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On classical solutions of SU(3) gauge field equations

by

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RÉSUMÉ. — Une classe de solutions statiques pour les champs de jauge classiques est étudiée. On considère le cas non abélien avec SU(3) pour groupe de symétrie. Utilisant systématiquement certaines propriétés du sous-groupe O(3) et des générateurs quadrupolaires, une forme générale des champs de jauge W_μ est construite. Le tenseur covariant des champs $F_{\mu\nu}$ a alors la même structure. Ceci permet de trouver, d'une façon remarquablement simple, les contraintes dues aux équations du mouvement. Quelques solutions exactes singulières à l'origine, sont données. On introduit ensuite un octet de mésons scalaires. Une condition de jauge est naturellement suggérée par ce cas. Finalement, on relie nos résultats aux cas particuliers déjà connus.

SUMMARY. — Static classical solutions of SU(3) gauge field equations are studied. The roles of the O(3) subgroup and of the quadrupole generators are discussed systematically. The general form thus obtained leads, throughout, to a high degree of symmetry in the results. This brings in some simplifying features. An octet of scalar mesons is finally added. Certain classes of exact solutions are given that are singular at the origin. A generalized gauge condition is pointed out. The relation of the general form to known particular cases is discussed.

1. INTRODUCTION

Classical solutions of Yang-Mills field equations have recently attracted considerable attention [*1*] [*2*], ..., [*7*]. Non abelian gauge fields, with which we will be concerned in this paper, possess certain very special types of solutions (e. g. magnetic monopoles). More generally suitable classical solutions may, hopefully, be utilised as starting points for quantum theoretical study of bound states. As usual we will consider *only static solutions.*

Most of the previous studies have been confined to SU(2). SU(3) has been introduced however as the next evident step [*2*] [*6*]. In this paper we will be concerned with a generalization of the work of Wu and Wu [*2*] for SU(3). This will include the gauge field part of the generalized SU(2) solution of reference 5, where, however, SU(2) will be replaced by O_3, having the same Lie algebra.

As will be seen, it is the subgroup O(3) *of* SU(3) *that plays a crucial role, so far as this class of solutions are concerned, and not* SU(2). After having exhibited our solutions we will return to this point in the concluding section.

As is well-known, with respect to the O(3) subgroup the remaining five generators of SU(3) transform as quadrupole operators. These angular transformation properties will be exploited systematically.

To make contact with the familiar λ-matrices of Gell-Mann let us first write a few things in that basis. We will not need them after wards, since all the necessary commutators can be exhibited very systematically directly in terms of certain elements of the solutions (see the appendix).

Let L_i and Q_j denote the O(3) and the quadrupole generators respectively. For example, identifying L_3 with λ_2, one may define the circular components as

$$
\begin{aligned}
&L_3 = L_0 = \lambda_2, \quad L_{\pm 1} = \mp \frac{1}{\sqrt{2}}(L_1 \pm iL_2) = \mp \frac{1}{\sqrt{2}}(\lambda_7 \mp i\lambda_5) \\
&Q_0 = \lambda_8, \quad Q_{\pm 1} = \pm \frac{1}{\sqrt{2}}(\lambda_4 \pm i\lambda_6), \quad Q \pm 2 = -\frac{1}{\sqrt{2}}(\lambda_3 \pm i\lambda_1)
\end{aligned}
\tag{1.1}
$$

2. SOLUTIONS FOR THE SPACE COMPONENTS OF THE GAUGEFIELD

Let us now consider only the space components of the 4-vector gauge field (W_1, W_2, W_3) with $W_0 = 0$. We will add the W_0 component and scalar mesons in the next section.

In the λ-basis our proposed form for W_3 is

$$
\begin{aligned}
W_3 = {} & a(r)\lambda_2 + b(r)(x_1\lambda_5 + x_2\lambda_7) + c(r)x_3(x_1\lambda_7 - x_2\lambda_5 + x_3\lambda_2) \\
& + d(r)\left(x_1\lambda_4 + x_2\lambda_6 - 2x_3\frac{\lambda_8}{\sqrt{3}}\right) \\
& + e(r)\left\{(x_2^2 - x_1^2)\lambda_1 + x_2x_3\lambda_4 - x_3x_1\lambda_6 + 2x_1x_2\lambda_3\right\} \\
& + f(r)\left\{2(x_1x_2\lambda_1 + x_3x_1\lambda_4 + x_2x_3\lambda_6)\right. \\
& \left. \qquad + (x_1^2 - x_2^2)\lambda_3 + (r^2 - 3x_3^2)\frac{\lambda_8}{\sqrt{3}}\right\}x_3
\end{aligned}
\tag{2.1}
$$

$$
\equiv a(r)A_3 + b(r)B_3 + c(r)C_3 + d(r)D_3 + e(r)E_3 + f(r)F_3, \quad \text{say.} \tag{2.2}
$$

