existence of a regular solution of the Schrodinger equation satisfying (8b), it would be sufficient to assume only the local integrability of W at the origin. However, when we introduce the Jost function, and demand that it should have properties quite similar to the case of regular potentials satisfying (10), we have to use also (1), or $W \in L_1(0, \infty)$. The most general conditions under which our results are valid are then as follows: 1) Absolute integrability of V on $[a, \infty)$, for each $a > 0$; 2) $W \in L_1(0, \infty)$; 3) Condition (7*b*).

We have considered in this paper one singular point, which is at the origin. However, from the analysis of the next section, it will become clear that singularities of the same kind (roughly speaking, simple integrability of V from left and right) situated at finite distances can be accommodated as well. The general case of n dimensions has been studied by Ginibre and Combescure (this Journal, the following paper).

2. SOLUTIONS OF THE RADIAL EQUATION

We wish to show that, under the hypotheses stated previously, namely the absolute local integrability of the potential V for $r > 0$, and $W \in L_1(0, 1)$, and (7*b*), there exist a solution satisfying the boundary conditions (8*b*). If such a solution exists, it must satisfy the well-known Volterra equation [3] [4]

$$\varphi(k, r) = \frac{\sin kr}{k} + \int_0^r \frac{\sin k(r - r')}{k} V(r')\varphi(k, r')dr' \tag{12}$$

where $k = \sqrt{E}$. Since the potential is short range, we shall cut it at $r = 1$ and replace it by 0 for $r > 0$. This tail can always be treated by perturbation as long as it satisfies (1), and does not lead to any modification of the results. Intergrating by parts, and using (7*a*, *b*), we find (²)

$$\begin{aligned} \varphi(k, r) = {} & \frac{\sin kr}{k} \\ & + \int_0^r W(r') \left[\cos k(r - r')\varphi(k, r') - \frac{\sin k(r - r')}{k} \varphi'(k, r') \right] dr' \end{aligned} \tag{13a}$$

$$\begin{aligned} \varphi'(k, r) = {} & \cos kr + W(r)\varphi(k, r) \\ & - \int_0^r W(r')[k \sin k(r - r')\varphi(k, r') + \cos k(r - r')\varphi'(k, r')]dr' \end{aligned} \tag{13b}$$

Before showing the existence and uniqueness of the solution of these two coupled Volterra equations, let us first notice that $rW^2 \in L_1(0, 1)$. This is obvious since rW, by (7*b*), is continuous and bounded in $[0, 1]$. Therefore, rW^2 is in $L_1(0, 1)$ if W is so.

It is then obvious that by iterating (13*a* and *b*), one always gets, at each

(²) Because of the boundary conditions (8*b*), $\varphi = 0(r)$ near the origin. Therefore, $\varphi W \to 0$ as $r \to 0$, in accordance with the assumption (7*b*).

step, integrals of absolutely integrable functions. The usual procedure for proving the absolute and uniform convergence of the series thus obtained should then go through as for regular potentials. For these potentials, one starts from the bound

$$|\sin z| < C\frac{|z|}{1+|z|}e^{|\mathrm{Im}\, z|} \tag{14}$$

which one uses both for the inhomogeneous term and inside the integral in (12). The simplest way to obtain (14) is to notice that

$$e^{-|\mathrm{Im}\, z|}\,|\sin z| < \begin{cases} \frac{1}{2}C|z| & |z|<1 \\ \frac{1}{2}C & |z|>1 \end{cases} \tag{15}$$

where C is an appropriate constant. Now using

$$\frac{1}{2} < \frac{1}{1+|z|} \qquad |z|<1$$

and

$$\frac{1}{2} < \frac{|z|}{1+|z|} \qquad |z|>1$$

we get (14). In our case, since we are dealing with the coupled equation (13*a*, *b*), we shall use separately the two bounds (15) for $|kr|<1$ and $|kr|>1$, together with

$$|\cos z| < Ce^{|\mathrm{Im}\, z|} \tag{16}$$

Given $k \neq 0$, and $r_0 = |k|^{-1}$, we have the two cases $r<r_0$ and $r>r_0$. First consider $r<r_0$. Equations (13*a*, *b*) lead to

$$|\varphi| \leq Cre^{|\mathrm{Im}\, k|r} + \int_0^r dr'\,|W(r')|\,[Ce^{|\mathrm{Im}\, k|(r-r')}|\varphi| + C(r-r')e^{|\mathrm{Im}\, k|(r-r')}|\varphi'|]$$

$$|\varphi'| \leq Ce^{|\mathrm{Im}\, k|r} + |W|\,|\varphi| + \int_0^r dr'[Ce^{|\mathrm{Im}\, k|(r-r')}k^2(r-r')|\varphi| + Ce^{|\mathrm{Im}\, k|(r-r')}|\varphi'|]\,|W(r')|$$

