ANNALES DE L'I. H. P., SECTION A

E. RUDOLPH

Electromagnetic radiation damping of charges in external gravitational fields (weak field, slow motion approximation)

Annales de l'I. H. P., section A, tome 23, nº 2 (1975), p. 113-136
<http://www.numdam.org/item?id=AIHPA_1975__23_2_113_0>

Ann. Inst. Henri Poincaré,
Vol. XXIII, n° 2, 1975, p. 113-136.

Section A :
Physique théorique.

Electromagnetic radiation damping of charges in external gravitational fields (weak field, slow motion approximation)

by

E. RUDOLPH [1]
Max Planck Institut für Physik und Astrophysik
Föhringerring 6, München 40, Germany

ABSTRACT. — As a model for gravitational radiation damping of a planet the electromagnetic radiation damping of an extended charged body moving in an external gravitational field is calculated in harmonic coordinates using a weak field, slow-motion approximation. Special attention is paid to the case where this gravitational field is a weak Schwarzschild field. Using Green's function methods for this purpose it is shown that in a slow-motion approximation there is a strange connection between the tail part and the sharp part: Radiation reaction terms of the tail part can cancel corresponding terms of the sharp part. Due to this cancelling mechanism the lowest order electromagnetic radiation damping force in an external gravitational field in harmonic coordinates remains the flat space Abraham Lorentz force. We demonstrate in our simplified model that a naive slow-motion approximation may easily lead to divergent higher order terms. We show that this difficulty does not arise up to the considered order.

INTRODUCTION

The final aim of this and the subsequent paper is the calculation of the equations of motion of a planet orbiting a black hole. The problem becomes

(1) Main part of this work was done during an European fellowship in the Dept. of Astrophys. Oxford and the Dept. of Appl. Math. and Theor. Phys., Cambridge.

simpler if we replace the black hole by a noncollapsed star. In both cases the planet is moving under the influence of the total gravitational (grav.) field which is due both to the black hole (central star) and to the planet itself. In a consistent approximation (appr.) scheme we therefore have to calculate the total grav. field and to solve the equations of motion alternatively.

The appr. scheme we want to choose is the slow-motion (slow-mot.), weak field expansion. On a more advanced stage it may be necessary to apply different methods (e. g. the fast motion appr.). It depends on the special problem whether in zeroth appr. we choose the flat space or a given background metric (e. g. a black hole metric).

The mathematical tools we want to use for the calculation of the grav. field are Green's function methods in curved space [*1*] [*2*] [*3*] [*4*] [*5*] [*6*] and the method of matching asymptotic expansions (singular perturbation theory) [*7*] [*8*] [*9*] [*10*].

For the determination of the equations of motion of the planet it seems to us to be appropriate to treat the planet as an extended body and to discuss the limit of a pointlike source only afterwards. Thus we avoid the difficulties connected with pointlike sources but are confronted with the problem to describe « rigid » bodies in General Relativity (G. R.) [*11*] [*12*]. For the investigation of the equations of motion it is helpful to distinguish between conservative and nonconservative radiation (rad.) damping (damp.) forces working on the planet. Since nonconservative forces are connected with grav. rad. they deserve special attention. A slow mot. (inner) expansion yields a natural splitting into conservative and nonconservative forces.

Slow mot. appr. in G. R. have been investigated recently mainly by S. Chandrasekhar and coworkers [*13*] and by W. L. Burke [7] and K. Thorne. Presently W. G. Dixon [*14*] [*15*] and J. Bird [*16*] in Cambridge try to make these calculations more rigorous using Green's function methods in curved space. In this approach Green's function techniques (in a background metric) are used for imposing the correct boundary condition ([2]).

In connection with W. G. Dixon's programme we want to study in this paper a simpler problem which may serve as a model for grav. rad. damp. Using Green's function methods we want to calculate in a slow. mot. appr. the electromagnetic (el. magn.) rad. damp. of an extended charged body orbiting a (neutral) non-collapsed star. We only take into account the grav. field of this star (as an external grav. field) and disregard the grav. field of the charged body and of its el. magn. field. Our calculation of the

---

([2]) It is an open question whether instead of using curved space retarded Green's functions one could not use flat space retarded Green's functions in each order of such an appr. scheme as well and get the same result. This is in fact the usual approach. In some simple cases it can be shown that both approaches are equivalent.

el. magn. rad. damp. exactly corresponds to the wellknown derivation of the Abraham-Lorentz force in flat space [*17*].

We look at el. rad. damp. in an external grav. field mainly as a model for grav. rad. damp. El. magn. rad. damp. of celestial bodies itself does not play a significant role since they are neutral up to a very good appr. Our calculations still describe the non-relativistic rad. damp. of charged particles in a grav. field which is a rather academic problem however.

El. rad. damp. of pointlike charges in a external grav. field has been investigated first by B. S. De Witt and R. W. Brehme [*18*] [*19*] [*20*]. C. M. De Witt and B. S. De Witt [*21*] [*22*] [*23*] have discussed the non-relativistic limit choosing for the grav. field a weak Schwarzschild field. In comparison with these papers we do all calculations for extended charge distributions and we do not discuss the non-relativistic limit of a relativistic formula but use slow. mot. appr. from the very beginning. From this point of view el. magn. rad. damp. in a background metric looks very different.

We first want to briefly report De Witt and De Witt's stand-point. According to their paper the sharp part of the Green's function does not yield any contribution to rad. damp. of a pointlike charge. Therefore the tail alone is responsible for rad. damp. and yields (after a very complicated calculation) simply the well-known Abraham Lorentz force in the considered order. This result is really miraculous!

In the case of an extended charge the situation is quite different. In the considered slow mot. appr. we get a rad. damp. force $\overset{(3)}{\mathscr{R}}{}^i(x)^{(\#)}$ (acting on one point of the charge distribution) due to the sharp part and a force $\overset{(3)}{\mathscr{R}}{}^i(x)^{(b)}$ due to the tail part. $\overset{(3)}{\mathscr{R}}{}^{i(\#)}$ can be split into a part due to the flat space, namely the Abraham Lorentz force $\overset{(3)}{\mathscr{R}}{}^{i(\#F)}$ and a correction term $\overset{(3)}{\mathscr{R}}{}^{i(\#C)}$ due to the background metric. Only in the limit of a pointlike charge $\overset{(3)}{\mathscr{R}}{}^{i(\#F)}$ is cancelled by $\overset{(3)}{\mathscr{R}}{}^{i(\#C)}$. The tail yields a contribution $\overset{(3)}{\mathscr{R}}{}^{i(b)}$ which cancels $\overset{(3)}{\mathscr{R}}{}^{i(\#C)}$ for an arbitrary charge distribution. From our point of view the Abraham Lorentz force $\overset{(3)}{\mathscr{R}}{}^{i(\#F)}$ which we finally obtain is therefore due to the sharp part.

However in the limit of a pointlike charge the very strange situation occurs that we get (apart from the sign) the same result for $\overset{(3)}{\mathscr{R}}{}^{i(\#F)}$, $\overset{(3)}{\mathscr{R}}{}^{i(\#C)}$ and $\overset{(3)}{\mathscr{R}}{}^{i(b)}$, namely the Abraham Lorentz force. Therefore we can either say that

$\overset{(3)}{\mathscr{R}}{}^{i(\#F)}$ is cancelled by $\mathscr{R}^{i(\#C)}$ thus leaving us with
$\overset{(3)}{\mathscr{R}}{}^{i(b)}$ (De Witt and De Witt's standpoint) or that
$\overset{(3)}{\mathscr{R}}{}^{i(\#C)}$ is cancelled by $\overset{(3)}{\mathscr{R}}{}^{i(b)}$ thus leaving us with

$\overset{(3)}{\mathcal{R}}{}^{i(\#F)}$ (our standpoint). Our standpoint obviously is more general and more natural.

The paper is divided into four chapters:

In the first chapter we give a brief summary of the properties of the vector Green's function in a background metric.

In the second chapter we calculate the Green's function using an expansion of the background metric in a weakness parameter. We assume that the background metric is stationary. We specialize the most important results for the Schwarzschild metric.

A retardation expansion of the sharp part is performed in the first part of the third chapter. Using this expansion we calculate the contribution of the sharp part to radiation damping of a charged body to third order in the expansion parameter. The limit of a pointlike charge is discussed.

In the second part of the third chapter we do the analogous calculation for the tail term.

In the fourth chapter we discuss the slow mot. expansion of the tail in detail. We show that different asymptotic expansions for small and for large retardation times of the tail have to be used.

