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Vacuum-like state analysis of the representations of the para-Fermi operators

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by

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ABSTRACT. — Using the underlying Lie-algebraical structure of a given number n of para-Fermi operators (PFO), we study the set of all finite dimensional representations of these operators. We determine the subspace of all vacuum-like states, i. e., vectors from the representation space on which the para-Fermi annihilation operators vanish and show that this space carries an irreducible representation of the algebra SU(n). We write down an explicit formula for the number of the linearly independent vacuum-like states which appear within an arbitrarily given irreducible representation of PFO, and discuss their multiplicities. Finally, we compare our results with the corresponding ones obtained in the recent paper of Bracken and Green.

1. INTRODUCTION

Let a_i, b_i, $i = 1, \ldots, n$ be n pairs of para-Fermi annihilation and creation operators, i. e., a set of $2n$ linear operators defined in a space W and satisfying the operator identities [1] ($[x, y] \equiv xy - yx$):

$$[[a_p, b_q], a_r] = 2\delta_{qr}a_p$$
$$[[a_p, b_q], b_r] = -2\delta_{pr}b_q$$
$$[[b_p, b_q], a_r] = 2(\delta_{qr}b_p - \delta_{pr}b_q) \qquad (1)$$
$$[[a_p, a_q], b_r] = 2(\delta_{qr}a_p - \delta_{pr}a_q)$$
$$[[a_p, a_q], a_r] = [[b_p, b_q], b_r] = 0$$

In the present paper we study all finite dimensional spaces W, irreducible under the operators a_i, b_i mainly with the purpose of analyzing the properties of the vectors from the representation space W on which all para-Fermi annihilation operators $a_1, \ldots, a_n$ vanish. Following the terminology of Bracken and Green [2] we call these vectors « reservoir states » or « vacuum-like states ». We find an explicit formula for the dimension of the subspace $V \subset W$ spanned by all such states and show that V carries an irreducible representation of the algebra SU(n).

The reason we do not discuss the analogous problem for the para-Bose operators is a purely technical one. It is due to the circumstance that to determine V we make an essential use of the underlying Lie-algebraical structure of the para-Fermi operators (PFO), namely of the fact that every irreducible representation of the PFO can be extended to an irreducible representation of the classical Lie algebra B_n [with compact form SO(2n + 1)] and *vice versa*. This allows us in the case of PRO operators to reduce the whole problem to a purely Lie-algebraical one, whereas this seems to be impossible in the para-Bose case since in the structure relations (1) one of the commutators is an anticommutator.

What kind of representations of PFO are relevant for physics ? The answer to this question depends on the physical meaning ascribed to the operators. If PFO operators create real particles then only the ordinary (hereafter called also canonical) representations [3], corresponding to a single vacuum state, should be considered. This is not the case if the vacuum is degenerated and some authors even indicate that models with degenerate vacuum could provide a possibility of overcoming some of the difficulties in the quantum field theory [4]. The non-canonical representations arise also in a natural way in the case the real particles are considered as composite ones as, for example, in the quark model. Supposing, for instance, that the particles are built out of different spin $\frac{1}{2}$ objects, it is easy to show that they are para-Fermions and there is *a priori* no reason to demand that they transform according to only canonical representations.

The first physical application of non-canonical representations, namely those which are a direct product of two or three representations of Fermi fields, was considered by Govorkov [5] in 1968. He showed how, in this case, one can introduce internal degrees of freedom and wrote down explicit expressions for the isospin operators. He did not succeed to give, however, explicit expressions for the SU(3) generators. Although in the meantime there appeared several investigations on the algebraic properties of the ordinary representations of PFO [6] and their physical applications [7]; Govorkov's idea was properly realized only in the recent paper [2] of Bracken and Green.

The former authors proposed the consideration of a generalized para-statistics algebra of order p, which makes use of all representations of PFO obtained from the Green's ansatz [1]. That is to say, the representation is a reducible one, realized in a direct product space H_p of p irreducible spaces, each one characterized by a highest weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ with respect to $SO(2n + 1)$. Among the several results concerning the group structure of H_p, Bracken and Green have established some of the properties of the vacuum-like states. They have shown that within a given representation ([2]) of $SO(2n + 1)$ all representations of $U(n)$ containing reservoir states appear only once and all of them are contained in one $SO(2n)$ representation. We shall prove a stronger statement, namely that all reservoir states are contained in one $SU(n)$ representation ([3]) and it contains only reservoir states. Furthermore we show that in the general case the weights corresponding to the vacuum-like states are not simple. This result disagrees with those stated in Ref. [1]. In order to introduce the notations and to make the paper reasonably self-consistent, we collect in the next section some definitions and properties of the Lie algebras as well as their representations.

