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E. IHRIG
D. K. SEN

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Section A :
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Uniqueness of Timelike Killing Vector Fields (*)

by

E. IHRIG and D. K. SEN
Department of Mathematics, University of Toronto,
Toronto, Canada, M5S 1A1

ABSTRACT. — Under certain weak, physically reasonable conditions it is shown that a space-time can have at most one timelike hypersurface orthogonal killing vector field (to within a constant multiple).

RÉSUMÉ. — Sous certaines faibles conditions physiques on démontre qu'un espace-temps admet, à une constante près, un seul champ de vecteur de Killing hypersurface orthogonal orienté dans le temps.

1. INTRODUCTION

An interesting problem in General Relativity is to determine under what conditions is a timelike killing vector unique in a space-time. This is equivalent to asking when time in a stationary space-time is essentially unique (**). Although one would hope this to be always the case, one sees in theorem 1 that this is essentially always not the case. Thus we see that we must restrict the problem somewhat if we hope to get a positive uniqueness result. The first reasonable restriction is to consider static space-times; that is, to restrict our attention to timelike hypersurface orthogonal vector fields. Another assumption that seems reasonable concerning our static space-time is that its hypersurface orthogonal timelike killing vector

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(**) There is always an ambiguity in time due to a choice of coordinates (see [*2*]).

field lie in the center of the Lie Algebra of the isometry group. Physically we think of the isometry group of a static space-time as being built up of a timelike isometry plus spacelike isometries. This assumption says that the timelike isometry does not interfere with spacelike ones, that is, that it commutes with them. However these assumptions alone are not enough to give a positive result as illustrated in example 5. But if we add a mild energy condition we do find a result which is given in theorem 3.

Although physically the only surprising aspect of this result is that so many conditions are necessary, mathematically it is surprising that one can get any result of this nature at all. There is no analogous result possible in Riemannian geometry. This is very unusual because it seems that the properties of killing vector fields are relatively independent of signature. It is not clear why the signature plays such a dominant role in this case, and this situation presents an interesting mathematical problem. The immediate reason for the success of our theorem is clear however. If one is given two suitable killing vector fields one can construct a third which has desired properties concerning its direction. What is needed is a killing vector field with constant length in order to get an energy condition. In the pseudo-Riemannian case one can make sure a vector field will have constant length by having it point in a null direction. The length will then be zero. In the Riemannian case there is no such association of length with direction.

2. UNIQUENESS THEOREMS

We start with a theorem that motivates the extra assumptions needed for our main theorem. It is essentially a local theorem, but we wish to restrict ourselves to only local considerations since this is what is of interest in considering stationary space-times.

Let M be a space-time with metric $\langle\, , \rangle$ of signature $-1, 1, 1, 1$.

1. THEOREM. — Suppose M has a timelike killing vector field v and any other killing vector field w independent of v. Let $S \subset M$ such that $\bar{S}$ is compact. Then there are two independent timelike killing vector fields defined on S.

Proof. — Let λ_0 be a constant. Then $v + \lambda_0 w$ is a killing field. We just need to pick λ_0 so that $v + \lambda_0 w$ will be timelike on S. Since $\bar{S}$ is compact there exists positive numbers A_i such that the following inequalities hold on $\bar{S}$:

$$\langle w, w \rangle < A_1, \quad \langle v, w \rangle < A_2 \quad \text{and} \quad \langle v, v \rangle < -A_3 < 0$$

Thus

$$\langle v + \lambda_0 w, v + \lambda_0 w \rangle < -A_3 + 2\lambda_0 A_2 + \lambda_0^2 A_1 .$$

Since $-A_3 < 0$ there is a λ_0 suffiently close to zero, but not equal to zero, so that the right hand side above remains negative. For this choice of λ_0 we will have $v + \lambda_0 w$ timelike on S. Q. E. D.

The theorem has the following immediate corollary.

2. COROLLARY. — Suppose M has a timelike killing vector field and another independent killing vector field. Then every point has a neighborhood in which there are two independent timelike killing vector fields.

