ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 24, nº 1 (1976), p. 17-30
<http://www.numdam.org/item?id=AIHPA_1976__24_1_17_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXIV, n° 1, 1976, p. 17-29. *Physique théorique.*

Spectral and scattering theory for the Schrödinger operator with strongly oscillating potentials

by

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ABSTRACT. — We consider the Schrödinger operator in $L^2(\mathbb{R}^n)$ with potential $V = \nabla . W$ where W is a real $\mathbb{R}^n$ valued function such that (1) the local singularities of W^2 are controlled in a suitable sense by the kinetic energy, (2) W tends to zero at infinity faster than r^{-1}. We define the Hamiltonian by a method of quadratic forms and derive the usual results of scattering theory: the negative spectrum is discrete and finite, the absolutely continuous spectrum is $[0, \infty)$, the continuous singular spectrum is empty, the wave operators exist and are asymptotically complete.

1. INTRODUCTION

The spectral and scattering theory of the Schrödinger operator $H = H_0 + V$, where $H_0 = -\Delta$ is the Laplacian in $\mathbb{R}^n$ and V a real potential, has reached a very satisfactory state for a large class of potentials (See for instance [*14*] or the lectures by Amrein in [*4*] and the references therein quoted). Under suitable and general assumptions on V, one can prove some or all of the following properties:

(1) H is defined as a self-adjoint operator in $\mathscr{H} = L^2(\mathbb{R}^n)$ with a reasonable degree of uniqueness.

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(2) The essential spectrum $\sigma_e(\mathrm{H})$ of H is the positive real axis.
(3) The (negative) discrete spectrum of H is finite.
(4) The positive point spectrum $\sigma_p(\mathrm{H})$ is empty.
(5) The continuous singular spectrum $\sigma_{cs}(\mathrm{H})$ is empty.
(6) The wave operators $\Omega_\pm$ exist as strong limits:

$$\Omega_\pm = \mathrm{s.}\lim_{t\to\mp\infty} \exp(it\mathrm{H}) \exp(-it\mathrm{H}_0) \tag{1.1}$$

(7) The wave operators are asymptotically complete, i. e.

$$\mathcal{R}(\Omega_+) = \mathcal{R}(\Omega_-) = \mathcal{H}_{ac} \tag{1.2}$$

where $\mathcal{H}_{ac}$ is the subspace of absolute continuity of H.

In a large number of works on the subject, the results are derived under assumptions on V that involve only its absolute value $|\mathrm{V}|$. Typically, (1) follows from some local regularity condition, (2) from the condition that $|\mathrm{V}|$ tends to zero when $r = |x|$ tends to infinity, (3) from the condition that $|\mathrm{V}|$ tends to zero at infinity faster than r^{-2}, and (4) to (7) from the condition that $|\mathrm{V}|$ tends to zero at infinity faster than r^{-1}. Some of these results have been extended to other classes of potentials, including long range but non oscillating potentials [*2*] [*14*] [*8*], very singular but predominantly repulsive potentials [*11*] [*20*] [*23*] or very singular attractive potentials [*3*] [*16*].

It is only recently however that the same problems were considered for potentials that may have very large and possibly very singular positive and negative parts, in particular that may oscillate wildly, but for which important cancellations occur between positive and negative parts [*5*] [*15*] [*17*] [*24*]. The main point that emerges from these investigations is that the results listed under (1) to (7) carry over to potentials V that are in some sense the derivatives of some function W such that W satisfies conditions similar to those imposed on $|\mathrm{V}|$ in previous investigations. In [*24*], Skriganov considers the n-dimensional case with potentials that are locally regular, but may have large oscillations at infinity. For these potentials, he derives the results (1) to (7) by appealing either to results on partial differential equations or drawing upon the classical work of Ikebe [*7*]. In [*5*], Baeteman and Chadan derive the same results for radial potentials with W locally integrable and decreasing faster than r^{-1} at infinity.

In the present paper, we extend most of the results of [*5*] and [*24*] to the n-dimensional case with potentials V such that

(*a*) W may be locally singular, its singularities being controlled in a suitable sense by the kinetic energy.

(*b*) W decreases faster than r^{-1} at infinity in a suitable sense. We then derive properties (1) to (7) (except for (4)) by standard Hilbert space methods. In particular, we use the method of Birman [*6*] and Schwinger [*21*] to prove (3), the method of Agmon [*1*] based on *a priori* estimates in weighted Hilbert spaces to prove (5), and the method of Kato [*10*] and Lavine [*13*] using smooth operators to prove (6) and (7).