I. PROPERTIES OF THE EXACT VECTOR GREEN'S FUNCTION IN A BACKGROUND METRIC

For the calculation of the el. magn. field of a charge distribution we need the vector Green's function. The retarded Green's function is defined by the following equation [*18*]:

$$\Box_2 G_{R_{\mu_2\nu_1}}(x_2, x_1) - R_{\mu_2}{}^{\varkappa_2} G_{R_{\varkappa_2\nu_1}}(x_2, x_1) = + \frac{1}{\sqrt{-g}} \delta^4(x_2, x_1) \bar{g}_{\mu_2\nu_1}(x_2, x_1) \quad (I.1)$$

and the condition that $G_{R_{\mu_2\nu_1}}$ is nonzero only on and inside of the forward light cone of x_1. If x_2 is in a normal neighborhood of x_1 it has the form (3):

$$G_{R_{\mu_2\nu_1}}(x_2, x_1) = \frac{1}{4\pi}[\Delta^{1/2}(x_2, x_1)\bar{g}_{\mu_2\nu_1}(x_2, x_1)\delta_R(\Omega(x_2, x_1)) - v_{\mu_2\nu_1}(x_2, x_1)\Theta_R(-\Omega(x_2, x_1))] \quad (I.2)$$

(3) Existence and uniqueness of Green's functions in the form (I.2) usually is guaranteed only in a normal neighborhood. In this paper we use Green's functions in a rather naive manner also in global regions of space time. In any case a normal neighborhood would be too restrictive. We hope that our calculations can be justified especially since we assume weak fields everywhere.

As usually we want to distinguish between the sharp part and the tail part of the Green's function using P. C. Waylen's notation [*24*]:

$$\text{sharp part:} \qquad G_{R_{\mu_2\nu_1}}^{(\#)} = \frac{1}{4\pi}[\Delta^{1/2}\bar{g}_{\mu_2\nu_1}\delta_R(\Omega)] \; (a)$$

$$\text{tail part:} \qquad G_{R_{\mu_2\nu_1}}^{(b)} = -\frac{1}{4\pi}[v_{\mu_2\nu_1}\Theta_R(-\Omega)] \; (b) \qquad \text{(I.3)}$$

We first discuss the sharp part. Δ, Ω and $\bar{g}_{\mu_2\nu_1}$ are bitensors; μ_1 refers to x_1, μ_2 to x_2. Ω is the world function

$$\Omega = \frac{1}{2} S^2(x_2, x_1) \qquad \text{(I.4)}$$

where Δ is the geodesic distance. $\delta_R(\Omega)$ is the retarded δ-fkt whose support is the forward light cone through x_1.

$\bar{g}_{\mu_2\nu_1}(x_2, x_1)$ is the parallel propagator which reduces to the metric tensor $g_{\mu\nu}$ if $x_2 = x_1$. Δ is a determinant defined by:

$$\Delta = -\,|\,\Omega_{;\mu_2;\nu_1}(x_2, x_1)\,| \cdot \frac{1}{g^{1/2}(x_1)g^{1/2}(x_2)} \qquad \text{(I.5)}$$

For Ω and $\bar{g}_{\mu_2\nu_1}$ we list some important relations which will be useful for our approximative calculation later on:

$$\Omega_{;\mu_1}g^{\mu_1\nu_1}\Omega_{;\nu_1} = 2\Omega, \; \Omega_{;\mu_2}g^{\mu_2\nu_1}\Omega_{;\nu_1} = 2\Omega \qquad \text{(I.6)}$$

$$\bar{g}_{\mu_2\alpha_1;\nu_2}g^{\nu_2\beta_2}\Omega_{;\beta_2} = 0. \qquad \bar{g}_{\mu_2\alpha_1;\beta_1}g^{\beta_1\gamma_1}\Omega_{;\gamma_1} = 0 \qquad \text{(I.7)}$$

In order to understand the role of the tail term, which is nonzero also inside the light cone, we calculate:

$$\Box_2 G_{R_{\mu_2\nu_1}}^{(\#)} - R_{\mu_2}{}^{\varkappa_2}G_{R_{\varkappa_2\nu_1}}^{(\#)} = \frac{1}{4\pi}[\Box_2(\Delta^{1/2}\bar{g}_{\mu_2\nu_1}) - R_{\mu_2}{}^{\varkappa_2}\Delta^{1/2}\bar{g}_{\varkappa_2\nu_1}]\delta_R(\Omega) + \bar{g}_{\mu_2\nu_1}\frac{1}{\sqrt{-g}}\delta^4(x_2, x_1) \qquad \text{(I.8)}$$

Apart from the δ^4-function we obtain a term which has to be compensated by the tail. One can therefore derive an integral equation for the tail term ([4]):

$$G_{R_{\mu_2\alpha_1}}^{(b)}(x_2, x_1) = -\int_{\infty} G_{R_{\mu_2}}{}^{\nu_3}(x_2, x_3)\frac{1}{4\pi}[\Box_3(\Delta^{1/2}(x_3, x_1)\bar{g}_{\nu_3\alpha_1}(x_3, x_1)) - R_{\nu_3}{}^{\varkappa_3}\bar{g}_{\varkappa_3\alpha_1}(x_3, x_1)\Delta^{1/2}(x_3, x_1)]\delta_R(\Omega(x_3, x_1))\sqrt{-g}d^4x_3 \qquad \text{(I.9)}$$

This integral equation is very useful for approximative solutions.

([4]) We have dropped the surface term in (I.9) which appears in a general solution of (I.8) by means of Green's functions [*25*]. In fact we impose the boundary condition by assuming that this surface term vanishes at past infinity. Retarded solutions in this paper automatically mean that the surface term is zero.

II *a*. WEAK FIELD EXPANSION OF THE VECTOR GREEN'S FUNCTION IN A STATIONARY BACKGROUND METRIC

In general we cannot hope to find an exact solution for the vector Green's function in a curved space. Thus we have to look for suitable appr. methods. One possible appr. method is the weak field expansion (5) where one expands the background metric in a certain coordinate system in powers of a weakness parameter and assumes a corresponding power series expansion of the Green's function. We want to point out that this appr. is based essentially on an expansion of the gauge variant metric tensor. Since we have not found a suitable appr. method using gauge invariant quantities only we perform all (coordinate dependent) calculations in a fixed coordinate system. However, these calculations have to be looked at as an intermediate step only since one still has to extract observable quantities afterwards.

In a weak field expansion of the Green's function we have to treat the sharp part and the tail part on the same footing. There is no *a priori* reason why we may neglect the tail term since both parts are of the same order in the weakness parameter.

We now want to expand the background metric in a weakness parameter ε. Although we are interested mainly in the weak Schwarzschild metric and in the weak Kerr metric (both with a regular interior solution since a black hole metric cannot be expanded in a weakness parameter everywhere) we want to include more general cases. We assume that the metric is stationary and asymptotically flat and that it corresponds to a solution of Einstein's equations (in a slow mot. appr.) with a source which is restricted to a certain finite region.

In order to justify our expansion of the background metric we want

(5) Throughout this paper expansions have to be regarded as asymptotic expansions with the definition [7] [8] [9] [*10*]:

$$f(x, \varepsilon) \sim \sum_{k=0}^{\infty} f_k(x, \varepsilon)$$

$$\lim \left(f(x, \varepsilon) - \sum_{k=0}^{m} f_k(x, \varepsilon) \right) / \gamma_m(\varepsilon) = 0$$

with the asymptotic sequence $\gamma_n(\varepsilon)$. ($\gamma_{n+1} = o(\gamma_n)$). In most cases we choose ε^n for this sequence. We have to remark however that in most cases we can show only that

$$f_{n+1}(x, \varepsilon) = o(\varepsilon^n)$$

and that $\sum_{n=0}^{m} f_n(x, \varepsilon)$ fulfills certain defining equations up to m-th order in ε.

to make some remarks concerning the slow mot. weak field appr. in G. R. W. G. Dixon [27] has suggested a general rule to determine the orders of magnitude in such an expansion. It is based on the observation that a solution of the Newtonian equations can be transformed into another solution by the transformation:

$$\begin{aligned} x^0 &\to \varepsilon^{-1} x^0 \\ x^i &\to x^i \end{aligned} \tag{II.1}$$

$$\begin{aligned} \rho_M &\to \varepsilon^2 \rho_M \\ v^i &\to \varepsilon v^i \\ \mathcal{U} &\to \varepsilon^2 \mathcal{U}, \quad \text{etc.} \end{aligned} \tag{II.2}$$

In (II.1) only the time coordinate is transformed. In (II.2) ρ_M is the mass density, v^i the velocity and U the Newtonian grav. potential.

The transformation (II.1), (II.2) defines a whole family of solutions depending on ε which is necessary for the determination of the orders of magnitude. We also want to use the transformation (II.1) to define the orders of magnitude in a slow mot. appr. in G. R. Furthermore we want to start in lowest order of such an appr. scheme with a family of solutions of the Newtonian equations ([6]) (defined by (II.1) and (II.2)).

We therefore make the ansatz for a stationary metric:

$$\begin{aligned} g_{00} &= 1 + \overset{(2)}{g}_{00} + \overset{(4)}{g}_{00} + \dots \\ g_{ij} &= -\delta_{ij} + \overset{(2)}{g}_{ij} + \overset{(4)}{g}_{ij} + \dots \\ g_{i0} &= + \overset{(3)}{g}_{i0} + \dots \\ g^{00} &= 1 - \overset{(2)}{g}{}^{00} + \dots \\ g^{ij} &= -\delta^{ij} - \overset{(2)}{g}{}^{ij} + \dots \\ g^{i0} &= -\overset{(3)}{g}{}^{i0} + \dots \end{aligned} \tag{II.3}$$

We want to use the convention that in such a series expansion (upon the flat space) lowering and raising of indices refer to the Minkowski metric.