2. PRELIMINARIES AND NOTATIONS

Let R be a semi-simple complex Lie algebra and H be its Cartan sub-algebra with basis $\omega_1, \ldots, \omega_n$. For arbitrary $x, y \in R$ denote by (x, y) the bilinear Cartan-Killing form defined on R as

$$(x, y) = \operatorname{Tr} adx \, ady \qquad (2)$$

where adx is a linear operator in $R : (adx)z = [x, z]$, $[x, z]$ commutator in R, $x, z \in R$. The basis $\omega_1, \ldots, \omega_n, l_{h_1}, \ldots, l_{h_K}$ in R can be chosen such that for arbitrary $h \in H$

$$[h, l_{h_i}] = (h, h_i)l_{h_i} \qquad i = 1, \ldots, n \qquad (3)$$

where $h_i \in H$ and the correspondence $h_i \rightarrow l_{h_i}$ is one to one. The vectors h_i, l_{h_i} $(i = 1, \ldots, K)$ are called roots and root vectors of R accordingly. The Cartan-Killing form defines a scalar product in the space H^r which is the real linear envelope of all roots; $H = H^r + iH^r$. The roots and the metric properties of H^r define up to an isomorphism the algebra R. Let

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([2]) Unless otherwise stated by representation we mean finite-dimensional irreducible representation.

([3]) It could be worth mentioning that this terminology is not exact, however, it is accepted in the physical literature. The reservoir state belongs to the subspace in which the representation of $SU(n)$ is realized. More precisely, one should say that all reservoir states belong to one irreducible $SU(n)$ module.

from now on $\omega_1, \ldots, \omega_n$ be an orthogonal basis in H^r (and hence a basis in H). The root h_i is said to be positive (negative) if its first non-zero co-ordinate is positive (negative). The simple roots, i. e., those positive roots which are not the sum of two other positive roots, constitute a basis in H. Any positive (negative) root is a linear combination of simple roots with positive (negative) integer coefficients. Consider a finite-dimensional representation π of R and let $\hat{h}$ be the operator corresponding to $h \in H$. The basis $x_1, \ldots, x_N$ in the representation space W can be always chosen such that

$$\hat{h}.x_i = (h, \alpha_i)x_i, \qquad \alpha_i \in H, \qquad i = 1, \ldots, N \tag{4}$$

so that to every basic vector $x_i \in W$ there corresponds an image $\alpha_i \in H$. The vectors x_i are the weight vectors and their images α_i — the weights of the representation π. The mapping $\tau : x_i \rightarrow \alpha_i$ is surjective and the number of vectors $\tau^{-1}(\alpha_i)$ is called multiplicity of the weight α_i. Let $l_\alpha \in R$ be a root vector and α_i be the weight of x_i. Then $\hat{l}_\alpha.x_i$ is either zero or a weight vector with weight $\alpha + \alpha_i$. In every representation (see the footnote on p. 51), π there exists a unique weight vector x_Λ with properties $\hat{l}_\alpha.x_\Lambda = 0$ for all operators $\hat{l}_\alpha$ corresponding to positive roots α. The weight Λ of x_Λ is the highest weight of π. The representation space W is spanned on all vectors

$$\hat{l}_{\alpha'}.\hat{l}_{\alpha''} \ldots \hat{l}_{\alpha^{(m)}}.x_\Lambda \tag{5}$$

where $\alpha', \ldots, \alpha^{(m)}$ are negative roots. Therefore an arbitrary weight λ is of the form

$$\lambda = \Lambda - \sum_i K_i \alpha_i \tag{6}$$

with K_i positive integers and sum over positive (or only simple) roots.