The paper is organized as follows. In section (2) we introduce various definitions and notations, define the Hamiltonian by a method of quadratic forms, and show that $\sigma_e(\mathrm{H}) \subset [0, \infty)$. In section (3) we prove the finiteness of the discrete spectrum. In section (4) we derive properties (5), (6) and (7). The assumptions on W are slightly different from section to section and will be stated when needed.

2. THE HAMILTONIAN AND ITS ESSENTIAL SPECTRUM

We consider the operator $\mathrm{H} = \mathrm{H}_0 + \mathrm{V}$ in $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n)$, where $\mathrm{H}_0 = -\Delta$ and V is a real potential to be specified below. We shall need the space $\overline{\mathcal{H}} = \mathrm{L}^2(\mathbb{R}^n) \otimes \mathbb{C}^n$ of square integrable functions from $\mathbb{R}^n$ to $\mathbb{C}^n$. We shall denote by $\|\ .\ \|$ and $\langle\ .\ ,\ .\ \rangle$ the norm and scalar product both in $\mathcal{H}$ and $\overline{\mathcal{H}}$. Which one occurs will be clear from the context. In all that follows, we denote by W a real measurable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Multiplication by W defines an operator from $\mathcal{H}$ to $\overline{\mathcal{H}}$, also denoted by W. Similarly, we denote by ∇ the operator $\varphi \to (\partial_j \varphi)$ from $\mathcal{H}$ to $\overline{\mathcal{H}}$. Multiplication by the real measurable function $\mathrm{W}^2 = \sum_{j=1}^{n} \mathrm{W}_j^2$ defines a self-adjoint positive operator in $\mathcal{H}$, also denoted by W^2. We recall that there is a one-to-one correspondence between positive self-adjoint operators and closed positive quadratic forms, such that the domain of the closed form associated with the operator A is $\mathcal{D}(\sqrt{\mathrm{A}})$ ([*9*], chap. VI). This domain will be called the form domain of A and will be denoted Q(A). We shall in general use the same notation for the operator and the associated quadratic form.

We want to define the potential V formally by $\mathrm{V} = \nabla . \mathrm{W} = \sum_{j=1}^{n} \partial_j \mathrm{W}_j$. We need the following lemma.

LEMMA 2.1. — Suppose that:

$$4\mathrm{W}^2 \leq a^2(\mathrm{H}_0 + 2b) \tag{2.1}$$

as quadratic forms on $\mathrm{Q}(\mathrm{H}_0)$, with $b \geq 0$ and $a \geq 0$. Then the equation

$$\langle \varphi, \mathrm{V}\psi \rangle = - \langle \nabla\varphi, \mathrm{W}\psi \rangle - \langle \mathrm{W}\varphi, \nabla\psi \rangle \tag{2.2}$$

defines a quadratic form V with domain $\mathrm{Q}(\mathrm{H}_0)$, and this form satisfies:

$$\pm \mathrm{V} \leq a(\mathrm{H}_0 + b) \tag{2.3}$$

Proof. — The first statement is obvious. The inequality (2.3) follows from the estimate

$$\begin{aligned} |\langle \varphi, V\varphi \rangle| = 2\,|\,\mathrm{Re}\,\langle \nabla\varphi, W\varphi \rangle| &\le 2\,\|\nabla\varphi\|\ \ \|W\varphi\| \\ &\le a\{\langle \varphi, H_0\varphi \rangle \langle \varphi, (H_0 + 2b)\varphi \rangle\}^{1/2} \\ &\le a\langle \varphi, (H_0 + b)\varphi \rangle \end{aligned}$$

We assume from now on that W satisfies (2.1) for some $a < 1$. It follows from (2.3) that $H = H_0 + V$ can be defined by the sum of the quadratic forms by a well-known perturbation method. (See [*22*], theorem II.7, p. 41). In particular, H is self-adjoint semi-bounded, and $Q(H) = Q(H_0)$.

REMARK 2.1. — One can consider that the potential V is defined by $V = \nabla . W$ in the sense of distributions. The previous method can therefore accommodate potentials that are not functions. For instance for $n = 1$, with $W(x) = x/2\,|\,x\,|$, one obtains $V = \delta(x)$.