The radial factors ($a(r), \ldots, f(r)$) are functions of $r^2(=(x_1^2 + x_2^2 + x_3^2))$ only. It will be noted that using (1.1), C. G. coefficients and solid spherical harmonics, we have

$$
\begin{aligned}
A_3 &= L_0 \\
B_3 &\approx (1m, 1-m \mid 10)\mathcal{Y}_m^{(1)} L_{-m} \\
D_3 &\approx (1m, 2-m \mid 10)\mathcal{Y}_m^{(1)} Q_{-m} \\
E_3 &\approx (2m, 2-m \mid 10)\mathcal{Y}_m^{(2)} Q_{-m} \\
F_3 &\approx \mathcal{Y}_0^{(1)}((2m, 2-m \mid 00)\mathcal{Y}_m^{(2)} Q_{-m}) \equiv \mathcal{Y}_0^{(1)}\hat{F} \\
C_3 &\approx \mathcal{Y}_0^{(1)}((1m, 1-m \mid 00)\mathcal{Y}_m^{(1)} L - m) \equiv \mathcal{Y}_0^{(1)}\hat{C} .
\end{aligned} \tag{2.3}
$$

Let us also note that

$$
\begin{aligned}
(2m, 1-m \mid 10)\mathcal{Y}_m^{(2)} L_{-m} &\approx \left(C_3 - \frac{1}{3} r^2 A_3\right) \\
(3m, 2-m \mid 10)\mathcal{Y}_m^{(3)} Q_{-m} &\approx (5F_3 - 2r^2 D_3) .
\end{aligned} \tag{2.4}
$$

Hence they provide no new terms. As will become evident later on it is more convenient for our purposes to use the forms C_3 and F_3.

The angular structure of our proposed form for W_3 is now evident, namely *we have used all possible combinations of the $\mathcal{Y}$'s with the λ's using C. G. coefficients of the form*

$$
(j_1 m, j_2 - m \mid 10) ; \begin{cases} j_1 = 0, 1, 2, 3 \\ j_2 = 1, 2. \end{cases} \tag{2.5}
$$

This is the key to our solutions. The next point to note is that *given one component W_i ($i = 1, 2, 3$) the other two are obtained from it by simultaneous cyclic permutations of the L's, the Q's and the x's.* The resulting explicit forms for W_1, W_2 are given, in the λ' basis, for the sake of completeness, in the appendix.

Thus denoting,

$$
\vec{A} \equiv (A_1, A_2, A_3)
$$

and so on, we obtain (see appendix),

$$
\vec{W} = a(r)\vec{A} + b(r)\vec{B} + c(r)\vec{C} + d(r)\vec{D} + e(r)\vec{E} + f(r)\vec{F} \tag{2.6}
$$

where

$$
\vec{C} \equiv \vec{x}\hat{C}, \qquad \vec{F} \equiv \vec{x}\hat{F} .
$$

Henceforward we will write everything directly in terms of ($\vec{A}, \ldots, \vec{F}$).

All the necessary algebra has been displayed systematically in the appendix, making the symmetries very evident. Using them we easily obtain the field tensors and the equations of motion.

For pure Yang-Mills fields we have, in the matrix notation,

$$
F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu] \tag{2.7}
$$

and

$$
\partial^\mu F_{\mu\nu} + ig[W^\mu, F_{\mu\nu}] = 0 . \tag{2.8}
$$

Henceforth we will put

$$g = 1$$

For arbitrary g we have only to multiply our solutions by $1/g$ (In [4] and [5], for example, $g = -e$).

In this section we are considering the static case with, in addition,

$$W_0 = 0 \tag{2.9}$$

Defining

$$\mathscr{F} \equiv (F_{23}, F_{31}, F_{12}) \tag{2.10}$$

and using (2.7) and the results of the appendix we obtain,

$$\vec{\mathscr{F}} = A(r)\vec{A} + B(r)\vec{B} + C(r)\vec{C} + D(r)\vec{D} + E(r)\vec{E} + F(r)\vec{F} \tag{2.11}$$

where, with $a'(r) \equiv \frac{d}{dr} a(r)$ and so on and defining

$$\tilde{b}(r) \equiv \left(b(r) - \frac{1}{r^2} \right)$$

$$\begin{aligned}
A &= (r\tilde{b}' + 2\tilde{b}) - \{ a(a + cr^2) + 2dr^2(d + fr^2) \} \\
D &= (re' + 3e) - \{ d(a + cr^2) + 2a(d + fr^2) \} \\
B &= -\frac{1}{r^2}(ra' + a) - \{ \tilde{b}(a + cr^2) + 2er^2(d + fr^2) \} \\
E &= -\frac{1}{r^2}(rd' + 2d) - \{ e(a + cr^2) + 2\tilde{b}(d + fr^2) \} \\
(r^2C + A) &= -\left\{ (a^2 + d^2r^2) + r^2(\tilde{b}^2 + e^2r^2) - \frac{1}{r^2} \right\} \\
(r^2F + D) &= -3(ad + r^2\tilde{b}e).
\end{aligned} \tag{2.12}$$

(For brevity we have suppressed the argument (r) in $A(r)$, $a(r)$, etc. For the purposes of section 4 this should not be forgotten).

We note the following important fact. Taking *the general form* (2.6), *we have introduced a complete set or basis*, in the sense that $\vec{\mathscr{F}}$ has the same general form as $\vec{W}$. This will also be true for the left hand side of the equation of motion (2.8).