Writing now

$$\Phi = |\varphi|\,e^{-|\mathrm{Im}\, k|r}$$
$$\Psi = |\varphi'|\,e^{-|\mathrm{Im}\, k|r}$$

and using

$$k^2(r-r') < k^2r < k < \frac{1}{r} < \frac{1}{r'}$$

we get

$$\Phi \leq Cr + Cr\int_0^r dr'\,|W(r')|\left[\frac{\Phi}{r'} + \Psi\right]$$

and

$$\Psi \leq C + C|W|\Phi + C\int_0^r dr'\,|W(r')|\left[\frac{\Phi}{r'} + \Psi\right]$$

whose solutions are easily obtained by transforming them into differential inequalities. One then gets

$$|\varphi| < \mathrm{M}(k, r) = \mathrm{C}r e^{|\mathrm{Im}\, k|r} e^{\int_0^r [2\mathrm{C}|\mathrm{W}| + \mathrm{C}^2 t \mathrm{W}^2] dt} \tag{17a}$$

and

$$|\varphi'| < \mathrm{N}(k, r) = \frac{\mathrm{M}'(k, r)}{1 + \mathrm{C}r|\mathrm{W}|} \tag{17b}$$

It can easily be seen that they are also true for $k = 0$ (just make $k = 0$ in (13a, b)). These bounds, when used in (13a, b), at once show the existence, for $r < r_0$, of the solution φ satisfying the boundary conditions (8b). That this solution is unique, follows from the standard procedure, and does not need to be elaborated here.

Now consider the case $r > r_0$. This time we have

$$\varphi = \varphi(r_0) \cos k(r - r_0) + \frac{1}{k} \sin k(r - r_0)\varphi'(r_0) - \mathrm{W}(r_0)\varphi(r_0)$$
$$\times \frac{\sin k(r - r_0)}{k} + \int_{r_0}^{r} \mathrm{W}(r') \left[\cos k(r - r')\varphi - \frac{\sin k(r - r')}{k} \varphi' \right] dr'$$

and

$$\varphi' = -k\varphi(r_0) \sin k(r - r_0) + \cos k(r - r_0)\varphi'(r_0) - \mathrm{W}(r_0)\varphi(r_0) \cos k(r - r_0)$$
$$+ \mathrm{W}(r)\varphi(r) - \int_{r_0}^{r} \mathrm{W}(r')[k \sin k(r - r')\varphi + \cos k(r - r')\varphi'] dr'$$

Using now the second bound in (15), and the previous bounds obtained for φ and φ', at $r = r_0$, $\mathrm{M}(k, r_0)$ and $\mathrm{N}(k, r_0)$, we find that the inhomogeneous terms in the above integral equations are bounded, respectively, by

$$\mathrm{C}r_0 e^{|\mathrm{Im}\, k|r} e^{\int_0^{r_0} (2\mathrm{C}|\mathrm{W}| + \mathrm{C}^2 t \mathrm{W}^2) dt}$$

and

$$\mathrm{C} e^{|\mathrm{Im}\, k|r} e^{\int_0^{r_0} (2\mathrm{C}|\mathrm{W}| + \mathrm{C}^2 t \mathrm{W}^2) dt}$$

Proceeding now exactly as before to convert our integral equations into integral and then differential inequalities, and solving them, we find

$$|\varphi| < \tilde{\mathrm{M}}(k, r) = \mathrm{M}(k, r_0) e^{\int_{r_0}^{r} \left(2\mathrm{C}|\mathrm{W}| + \frac{\mathrm{C}^2\mathrm{W}^2}{|k|}\right) dt} e^{|\mathrm{Im}\, k|(r - r_0)}$$