On the other hand, the present situation covers cases of highly singular oscillating potentials, both locally and at infinity. We give two examples that satisfy all the assumptions made in the rest of the paper (Here ε is any strictly positive number).

$$\begin{cases} W = x(1 + r^2)^{-(1+\varepsilon)} \sin(e^r) \\ V = r(1 + r^2)^{-(1+\varepsilon)} e^r \cos e^r + O((1 + r^2)^{-(1+\varepsilon)}) \end{cases}$$

$$\begin{cases} W = x(1 + r^2)^{-(1+\varepsilon)} \sin(e^{1/r}) \\ V = -\,r^{-1}(1 + r^2)^{-(1+\varepsilon)} e^{1/r} \cos(e^{1/r}) + O((1 + r^2)^{-(1+\varepsilon)}) . \end{cases}$$

REMARK 2.2. — One can easily give sufficient conditions on W to ensure that (2.1) holds with $a < 1$. For instance it is sufficient that $W \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with uniform bound:

$$\int_{|x-y|\le 1} (W^2(y))^{p/2} d^n y \equiv m(x) \le M \tag{2.4}$$

with M not depending on x and with $p = 2$ for $n = 1$, $p > 2$ for $n = 2$ and $p = n$ for $n \ge 3$. Since for any n the condition $W \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ with uniform bound is necessary, the previous condition is necessary and sufficient for $n = 1$.

For $n = 3$, the condition can be weakened to a local Rollnik condition (see [*22*]) with uniform bound:

$$\int_{\substack{|x_1 - x| \le 1 \\ |x_2 - x| \le 1}} W^2(x_1) W^2(x_2) \,|\, x_1 - x_2 \,|^{-2} d^3x_1 d^3x_2 \equiv m'(x) \le M' \tag{2.5}$$

with M' independent of x.

For future use we also define

$$H_1 = -(\nabla + W).(\nabla - W) \tag{2.6}$$

Clearly under the assumption (2.1), H_1 is defined as a quadratic form with domain $Q(H_0)$ and is positive. Furthermore

$$H = H_1 - W^2 \tag{2.7}$$

The form H_1 satisfies the following estimates.

LEMMA 2.2. — Let W satisfy (2.1) with $a < 1$. Then, as quadratic forms on $Q(H_0)$, H_0, H_1 and W^2 satisfy:

$$W^2 \leq \left(\frac{a}{2-a}\right)^2 [H_1 + (2-a)b] \tag{2.8}$$

$$(2-a)^2 H_0 - (2-a)2ab \leq 4H_1 \leq (2+a)^2 H_0 + (2+a)2ab \tag{2.9}$$

Proof. — Let $\alpha > 0$. It follows from

$$-(\nabla \pm 2\alpha W)(\nabla \mp 2\alpha W) \geq 0$$

that

$$\pm V \leq 2\alpha W^2 + (2\alpha)^{-1} H_0 .$$

Therefore

$$H_1 = H_0 + V + W^2 \lesseqgtr H_0(1 \pm (2\alpha)^{-1}) + W^2(1 \pm 2\alpha) .$$

From this and from (2.1), it follows by elimination of W^2 that for $2\alpha > 1$:

$$H_1 \lesseqgtr H_0\left(1 + \frac{a^2}{4} \pm \left(\frac{1}{2\alpha} + \frac{\alpha a^2}{2}\right)\right) + (1 \pm 2\alpha)\frac{a^2 b}{2} .$$

Taking for α the optimal value $\alpha = a^{-1}$ yields (2.9).

The inequality (2.8) is proved similarly by eliminating H_0.

It follows from lemma 2.2 that the quadratic form H_1 with domain $Q(H_0)$ is closed, and therefore defines a positive self-adjoint operator with form domain $Q(H_1) = Q(H_0)$. Furthermore, because of (2.8), H can equivalently be defined as the sum of quadratic forms (2.7) by the same perturbation argument that was used to define it as $H_0 + V$.

We now turn to the study of the essential spectrum of H. We introduce the resolvent operators $R_i(\lambda) = (H_i - \lambda)^{-1}$, where i stands for 0, 1 or nothing. In terms of these operators, condition (2.1) with $a < 1$ states that the operator $WR_0(\lambda)W$ acting in $\mathscr{H}$ is bounded with norm less than 1/4 for λ real negative sufficiently large. In order to obtain interesting results on the essential spectrum of H, we shall assume in addition that this operator is compact for some $\lambda < 0$, or equivalently for all λ in the resolvent set $\mathbb{C}\backslash[0, \infty)$ of H_0. One sees easily that this assumption implies that $\| WR_0(\lambda)W \|$ tends to zero when $\mathrm{Re}\,\lambda \to -\infty$, and therefore implies (2.1), where moreover a can be taken arbitrarily small by taking b sufficiently large.

PROPOSITION 2.1. — Let W be such that $WR_0(\lambda)W$ is compact for some $\lambda < 0$. Then $\sigma_e(H) \subset [0, \infty)$.

Proof. — The proof uses standard methods and will only be sketched briefly. One first considers the operator $WR_1(\lambda)W$.

LEMMA 2.3. — Let W satisfy the assumption of proposition 2.1. Then the operator $WR_1(\lambda)W$ is bounded, compact and analytic in λ for $\lambda \in \mathbb{C}\backslash[0, \infty)$. Moreover $\| WR_1(\lambda)W \| \to 0$ when $\operatorname{Re} \lambda \to -\infty$.