Now we want to calculate in a weakness expansion the world function Ω, the determinant Δ and the parallel propagator $\bar{g}_{\mu\alpha}$ which are essential elements of the vector Green's function. We make the ansatz:

$$\begin{aligned} \Omega &= \overset{(0)}{\Omega} + \overset{(2)}{\Omega} + \overset{(3)}{\Omega} + \dots \\ \overset{(0)}{\Omega} &= \frac{1}{2}\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu, \quad \Delta x^\mu = x_2^\mu - x_1^\mu \end{aligned} \tag{II.4}$$

([6]) Following the usual custom we want to assume in this paper that we are dealing with bound systems since the ordering $M = O(\varepsilon^2) v = O(\varepsilon)$ given above is appropriate for such a situation. However, it may be possible to apply this slow mot. appr. scheme also to unbound motion in some cases.

Relation (I.6) yields:

$$\overset{(i)}{\Omega} = \overset{(i)}{\Omega}_{\mu\nu}\Delta x^\mu \Delta x^\nu \qquad i = 2, 3$$
$$\overset{(i)}{\Omega}_{\mu\nu} = \frac{1}{2}\int_0^1 \overset{(i)}{g}_{\mu\nu}(x(\omega))d\omega \tag{II.5}$$
$$x^\mu = x_1^\mu + \omega \Delta x^\mu$$

In (II.5) we obviously integrate along a straight line and this simplifies our calculations considerably. To show that (II.5) fulfills (I.6) in the considered order we need to know the derivatives of $\overset{(i)}{\Omega}$. We have:

$$\frac{\partial}{\partial x_2^\varkappa}\overset{(i)}{\Omega} = \int_0^1 \overset{(i)}{g}_{\varkappa\nu}(x(\omega))d\omega\Delta x^\nu + \frac{1}{2}\int_0^1 \overset{(i)}{g}_{\mu\nu,\varkappa}(x(\omega))d\omega\Delta x^\mu\Delta x^\nu \qquad i = 2, 3$$
$$\frac{\partial}{\partial x_1^\varkappa}\overset{(i)}{\Omega} = -\int_0^1 \overset{(i)}{g}_{\varkappa\nu}(x(\omega))d\omega\Delta x^\nu + \frac{1}{2}\int_0^1 \overset{(i)}{g}_{\mu\nu}(x(\omega))(1-\omega)d\omega\Delta x^\mu\Delta x^\nu \tag{II.6}$$

The sharp part of the Green's function (I.3) (*a*) contains the factor $\delta_R(\Omega)$. A transformation of variables shows that we can replace $\delta_R(\Omega)$ by:

$$\delta_R(\Omega) \rightarrow \frac{\delta(\Delta x^0 - \delta x^0)}{|\Gamma|} \tag{II.7}$$

with

$$\Gamma = \frac{\partial}{\partial x_1^0}\Omega\,|_{\Omega=0} \tag{II.8}$$

δx^0 is the retardation time for which we make the ansatz:

$$\delta x^0 = \overset{(0)}{\delta} x^0 + \overset{(2)}{\delta} x^0 + \overset{(3)}{\delta} x^0 + \dots \tag{II.9}$$

We obtain using (II.5):

$$\overset{(0)}{\delta} x^0 = \sqrt{\Delta x^i \Delta x^i} \overset{!}{=} R$$
$$\overset{(2)}{\delta} x^0 = -[\overset{(2)}{\Omega}_{ij}\Delta x^i \Delta x^j]\cdot\frac{1}{R} - \overset{(2)}{\Omega}_{00}R \tag{II.10}$$
$$\overset{(3)}{\delta} x^0 = -2\overset{(3)}{\Omega}_{0i}\Delta x^i$$

We get for Γ:

$$\Gamma = \overset{(0)}{\Gamma} + \overset{(2)}{\Gamma} + \overset{(3)}{\Gamma} + \dots \tag{II.11}$$

with

$$\overset{(0)}{\Gamma} = -R$$
$$\overset{(2)}{\Gamma} = -\overset{(2)}{\Omega}_{00}R + [\overset{(2)}{\Omega}_{ij}\Delta x^i\Delta x^j]\cdot\frac{1}{R} \tag{II.12}$$
$$\overset{(3)}{\Gamma} = 0$$

Using the definition for Δ (I.5) we immediately obtain with (II.3) and (II.4):

$$\begin{aligned}
&\Delta = \overset{(0)}{\Delta} + \overset{(2)}{\Delta} + \overset{(3)}{\Delta} + \dots \\
&\overset{(0)}{\Delta} = 0 \\
&\sqrt{-g} = 1 + \frac{1}{2}\overset{(2)}{g}{}_{\mu}{}^{\mu} + \frac{1}{2}\overset{(3)}{g}{}_{\mu}{}^{\mu} + \dots \\
&\overset{(i)}{\Delta} = -\left\{\frac{1}{2}[\overset{(i)}{g}{}_{\mu}{}^{\mu}(x_1) + \overset{(i)}{g}{}_{\mu}{}^{\mu}(x_2)] + \eta^{\mu\alpha}\cdot\overset{(i)}{\Omega}_{,\mu_1,\alpha_2}\right\} \qquad i = 2,3
\end{aligned} \tag{II.13}$$

If we put $\overset{(i)}{\Omega}$ from (II.5) into (II.13) it turns out that all expressions become considerably simplified if we use harmonic coordinates:

$$\begin{aligned}
&\partial_\mu[g^{\mu\nu}\sqrt{-g}] = 0 \\
&\partial_\mu \overset{(i)}{g}{}^{\mu\nu} = \frac{1}{2}\partial^\nu \overset{(i)}{g}{}_{\mu}{}^{\mu}
\end{aligned} \tag{II.14}$$

In the following we want to choose a harmonic coordinate system defined by (II.14).

We then obtain for $\overset{(i)}{\Delta}$:

$$\begin{aligned}
&\overset{(i)}{\Delta} = \overset{(i)}{\Delta}_{\varkappa\lambda}\Delta x^\varkappa\cdot\Delta x^\lambda \\
&\overset{(i)}{\Delta}_{\varkappa\lambda} = -\frac{1}{2}\int_0^1 \eta^{lm}\partial_l\partial_m[\overset{(i)}{g}_{\varkappa\lambda}](\omega - \omega^2)d\omega \qquad i = 2, 3
\end{aligned} \tag{II.15}$$

We remark that $-\frac{1}{2}\eta^{lm}\partial_l\partial_m[\overset{(i)}{g}_{\varkappa\lambda}]$ is exactly the Ricci tensor in this appr.:

$$\overset{(i)}{R}_{\mu\nu} = -\frac{1}{2}\eta^{lm}\partial_l\partial_m\overset{(i)}{g}_{\mu\nu} \tag{II.16}$$

We go over to the calculation of the parallel propagator. Relation (I.7) yields:

$$\begin{aligned}
&\bar{g}_{\mu_2\alpha_1}(x_2, x_1) = \eta_{\mu\alpha} + \overset{(2)}{\bar{g}}_{\mu\alpha}(x_2, x_1) + \overset{(3)}{\bar{g}}_{\mu\alpha}(x_2, x_1) + \dots \\
&\overset{(i)}{\bar{g}}_{\mu\alpha} = \overset{(i)}{\bar{g}}_{S\mu\alpha} + \overset{(i)}{\bar{g}}_{A\mu\alpha} \\
&\overset{(i)}{g}_{S\mu\alpha} = \frac{1}{2}[\overset{(i)}{g}_{\mu\alpha}(x_2) + \overset{(i)}{g}_{\mu\alpha}(x_1)] \\
&\overset{(i)}{\bar{g}}_{A\mu\alpha} = \overset{(i)}{\bar{g}}_{\mu\alpha\varkappa}\Delta x^\varkappa \qquad i = 2, 3 \\
&\overset{(i)}{\bar{g}}_{\mu\alpha\varkappa} = \frac{1}{2}\int_0^1 [\overset{(i)}{g}_{\varkappa\alpha,\mu} - \overset{(i)}{g}_{\varkappa\mu,\alpha}]d\omega
\end{aligned} \tag{II.17}$$

$$G_{R_{\mu_2\alpha_1}}^{(\#)}(x_2, x_1) = \overset{(0)}{G}{}_{R_{\mu\alpha}}^{(\#)}(x_2, x_1) + \overset{(2)}{G}{}_{R_{\mu\alpha}}^{(\#)}(x_2, x_1) + \overset{(3)}{G}{}_{R_{\mu\alpha}}^{(\#)}(x_2, x_1) + \dots \tag{II.18}$$

with (7):

$$\overset{(i)}{G}_{R_{\mu\alpha}}^{(*)}(x_2, x_1) = \frac{1}{4\pi}\left[\frac{1}{2}\overset{(i)}{\Delta}\eta_{\mu\alpha}\delta_R(\overset{(0)}{\Omega}) + \overset{(i)}{\bar{g}}_{\mu\alpha}\delta_R(\overset{(0)}{\Omega}) + \eta_{\mu\alpha}\delta'_R(\overset{(0)}{\Omega})\overset{(i)}{\Omega} + \dots\right] \qquad i = 2, 3 \quad \text{(II.19)}$$