In fact the space-space part of (2.8) can be written as

$$\begin{aligned}
0 &= \vec{\nabla} \times \vec{\mathscr{F}} + i(\vec{W} \times \vec{\mathscr{F}} + \vec{\mathscr{F}} \times \vec{W}) && (2.13) \\
&= \tilde{A}(r)\vec{A} + \tilde{B}(r)\vec{B} + \tilde{C}(r)\vec{C} + \tilde{D}(r)\vec{D} + \tilde{E}(r)\vec{E} + \tilde{F}(r)\vec{F}, && (2.14)
\end{aligned}$$

where *the $\tilde{A}$'s can be written down without any calculation whatsoever* from the following simple rule.

In the expressions (2.12) for the A's substitute

$a \rightarrow A$, etc. for the terms linear in a's

and

$a^2 \rightarrow 2aA$, $ab \rightarrow (aB + bA)$ for the terms bilinear in the a's.

Thus

$$\tilde{A} = (B'r + 2B) - \{ 2aA + (aC + cA)r^2 + 4dDr^2 + 2(dF + fD)r4 \} \quad (2.15)$$

and so on (Suppressing again temporarily the arguments (r)).

Next we have to impose the conditions

$$\tilde{A}(r) = \tilde{B}(r) = \tilde{C}(r) = \tilde{D}(r) = \tilde{E}(r) = \tilde{F}(r) = 0. \quad (2.16)$$

Thus we find that in our technique the Lagrangian equations of motion are obtained without extra labour. However, one can also quite easily express the Lagrangian directly as an integral of the radial factors (A's) and eventually apply the variational principle directly on it [4] [5] to obtain an extremum of the energy.

We obtain (for the present case)

$$\int d\vec{x}\mathscr{L}(x) = \frac{1}{2}\,\mathrm{Tr} \int \left(-\frac{1}{2}\vec{\mathscr{F}}^2 \right)$$
$$= (-4\pi) \int dr\, r^2[2\{ (A^2(r) + r^2D^2(r)) + r^2(B^2(r) + r^2E^2(r)) \}$$
$$+ (A(r) + r^2C(r))^2 + \frac{4}{3}r^2(D(r) + r^2F(r))^2]. \quad (2.17)$$

3. GENERALIZATIONS ($W_0 \neq 0$ AND SCALAR MESONS):

A. $W_0 \neq 0$

Let us now suppose that in addition to $\vec{W}$ (2.6) we have

$$W_0 = c_0(r)\hat{C} + f_0(r)\hat{F}. \quad (3.1)$$

Then (using again the results of the appendix)

$$(F_{10}, F_{20}, F_{30}) \equiv \vec{\mathscr{F}}_{(0)} = \vec{\nabla}W_0 + i[\vec{W}, W_0]$$
$$= (c_0\tilde{b} + 2f_0er^2)(\vec{C} - r^2\vec{A}) + (c_0a + 2f_0dr^2)\vec{B}$$
$$+ (c_0e + 2f_0\tilde{b})(\vec{F} - r^2\vec{D}) + (c_0d + 2f_0a)\vec{E}$$
$$+ \frac{1}{r^2}(rc_0' + c_0)\vec{C} + \frac{1}{r^2}(rf_0' + 2f_0)\vec{F} \qquad (c_0 \equiv c_0(r), \text{ etc.}) \quad (3.2)$$
$$\equiv A_0\vec{A} + B_0\vec{B} + C_0\vec{C} + D_0\vec{D} + E_0\vec{E} + F_0\vec{F}, \quad \text{say} \quad (A_0 \equiv A_0(r), \text{ etc.}). \quad (3.3)$$

The additional contribution (from W_0) to the equations of motion for consists of the term (since $\partial_0\vec{\mathscr{F}}_{(0)} = 0$)

$$(-i)[W_0, \vec{\mathscr{F}}_{(0)}] = (c_0A_0 + 2f_0D_0r^2)\vec{B} + (c_0D_0 + 2f_0A_0)\vec{E}$$
$$+ (c_0B_0 + 2f_0E_0r^2)(\vec{C} - r^2\vec{A}) + (c_0E_0 + 2f_0B_0)(\vec{F} - r^2\vec{D}). \quad (3.4)$$

The equations of motions for $\vec{\mathscr{F}}_{(0)}$

$$\vec{\nabla}\cdot\vec{\mathscr{F}}_{(0)} + i(\vec{W}\cdot\vec{\mathscr{F}}_{(0)} - \vec{\mathscr{F}}_{(0)}\cdot\vec{W}) = 0 \quad (3.5)$$

lead to

$$\tilde{C}_0(r)\hat{C} + \tilde{F}_0(r)\hat{F} = 0 \tag{3.6}$$

where

$$\begin{aligned}\tilde{C}_0 &= \frac{1}{r} A'_0 + (rC'_0 + 4C_0) - 2\{ aB_0 - A_0 b) + r^2(dE_0 - D_0 e) \} \\ \tilde{F}_0 &= \frac{1}{r} D'_0 + (rF'_0 + 5F_0) - 3\{ (aE_0 - A_0 e) + (dB_0 - D_0 b) \}. \\ &\qquad (\tilde{C}_0 \equiv \tilde{C}_0(r), \text{ etc.})\end{aligned} \tag{3.7}$$

Thus again we have exactly as many constraint equations as radial parameters ($\tilde{C}_0 = 0 = \tilde{D}_0$).