Proof of lemma 2.3. — As in the case of the operator $WR_0(\lambda)W$, the second statement follows from the first and it suffices to establish compactness for some $\lambda < 0$. Now from lemma 2.2, one obtains

$$H_1 \geq c(H_0 - d)$$

for some $c > 0$, $d \geq 0$. Therefore

$$0 \leq WR_1(\lambda)W \leq c^{-1}WR_0(\lambda/c + d)W \tag{2.10}$$

for $\lambda < -cd$. The last member of (2.10) is compact by assumption, and therefore the second member is compact by lemma 2.4 below.

LEMMA 2.4. — Let A and B be self-adjoint operators with $0 \leq A \leq B$ and B compact. Then A is compact.

Proof of lemma 2.4. — Let P be a finite rank spectral projector of B satisfying $BP = PB$ and $\| (1 - P)B \| \leq \varepsilon$.

Then

$$\| (1 - P)A(1 - P) \| \leq \| (1 - P)B(1 - P) \| \leq \varepsilon$$

Furthermore, for all φ and ψ:

$$\begin{aligned} | \langle (1 - P)\varphi, AP\psi \rangle |^2 &\leq \langle (1 - P)\varphi, A(1 - P)\varphi \rangle \langle P\psi, AP\psi \rangle \\ &\leq \| A \| \quad \| \psi \|^2 \| (1 - P)A(1 - P) \| \quad \| \varphi \|^2 \end{aligned}$$

so that

$$\| (1 - P)AP \|^2 \leq \varepsilon \| A \| .$$

Therefore

$$\| A(1 - P) \|^2 \leq \varepsilon^2 + \varepsilon \| A \| .$$

Therefore A is the norm limit of finite rank operators, and is therefore compact

End of the proof of proposition 2.1.

From the fact that $\| WR_1(\lambda)W \| \to 0$ for $\operatorname{Re} \lambda \to -\infty$, and from elementary algebra, it follows that for $\operatorname{Re} \lambda$ negative and sufficiently large

$$R(\lambda) = R_1(\lambda) + R_1(\lambda)W[1 - WR_1(\lambda)W]^{-1}WR_1(\lambda) \tag{2.11}$$

where the operator $(1 - WR_1(\lambda)W)^{-1}$ is defined by a (norm convergent) power series, and where both members are analytic in λ. Since $WR_1(\lambda)W$ is compact and analytic in λ for all $\lambda \in \mathbb{C}\backslash[0, \infty)$, it follows from the analytic Fredholm theorem (see [*18*], p. 201) that the RHS of (2.11) is meromorphic there with compact residues. From this it follows that $\sigma_e(H) \subset [0, \infty)$.

REMARK 2.3. — One can easily give sufficient conditions on W that ensure compactness of $WR_0(\lambda)W$ for $\lambda < 0$. For instance it is sufficient that $W \in L^p_{loc}(\mathbb{R}^n)$ for the same p (depending on n) as in remark 2.2, and that W satisfies the estimate (2.4) where in addition $m(x)$ tends to zero when $r \to \infty$. For $n = 3$, this condition can be weakened to a local Rollnik condition with uniform estimate (2.5), where in addition $m'(x)$ tends to zero when $r \to \infty$.

3. FINITENESS OF THE DISCRETE SPECTRUM

We have seen in section 2 that under suitable assumptions the essential spectrum of H is contained in $[0, \infty)$ so that the negative spectrum is discrete. In this section we prove that if in addition W tends to zero at infinity faster than r^{-1}, then the discrete spectrum is finite (in physical terms: H has a finite number of negative energy bound states). This is the extension to the present situation of the argument of Birman and Schwinger, which proves the finiteness of the discrete spectrum for ordinary potentials V that decrease faster than r^{-2} at infinity. In order to formulate the result, we need an additional assumption and definition. We shall assume that there exists a locally integrable function $U(x)$ such that $W = \nabla U$ in the sense of distributions, so that $V = \Delta U$. Then:

PROPOSITION 3.1. — Let $W = \nabla U$, assume that U is a bounded function and that the operator $WR_0(0)W$ is compact. Then H has only a finite number of negative eigenvalues (including $\lambda = 0$ with its multiplicity).

REMARK 3.1. — From lemma 2.4 and the inequality

$$0 \leq WR_0(\lambda)W \leq WR_0(0)W \tag{3.1}$$

for all $\lambda \leq 0$, it follows that compactness of $WR_0(0)W$ is a stronger assumption than that of proposition 2.1.