Finally we want to perform a weak field calculation for the tail term: Expanding $G_{R_{\mu_2\alpha_1}}^{(b)}$ in (I.9) we obtain:

$$\begin{aligned} G_{R_{\mu_2\alpha_1}}^{(b)} &= \overset{(2)}{G}_{R_{\mu\alpha}}^{(b)} + \overset{(3)}{G}_{R_{\mu\alpha}}^{(b)} + \dots \\ R_{\mu\nu} &= \overset{(2)}{R}_{\mu\nu} + \overset{(3)}{R}_{\mu\nu} + \dots \end{aligned} \qquad \text{(II.20)}$$

$$\begin{aligned} \overset{(i)}{G}_{R_{\mu\alpha}}^{(b)} = & -\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} \delta_R(\overset{(0)}{\Omega}(x_2, x_3)) \\ & \times \frac{1}{4\pi}\left\{\overset{(0)}{\Box}_3\left(\frac{1}{2}\overset{(i)}{\Delta}(x_3, x_1)\eta_{\mu\alpha}\right) + \overset{(i)}{\bar{g}}_{\mu\alpha}(x_3, x_1) - \overset{(i)}{R}_{\mu\alpha} - \partial^\lambda\overset{(i)}{\Gamma}_{\alpha|\mu\lambda}(x_3)\right\} \\ & \times \delta_R(\overset{(0)}{\Omega}(x_3, x_1))d^4x_3 \qquad i = 2, 3 \end{aligned} \qquad \text{(II.21)}$$

In (II.21) we have used the expansion of the d'Alembert operator:

$$\Box = \overset{(0)}{\Box} + \overset{(2)}{\Box} + \overset{(3)}{\Box} + \dots \qquad \text{(II.22)}$$

Contrary to (I.9) we have integrated over a finite volume in (II.21). ($|\vec{x}_3| \leq \lambda$, λ is the characteristic wavelength of the source to be introduced later on). We use this cutoff because our expansion of the tail is not uniform for arbitrarily large $\vec{x}_3$. In Chapter IV we shall discuss this point in detail and we also shall try to justify the cutoff.

(II.21) immediately suggests the interpretation that the tail is due to a backscatter from the background metric. Now we can show that to the considered order the backscatter is only due to the region where $R_{\mu\nu} \neq 0$ that means to the matter region (In the full theory the backscatter vanishes only if $R_{\mu\nu\kappa\lambda} = 0$).

We get in harmonic coordinates:

$$\partial^\lambda\overset{(i)}{\Gamma}_{\kappa|\mu\lambda} = \frac{1}{2}\eta^{ij}\partial_i\partial_j\overset{(i)}{g}_{\kappa\mu} \qquad i = 2, 3 \qquad \text{(II.23)}$$

Therefore with (II.16):

$$\overset{(i)}{R}_{\mu\alpha} + \partial^\lambda\overset{(i)}{\Gamma}_{\alpha|\mu\lambda} = 0 \qquad i = 2, 3 \qquad \text{(II.24)}$$

(7) Taylor expansions of distributions certainly make sense only if the test functions admit similar expansions. It can be shown that our expansions make sense if the source describes a sufficiently smooth world tube in space time.

We know from (II.15), (II.16) already that $\overset{(i)}{\Delta}$ vanishes in empty space. It remains to calculate:

$$\overset{(0)}{\square}_3 \overset{(i)}{\bar{g}}_{\mu\alpha}(x_3, x_1) = \left[-\frac{1}{2}\eta^{ij}\partial_{i_3}\partial_{j_3}\overset{(i)}{g}_{\mu\alpha}(x_3) \right] - \frac{1}{2}\int_0^1 \eta^{ij}\partial_i\partial_j[\overset{(i)}{g}_{\varkappa\alpha,\mu} - \overset{(i)}{g}_{\varkappa\mu,\alpha}]\omega^2 d\omega \Delta x^\varkappa \qquad i = 2, 3 \quad \text{(II.25)}$$

With (II.16) it is obvious that the expressions (II.25) vanish in empty space.

Although only the matter region is responsible for the backscatter in the considered order there is also a backscatter from outside the matter region, namely if the straight line $\overline{x_1 x_3}$ crosses this region. We shall come back to this point in chapter IV.

II *b*. SPECIALIZATION FOR THE WEAK SCHWARZSCHILD FIELD

Now we want to apply our results to the weak Schwarzschild metric: in harmonic coordinates the weak Schwarzschild metric has the form (with the convention $c = 1$):

$$\begin{aligned} \overset{(0)}{g}_{\mu\nu} &= \eta_{\mu\nu} \\ \overset{(2)}{g}_{00} &= -2\mathcal{U}(r) \\ \overset{(2)}{g}_{ij} &= -2\mathcal{U}(r)\delta_{ij} \\ \overset{(2)}{g}_{\mu}{}^{\mu} &= 4\mathcal{U}(r), \qquad g_{0i} = 0 \end{aligned} \qquad \text{(II.26)}$$

with

$$\eta^{ij}\partial_i\partial_j\mathcal{U}(r) = 4\pi\rho_M \qquad \text{(II.27)}$$

ρ_M is the mass density of the spherically symmetric matter distribution which we assume to be nonzero only within a sphere of radius r_{ρ_M}.

Since the metric shall be weak everywhere we have to require that r_{ρ_M} is much larger than the Schwarzschild radius of the central mass M.

Outside the matter region we obtain for the metric

$$\mathcal{U}(r) = \frac{M}{r} \qquad \text{(II.28)}$$

With the metric (II.26) we get for (II.10):

$$\overset{(2)}{\delta} x^0 = \int_0^1 2\mathcal{U}(r) d\omega R \qquad \text{(II.29)}$$

and for (II.12)

$$\overset{(2)}{\Gamma} = 0 \tag{II.30}$$

(II.15) yields:

$$\overset{(2)}{\Delta} = -\int_0^1 4\pi\rho(\omega - \omega^2)d\omega((\Delta x^0)^2 + R^2) \tag{II.31}$$

and we get for the parallel propagator using (II.17):

$$\begin{aligned}
\overset{(2)}{\bar{g}}_{00} &= -[\mathcal{U}(r_2) + \mathcal{U}(r_1)] \\
\overset{(2)}{\bar{g}}_{0i} &= \int_0^1 \frac{\partial}{\partial x^i}\mathcal{U}(r)d\omega(\Delta x^0) = -\overset{(2)}{g}_{i0} \\
\overset{(2)}{\bar{g}}_{ij} &= -\delta_{ij}[\mathcal{U}(r_1) + \mathcal{U}(r_2)] \\
&\quad + \int_0^1 \frac{\partial}{\partial x^i}\mathcal{U}(r)d\omega\Delta x^j - \int_0^1 \frac{\partial}{\partial x^j}\mathcal{U}(r)d\omega\Delta x^i
\end{aligned} \tag{II.32}$$

III *a*. CALCULATION OF THE RADIATION DAMPING FORCE OF AN ORBITING CHARGE DUE TO THE SHARP PART OF THE VECTOR GREEN'S FUNCTION

(Retardation (slowness) expansion of the sharp part).

In addition to the weak field expansion described in the last chapter we now want to perform a retardation expansion of the sharp part. This retardation expansion in fact is a slowness expansion in a parameter v_0 where v_0 is the average velocity of the orbiting charge. But we know already from our general considerations about slow mot. appr. in G. R. that

$$v_0 = O(\varepsilon) \tag{III.1}$$

where ε is the expansion parameter introduced in chapter II. Therefore for the slowness—weak field expansion we want to use one parameter ε only. Nevertheless we perform the slowness and the weak field expansion one after another because this enables us to impose the boundary condition without going to infinity. This approach yields a deep insight into the weak field, slow. mot. appr. in curved space. As will turn out later on it is very important to understand the role of the sharp part and of the tail in such an appr. On the other hand it may have computational advantages to do the weakness and the slowness expansion simultaneously. This approach was discussed in G. R. mainly by W. L. Burke [7]. In this case one has to use matched asymptotic expansions in order to impose the correct boundary condition. In this paper we only make use of the first method.

We now want to define our problem in detail. We assume than an extended charged body is orbiting in an external grav. field outside the matter region $R_{\mu\nu} \neq 0$. To be precise the charge is moving under the influence of the external grav. field (e. g. the Schwarzschild field) and its own el. magn. field.