Using now that fact that $|k| = r_0^{-1} > t^{-1} > r^{-1}$, we see that the integral in the above expression can be converted into an integral identical to that which figures in $\mathrm{M}(k, r)$. Taking now into account this fact, and using the same remark to go from (15) to (14), we finally get, for all r,

$$|\varphi| < \frac{\mathrm{C}r}{1 + |k|r} e^{|\mathrm{Im}\, k|r} \tag{18a}$$

where the factor

$$\int_0^1 (2\mathrm{C}|\mathrm{W}| + r\mathrm{W}^2\mathrm{C}^2) dr$$

has been absorbed into C. Similarly

$$|\varphi'| < Ce^{|\mathrm{Im}\,k|r} \tag{18b}$$

These bounds are valid for all $r \geq 0$ and finite k. Notice that they are identical to the bounds for regular potentials [3] [4]. Moreover, if we now add a regular tail to our potential, the bounds will keep the same form.

From the above results, we obtain, exactly as for regular potentials [3] [4], that φ for each $r \geq 0$, is an entire function of k of order 1 and type r, and has the asymptotic property (3)

$$\varphi \underset{|k| \to \infty}{=} \frac{\sin kr}{k} + \frac{e^{|\mathrm{Im}\,k|r}}{k} o(1) \tag{19}$$

Since we are dealing with short range potentials, we can also introduce the Jost solution $f(k, r)$ which satisfies

$$\lim_{r \to \infty} e^{-ikr} f(k, r) = 1 \tag{20a}$$

$$\lim_{r \to \infty} | f' - ike^{ikr} | = 0 \tag{20b}$$

It exists for all $r > 0$, and is analytic in $\mathrm{Im}\, k > 0$, continuous in $\mathrm{Im}\, k \geq 0$ because of (1) [3] [4]. Similarly, we can define the Jost function through the Wronskian

$$\mathrm{F}(k) = \mathrm{W}[f, \varphi] \tag{21}$$

which, because of (8b), is given also by

$$\mathrm{F}(k) = f(k, 0) \tag{22}$$

and has the integral representation

$$\mathrm{F}(k) = 1 + \int_0^\infty e^{ikr}\mathrm{V}(r)\varphi(k, r)dr = 1 - \int_0^\infty e^{ikr}\mathrm{W}(r)[ik\varphi + \varphi']dr \tag{23}$$

where (4)

$$\mathrm{W}(r) = - \int_r^\infty \mathrm{V}(t)dt \tag{24}$$

Of course we assume now that W is L_1 on the whole axis $r \geq 0$. The Jost function is analytic in $\mathrm{Im}\, k > 0$, continuous in $\mathrm{Im}\, k \geq 0$, and has, because of (18a, b), the asymptotic form ($\mathrm{Im}\, k \geq 0$).

$$\mathrm{F}(k) \underset{|k| \to \infty}{=} 1 + o(1) \tag{25}$$

The wave-function φ, being an even function of k, can now be written, for real values of k,

$$\varphi(k, r) = \frac{1}{2ik}[\mathrm{F}(-k)f(k, r) - \mathrm{F}(k)f(-k, r)] \tag{26}$$

(3) Just use (18a, b), (14) and (16) in (13a).

(4) Since $\varphi = 0(r)$ near the origin, and because of (6), there is really no need to intergrate by parts the integral in (23).

Because of

$$\varphi(-k, r) = \varphi(k, r) = [\varphi(k^*, r)]^* \tag{27}$$

and

$$f(-k^*, r) = [f(k, r)]^*, \qquad \text{Im } k \geq 0 \tag{28}$$

We also have

$$F(-k^*) = [F(k)]^*, \qquad \text{Im } k \geq 0 \tag{29}$$

Using now (20*a*) in (26), we get for k real, the asymptotic form

$$\varphi(k, r) \underset{r \to \infty}{=} \frac{|F(k)|}{k} \sin(kr + \delta) + o(1) \tag{30}$$

where the phase-shift δ is defined by

$$\delta(k) = -\operatorname{Arg} F(k) \tag{31}$$

Continuing our analysis we find, exactly as for regular potentials, and by absolutely identical methods, that the Jost function F does not vanish on the real axis, except perhaps at $k = 0$ (eventual « bound » state at zero energy), that its zeros in Im $k > 0$ are situated on the imaginary axis, are simple, and finite in number. There is a one to one correspondance between these zeros and bound states. We know also, as was said before, that there are no bound states with positive energies. We see therefore that everything is quite identical to the regular case.