REMARK 3.2. — For $n \geq 3$, one can easily give sufficient conditions on W to ensure compactness of $WR_0(0)W$. For instance, it is sufficient that $W \in L^n(\mathbb{R}^n)$. For $n = 3$, this can be weakened to the condition that W^2 belongs to the Rollnik class $\mathcal{R}$, in which case $WR_0(0)W$ is moreover a Hilbert-Schmidt operator. Alternative but similar conditions have been given by Birman [*6*]. For $n = 1$ or $n = 2$, $WR_0(0)W$ is not bounded in general, and *a fortiori* not compact, even for smooth W with compact support (See however note added in proof).

Proof of proposition 3.1. — The proof is similar to that of Schwinger for ordinary potentials and will only be sketched briefly. (See [*22*], p. 86 for details). We consider the eigenvalue problem

$$(H_1 - gW^2)\psi = \lambda\psi \tag{3.2}$$

for $0 \leq g \leq 1$, $\psi \in Q(H_0)$ and $\lambda < 0$.

It follows from the mini-max principle that the eigenvalues λ are continuous and strictly decreasing functions of g. From this it follows that the number $N(\lambda)$ of eigenvalues of H smaller than or equal to λ, each counted with its multiplicity, is equal to the number of values of g in the interval (0, 1] such that (3.2) has a solution. Now (3.2) is equivalent to

$$WR_1(\lambda)W\varphi = g^{-1}\varphi \tag{3.3}$$

where $\varphi = W\psi \in \mathscr{H}$. Therefore $N(\lambda)$ is equal to the number of eigenvalues of $WR_1(\lambda)W$ greater than or equal to one. Since

$$0 \leq WR_1(\lambda)W \leq WR_1(0)W, \tag{3.4}$$

this is smaller than the number of eigenvalues of $WR_1(0)W$ greater than or equal to one, which is finite provided we can show that $WR_1(0)W$ is compact. Now

$$WR_1(0)W = We^U(-\nabla e^{2U}\nabla)^{-1}We^U \tag{3.5}$$

Since U is bounded above, e^U is bounded. Since U is bounded below,

$$0 \leq W(-\nabla e^{2U}\nabla)^{-1}W \leq \exp[-2 \operatorname{Inf} U]WR_0(0)W. \tag{3.6}$$

Therefore $WR_1(0)W$ is compact by lemma 2.4.

Therefore the number of strictly negative eigenvalues is finite. The inclusion of the eigenvalue $\lambda = 0$ with its multiplicity follows from the fact that solutions of the equation $(H_1 - W^2)\psi = 0$ give rise to solutions of the equation

$$WR_1(0)W\varphi = \varphi \tag{3.7}$$

The assumption that U is bounded in proposition 3.1 seems unnatural. In fact, we describe below a special case where it is not needed, namely that of dimension $n = 3$ with W^2 satisfying the Rollnik condition.

PROPOSITION 3.2. — Let $n = 3$ and $W^2 \in \mathscr{R}$ with

$$w^2 \equiv \| W^2 \|_{\mathscr{R}}^2 = (4\pi)^{-2} \int d^3x_1 d^3x_2 W^2(x_1)W^2(x_2) \mid x_1 - x_2 \mid^{-2} \tag{3.8}$$

Then the number N of (strictly) negative eigenvalues of H (with their multiplicities) is finite, and bounded by

$$N \leq 16w^2(1 + w)^2 \tag{3.9}$$

Proof. — We now consider the eigenvalue problem

$$(H_0 + gV)\psi = \lambda\psi \tag{3.10}$$

for $0 \leq g \leq 1$, $\psi \in Q(H_0)$ and $\lambda < 0$. By the same argument as above, $N(\lambda)$ is equal to the number of values of g in the interval (0, 1] for which (3.10) has a solution. Now (3.10) is equivalent to the equation

$$A\Phi = g^{-1}\Phi \tag{3.11}$$

where $\Phi = (\varphi_1, \varphi_2) \in \mathscr{H} \oplus \mathscr{H}$, $\varphi_1 = \nabla\psi$, $\varphi_2 = W\psi$ and A is the operator:

$$A(\lambda) = \begin{pmatrix} \nabla R_0(\lambda)W & -\nabla R_0(\lambda)\nabla \\ WR_0(\lambda)W & -WR_0(\lambda)\nabla \end{pmatrix} \tag{3.12}$$

Therefore $N(\lambda)$ is smaller than the number of eigenvalues of $A(\lambda)$ in $[1, \infty)$. Now for any $\lambda \leq 0$, one sees easily that $A(\lambda)^2$ is compact and that $A(\lambda)^3$ is a Hilbert-Schmidt operator. Therefore

$$N(\lambda) \leq \| A(\lambda)^3 \|_{HS}^2$$

An elementary but tedious computation yields:

$$\| A(\lambda)^3 \|_{HS}^2 = \mathrm{Tr}\, [R_0(VR_0)^2(H_0 + W^2)]^2 \leq 16w^2(1 + w)^2 .$$

This is bounded uniformly in λ for $\lambda \leq 0$, which proves proposition 3.2.