Therefore we get the following equations of motion for the charge distribution:

$$\nabla_{\varkappa} T^{\mu\varkappa} + F^{\mu}_{\cdot\nu} j^{\nu} = 0 \qquad \mathscr{R}^{\mu} \overset{!}{=} F^{\mu}_{\cdot\nu} j^{\nu} \tag{III.2}$$

with the field equations for $F^{\mu\nu}$:

$$\nabla_{\nu} F^{\mu\nu} = 4\pi j^{\nu} \qquad j^{0} = \frac{\rho_e}{\sqrt{-g}} \qquad j^{i} = \frac{\rho_e v^{i}}{\sqrt{-g}} \tag{III.3}$$

We have to add certain initial or boundary conditions to the equations (III.2) and (III.3) and to specify the structure of $T^{\mu\varkappa}$ in order to get a uniquely defined problem. The « physical » situation requires that we have to impose initial conditions on the charged body and boundary condition on the el. magn. field $F^{\mu\nu}$. That means we assume that certain initial values are given for $T^{\mu\nu}$ and j^{μ} of the charged body at $x^0 = 0$. For the el. magn. field we want to require that we have outgoing rad. only or that the el. magn. field is purely retarded (8) (with respect to the background metric). In this paper we make use of the second condition only.

We want to determine (using the equations (III.2), (III.3) and the initial and boundary condition) the motion of the charged body in a neighborhood of $x^0 = 0$ or the force working on the charged body at $x^0 = 0$. In this paper we are concerned with the calculation of the rad. damp. force only.

Although the problem formally seems to be well defined now in practice there are still serious problems due to the tail of the retarded Green's function. The force acting on the charge at $x^0 = 0$ depends on the whole past history of the charge. However, since the tail usually falls down rapidly for large retardation times it is not necessary to know the whole past history of the charge for an approximative calculation. We use the following appr. method:

Before we perform a slow. mot. appr. we assume an expansion of the solution of (III.2), (III.3) in powers of the el. magn. coupling constant (or a related dimensionless parameter $\varkappa$ typical for rad. damp.) in a neighborhood of x^0 (9). In the limit of a vanishing $\varkappa$ we obtain an undamped

(8) It is an open question whether both conditions are equivalent.

(9) This appr. is familiar from el. magn. rad. damp. calculations in flat space [*28*]. It makes sense only in a neighborhood of $x^0 = 0$. However because of the tail the rad. damp. force at $x^0 = 0$ depends on the whole past of the charge and therefore the method only works if the tail falls down sufficiently for large retardation times and if the damping is not too strong. If we calculate grav. rad. damp. using a slow mot. appr. described in chapter II we automatically perform an analogous expansion. In this case it is sufficient to use one expansion parameter only.

orbit for the charged body (zeroth order). We want to make a first order calculation in $\varkappa$ for the damp. force. That means we take the charged body orbiting on an undamped orbit as the source in (III.3) and calculate $F^{\mu}_{\cdot\nu}$ and $F^{\mu}_{\cdot\nu}j^{\nu}$ at $x^0 = 0$. In the following a calculation of the damp. force automatically means that we are working to first order in $\varkappa$.

After these introductory remarks we now come to the solution of Maxwell's equations (III.3) using a slow mot., weak field expansion. We first solve the field equations (III.3) by means of the Green's function in a weak background metric (II.18), (II.19), (II.20), (II.21).

With

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{III.4}$$

(III.3) can be rewritten in the form:

$$\Box A^{\mu} - R^{\mu}_{\cdot\varkappa}A^{\varkappa} = -4\pi j^{\mu} \tag{III.5}$$

using the Lorentz condition in curved space:

$$\nabla_{\mu}A^{\mu} = 0 \tag{III.6}$$

The vector Green's function immediately yields a solution of (III.5):

$$\begin{aligned} A_R{}^{\mu}(x_2) &= -4\pi \int_{\infty} G_R{}^{\mu}_{\cdot\nu}(x_2, x_1)j^{\nu}(x_1)\sqrt{-g}\,d^4x_1 \\ A_R{}^{\mu(\#)}(x_2) &\overset{!}{=} -4\pi \int_{\infty} G_R{}^{\mu}_{\cdot\nu}{}^{(\#)}(x_2, x_1)j^{\nu}(x_1)\sqrt{-g}\,d^4x_1 \\ A_R{}^{\mu(b)}(x_2) &\overset{!}{=} -4\pi \int_{\infty} G_R{}^{\mu}_{\cdot\nu}{}^{(b)}(x_2, x_1)j^{\nu}(x_1)\sqrt{-g}\,d^4x_1 \end{aligned} \tag{III.7}$$

(III.7) also fulfills the Lorentz condition (III.6) if we take into account the covariant current conservation:

$$\nabla_{\mu}j^{\mu} = 0 \tag{III.8}$$

Before we perform the retardation expansion of $A_R{}^{\mu}$ in (III.7) in a weak background we want to briefly review the corresponding expansion in flat space [*17*]:

$$\begin{aligned} A_R{}^0(x_2^0, \vec{x}_2) &= -\int \frac{\rho_e(x_2^0 - R, \vec{x}_1)}{R} d^3\vec{x}_1 \\ A_R{}^i(x_2^0, \vec{x}_2) &= -\int \frac{\rho_e v^i(x_2^0 - R, \vec{x}_1)}{R} d^3\vec{x}_1 \end{aligned} \tag{III.9}$$

A retardation expansion yields:

$$A_R{}^0(x_2^0, \vec{x}_2) = \overset{(0)}{A}_R{}^0 + \overset{(1)}{A}_R{}^0 + \overset{(2)}{A}_R{}^0 + \dots \tag{III.10}$$

with

$$\overset{(0)}{A}_R{}^0 = -\int \frac{\rho_e(x_2^0, \vec{x}_1)}{R} d^3\vec{x}_1$$

$$\overset{(1)}{A}_R{}^0 = +\frac{\partial}{\partial x_2^0}\int \rho_e(x_2^0, \vec{x}_1)d^3\vec{x}_1 \qquad \text{(III.11)}$$

$$\overset{(2)}{A}_R{}^0 = -\frac{1}{2}\frac{\partial^2}{(\partial x_2^0)^2}\int R\rho_e(x_2^0, \vec{x}_1)d^3\vec{x}_1$$

and

$$A_R{}^i(x_2^0, \vec{x}_1) = \overset{(0)}{A}_R{}^i + \overset{(1)}{A}_R{}^i + \overset{(2)}{A}_R{}^i + \dots \qquad \text{(III.12)}$$

$$\overset{(0)}{A}_R{}^i = -\int \frac{\rho_e v^i(x_2^0, \vec{x}_1)}{R} d^3\vec{x}_1$$

$$\overset{(1)}{A}_R{}^i = +\frac{\partial}{\partial x_2^0}\int \rho_e v^i(x_2^0, \vec{x}_1)d^3\vec{x}_1 \qquad \text{(III.13)}$$

$$\overset{(2)}{A}_R{}^i = -\frac{1}{2}\frac{\partial^2}{(\partial x_2^0)^2}\int R\rho_e v^i(x_2^0, \vec{x}_1)d^3\vec{x}_1$$

As mentioned above in this paper we are concerned with rad. damp. forces only. Rad. damp. terms are sensitive to the outgoing rad. condition, they change their sign if we replace retarded by advanced solutions.

The lowest order rad. damp. term in (III.11) is $\overset{(1)}{A}_R{}^0$. This term however, vanishes in flat space because of charge conservation. In flat space the first rad. damp. term which is nonzero is the wellknown Abraham Lorentz force [*17*] which we obtain to third order in ε:

$$\overset{(3)}{F}{}^i_{\cdot 0} = -\frac{2}{3}\frac{d^3}{(dx^0)^3}\delta^i \qquad \text{(III.14)}$$

$F^i_{\cdot 0}$ is the electric field strength and δ^i the dipole moment.

We want to perform an analogous retardation expansion of the sharp part of the vector Green's function in a weak background metric. We obtain for the rad. damp. force to third order in ε the flat space Abraham Lorentz force $\mathscr{R}^{i(\#F)}$ acting on the charge density $\rho_e(x)$ and certain correction terms $\mathscr{R}^{i(\#C)}$ due to the background metric.

$$\mathscr{R}^{i(\#)} = F^i_{\cdot\nu}{}^{(\#)} \times j^\nu$$

$$\overset{(3)}{\mathscr{R}}{}^{i(\#F)} = -\frac{2}{3}\frac{d^3}{(dx^0)^3}\delta^i \cdot \rho_e \qquad \text{(III.15)}$$

$$\overset{(3)}{\mathscr{R}}{}^{i(\#C)} = \overset{(3)}{F}{}^i_{\cdot 0}{}^{(\#C)} \times \rho_e + \overset{(2)}{F}{}^i_{\cdot j}{}^{(\#C)} \times \rho_e v^j$$

$$\overset{(3)}{\mathrm{F}}{}^{i}_{\cdot 0}{}^{(\#C)} = \partial^{i_2}\left\{ \int_\infty \left(\frac{\overset{(2)}{\delta x^0}}{\mathrm{R}} + \frac{\overset{(2)}{\Gamma}}{\mathrm{R}} \right) \frac{\partial}{\partial x_2^0} \rho_e(\vec{x}_2, \vec{x}_1) d^3\vec{x}_1 \right. \quad (a)$$

$$+ \int_\infty \overset{(2)}{g}_{00} \frac{\partial}{\partial x_2^0} \rho_e(x_2^0, \vec{x}_1) d^3\vec{x}_1 \quad (b)$$

$$\left. + \int_\infty \overset{(2)}{g}_{j00} \rho_e(x_2^0, \vec{x}_1) v^j d^3\vec{x}_1 \right\} \quad (c)$$

$$+ \frac{\partial}{\partial x_2^0} \int_\infty \overset{(2)}{g}{}^{i}_{\cdot 00} \rho_e(x_2^0, \vec{x}_1) d^3\vec{x}_1 \quad (d) \qquad \text{(III.16)}$$

$$\overset{(2)}{\mathrm{F}}{}^{i}_{\cdot j}{}^{(\#C)} = -\partial^{i_2} \int_\infty \overset{(2)}{g}_{j00} \rho_e(x_2^0, \vec{x}_1) d^3\vec{x}_1$$

$$+ \partial_{j_2} \int_\infty \overset{(2)}{g}{}^{i}_{\cdot 00} \rho_e(x_2^0, \vec{x}_1) d^3\vec{x}_1 \quad (e)$$