Now we proceed to give our positive result concerning the uniqueness of timelike killing vector fields. A general reference that is useful in our proof is Kobayashi and Nomizu [*1*].

3. THEOREM. — Let M be a space-time which satisfies the following energy condition: for any null killing vector field w on M there is a point $m \in M$ such that

$$\mathrm{Ric}_m(w, w) \neq 0 .$$

Suppose M has a timelike hypersurface orthogonal killing vector field v_0 that commutes with every other killing vector field. Then any other timelike hypersurface orthogonal killing vector field is a constant multiple of v_0.

Proof. — Suppose we are given two vector fields v_0 and v_2 as described in the theorem. Suppose also they are independent at m_0. We will construct a null killing vector field w and show that $\mathrm{Ric}_m(w, w) = 0$. We start by constructing a spacelike killing vector field orthogonal to v_0. Let

$$v_1 = \lambda v_2 + v_0$$

where

$$\lambda = - \langle v_0, v_0 \rangle / \langle v_0, v_2 \rangle$$

To show that v_1 is a killing vector we need only show λ is constant. Let w_1 be any vector field such that $\langle w_1, v_0 \rangle = \langle w_1, v_2 \rangle = 0$. Since $\langle v_1, v_0 \rangle = 0$ we have

$$0 = \langle [v_1, w_1], v_0 \rangle = - w_1(\lambda) \langle v_0, v_2 \rangle$$

and $w_1(\lambda) = 0$ since $\langle v_0, v_2 \rangle \neq 0$ (both v_0 and v_2 are timelike). If we show $v_0(\lambda) = v_2(\lambda) = 0$ then we will have λ a constant as desired since every vector is a linear combination of v_0, v_2 and some w_1. We have $(i, j = 0, 2)$, $v_i \langle v_0, v_2 \rangle = v_i \langle v_0, v_0 \rangle = 0$ by an application of $L_{v_i} g = 0$ using $[v_i, v_j] = 0$. This applied to our definition of λ gives our desired result.

We now use a similar argument to define our null killing vector field. First, for convenience we normalize v_0 and v_1 at a point m_0 by dividing by $\sqrt{- \langle v_0, v_0 \rangle_{m_0}}$ and $\sqrt{\langle v_1, v_1 \rangle_{m_0}}$ respectively. We will show

$$\langle v_0, v_0 \rangle = - \langle v_1, v_1 \rangle .$$

To do this we define γ by $\langle v_0, v_0 \rangle = \gamma \langle v_1, v_1 \rangle$ and show γ is constant. Since γ at m_0 is -1 this will prove our result. Observe that

$$\langle v_0 + \gamma v_1, v_2 \rangle = 0 .$$

Thus $\langle [w_1, v_0+\gamma v_1], v_2 \rangle = 0$ since $\langle w_1, v_2 \rangle = 0$. This gives $w_1(\gamma)\langle v_0, v_2 \rangle = 0$ so $w_1(\gamma) = 0$ as before. Again using the killing equations we find $v_i(\gamma) = 0$ for all i and γ is constant.

We may now define our null killing vector field w by

$$w = v_0 - v_1 .$$

It remains to show $\mathrm{Ric}_m(w, w) = 0$. We observe since $\langle w, w \rangle$ is constant and w is a killing vector field we have

$$0 = \mathrm{Ric}(w, w) + \mathrm{trace}(A_w A_w)$$

where $A_w v = -D_v w$. To compute this trace we must pick a suitable orthonormal bases of $T_m(M)$. We note that $\{ v \mid \langle v_0, v \rangle = 0$ and $\langle v_1, v \rangle = 0 \} = D$ is hypersurface generating since it is the intersection of $\{ v \mid \langle v_0, v \rangle = 0 \}$ and $\{ v \mid \langle v_2, v \rangle = 0 \}$, both of which are hypersurface generating. Thus we may pick $\{ v_i \mid i = 2, \ldots, n-1 \}$ such that v_i are orthonormal at m, v_i generate D, and $[v_i, v_j] = 0$ where $i, j = 2, \ldots, n-1$. In this bases we find

$$-\mathrm{trace}(A_w A_w) = \sum_{i,j} \sigma(i)\sigma(j) \langle D_{v_i} w, v_j \rangle \langle v_j, D_{v_i} w \rangle$$

where

$$\sigma(i) = \begin{cases} -1 & \text{if } i = 0 \\ 1 & \text{if } i \neq 0 \end{cases}$$