4. POSITIVE SPECTRUM AND SCATTERING THEORY

In this section, we shall derive the properties listed under (5), (6) and (7) in the introduction, and complete the proof of (2). We use the method of Agmon based on *a priori* estimates in weighted Hilbert spaces to prove property (5). Properties (6) and (7) follow easily by the method of smooth operators of Kato and Lavine. Since all these methods apply to the present case with very few modifications, the exposition will be sketchy and some of the proofs will be omitted. The reader is referred to [*19*], chap. XIII for details.

Let α and β be real numbers. We define the following auxiliary Hilbert spaces

$$L_\alpha^2(\mathbb{R}^n) = \{ \varphi : \| \varphi \|_\alpha^2 \equiv \| (1 + x^2)^{\alpha/2} \varphi \|^2 < \infty \} \tag{4.1}$$

and

$$\mathscr{H}_\alpha^\beta = \{ \varphi : \| \varphi \|_{\alpha,\beta}^2 \equiv \| (1 + H_0)^{\beta/2}(1 + x^2)^{\alpha/2} \varphi \|^2 < \infty \} \tag{4.2}$$

We recall that $\| \,.\, \|$ denotes the norm in $\mathscr{H} = L^2(\mathbb{R}^n)$.

We now state the results.

PROPOSITION 4.1. — Let $\alpha > 1/2$ and suppose that the operator

$$W(1 + r^2)^\alpha R_0(-1)W(1 + r^2)^\alpha$$

is compact. Then:

(1) The positive point spectrum of H is a discrete subset $\mathscr{E}$ of $(0, \infty)$ (with possibly 0 as an accumulation point), and is bounded above. Each eigenvalue has finite multiplicity.

(2) The continuous singular spectrum is empty $\sigma_{cs}(H) = \emptyset$.

(3) For any compact subinterval $[a, b] \subset (0, \infty)\backslash\mathscr{E}$, the operator $R(\lambda)$

is a bounded operator from $\mathscr{H}_\alpha^{-1}$ to $\mathscr{H}_{-\alpha}^1$, with norm uniformly bounded in λ for $a \leq \mathrm{Re}\,\lambda \leq b$.

(4) The wave operators as defined by (1.1) exist and are asymptotically complete in the sense of (1.2). The absolutely continuous spectrum of H is $[0, \infty)$.

Proof. — The main point of the proof is to get a sufficient control of $R(\lambda)$ for real positive λ. The free resolvent $R_0(\lambda)$ satisfies the following properties.

LEMMA 4.1. — Let $\alpha > 1/2$. Then:

(1) $R_0(\lambda)$ is a bounded operator from $\mathscr{H}_\alpha^{-1}$ to $\mathscr{H}_{-\alpha}^1$ with norm uniformly bounded in λ for $0 < a \leq \mathrm{Re}\,\lambda \leq b$.

(2) As an operator from $\mathscr{H}_\alpha^{-1}$ to $\mathscr{H}_{-\alpha}^1$, $R_0(\lambda)$ is norm continuous in λ for $\lambda \neq 0$.

Proof. — See [*19*]

We now turn to $R(\lambda)$. The potential V can be written as

$$V = [\nabla, W] = A^*B + B^*A \tag{4.3}$$

where

$$\begin{cases} A = -(1+r^2)^{-\alpha/2}\nabla \\ B = (1+r^2)^{\alpha/2}W \\ A^* = \nabla(1+r^2)^{-\alpha/2} \\ B^* = W(1+r^2)^{\alpha/2} \end{cases} \tag{4.4}$$

A and B are operators from $\mathscr{H}$ to $\overline{\mathscr{H}}$, A^* and B^* from $\overline{\mathscr{H}}$ to $\mathscr{H}$. We want to construct $R(\lambda)$ as

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)(A^*B + B^*A)R(\lambda) \tag{4.5}$$

or equivalently

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)(A^*B^*)(\mathbb{1} + F(\lambda))^{-1}\begin{pmatrix} B \\ A \end{pmatrix} R_0(\lambda) \tag{4.6}$$

Here $\begin{pmatrix} B \\ A \end{pmatrix}$ is an operator from $\mathscr{H}$ to $\overline{\mathscr{H}} \oplus \overline{\mathscr{H}}$, (A^*B^*) from $\overline{\mathscr{H}} \oplus \overline{\mathscr{H}}$ to $\mathscr{H}$, and

$$F(\lambda) = \begin{pmatrix} B \\ A \end{pmatrix} R_0(\lambda)(A^*B^*) = \begin{pmatrix} BR_0A^* & BR_0B^* \\ AR_0A^* & AR_0B^* \end{pmatrix} \tag{4.7}$$

is an operator in $\overline{\mathscr{H}} \oplus \overline{\mathscr{H}}$.