With (II.10) and (II.12) we obtain in (III.16) (*a*):

$$\frac{\overset{(2)}{\delta x^0}}{\mathrm{R}} + \frac{\overset{(2)}{\Gamma}}{\mathrm{R}} = - \int_0^1 \overset{(2)}{g}_{00} d\omega \qquad \text{(III.17)}$$

We want to discuss the various terms in (III.16): (*a*) is due to the modification of the retardation time δx^0 and of the denominator (II.8) in curved space. (*b*), (*c*), (*d*), (*e*) are due to the parallel propagator in curved space. A simple consideration shows that (*b*) and (*e*) vanish.

Up to now we have not made any special assumption about the size of the charged body. Now we want to discuss the limit of a point like charge. To this end we make use of the continuity equation and of Gaussian's theorem in (III.16). We then obtain for (III.16) (still for an extended charge distribution):

$$\begin{aligned} \overset{(3)}{\mathrm{F}}{}^{i}_{\cdot 0}{}^{(\#C)} = & - \int_\infty \int_0^1 \omega(1-\omega)\partial_j \partial^i \overset{(2)}{g}_{00} d\omega \\ & \times \rho_e(x_2^0, \vec{x}_1) v^j(x_2^0, \vec{x}_1) d^3\vec{x}_1 \\ & + \frac{1}{2} \int_\infty \int_0^1 \omega \partial_j \partial^i \overset{(2)}{g}_{00} d\omega \\ & \times \rho_e(x_2^0, \vec{x}_1) v^j(x_2^0, \vec{x}_1) d^3\vec{x}_1 \\ & + \frac{1}{2} \int_\infty \int_0^1 (1-\omega) \partial^i \partial_j \overset{(2)}{g}_{00} d\omega \\ & \times \rho_e(x_2^0, \vec{x}_1) v^j(x_2^0, \vec{x}_1) d^3\vec{x}_1 \end{aligned} \qquad \text{(III.18)}$$

In order to obtain the limit of a pointlike charge we expand $\partial_i \partial_j \overset{(2)}{g}_{00}$

in the neighbourhood of z^i in a Taylor expansion and keep the zeroth order term only. We then are able to integrate along the straight line from $\vec{x}_1$ to $\vec{x}_2$ in (III.18) and get:

Limit of a pointlike charge:

$$\overset{(3)}{\mathrm{F}}{}^{i}{}_{\cdot 0}{}^{(*C)} = \frac{2}{3}e\left[\frac{1}{2}\partial^i\partial_j \overset{(2)}{g}_{00}(\vec{z})\right]v^j = -\frac{2}{3}e[\partial_j \overset{(2)}{\Gamma}{}^{i}{}_{\cdot 00}(\vec{z})]v^j \qquad \text{(III.19)}$$

It can be shown that to the considered order in ε the correction term (III.19) together with the Abraham-Lorentz force for a pointlike particle can be written in the form:

$$\left(\overset{(3)}{\mathrm{F}}{}^{i}{}_{\cdot 0}{}^{(*C)} - \frac{2}{3}e\frac{d^3}{(dx_2^0)^3}z^i\right) = -\frac{2}{3}e\frac{\delta^3}{\delta\tau^3}z^i \qquad \text{(III.20)}$$

Hence it follows that the reaction force due to the sharp part vanishes if the particle moves on a geodesic. A point charge therefore could fall freely on a geodesic if it would not be for the tail [18].

III *b*. CALCULATION OF THE RADIATION DAMPING FORCE OF AN ORBITING CHARGE DUE TO THE TAIL PART OF THE VECTOR GREEN'S FUNCTION

(Retardation (slowness) expansion of the tail part)

Using our integral formula (II.21) we now want to do an analogous calculation of the rad. damp. force due to the tail. We shall show that the additional rad. damp. terms (III.16) are just cancelled by the corresponding terms of the tail.

With (II.24) we obtain for the tail formula (II.21):

$$\begin{aligned}\overset{(i)}{\mathrm{G}}{}_{\mathrm{R}_{\mu\alpha}}^{(b)}(x_2, x_1) = -\frac{1}{4\pi}\int_{|\vec{x}_3|\le\lambda} & \delta_{\mathrm{R}}(\overset{(0)}{\Omega}(x_2, x_3)) \\ & \times \frac{1}{4\pi}\overset{(0)}{\Box}_3\left\{\eta_{\mu\alpha}\frac{1}{2}\overset{(i)}{\Delta}(x_3, x_1) + \overset{(i)}{\bar{g}}_{\mu\alpha}(x_3, x_1)\right\} \\ & \times \delta_{\mathrm{R}}(\overset{(0)}{\Omega}(x_3, x_1))d^4x_3 \qquad i = 2, 3\end{aligned} \qquad \text{(III.21)}$$

After a simple transformation of (III.21) using (II.7) and the explicit formula for $\overset{(i)}{\bar{g}}_{\mu\alpha}$ and $\overset{(i)}{\Delta}$ we obtain:

$$\begin{aligned}\overset{(2)}{\mathrm{G}}{}_{\mathrm{R}_{\mu\alpha}}^{(b)}(x_2, x_1) = -\frac{1}{4\pi}\int_{|\vec{x}_3|\le\lambda} & d^3\vec{x}_3\frac{\delta(x_2^0 - x_1^0 - \mathrm{R}_{2|3} - \mathrm{R}_{1|3})}{\mathrm{R}_{2|3} - \mathrm{R}_{1|3}} \\ & \times \frac{1}{4\pi}\left\{\eta_{\mu\alpha}\eta^{ij}\partial_{i_3}\partial_{j_3}\left[\frac{1}{2}\overset{(2)}{\Delta}(x_3, x_1)\right]_{\Delta x^0 = \mathrm{R}_{1|3}}\right.\end{aligned}$$

$$- \eta_{\mu\alpha} \int_0^1 \eta^{ij}\partial_i\partial_j \overset{(2)}{g}_{00}\omega^2 d\omega$$

$$+ [\eta^{ij}\partial_{i_3}\partial_{j_3} \overset{(2)}{\bar{g}}_{\mu\alpha}(x_3, x_1)]_{\Delta x^0 = R_{1|3}} \Big\} \quad \text{(III.22)}$$

$$\overset{(3)}{G}{}^{(b)}_{R_{\mu\alpha}}(x_2, x_1) = -\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} d^3\vec{x}_3 \frac{\delta(x_2^0 - x_1^0 - R_{2|3} - R_{1|3})}{R_{2|3} - R_{1|3}}$$

$$\times \frac{1}{4\pi}\Big\{ + \frac{1}{2}[\eta^{ij}\partial_{i_3}\partial_{j_3}\overset{(3)}{\Delta}(x_3, x_1)]_{\Delta x^0 = R_{1|3}}$$

$$+ [\eta^{ij}\partial_{i_3}\partial_{j_3}\overset{(3)}{\bar{g}}_{\mu\alpha}]_{\Delta x^0 = R_{1|3}}\Big\}$$

with

$$R_{2|3} = |\vec{x}_2 - \vec{x}_3|, \qquad R_{1|3} = |\vec{x}_1 - \vec{x}_3| \quad \text{(III.23)}$$