Also notice that, because of (25),

$$\lim_{k \to +\infty} \delta(k) = 0 \tag{32}$$

If the potential is regular and non-oscillating near the origin, it is known that for large values of k, the phase-shift keeps a constant sign, opposite to that of the potential. For oscillating potentials, however, the phase-shift, while going to zero as before, oscillates indefinitely with a period related to that of the potential. We shall see examples of this phenomenon in the next section.

The Levinson theorem [*3*] [*4*], which relates the variation of the phase of the Jost function along the real axis to the number of its zeros inside the holomorphy domain Im $k > 0$ is again obviously valid in our case, and reads

$$\delta(0) - \delta(\infty) = n\pi \tag{33}$$

where n is the number of bound states. In case there is an almost bound state at E = 0, one has of course to add $\pi/2$ to the right hand side of (33).

Having shown that everything concerning the Jost function is exactly the same as for regular potentials, it is obvious that there is no difficulty to carry out the eigenfunction expansion à la Titchmarsh, as it has been done by Jost and Kohn and others [*3*] [*4*]. The completeness of the set $\{ \varphi(E, r) \}$ when E runs over the whole spectrum can be written then, symbolically

$$\int \varphi(E, r)\varphi(E, r')d\rho(E) = \delta(r - r') \tag{34a}$$

$$\frac{d\rho}{dE} = \begin{cases} \frac{1}{\pi} | F(k) |^{-2} \sqrt{E} & E \geq 0 \\ \sum_{j=1}^{n} C_j \delta(E - E_j) & E < 0 \end{cases} \tag{34b}$$

where $\{ E_j \}$ are the energies of the bound states, and C_j their normalization constants

$$C_j^{-1} = \int_0^\infty \varphi_j^2(r)dr \tag{34c}$$

According to the general theory of differential equations, for each value of k, the roots of φ on the r-axis are isolated. Let r_1 (> 0) be the closest root to the origin. A second and linearly independent solution of (8a) is given by

$$\Psi = \varphi \int_a^r \varphi^{-2}(t)dt , \qquad 0 < a < r_1 \tag{35}$$

Because of (8b), it is obvious that $\Psi(0) = -1$. Since the Wronskian of the two solutions is

$$W[\varphi, \Psi] = 1 \tag{36}$$

it follows that

$$\lim_{r \to 0} r\Psi'(r) = 0 \tag{37}$$

All solutions, in particular the Jost solution f, being a combination of these two solutions, it is obvious that we have also, exactly as in the regular case,

$$\lim_{r \to 0} rf'(k, r) = 0 \tag{38}$$

This justifies the definition (22) of the Jost function.

At any rate, the origin is a limit-circle case since all solutions are L_2. There is no difficulty in defining the free and total Hamiltonian

$$H_0 = -\frac{d^2}{dr^2} , \tag{39a}$$

$$H = -\frac{d^2}{dr^2} + V(r) \tag{39b}$$

as self-adjoint operators in the Hilbert space L_2 $(0, \infty)$, exactly as for a regular potential, by adding the boundary condition

$$\varphi(0) = 0 \tag{39c}$$

The existence of the Mϕller wave operators as strong limits follows from a recent powerful theorem of Amrein and Georgescu [7], which is as follows. Let V belong to L_1 ([a, ∞)) for each $a > 0$. Let H be the self-adjoint operator defined by the differential expression (39b) and the boundary condition (39c). Suppose that the essential spectrum of one of the self-adjoint operators defined by the differential expression $-d^2/dr^2 + V$ in L_2 (0, 1) is empty. Then

a) The spectrum of H is simple.

b) The essential spectrum of H is $[0, \infty)$.

c) H as no singularly continuous spectrum.

d) H has no positive eigenvalues.

e) Furthermore, let V and $\tilde{V}$ be such that each of them verifies the hypotheses given above. Let H be a self-adjoint extension of $-d^2/dr^2 + V$, and $\tilde{H}$ a self-adjoint extension of $-d^2/dr^2 + \tilde{V}$. Then the wave operators

$$\Omega_{\pm} = s. \lim_{t \to \pm\infty} e^{iHt} e^{-i\tilde{H}t} E_{a.c.}(\tilde{H}) \tag{40}$$

exist and are asymptotically complete

$$\Omega_{\pm}\Omega_{\pm}^{*} = E_{a.c.}(H) \tag{41}$$

and it suffices to set $\tilde{V} = l(l+1)/r^2$ to obtain the existence and the asymptotic completeness of the wave operators for each partial-wave.