The key of the proof is the following property.

LEMMA 4.2. — A is a bounded operator from $\mathscr{H}_{-\alpha}^1$ to $\overline{\mathscr{H}}$. B is a compact operator from $\mathscr{H}_{-\alpha}^1$ to $\overline{\mathscr{H}}$.

Proof. — We first consider A. Let $\varphi \in \mathscr{H}_{-\alpha}^1$. Then

$$\begin{aligned} \| A\varphi \|^2 &= \langle \nabla\varphi, (1+r^2)^{-\alpha}\nabla\varphi \rangle \\ &= \| \varphi \|_{-\alpha,1}^2 - \| \varphi \|_{-\alpha}^2 + \langle \varphi, (1+r^2)^{-\alpha/2}[\Delta(1+r^2)^{-\alpha/2}]\varphi \rangle \end{aligned} \tag{4.8}$$

where the factor in the square bracket is the Laplacian of the function $(1 + r^2)^{-\alpha/2}$ and where we have used the identity

$$-\nabla(1+r^2)^{-\alpha}\nabla = (1+r^2)^{-\alpha/2}H_0(1+r^2)^{-\alpha/2} + (1+r^2)^{-\alpha/2}[\Delta(1+r^2)^{-\alpha/2}] \quad (4.9)$$

It follows immediately from (4.8) that

$$\| A\varphi \|^2 \leq \| \varphi \|^2_{-\alpha,1} + c \| \varphi \|^2_{-\alpha} \leq (1 + c) \| \varphi \|^2_{-\alpha,1} \quad (4.10)$$

for some constant c.

We next consider B. Let $\varphi \in \mathcal{H}^1_{-\alpha}$. Then

$$\begin{aligned} B\varphi &= W(1 + r^2)^{\alpha/2}\varphi \\ &= [W(1 + r^2)^{\alpha}(1 + H_0)^{-1/2}][(1 + H_0)^{1/2}(1 + r^2)^{-\alpha/2}]\varphi \end{aligned} \quad (4.11)$$

The operator in the first bracket is compact by assumption, while that in the second bracket maps $\mathcal{H}^1_{-\alpha}$ onto $\mathcal{H}$ unitarily. This completes the proof.

Combining lemmas 4.1 and 4.2, we obtain the following properties of $F(\lambda)$. We denote by $\bar{\mathbb{C}}$ the closed cut plane, i. e. the complex plane cut along $[0, \infty)$, including the cut counted twice. Then:

LEMMA 4.3.

(1) $F(\lambda)$ is a bounded operator in $\mathcal{H} \oplus \mathcal{H}$ and is norm continuous in λ for $\lambda \in \bar{\mathbb{C}}\backslash\{0\}$. Its norm is bounded uniformly in λ on the compact subsets of $\bar{\mathbb{C}}\backslash\{0\}$.

(2) $F(\lambda)$ is analytic in λ for $\lambda \in \mathbb{C}\backslash[0, \infty)$.

(3) F^2 is compact for all $\lambda \in \bar{\mathbb{C}}\backslash\{0\}$ and $\| F(\lambda)^2 \| \to 0$ when $|\lambda| \to \infty$.

We can then apply the analytic Fredholm theorem ([*18*], p. 201) to invert the operator $\mathbb{1} + F(\lambda)$:

LEMMA 4.4. — Let $\mathcal{E} \subset [0, \infty)$ be the set of positive λ for which the homogeneous equation

$$[\mathbb{1} + F(\lambda)]\Phi = 0 \quad (4.12)$$

has a solution in $\mathcal{H} \oplus \mathcal{H}$. Then

(1) $\mathcal{E} \cup \{0\}$ is a bounded closed set of Lebesgue measure zero.

(2) For any compact subinterval $[a, b] \subset (0, \infty)\backslash\mathcal{E}$, $R(\lambda)$ is a bounded operator from $\mathcal{H}_\alpha^{-1}$ to $\mathcal{H}^1_{-\alpha}$ with norm uniformly bounded in λ for $a \leq \operatorname{Re} \lambda \leq b$.