Also, exactly as before (see (2.17))

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\int d\vec{x}\left(\frac{1}{2}\vec{\mathscr{F}}^2_{(0)}\right) &= (4\pi)\int dr\, r^2[2\{ (A_0^2(r) + r^2 D_0^2(r)) + r^2(B_0^2(r) + r^2E_0^2(r)) \} \\ &\quad + (A_0(r) + r^2C_0(r))^2 + \frac{4}{3} r^2(D_0(r) + r^2F_0(r))^2 \Big]\end{aligned} \tag{3.8}$$

B. Scalar octet.

An octet of scalar mesons Φ ($\equiv \Phi_i\lambda_i$) can be added to play a role closely analogous to that of W_0, except for the effect of the scalar potential term $V(\Phi)$.

The terms added to the Lagrangian are

$$\frac{1}{4}\,\mathrm{Tr}\{ (D_\mu\Phi)(D^\mu\Phi) \} - V(\Phi). \tag{3.9}$$

We will assume,

$$\Phi = c_\varphi(r)\hat{C} + f_\varphi(r)\hat{F} \tag{3.10}$$

so that in

$$D_\mu\Phi \equiv \partial_\mu\Phi + i[W_\mu, \Phi] \tag{3.11}$$

$$D_0\Phi = 0 \tag{3.12}$$

and

$$\vec{D}\Phi = A_\varphi(r)\vec{A} + \ldots + F_\varphi(r)\vec{F} \tag{3.13}$$

where $(A_\varphi, \ldots, F_\varphi)$ are obtained from $(A_0, \ldots, F_0)$ defined in (3.2) and (3.3) by the substitution

$$(c_0, f_0) \rightarrow (c_\varphi, f_\varphi). \tag{3.14}$$

The expressions $[\Phi, \vec{D}\Phi]$, $\vec{D}.(\vec{D}\Phi)$ and $\mathrm{Tr}\int d\vec{x}(\vec{D}\Phi)^2$ are also similarly obtained directly from the corresponding expressions for W_0 ((3.4)-(3.8)). Lastly, assuming $V(\Phi)$ to be a function of ($\mathrm{Tr}\,\Phi^2$) we note

$$\frac{1}{2}\,\mathrm{Tr}\,\Phi^2 = \left(r^2c_\varphi^2(r) + \frac{4}{3} r^4 f_\varphi^2(r)\right). \tag{3.15}$$

4. « ASYMPTOTIC » SOLUTIONS

Needless to say that we will not attempt any general study of the possible solutions of the set of coupled non-linear equations for the radial form factors.

Certain consequences of simple asymptotic behaviours can however be easily extracted. In fact, one can easily construct certain families of solutions which are *exact solutions* everywhere, except that there is a singularity at the origin. Such exact solutions are useful for indicating suitable asymptotic ($r \to \infty$) properties of non-singular solutions leading to finite energies [4] [5] [7]. They can, for example, lead to simple trial functions for finite energy solutions. It is only in this sense that they will be termed asymptotic.

Even in this class we will concentrate exclusively on a few cases which simplify things enormously from the very beginning.

Let us start with the case of section 2 (only $\vec{W}$ with $W_0 = 0 = \Phi$). Let us put,

$$a(r) = \frac{a}{r}, \quad b(r) = \frac{b}{r^2}, \quad c(r) = \frac{c}{r^3}, \quad d(r) = \frac{d}{r^2}, \quad e(r) = \frac{e}{r^3}, \quad f(r) = \frac{f}{r^4}, \tag{4.1}$$

where now $(a, b, \ldots, f)$ are now *constants*.

(In the previous sections we have denoted $a(r), \ldots$ itself by $a, \ldots$. *Now the a's are distinct.* We hope that this causes no confusion).

From (2.12) and (4.1) we obtain,

$$\begin{aligned}
- r^2 A(r) &= a(a + c) + 2d(d + f) \\
- r^3 D(r) &= d(a + c) + 2a(d + f) \\
- r^3 B(r) &= (b - 1)(a + c) + 2e(d + f) \\
- r^4 E(r) &= e(a + c) + 2(b - 1)(d + f) \\
- r^2(r^2 C(r) + A(r)) &= (a^2 + d^2 + (b - 1)^2 + e^2 - 1) \\
- r^3(r^2 F(r) + D(r)) &= 3(ad + (b - 1)e) .
\end{aligned} \tag{4.2}$$

The forms of the above equations single out as particularly simple the case

$$a + c = 0 = d + f. \tag{4.3}$$

This is the case we will study.

Immediately (*excluding always the singular point* $r = 0$)

$$\begin{aligned}
A(r) = D(r) &= B(r) = E(r) = 0 \\
- r^4 C(r) &= (a^2 + d^2 + (b - 1)^2 + e^2 - 1) \equiv - C, \quad \text{say;} \\
- r^5 F(r) &= 3(ad + (b - 1)e) \equiv - F, \quad \text{say.}
\end{aligned} \tag{4.4}$$

(C and F are now *constants*).