We now want to calculate $\overset{(3)}{\mathcal{R}}{}^{i(b)}$ by expanding (III.22) in a retardation expansion. Up to third order we get for the rad. damp. force:

$$\mathcal{R}^{\mu(b)} = F^{\mu}_{\cdot\nu}{}^{(b)} \times j^{\nu}$$

$$\overset{(3)}{\mathcal{R}}{}^{j(b)} = \overset{(3)}{F}{}^{j}_{\cdot 0}{}^{(b)} \times \rho_e + \overset{(2)}{F}{}^{j}_{\cdot k}{}^{(b)} \times \rho_e v^k \quad \text{(III.24)}$$

$$\overset{(3)}{F}{}^{i}_{\cdot 0}{}^{(b)}$$

$$= \partial^{i_2}\left\{\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \left[\frac{\frac{\partial}{\partial x_2^0}\rho_e(x_2^0, \vec{x}_1)}{R_{1|3}} + \frac{\frac{\partial}{\partial x_2^0}\rho_e(x_2^0, \vec{x}_1)}{R_{2|3}}\right]\right\}$$

$$\times \eta^{ij}\partial_{i_3}\partial_{j_3}\Big\{ - \frac{1}{2}[\overset{(2)}{\Delta}(\vec{x}_3, \vec{x}_1)]_{\Delta x^0 = R_{1|3}} \quad (a)$$

$$+ \int_0^1 \overset{(2)}{g}_{00} d\omega \Big\} \quad (b)$$

$$+ \partial^{i_2}\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \frac{\rho_e(x_2^0, \vec{x}_1)}{R_{2|3}} \times \eta^{ij}\partial_{i_3}\partial_{j_3}(\overset{(3)}{\Delta}_{k_0}\Delta x^k) \quad (c)$$

$$- \partial^{i_2}\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \left[\frac{\frac{\partial}{\partial x_2^0}\rho_e(x_2^0, \vec{x}_1)}{R_{1|3}} + \frac{\frac{\partial}{\partial x_2^0}\rho_e(x_2^0, \vec{x}_1)}{R_{2|3}}\right]$$

$$\times \eta^{ij}\partial_{i_3}\partial_{j_3}\overset{(2)}{\bar{g}}_{00}(x_3, x_1) \quad (d) \quad \text{(III.25)}$$

$$- \partial^{i_2}\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \frac{\rho_e(x_2^0, \vec{x}_1)v^l}{R_{2|3}} \times \eta^{kj}\partial_{k_3}\partial_{j_3}\overset{(2)}{g}_{l00}(x_3, x_1) \quad (e)$$

$$- \frac{\partial}{\partial x_2^0}\frac{1}{4\pi}\int_{|\vec{x}_3|\leq\lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \frac{\rho_e(x_2^0, \vec{x}_1)}{R_{2|3}} \times \eta^{ik}\partial_{i_3}\partial_{k_3}\overset{(2)}{g}{}^{i}_{\cdot 00}(x_3, x_1) \quad (f)$$

$$\overset{(2)}{\mathrm{F}}{}^{i}_{\cdot k}{}^{(b)}$$

$$= \partial^{i_2} \frac{1}{4\pi} \int_{|\vec{x}_3| \le \lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \left[\frac{\rho_e(x_2^0, \vec{x}_1)}{\mathrm{R}_{2|3}} \times \eta^{ij} \partial_{i_3} \partial_{j_3} \overset{(2)}{g}_{k00}(x_3, x_1) \right] \quad (g)$$

$$- \partial_{k_2} \frac{1}{4\pi} \int_{|\vec{x}_3| \le \lambda} d^3\vec{x}_3 \int_\infty d^3\vec{x}_1 \left[\frac{\rho_e(x_2^0, \vec{x}_1)}{\mathrm{R}_{2|3}} \times \eta^{ij} \partial_{i_3} \partial_{j_3} \overset{(2)}{g}{}^{i}_{\cdot 00}(x_3, x_1) \right]$$

Using Gauss's theorem it is possible to show that the terms in (III.25) cancel the corresponding expressions in (III.16). In this chapter we simply neglect the surface terms which we obtain if we apply Gauss's theorem. In the next chapter we shall indicate how this neglection can be justified.

With this assumption it can be seen easily that to the considered order (III.25) (*a*), (*c*), (*d*) and (*g*) vanish and that (III.16) (*a*) is cancelled by (III.25) (*b*), (III.16) (*c*) is cancelled by (III.25) (*e*) and (III.16) (*d*) is cancelled by (III.25) (*f*). This surprising connection between tail and sharp part is typical for slow. mot. appr. in G. R.

We obtain the following final result in harmonic coordinates: the rad. damp. force $\overset{(3)}{\mathscr{R}}{}^i$ acting on an extended charge moving in a stationary background metric is the flat space Abraham-Lorentz force [*21*] [*22*] [*23*].

IV. INNER, OUTER AND INTERMEDIATE EXPANSION OF THE TAIL

In a slow. mot., weak field expansion of the tail one is confronted with one of the most delicate problems in slow. mot. appr. in G. R. It is wellknown that the retardation expansion only makes sense for retardation times small compared with the wavelength. From a mathematical point of view the retardation expansion is not a uniform asymptotic expansion for arbitrarily large retardation times. Therefore it is a familiar result of flat space electrodynamics that a retardation expansion is valid in the nearzone ($r \ll \lambda$) only whereas outside the nearzone one has to perform different expansions (intermediate and outer expansion). In curved space however one also has arbitrarily large retardation times in the immediate vicinity of the source because of the tail. In other words even if one is working in the nearzone only one is confronted with the farzone because of the backscatter. If one nevertheless performs the retardation expansion for arbitrarily large retardation times one obtains divergent or cutoff dependent expressions in higher order. This is understandable since the polynomials appearing in a retardation (Taylor) expansion blow up at infinity.

Fortunately it often turns out that the backscatter from the outer region is negligible up to the considered order. This seems to be the case also in our model calculations. Therefore even if one proceeds in a rather naive manner one probably obtains reasonable results up to the considered order.

It is more difficult to justify these appr. and the following considerations are a first attempt to clarify the situation. We want to point out that we do not regard these problems as solved.

In order to justify our appr. we have to show that the backscatter from the outer region ($|\vec{x}_3| \geq \lambda$) in (I.9) is negligible up to the considered order. Furthermore we have to show that (III.25) contains all rad. damp. terms up to third order in ε and that the surface terms which one obtains after the transformation of (III.25) are negligible. It is not difficult then to see that to the considered order (II.21), (III.21), (III.22) and (III.25) are cutoff independent.

We have to find a proper expansion of the tail (uniform for all retardation times or for all scattering points $\vec{x}_3$) by using an inner, intermediate and outer expansion. We want to remark that our problem is unusual even in singular perturbation theory [*7*] [*8*] [*9*] [*10*] where one usually needs several asymptotic expansions for imposing the correct boundary condition. However we have imposed the boundary condition already by using retarded Green's functions. For simplicity we want to restrict ourselves to the weak Schwarzschild background metric (II.26) with a constant matter density ρ (II.27). We define the three different expansions in the inner, intermediate and outer region:

$$\begin{array}{ll} \text{inner zone:} & 0 \leq r \leq r_{\mathrm{I}}, \\ \text{intermediate zone:} & r_{\mathrm{I}}^* \leq r^* \leq \lambda \\ \text{outer zone:} & \lambda \leq r^* < \infty \end{array} \tag{IV.1}$$

In (IV.2) we have introduced the « renormalized » radial coordinate:

$$r^* = r + 2\mathrm{M} \ln \left(\frac{r}{2\mathrm{M}} - \frac{1}{2} \right) \tag{IV.2}$$

The ordering in the three expansions is determined by the inner, intermediate and outer limit. The inner limit was defined in (II.1). In addition we define:

$$\begin{array}{ll} \text{intermediate limit:} & r^* \to \alpha(r^*, \varepsilon) r^* \\ \text{(intermediate region):} & x^0 \to \frac{1}{\varepsilon} x^0 \end{array}$$

$$\alpha(r^*, \varepsilon) = 1 + \left(\frac{1}{\varepsilon} - 1 \right) \left(\frac{r^* - r_{\mathrm{I}}^*}{\lambda - r_{\mathrm{I}}^*} \right) \tag{IV.3}$$

$$\begin{array}{ll} \text{Outer limit:} & r^* \to \frac{1}{\varepsilon} r^* \\ \text{(outer region):} & x^0 \to \frac{1}{\varepsilon} x^0 \end{array} \tag{IV.4}$$

It remains to find expansions which are asymptotic with respect to the inner, intermediate and outer limit and which are uniform in the corresponding regions. The retardation expansion performed in chapter III corresponds to the inner limit. A multipole expansion corresponds to the outer limit. The intermediate expansion again has to lie between the outer and inner expansion. In this paper we define a multipole expansion as a Taylor expansion also in the nonstatic case ([10]). In a multipole expansion usually one has to deal with one inner coordinate (inner limit) and one outer coordinate (outer limit). The multipole expansion in electrostatics in flat space is a familiar example: (x_2: outer coordinate, $\vec{x}_1$: inner coordinate)

$$\frac{1}{|\vec{x}_2 - \vec{x}_1|} \to \frac{1}{\left|\frac{1}{\varepsilon}\vec{x}_2 - \vec{x}_1\right|} = \varepsilon \frac{1}{|\vec{x}_2|} - \varepsilon^2 \frac{\partial}{\partial x_2^i} \frac{1}{|\vec{x}_2|} x_1^i + \ldots \quad \text{(IV.5)}$$