Using the following equations this sum can be seen to be zero:

a) $\langle D_{v_0} w, v_0 \rangle = 0$

b) $\langle D_{v_0} w, v_i \rangle = \langle D_{v_1} w, v_i \rangle$

c) $\langle D_{v_i} w, v_0 \rangle = \langle D_{v_i} w, v_1 \rangle$

d) $\langle D_{v_i} w, v_j \rangle = 0 \quad$ for $i, j \geq 2$.

a) follows since

$$\langle D_{v_0} v_0, v_0 \rangle = \frac{1}{2} v_0 \langle v_0, v_0 \rangle = 0$$

and

$$\langle D_{v_0} v_1, v_0 \rangle = \langle D_{v_1} v_0, v_0 \rangle = \frac{1}{2} v_1 \langle v_0, v_0 \rangle = 0 .$$

b) follows from *c*) since

$$\langle D_{v_0} w, v_i \rangle = -\langle A_w v_0, v_i \rangle = \langle v_0, A_w v_i \rangle = \langle v_0, D_{v_i} w \rangle$$

and the same for $\langle D_{v_1} w, v_i \rangle$.

c) can be written as $\langle D_{v_i} w, w \rangle = 0$. But

$$\langle D_{v_i} w, w \rangle = \frac{1}{2} v_i \langle w, w \rangle = 0$$

since $\langle w, w \rangle = 0$,

d) follows because $\langle D_{v_i}w, v_j \rangle = - \langle w, D_{v_i}v_j \rangle$. Also as in *b*) we have $\langle D_{v_i}w, v_j \rangle = - \langle D_{v_j}w, v_i \rangle$ and $- \langle D_{v_j}w, v_i \rangle = \langle w, D_{v_j}v_i \rangle$. Since $[v_i, v_j] = 0$ we thus have $\langle w, D_{v_i}v_j \rangle = - \langle w, D_{v_i}v_j \rangle$ as desired. Since the trace is zero then $\mathrm{Ric}_m(w, w) = 0$ as desired.

To illustrate how weak our energy condition is we give a few examples.

4. EXAMPLE :

a) Schwarzschild space satisfies the energy condition of our theorem since it has no null killing vector fields.

b) A space-time which is non-empty at at least one point $m \in M$ satisfies our energy condition (if the cosmological constant is zero).

Here non-empty at m means that Ric_m has its timelike eigenvalue greater than its spacelike ones.

We conclude with an example of the necessity of the above energy condition for our theorem.

5. EXAMPLE. — Let $\tilde{M}$ be $\mathbb{R}^4$; $ds^2 = g_{ij}dx^i dx^j$; with $g_{11} = - g_{00} > 0$, $g_{22}g_{33} - g_{23}g_{32} > 0$; $g_{0i} = 0$, $i \neq 0$; $g_{1i} = 0$, $i \neq 1$, and

$$\frac{\partial}{\partial x^0} g_{ij} = \frac{\partial}{\partial x^1} g_{ij} = 0.$$

Then we have two hypersurface orthogonal timelike killing vector fields $\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^0} + \frac{1}{2}\frac{\partial}{\partial x^1}$. Now define $M \simeq \tilde{M} \mid \sim$ where

$$(x^0, x^1, x^2, x^3) \sim (x^0, x^1 + n, x^2, x^3)$$

for all $n \in Z$. g_{ij} induces a space-time metric on M which satisfies all the desired properties since its isometry group will be $\mathbb{R}^2$ for a suitable choice of g_{ij}.

REFERENCES

[1] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, Vol. I, Interscience, 1963.

[2] L. LANDAU and E. LIFSHITZ, *The Classical Theory of Fields*, Pergamon, 1962.

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