Hence from (2.14)-(2.16), the coefficients of the equations of motion reduce to

$$\begin{aligned} -r^3\tilde{A}(r) &= aC + 2dF, & -r^4\tilde{B}(r) &= (b-1)C + 2eF, \\ -r^4\tilde{D}(r) &= dC + 2aF, & -r^5\tilde{E}(r) &= eC + 2(b-1)F, \\ (r^2\tilde{C}(r) + \tilde{A}(r)) &= 0 = (r^2\tilde{F}(r) + \tilde{D}(r)). \end{aligned} \tag{4.5}$$

Hence we have the following subcases.

(*i*) $\underline{C = F = 0}$ (« *pure gauge* »): $F_{\mu\nu} = 0$

$$\begin{aligned} a^2 + d^2 + (b-1)^2 + e^2 &= 1, \\ ad + (b-1)e &= 0. \end{aligned} \tag{4.6}$$

For SU(2), the pure gauge asymptotic behavious is utilized, for example in [*5*], with $a = d = e = 0$, $b = 2$.

(*ii*) $\underline{F = 0,\ C = 1}$

$$a = d = (b-1) = e = 0. \tag{4.7}$$

This is utilized, for example, in [*4*] for SU(2).

(*iii*) $\underline{(C, F) \neq 0}$

$$\begin{aligned} a = \pm d, \qquad e &= \pm (b-1), \\ 8(a^2 + e^2) &= 1. \end{aligned} \tag{4.8}$$

(Both upper or both lower signs are to be taken).

Thus even this simple choice leads to various possibilities.

Let us now introduce successively W_0 and Φ. Again we will only look for the simplest cases. The following results are quite easily obtained (always maintaining the constraint (4.3)).

(*iv*) Putting (compare [*7*])

$$c_0(r) = \frac{c_0}{r}, \qquad f_0(r) = \frac{f_0}{r^2} \tag{4.9}$$

with c_0, f_0 having the dimension of mass, one again obtains the case (*iii*) (this time with $W_0 \neq 0$ but $\vec{\mathscr{F}}_{(0)} = 0$) for

$$\begin{aligned} a = \pm d, \qquad (b-1) &= \pm e, \\ c_0 &= \mp 2f_0. \end{aligned} \tag{4.10}$$

(The upper or lower signs being taken throughout).

(*v*) Putting

$$c_0(r) = \frac{c_0}{r^2}, \qquad f_0(r) = f_0/r^3 \tag{4.11}$$

(with c_0, f_0 now dimensionless like the $a's$) and again with

$$\begin{aligned} a = \pm d, \qquad (b-1) &= \pm e, \\ c_0 &= \mp 2f_0 \neq 0, \end{aligned} \tag{4.12}$$

we obtain,

$$\begin{gathered} A_0(r) = D_0(r) = B_0(r) = E_0(r) = 0 \\ -r^4 C_0(r) = c_0, \qquad -r^5 F_0(r) = f_0, \\ \tilde{C}_0(r) = \tilde{F}_0(r) = 0. \end{gathered} \tag{4.13}$$

Thus all the equations of motions of motion are satisfied. But this time $\vec{\mathscr{F}}_{(0)}$ is non zero.

(*vi*) As for the scalar octet Φ we note from (3.15) that for

$$c_\varphi(r) = \frac{c_\varphi}{r}, \qquad f_\varphi(r) = \frac{f_\varphi}{r^2} \tag{4.14}$$

$$\frac{1}{2}\,\mathrm{Tr}\,\Phi^2 = \left(c_\varphi^2 + \frac{4}{3} f_\varphi^2\right) \tag{4.15}$$

is independent of r.

We will consider only this case and merely note that for

$$\begin{gathered} a = \pm\, d, \qquad (b-1) = \pm\, e, \qquad c_0 = \mp\, 2f_0, \\ c_\varphi = \mp\, 2\varphi \end{gathered} \tag{4.16}$$

we have only the further necessary constaint

$$\frac{\delta V(\Phi)}{\delta\Phi} = 0. \tag{4.17}$$

This leads to (with mass μ and quartic *c. c.* λ)

$$\left[\mu^2 - \lambda\left(c_\varphi^2 + \frac{4}{3} f_\varphi^2\right)\right] = 0$$

or

$$4f_\varphi^2 = c_\varphi^2 = \frac{3}{4}\mu^2/\lambda. \tag{4.18}$$

Thus c_φ, f_φ have the dimension of mass as they should.

The different cases of this section will serve as illustrations of some of the simplest types of exact solutions.

5. THE GAUGE CONDITION

Let us note that (from (2.6) and the appendix)

$$\begin{aligned} \vec{\nabla}\cdot\vec{W} &= \left\{\frac{1}{r}a'(r) + (rc'(r) + 4c(r))\right\}\hat{C} \\ &+ \left\{\frac{1}{r}d'(r) + (rf'(r) + 5f(r))\right\}\hat{F}. \end{aligned} \tag{5.1}$$

Hence (4.1) leads to

$$\vec{\nabla}\cdot\vec{W} = \frac{1}{r^3}(c-a)\hat{C} + \frac{1}{r^4}(f-2d)\hat{F}. \tag{5.2}$$

Thus if we impose (4.3)

$$a + c = 0 = d + f$$

then

$$\vec{\nabla} \cdot \vec{W} = -(\partial^\mu W_\mu) = 0$$

only for

$$a = c = d = f = 0, \tag{5.3}$$

and for any (b, e).