In curved space in our integral formula for the tail (I.9) $\vec{x}_1$ and $\vec{x}_2$ have to be treated as inner coordinates whereas $\vec{x}_3$ can be located in each of the three regions. Since in this chapter we are interested mainly in the backscatter from the intermediate and outer region (that means where $\vec{x}_3$ has to be treated as an intermediate or outer coordinate) we have to study the behaviour of the relevant expressions for large $|\vec{x}_3|$. Expecially we have to study the behaviour of $V_{\nu_3\alpha_1}(x_3, x_1)$ (appearing in (I.9)) for large $|\vec{x}_3|$ with:

$$\begin{aligned} V_{\nu_3\alpha_1}(x_3, x_1) \overset{!}{=} \ & \Box_3(\Delta^{1/2}(x_3, x_1)\bar{g}_{\nu_3\alpha_1}(x_3, x_1)) \\ & - R_{\nu_3}{}^{\varkappa_3}\bar{g}_{\varkappa_3\alpha_1}(x_3, x_1)\Delta^{1/2}(x_3, x_1) \end{aligned} \quad \text{(IV.6)}$$

Up to the considered order $V_{\mu_3\alpha_1}(x_3, x_1)$ can be approximated by $\overset{(2)}{V}_{\mu\alpha}(x_3, x_1)$ also in the outer region with:

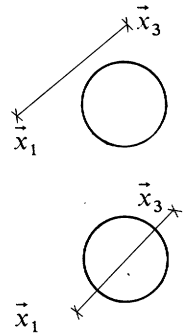
$$\overset{(2)}{V}_{\mu\alpha}(x_3, x_1) = \overset{(0)}{\Box}_3\left(\eta_{\mu\alpha}\frac{1}{2}\overset{(2)}{\Delta}(x_3, x_1) + \overset{(2)}{\bar{g}}_{\mu\alpha}(x_3, x_1)\right) \quad \text{(IV.7)}$$

It has to be shown that $\overset{(2)}{V}_{\mu\alpha}$ decreases sufficiently for large $|\vec{x}_3|$ so that the backscatter from the outer region is negligible. We want to demonstrate in a simple example how we have found the behaviour of the interesting quantities for large $|\vec{x}_3|$. We want to study $\overset{(2)}{\Delta}_{00}$ (defined in (II.15)) for the weak Schwarzschild case:

$$\overset{(2)}{\Delta}_{00}(x_3, x_1) = -\frac{1}{2}\int \rho_M(\omega - \omega^2)d\omega \quad \text{(IV.8)}$$

([10]) In this case we also perform a partial retardation expansion (with respect to the inner coordinate). In flat space the multipole expansion usually is introduced by using spherical harmonics.

We illustrate the situation by the following picture:



(a)

(IV.9)

(b)

In the case (a) where $\overline{x_1 x_2}$ does not cross the matter distribution (IV.8) vanishes. In the case (b) we can perform the integration along the straight line explicitely and obtain:

$$\overset{(2)}{\Delta}_{00}(x_3, x_1) = -\frac{1}{2}\rho_M\left(\frac{1}{2}(\omega_b^2 - \omega_a^2) - \frac{1}{3}(\omega_b^3 - \omega_a^3)\right) \quad \text{(IV.10)}$$

where ω_a and ω_b are the parameters of the straight line (II.5) belonging to the intersections points $\vec{x}_a$ and $\vec{x}_b$. From the parametrization of a straight line it immediately follows that $\overset{(2)}{\Delta}_{00}(x_3, x_1)$ falls down with ε^2 for $r^* \to \frac{1}{\varepsilon} r^*$ and fixed $\vec{x}_1$. Together with our other estimates it seems to be possible to show that the backscatter from the outer region is negligible to third order in ε. Similarly our extimates indicate that the other appr. mentioned in the beginning of this chapter are justified.

CONCLUSION AND OUTLOOK

In this paper we have investigated a simplified model for grav. rad. damp.: el. magn. rad. damp. of charges in external grav. fields. Using a slow mot. appr. we have tried to proceed as much as possible along lines developed in analogous problems in flat space electrodynamics. Our result even does not differ from the corresponding flat space result: In both cases we obtain the Abraham-Lorentz force as the lowest order el. rad. damp. force.

In comparison with flat space however we are confronted with three additional major problems in a background metric: firstly we have to perform all calculations in a special coordinate system. Our results are coordinate dependent and have no immediate physical significance. An extraction of observable quantities is a nontrivial task.

Secondly in curved space it is more problematic to impose the correct boundary condition on the el. magn. rad. than in flat space. In this paper we have used retarded Green's functions in curved space for this purpose. We leave a further clarification of the Green's function method and of the boundary value problem to further investigations.

Thirdly because of the backscatter (or the tail) the whole calculation differs from flat space considerably. Since we get arbitrarily large retardation times in the immediate neighborhood of the source we have to use several different asymptotic expansions in a proper slow mot. appr. even if we are working in the nearzone only. However, the original hope was to avoid the use of several expansions by applying Green's function methods. (Nevertheless up to a certain order this seems to be possible.)

Since one is concerned with singular perturbation theory in any case and since it is most natural to impose the outgoing rad. condition at infinity, Burke's method [7] (mentioned in chapter III) in some respects seems to be even more appropriate than the Green's functions approach. In addition this method probably has computational advantages. Our calculations show that the splitting into the tail part and into the sharp part is rather artificial. In Burke's approach there would be no distinction between terms coming from the tail and terms due to the sharp part. However, also this method has to be applied with care in curved space. An investigation of Burke's approach therefore will be one of our next problems. Of course in the future we shall be concerned also with grav. rad. damp. itself.

ACKNOWLEDGMENTS

I should like to express my gratitude to Dr. D. W. Sciama for suggesting the subject, introducing me into relativistic astrophysics and for continuous ecouragement. Most of the investigations presented in this paper were greatly influenced by Dr. W. G. Dixon and Mr. J. Bird whom I thank for valuable advice and for making their calculations available to me before publication. I am grateful to Dr. G. F. R. Ellis and the relativity group in Cambridge for their hospitality and many stimulating discussions. Furthermore I benefitted from discussions with Prof. J. Ehlers and Dr. J. Stewart. I thank Dr. B. Bramson, Dr. G. Börner, and Dr. D. Raine for critically reading my manuscripts. I am indebted to the Royal Society and the Deutsche Forschungsgemeinschaft for an exchange fellowship.

LITERATURE

[1] P. C. PETERS, *Phys. Rev.*, t. **146**, 1966, p. 938.
[2] P. C. PETERS, *Phys. Rev.*, **D 1**, 1970, p. 1559.
[3] P. C. PETERS, *Phys. Rev.*, **D 5**, 1972, p. 2476.
[4] P. C. PETERS, *Phys. Rev.*, **D 7**, 1973, p. 368.
[5] R. W. JOHN, *Ann. Phys.*, t. **29**, 1973, p. 15.

[6] K. P. CHUNG, *Nuovo Cim.*, t. **14 B**, 1973, p. 293.
[7] W. L. BURKE, *J. M. P.*, t. **13**, 1971, p. 401.
[8] J. D. COLE, *Perturbation Methods in Applied Mathematics*, 1970.
[9] M. VAN DYKE, *Perturbation Methods in Fluid Mechanics*, 1964.
[10] L. E. FRAENKEL, *Proc. Cambridge Phil. Soc.*, t. **65**, 1969, p. 209.
[11] W. G. DIXON, *Proc. Roy. Soc.*, t. **A 314**, 1970, p. 499 ; t. **A 319**, 1970, p. 509.
[12] W. G. DIXON, *G. R. G.*, t. **4**, 1973, p. 199.
[13] S. CHANDRASEKHAR et F. P. ESPOSITO, *Ap. J.*, t. **160**, 1970, p. 153.
[14] W. G. DIXON, *G. R. G.*, t. **3**, 1972, p. 227.
[15] W. G. DIXON, *Slow Motion Approximations in General Relativity II, Gauge Transformations*, Cambridge, prepr., 1972.
[16] J. M. BIRD, *The Scalar Wave Equation in the Weak Field Approximation*, Cambridge, prepr., 1973.
[17] L. D. LANDAU et E. M. LIFSHITZ, *Classical Field Theory*, 1966.
[18] B. S. De WITT et R. W. BREHME, *Ann. Phys.*, t. **9**, 1969, p. 220.
[19] R. BEIG, *Acta Phys. Austriaca*, t. **38**, 1973, p. 300.
[20] W. G. UNRUH, *Self Force on Charged Particles*, Berkeley, prepr., 1974.
[21] C. M. De WITT et B. S. De WITT, *Phys.*, t. **1**, 1964, p. 3.
[22] W. M. SCHMID, *Acta Phys. Austriaca*, t. **39**, 1974, p. 172.
[23] R. BEIG, *Lett. Nuovo Cim.*, t. **8**, 1973, p. 1007.
[24] P. C. WAYLEN, *Proc. Roy. Soc.*, t. **A 321**, 1971, p. 397.
[25] G. F. R. ELLIS et D. W. SCIAMA, Global and Nonglobal Problems in Cosmology, in *General Relativity*, Papers in Honour of J. L. SYNGE, 1972.
[26] J. STEWART et M. WALKER, *Perturbations of Spacetimes in General Relativity*, Munich, prepr., 1973.
[27] W. G. DIXON, *Weak Gravitational Fields*, Cambridge, prepr., 1972.
[28] G. N. PLASS, *Rev. Mod. Phys.*, t. **33**, 1961, p. 37.

(Manuscrit reçu le 3 mars 1974)