We know already the truth of the statements *a*)-*d*) through our study of the Jost function and its properties in the physical sheet of the E-plane, Im $k \geq 0$ ([5]). In order to verify the asymptotic completeness *e*), we only have to verify that the essential spectrum of $-d^2/dr^2 + V$ in L_2 (0, 1) is empty. To see this, let us consider the self-adjoint operator defined by the differential operator and the boundary conditions (39c) and

$$\varphi(1) \cos \alpha - \sin \alpha \varphi'(1) = 0\,, \qquad 0 \leq \alpha \leq \pi \tag{42}$$

As we saw before, the estimates (18a, b) show that the integrals in (13a) and (13b) are absolutely convergent for all finite values of r and k. The convergence is also obviously uniform in any compact of the k-plane. Therefore, φ and φ' are entire function of k. Therefore, the l.h.s. of (42) is an entire function of k, and its root are separated and form a denumerable set whose only accumulating point is infinity. Therefore, the self-adjoint operator defined in L_2 (0, 1) by the differential expression and the boundary condition only has a point spectrum. Therefore, we can directly apply the Amrein-Georgescu theorem, and the conclusions *a*)-*e*) hold.

The above analysis can be carried out without any difficulty for higher partial waves. The starting point is the analogue of the equation (12) in which sine and cosine are replaced by appropriate combinations of Bessel and Hankel functions. Integration by parts again leads to equations

([5]) Remember that we define now W by (24), and assume that $W \in L_1(0, \infty)$, together with (7b).

analogous to (13*a* et *b*), which are then solved by iteration as for regular potentials, exactly as we did above for $l = 0$. Using the well-known estimates and bounds for Bessel and Hankel functions [*3*] [*4*], one is able to show, under our assumptions on the potential, that everything is exactly as for regular potentials. In particular, that the Jost function, holomorphic in $\mathrm{Im}\, k > 0$ and continuous in $\mathrm{Im}\, k \geq 0$, contains all the necessary information about the spectrum and the asymptotic form of the wave-function for large r, that the connection between its roots and the bound states is normal, that these are finite in number and bounded below, and, finally, that one has an absolutely continuous spectrum on $[0, \infty)$, which is non-degenerate and leads to a completeness relation identical to (34*a*, *b* et *c*).

It is also straightforward to show that the hypotheses of the Amrein-Georgescu are verified, and that the Møller wave operators exist as strong limits and are complete.

As a final remark in this section, let us notice that to reach the above conclusions, there is really no need to assume $W \in L_1\ (0, \infty)$. The hypotheses of the Amrein-Georgescu theorem are satisfied if $V \in L_1\ ([a, \infty))$ and $W \in L_1\ (0, 1)$. The only property which is no longer true in general if we abandon (1) is the continuity of the Jost function at $k = 0$, and the occurrence of a true bound state at $E = 0$ [*8*].

3. INVERSE SCATTERING PROBLEM

In this section, we shall briefly discuss the Gel'fand-Levitan and Marchenko theories [*9*] [*10*] [*11*], assuming (1) and $W \in L_1\ (0, 1)$. Since the regular solution $\varphi(k, r)$ is an entire function of k of order 1 and type r, and the asymptotic estimate (19) holds, it is obvious that it satisfies the Gel'fand-Levitan representation

$$\varphi = \frac{\sin kr}{k} + \int_0^r K(r, r') \frac{\sin kr'}{k} dr' \tag{43}$$

Combining this with the completeness relation (34*a*), we are led, in a staightforward manner, exactly as for regular potentials, to the Gel'fand-Levitan integral equation

$$K(r, r') + G(r, r') + \int_0^r K(r, r'')G(r'', r')dr'' = 0 \tag{44a}$$

where

$$G(r, r') = \int_{-\infty}^{\infty} [d\rho(E) - d\rho_0(E)] \frac{\sin \sqrt{E}r}{\sqrt{E}} \frac{\sin \sqrt{E}r'}{\sqrt{E}} \tag{44b}$$

and

$$d\rho_0(E) = \begin{cases} \dfrac{1}{\pi} \sqrt{E} dE & E \geq 0 \\ 0 & E < 0 \end{cases} \tag{44c}$$

Therefore the same conclusions are reached concerning the existence and uniqueness of the solution of (44*a*), and we obtain the potential from the scattering data $d\rho(\mathrm{E})$, via (44*a*), by

$$\mathrm{V}(r) = 2\frac{d}{dr}\mathrm{K}(r, r) \tag{45}$$

Comparing this with (24), we see that $\mathrm{W}(r)$ and $\mathrm{K}(r, r)$ are identical up to an additive constant.