Thus we obtain the case of reference [2]. It is not essential to impose the transversality condition. But we find it interesting to generalize the gauge condition in the following way which is compatible with non zero values of a, c, d, f.

Using now (3.10), (4.14) and *an arbitrary constant* k, we obtain (for the non trivial case where (μ, λ) and hence (c_φ, f_φ) are non zero)

$$\left(\vec{\nabla} \cdot \vec{W} - \frac{k}{\mu} \vec{\nabla}^2 \Phi\right) = \frac{1}{r^3}\left\{(c - a) + \frac{2k}{\mu} c_\varphi\right\} \hat{C} + \frac{1}{r^4}\left\{(f - 2d) + \frac{6k}{\mu} f_\varphi\right\} \hat{F}. \tag{5.4}$$

Thus for

$$-\frac{2k}{\mu} = \frac{c - a}{c_\varphi} = \frac{f - 2d}{3f_\varphi} \tag{5.5}$$

we have the gauge condition,

$$\left(\partial \cdot W - \frac{k}{\mu} \Box \Phi\right) = 0. \tag{5.6}$$

This may be compared with the generalized covariant gauge condition used in perturbation treatment of Higgs-Kibble type of Lagrangians [8].

Such a gauge condition is of course not obligatory. But it is interesting to note that the simple case considered, namely

$$\begin{gathered} a + c = 0 = d + f \\ a = \pm d, \qquad (b - 1) = \pm e \\ c_0 = \mp 2f_0, \qquad c_\varphi = \mp 2f_\varphi \end{gathered} \tag{5.7}$$

fits in exactly with (5.6) which is however more general.

6. REMARKS

We have found a closed algebraic structure in terms of O_3 along with quadrupole generators and spherical harmonics. The calculations, apparently complicated to start with can thus be given a very systematic form, making the structural symmetries of the problem evident. This permits us to extract the constraints due to the group indices in an elegant fashion. We are left with the problem of solving the coupled non-linear radial equations. We have merely indicated the simplest types of « asymptotic » solutions. But the symmetries of the coupled equations with respect to the radial parameters may prove helpful in a more thorough investigation.

Let us now make a few remarks concerning some previously known particular cases in the context of our study.

In (2.6), if we put

$$a(r) = d(r) = c(r) = f(r) = 0 \tag{6.1}$$

we obtain the case of Wu and Wu [2] with the correspondence (using f and h of [2])

$$b(r) \rightarrow h, \qquad e(r) \rightarrow f. \tag{6.2}$$

(The imaginary factor of the O(3) generators λ_7, λ_5, λ_2 are to be taken into account in constructing our W_μ in the λ-basis from their D_i^{lm}). For the sake of comparison, let us consider this case separately. We obtain from (2.12) and (2.15) for

$$a(r) = d(r) = c(r) = f(r) = 0\,,$$
$$\tilde{A}(r) = \tilde{D}(r) = \tilde{C}(r) = \tilde{F}(r) = 0$$

identically and

$$\begin{aligned} -\tilde{B}(r) &= b''(r) + 4\frac{b'(r)}{r} + 7e^2(r)r^2 + 3b^2(r) - 7r^4b(r)e^2(r) - r^2b^3(r)\,, \\ -\tilde{E}(r) &= e''(r) + 6\frac{e'(r)}{r} + 14b(r)e(r) - 7r^2b^2(r)e(r) - r^4e^3(r)\,. \end{aligned} \tag{6.3}$$

Thus,

$$\tilde{B}(r) = 0, \qquad \tilde{E}(r) = 0\,,$$

gives us essentially the equations (18 b) and (18 a) of [2] respectively. But the relative signs of the terms do not always agree. Our version seems to be quite consistent and correct. Let us examine them further.

Putting

$$b(r) = \frac{b}{r^2}\,, \qquad e(r) = \frac{e}{r^3}$$

we obtain,

$$\begin{aligned} r^4\tilde{B}(r) &= (b-1)\{\,7e^2 + (b-1)^2 - 1\,\} = 0\,, \\ r^5\tilde{E}(r) &= e\{\,7(b-1)^2 + e^2 - 1\,\} = 0\,. \end{aligned} \tag{6.4}$$

Hence for $e = \pm(b-1)$ we get back the case (*iii*) of section 4 with $a = 0$. More probing is the fact that for

$$e = 0, \qquad \tilde{E}(r) = 0, \qquad r^4B(r) = b(b-1)(b-2)\,. \tag{6.5}$$

Thus for $b = 1, 2$ we get, for our pure O_3 case, the asymptotic SU(2) solutions of [4] and [5] respectively. This must, obviously, be case. (The relative signs of (18 b) of [2] lead to the solution $b = \left(\frac{3}{2} \pm \frac{\sqrt{17}}{2}\right)\Big)$. The highly symmetric forms of our general case also serve as checks.