REMARK 4.1 AND PROOF OF PROPOSITION 4.1.4. — Except for the additional information contained in propositions 4.1.1 and 4.1.2, lemma 4.4.2 is proposition 4.1.3. Propositions 4.1.1 and 4.1.2 will be proved below. At the present stage, we already obtain the statement of lemma 4.4.1 from a general result of Kuroda ([*12*], see also [*22*], p. 127). As a consequence, we are already in a position to prove proposition 4.1.4. Indeed it follows from lemmas 4.1.1 and 4.2 that A and B are H_0-smooth in the sense of Kato [*10*] and Lavine [*13*] on any interval $[a, b] \subset (0, \infty)$. It follows from lemmas 4.2 and 4.4.2 that A and B are H-smooth on any interval $[a, b] \subset (0, \infty)\backslash\mathscr{E}$, and from lemma 4.4.1 that the absolutely continuous spectrum of H is exhausted by a denumerable union of such intervals. This implies proposition 4.1.4.

It remains to prove propositions 4.1.1 and 4.1.2. One first shows that solutions of the homogeneous equation (4.12) are such that the Fourier transform of $A^*\varphi_1 + B^*\varphi_2$ vanishes on the energy shell. Let $\hat{}$ denote the Fourier transform and k the momentum variable. Then:

LEMMA 4.5. — Let $\Phi = (\varphi_1, \varphi_2) \in \mathscr{H} \oplus \mathscr{H}$ be a solution of (4.12) with $\lambda > 0$. Then

$$\widehat{(A^*\varphi_1 + B^*\varphi_2)}\,|_{k^2=\lambda} = 0 \tag{4.13}$$

Proof. — Let Φ be a solution of (4.12). Then

$$-\Phi = F(\lambda)\Phi = \lim_{\varepsilon \downarrow 0} F(\lambda + i\varepsilon)\Phi$$

by the continuity of $F(\lambda)$. Therefore

$$- 2 \operatorname{Re} \langle \varphi_1, \varphi_2 \rangle = - \left\langle \Phi, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi \right\rangle$$

$$= \lim_{\varepsilon \downarrow 0} \left\langle \Phi, \begin{pmatrix} AR_0(\lambda + i\varepsilon)A^* & AR_0(\lambda + i\varepsilon)B^* \\ BR_0(\lambda + i\varepsilon)A^* & BR_0(\lambda + i\varepsilon)B^* \end{pmatrix} \Phi \right\rangle$$

Taking the imaginary part, we obtain

$$\lim_{\varepsilon \downarrow 0} \varepsilon \, \| R_0(\lambda + i\varepsilon)(A^*\varphi_1 + B^*\varphi_2) \|^2 = 0 \tag{4.14}$$

Now by lemma 4.2, $A^*\varphi_1 + B^*\varphi_2 \in \mathscr{H}_\alpha^{-1}$. From this it follows that the restriction of its Fourier transform to the sphere $\Omega_\lambda = \{ k : k^2 = \lambda \}$ in momentum space belongs to $L^2(\Omega_\lambda)$ and is Hölder-continuous in norm as a function of λ [*1*] [*19*]. Therefore (4.13) is meaningful and follows immediately from (4.14).

The end of the proof of propositions 4.1.1 and 4.1.2 is identical with Agmon's for ordinary potentials and will be omitted.

REMARK 4.2. — From remarks 2.2 and 2.3, one easily obtains sufficient conditions on W to imply the assumption of proposition 4.1. For instance one can take $W(1 + r^2)^\delta \in L^p_{loc}(\mathbb{R}^n)$ with uniform bound, with the same values of p as in remark 2.2 and with $\delta > \alpha$. Intuitively, this means that W should tend to zero at infinity faster than r^{-1}.

ACKNOWLEDGMENTS

We are grateful to M. L. Baeteman and K. Chadan for communicating their results prior to publication, and especially to K. Chadan for several stimulating discussions.

Note added in proof

(1) Although $WR_0(0)W$ is not bounded for $n = 1$ or $n = 2$, finiteness of the discrete spectrum still holds in these cases under the following assumptions:

$n = 1$: $W \in L^2_{loc}(\mathbb{R})$, and

$$\lim_{r \to \infty} r \int_{|x| \geq r} W^2(x)dx = 0 .$$

$n = 2$: (cf. [*6*]) $W \in L^p_{loc}(\mathbb{R}^2)$ with $p > 2$, and

$$W = o(|x| \operatorname{Log} |x|)^{-1} \qquad \text{as} \qquad |x| \to \infty .$$

The proof is obtained by a straightforward modification of that of proposition 3.1

(2) Potentials of the type studied in this paper have also been considered by Schechter [*25*] [*26*]. In particular it is proved in [*25*] that $\sigma_e(H) = [0, \infty)$, and in [*26*] that the wave operators exist and are asymptotically complete.

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(Manuscrit reçu le 14 mai 1975)