The Marchenko formalism is also identical to that of the regular case. The S-matrix

$$\mathrm{S}(k) = \mathrm{F}(-k)/\mathrm{F}(k) \tag{46}$$

having properties identical to those of the regular case, there is no difficulty for proving the Marchenko representation of the Jost solution ($r \geq 0$, $\operatorname{Im} k \geq 0$)

$$f(k, r) = e^{ikr} + \int_r^\infty \tilde{\mathrm{K}}(r, t)e^{ikt}dt \tag{47}$$

and the Marchenko integral equation ($t \geq r$)

$$\tilde{\mathrm{K}}(r, t) = \mathrm{F}(r + t) + \int_r^\infty \tilde{\mathrm{K}}(r, s)\mathrm{F}(s + t)ds \tag{48a}$$

where

$$\mathrm{F}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [1 - \mathrm{S}(k)]e^{ikt}dt + \sum_{j=1}^{n} \tilde{\mathrm{C}}_j\delta(\mathrm{E} - \mathrm{E}_j), \qquad t > 0 \tag{48b}$$

The potential is again given by

$$\mathrm{V}(r) = 2\frac{d}{dr}\tilde{\mathrm{K}}(r, r') \tag{49}$$

and we see that now, according to (24),

$$\mathrm{W}(r) \equiv 2\tilde{\mathrm{K}}(r, r) \tag{50}$$

As an example, we can consider the simple case where there are no bound states and the phase-shift is given by

$$\delta(k) = -\int_0^\infty \gamma(t) \sin ktdt \tag{51a}$$

$$\gamma(t) = \frac{\sin (a/t)}{\sqrt{t}} \tag{51b}$$

This leads to

$$\delta(k) = \frac{-1}{2}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}} (\sin 2\sqrt{ak} - \cos 2\sqrt{ak} + e^{-2\sqrt{ak}}) \tag{52}$$

which shos that, in the limit $k \to \infty$, the phase-shift goes to zero while oscillating indefinitely. On the other hand, it is easily seen that [12]

$$
\begin{aligned}
|\mathrm{F}(k)|^{-2} &= \exp\left[-2\int_0^\infty \gamma(t)\cos ktdt\right] \\
&= \exp\left[-\sqrt{\frac{\pi}{2k}}\,(\sin 2\sqrt{ak} + \cos 2\sqrt{ak} - e^{-2\sqrt{ak}})\right] \qquad (53)
\end{aligned}
$$

It then follows from the Gel'fand-Levitan equation that the potential is given by (6)

$$
\mathrm{V}(r) \underset{r \to 0}{\cong} \frac{d}{dr}\frac{\sin(a/2r)}{\sqrt{r}} + 0\left(\frac{1}{r}\right) \qquad (54)
$$

This shows that the most singular part of the potential behaves like $r^{-5/2}$ near the origin. The Jost function itself is given by

$$
\mathrm{F}(k) = |\mathrm{F}(k)|\,e^{-i\delta} = \exp\left[\frac{1}{2}\sqrt{\frac{\pi}{2k}}(1 - i)(e^{2i\sqrt{ak}} - e^{-2\sqrt{ak}})\right] \qquad (55)
$$