Putting

$$d(r) = e(r) = f(r) = 0$$

we obtain essentially the general SU(2) solution of DHN [5], except that we have imbedded it into SU(3) through O(3). Can one use the SU(2) subgroup $(\lambda_1, \lambda_2, \lambda_3)$ in the same fashion? One solution is evident. We can add to the SU(2) solution a term proportional to λ_8. These two do not interfere. For example, for λ_8 we may choose a vortex like abelian solution [3]. So far we have not found any solution which generalizes the SU(2) ones by including also terms proportional to the generators transforming as spinors with respect to SU(2), namely $(\lambda_4, \lambda_5, \lambda_6, \lambda_7)$. Neither have we included fermions in our study though they have already been introduced in the SU(2) case [5]. We hope to study elsewhere, in a somewhat general fashion, the classical solutions of half integral spin fields coupled to gauge fields.

What about larger symmetry groups such as SU(4) and SU(6)? There is always the possibility of superposing known solutions for commuting subgroup.

But let us also note the following point. In the standard notation for the generators of unitary groups, the SU(4) generators M_i^4, M_4^i $(i = 1, 2, 3)$ transform as vectors with respect to the O_3 subgroup and M_4^4 as a scalar. This may possibly permit a generalization of our technique to SU(4).

Let us close with some brief remarks on the quadrupole generators of SU(3). Their role in the Elliott model [9] for nuclei is well known.

In the Gell-Mann-Okubo mass formula [10] the SU(3) symmetry breaking term is again a purely quadrupole operator with respect to the O(3) subgroup. In a basis completely symmetric with respect to I, U and V spins, where for example L_0, if one chooses to display it in the λ-basis, is

$$L_0 = \frac{1}{\sqrt{3}}(\lambda_2 - \lambda_5 + \lambda_7), \tag{6.6}$$

the famous mass operator (or (mass)^2 operator for mesons)

$$M_{(op)} = m_0 + m_1 Y + m_1\left(\vec{I}^2 - \frac{1}{4}Y^2\right) \tag{6.7}$$

can be expressed (using the cube roots of unity ω, ω^*) as

$$M_{(op)} = M_{(0)} + \left(\omega M_{(2)}^{+1} - \omega^* M_{(2)}^{-1}\right) - \frac{1}{\sqrt{2}}\left(\omega^* M_{(2)}^{+2} + \omega M_{(2)}^{-2}\right). \tag{6.8}$$

Here $M_{(0)}$ is a scalar and $M_{(2)}$ a quadrupole operator. For an octet (containing two O(3) multiplets $j = 1$, $j = 2$)

$$M_{(0)} = (m_0 + m_2), \qquad y = \frac{1}{2}m_2, \qquad z = m_1$$

where

$$y = \frac{\langle 1 \| \mathbf{M}_{(2)} \| 1 \rangle}{\sqrt{10}} = -\sqrt{\frac{3}{7}} \frac{\langle 2 \| \mathbf{M}_{(2)} \| 2 \rangle}{\sqrt{10}}$$

and

$$z = \sqrt{\frac{3}{10}} (i \langle 2 \| \mathbf{M}_{(2)} \| 1 \rangle) \tag{6.9}$$

are independent reduced matrix elements of $\mathbf{M}_{(2)}$ ($\langle 2 \| \mathbf{M}_{(2)} \| 1 \rangle$ being purely imaginary). For a decuplet ($j = 1, 3$) one can again easily obtain the well-known equispacing formula. Moreover by the simple substitution

$$\omega \rightarrow \omega + \zeta$$

where ζ is a small real parameter (ζ^2 being neglected), electromagnetic mass splittings can be included in a unified fashion.

We have briefly quoted these results from an old unpublished work of the present author in order to exhibit clearly the relation of the SU(3) quadrupole operators with the mass spectrum of hadrons.

It would be interesting to know the role played by our quadrupole radial parameters concerning masses corresponding to solutions with suitable physical properties.

APPENDIX

The rule stated in section 2 gives, in the λ-basis, the following expressions for W_1 and W_2. Defining,

$$\hat{C} \equiv (x_1\lambda_7 - x_2\lambda_5 + x_3\lambda_2), \tag{A.1}$$

$$\hat{F} \equiv 2(x_1x_2\lambda_1 + x_3x_1\lambda_4 + x_2x_3\lambda_6) + (x_1^2 - x_2^2)\lambda_3 + (r^2 - 3x_3^2) - \frac{\lambda_8}{\sqrt{3}} \tag{A.2}$$

we obtain

$$\begin{aligned} W_1 = {} & a(r)\lambda_7 - b(r)(x_2\lambda_2 + x_3\lambda_5) + c(r)x_1\hat{C} \\ & + d(r)\left(x_2\lambda_1 + x_3\lambda_4 + x_1\left(\lambda_3 + \frac{\lambda_8}{\sqrt{3}}\right)\right) \\ & + e(r)(x_3x_1\lambda_1 - x_1x_2\lambda_4 + (x_3^2 - x_2^2)\lambda_6 - x_2x_3(\lambda_3 - \sqrt{3}\lambda_8)) \\ & + f(r)x_1\hat{F} \\ \equiv {} & a(r)A_1 + b(r)B_1 + c(r)C_1 + d(r)D_1 + e(r)E_1 + f(r)F_1 . \end{aligned} \tag{A.3}$$

$$\begin{aligned} W_2 = {} & - a(r)\lambda_5 + b(r)(x_1\lambda_2 - x_3\lambda_7) + c(r)x_2\hat{C} \\ & + d(r)\left(x_1\lambda_1 + x_3\lambda_6 + x_2\left(- \lambda_3 + \frac{\lambda_8}{\sqrt{3}}\right)\right) \\ & + e(r)(- x_2x_3\lambda_1 + (x_1^2 - x_3^2)\lambda_4 + x_2x_1\lambda_6 - (\lambda_3 + \sqrt{3}\lambda_8)x_3x_1) \\ & + f(r)x_2\hat{F} \\ \equiv {} & a(r)A_2 + b(r)B_2 + \dots + f(r)F_2 . \end{aligned} \tag{A.4}$$