It is holomorphic in Im $k > 0$, continuous in Im $k \geq 0$, and satisfies (25). Another example is given by [13]

$$
d\rho(\mathrm{E}) = \frac{1}{\pi}\sqrt{\mathrm{E}}d\mathrm{E}\int_0^\infty dt\mathrm{H}(t)\cos kt \qquad (56a)
$$

where we choose again

$$
\mathrm{H}(t) = \frac{\sin(a/t)}{\sqrt{t}} \qquad (56b)
$$

Again, it can be seen that the potential near the origin is given by (6)

$$
\mathrm{V}(r) \underset{r \to 0}{\cong} \frac{d}{dr}\frac{\sin(a/2r)}{\sqrt{r}} + 0\left(\frac{1}{r}\right) \qquad (57)
$$

whereas the phase-shift for $k \to \infty$ oscillates and goes to zero like $k^{-\frac{1}{2}}$. Notice that in the above examples, the potential has a long tail: $\mathrm{V}(r) \sim r^{-5/2}$ as $r \to \infty$. This is the reason why $k = 0$ is a branch point for the Jost function. If one multiplies the r.h.s. of (51b) or (56b) by $e^{-\mu t}$, then the branch point is shifted to $k = -i\mu$, and $\mathrm{V} \sim e^{-2\mu}r$ as $r \to \infty$.

4. CONCLUDING REMARKS

From the analysis of section 2, it is clear that, in order to keep the usual conventional theory without any modification when dealing with potentials which are not necessarily repulsive near the origin, it is sufficient to

(6) Details and other examples will be given in a separate paper, to appear in *Nuclear Physics A*.

require only the absolute integrability of W at $r = 0$, which is a much weaker condition than the absolute integrability of rV.

Many of these potentials may be generated by superpositions of Yukawa potentials. For instance, we have

$$W = r^{-\frac{1}{2}} \frac{\sin}{\cos} (a/r) = \int_0^\infty C(\mu) e^{-\mu r} d\mu \tag{58a}$$

$$C(\mu) = (\pi\mu)^{-\frac{1}{2}} \frac{\sinh}{\cosh} (2a\mu)^{\frac{1}{2}} \frac{\sin}{\cos} (2a\mu)^{\frac{1}{2}} \tag{58b}$$

If one wishes a potential decreasing exponentially when r becomes very large, it is sufficient to integrate the above expressions only from m to ∞, $m > 0$. It is obvious that this does not change the singularity at the origin.

For such potentials, the Jost function is holomorphic in the entire k-plane cut from $-im/2$ to $-i\infty$, and (25) holds in this cut plane. Therefore, one would be able to write dispersion relations a la Martin for the Jost function, or dispersion relations and N/D equations for the partial-wave amplitudes [*4*]. These in turn can be used to solve inverse scattering problems by methos identical to those developped by Martin and others. We shall give more details on these in a separate paper.

There is also no difficulty in combining singularities considered in this paper with similar singularities at infinity considered in [*5*] [*6*], and have more general potential. Our analysis shows that, in order to have the double degeneracy of the continuous spectrum similar to that in the Pearson example, it is necessary that V, while oscillating violently near the origin, should be such that rV is not integrable there.

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REFERENCES

[*1*] D. B. PEARSON, *Helv. Phys. Acta*, t. **47**, 1974, p. 249.

[*2*] D. B. PEARSON, *Commun. math. Phys.*, t. **40**, 1975, p. 125. This paper contains a complete list of references to earler work.

[*3*] R. G. NEWTON, *Scattering theory of waves and particles*, New York, McGraw-Hill, 1966, chap. 12.

[*4*] V. De ALFARO and T. REGGE, *Potential Scattering*, Amsterdam, North-Holland, 1965, chap. 3-5. Notice the difference of notations ($k \to -k$) with us.

[*5*] V. B. MATVEEV, English translation. *Theoreticheskaya i Matematicheskaya Fizika*, t. **15**, 1974, p. 574.

[6] M. M. SKRIGANOV, *Tr. MIAN im V. A. Steklova*, t. **125**, 1973, p. 187.
[7] W. O. AMREIN and V. GEORGESCU, *Helv. Phys. Acta*, t. **47**, 1974, p. 517. Theorem 2, remark at the end of the main text, theorem 5 in the Appendix, and the remark after this theorem.
[8] V. BARGMANN, *Rev. Mod. Phys.*, t. **21**, 1949, p. 488 ; H. E. MOSES and S. F. TUAN, *Nuovo Cimento*, t. **13**, 1959, p. 197.
[9] Ref. [3], chap. 20, and references cited therein.
[10] Ref. [4], chap. 12.
[11] N. LEVINSON, *Phys. Rev.*, t. **89**, 1953, p. 755.
[12] Ref. [10], formulæ (12.33) and (12.34).
[13] Ref. [10], formula (12.32).

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