Let

$$\vec{W} = a(r)\vec{A} + b(r)\vec{B} + c(r)\vec{C} + d(r)\vec{D} + e(r)\vec{E} + f(r)\vec{F} . \tag{A.5}$$

We have the following relations

$$\begin{aligned} &\vec{B} = \vec{A} \times \vec{x}, \quad (\vec{C} - r^2\vec{A}) = \vec{B} \times \vec{x}, \quad \hat{C} = \vec{A}\cdot\vec{x} \\ &\vec{E} = \vec{D} \times \vec{x}, \quad (\vec{F} - r^2\vec{D}) = \vec{E} \times \vec{x}, \quad \hat{F} = \vec{D}\cdot\vec{x} . \end{aligned} \tag{A.6}$$

The following commutators are useful

$$\begin{aligned} &\vec{A} \times \vec{A} = i\vec{A}, \quad \vec{B} \times \vec{B} = i\vec{C}, \\ &\vec{D} \times \vec{D} = i(2r^2\vec{A} - \vec{C}), \quad \vec{E} \times \vec{E} = ir^2\vec{C} . \end{aligned} \tag{A.7}$$

$$\begin{aligned} &[\hat{C}, \hat{F}] = 0 , \\ &[\hat{C}, (\vec{A}, \vec{D}, \vec{B}, \vec{E})] = i(\vec{B}, \vec{E}, (\vec{C} - r^2\vec{A}), (\vec{F} - r^2\vec{D})) \\ &\qquad \text{(four commutators in order)} \\ &[\hat{F}, (\vec{A}, \vec{D}, \vec{B}, \vec{E})] = i2(\vec{E}, r^2\vec{B}, (\vec{F} - r^2\vec{D}), r^2(\vec{C} - r^2\vec{A})) . \end{aligned} \tag{A.8}$$

$$\{ [A_i, B_j] - [A_j, B_i] \} = iB_k \qquad (i, j, k = 1, 2, 3 \text{ in cyclic order})$$

i. e.

$$\vec{A} \times \vec{B} + \vec{B} \times \vec{A} = i\vec{B} ,$$

Similarly,

$$\begin{aligned} &\vec{A} \times \vec{D} + \vec{D} \times \vec{A} = i3\vec{D} , \\ &\vec{A} \times \vec{E} + \vec{E} \times \vec{A} = i\vec{E} , \\ &\vec{B} \times \vec{D} + \vec{D} \times \vec{B} = i2\vec{E} , \\ &\vec{B} \times \vec{E} + \vec{E} \times \vec{B} = i3\vec{F} , \\ &\vec{D} \times \vec{E} + \vec{E} \times \vec{D} = i2r^2\vec{B} . \end{aligned} \tag{A.9}$$

Such relations permit us to calculate in an intrinsic fashion independently of the λ-basis and quite simply.

For W_0 and its equations of motion the following ones are needed.

With summation over i $(= 1, 2, 3)$,

$$
\begin{aligned}
[C_i, (A_i, D_i, B_i, E_i)] &= 0\,, \\
[F_i, (A_i, D_i, B_i, E_i)] &= 0\,, \\
[A_i, (D_i, B_i, E_i)] &= i(0, 2\hat{C}, 3\hat{F})\,, \\
[B_i, (D_i, E_i)] &- i(-3\hat{F}, 0)\,, \\
[D_i, E_i] &= i2r^2\hat{C}\,.
\end{aligned}
\tag{A.10}
$$

Also,

$$
\begin{aligned}
&[(AA_i + \ldots + FF_i), (A_0A_i + \ldots + F_0F_i)] \\
&= i2\{(AB_0 - A_0B) + r^2(DE_0 - D_0E)\}\hat{C} + i3\{(AE_0 - A_0E) + (DB_0 - D_0B)\}\hat{F}\,.
\end{aligned}
\tag{A.11}
$$

Let us note that,

$$
\begin{gathered}
\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{D} = 0, \qquad \vec{\nabla} \times \vec{B} = 2\vec{A}, \qquad \vec{\nabla} \times \vec{E} = 3\vec{D}\,, \\
\vec{\nabla} \times \vec{C} = \vec{B}, \qquad \vec{\nabla} \times \vec{F} = 2\vec{E}\,;
\end{gathered}
$$

and

$$
\begin{gathered}
\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\vec{E} = 0\,; \qquad \vec{\nabla}\cdot\vec{C} = 4\hat{C}, \qquad \vec{\nabla}\cdot\vec{F} = 5\hat{F}\,; \\
\vec{\nabla}\hat{C} = \vec{A}, \qquad \vec{\nabla}\hat{F} = 2\vec{D}\,.
\end{gathered}
\tag{A.12}
$$

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