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of R. Then for an arbitrary weight λ the n-tuple $(\lambda_1, \ldots, \lambda_n)$ has integer co-ordinates defined as

$$\lambda_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \tag{7}$$

The n-tuple $(\Lambda_1, \ldots, \Lambda_n)$ corresponding to Λ has non-negative co-ordinators, and it defines the irreducible representation π up to equivalence. On the contrary, to every vectors $\Lambda \in H$ such that $\Lambda_1, \ldots, \Lambda_n$ defined from (7) are non-negative integers, there corresponds an irreducible representation of R. Thus, there exists a one-to-one correspondence between the irreducible representations of R and the set $(\Lambda_1, \ldots, \Lambda_n)$ of non-negative integers. We call $\lambda_1, \ldots, \lambda_n$ canonical co-ordinates of λ. It is always possible to choose a basis $F = \{ f_i \,|\, i = 1, \ldots, n \}$ in H such that the co-ordinates of every weight λ in F will coincide with its canonical co-ordinates.

An important property of the set Γ of all weights is its invariance

under the Weyl group S which is a group of transformations of H^r. $S = \{ S_{\alpha_i} \mid \alpha_i - \text{root of R} \}$ is a finite group, its elements S_{α_i} labelled by the roots α_i of R and defined as follows:

$$S_{\alpha_i} h = h - \frac{2(h, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i, \qquad h \in H^r \tag{8}$$

The set Γ of all weights is characterized by the following statement: if $\lambda \in \Gamma$, then $S_{\alpha_i}.\lambda = \lambda - j\alpha_i \in \Gamma$, j integer and Γ contains also all weights

$$\lambda, \lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \lambda - j_{\alpha_i} \tag{9}$$

3. DETERMINATION OF THE RESERVOIR STATES

In the present section we determine all vacuum-like states and derive a formula for the number of the linearly independent states within a given representation of PFO. Consider the set of n pairs of para-Fermi annihilation and creation operators a_i, b_i, $i = 1, \ldots, n$, which satisfy the structure relations (1).

The elements a_i, b_i, $[a_i, b_j]$, $[a_p, a_q]$, $[b_p, b_q]$, $p < q$, $i, j, p, q = 1, \ldots, n$, constitute a basis of a Lie algebra R which, over the field of complex numbers, is isomorphic to the classical algebra B_n [8], whereas, as a real algebra, this is $SO(n, n + 1)$ [9]. The set of all finite dimensional irreducible representations of the operators a_i, b_i, $i = 1, \ldots, n$ is completely determined from the observations that every finite dimensional irreducible representation of B_n defines an irreducible representation of PFO and *vice versa* [9] and the set of all finite dimensional irreducible representations of B_n is known [10].

DÉFINITION. — A vector $w \in W$ on which all annihilation operators a_i, $i = 1, \ldots, n$, vanish, i. e.,

$$a_i.w = 0 \tag{10}$$

is called a reservoir state. We wish to find the number of all linearly independent vectors in W satisfying (10), i. e., the dimension of the subspace $V \subset W$ spanned on all reservoir states. We first reduce this problem to a purely Lie algebraical one by proving the following Lemma:

LEMMA. — The basis in the Cartan subalgebra H of the para-Fermi algebra R of n pairs of PFO can be chosen in such a way that all para-Fermi annihilation (creation) operators $a_i(b_i)$, $i = 1, \ldots, n$ belong to the system of positive (negative) root vectors.

Proof. — Consider as H the commutative subalgebra of R given by all finite linear combinations of the elements $[a_i, b_i]$, $i = 1, \ldots, n$ and choose, as a basis in H, the vectors

$$\omega_i = \frac{[a_i, b_i]}{8n - 4} \qquad i = 1, \ldots, n \tag{11}$$

It can be verified that, with respect to the Cartan-Killing form, this basis is an orthogonal one:

$$(\omega_i, \omega_j) = \frac{\delta_{ij}}{4n - 2} \tag{12}$$

Let

$$\Sigma = (\pm \omega_i \pm \omega_j, \pm \omega_i \,|\, i, j = 1, \ldots, n) \tag{13}$$

Denote

$$\begin{aligned}
e_{\omega i} &= a_i \\
e_{-\omega_i} &= b_i \\
e_{\omega_i + \omega_j} &= [a_i, a_j] \\
e_{-\omega_i - \omega_j} &= [b_i, b_j] \\
e_{\omega_i - \omega_j} &= [a_i, b_j]
\end{aligned} \tag{14}$$

The vectors ω_i, e_α, $i = 1, \ldots, n$, $\alpha \in \Sigma$ introduce a new basis in the space of the para-Fermi algebra. Using the commutation relations (1) and the definition (2), after some calculations we obtain

$$\begin{aligned}
[h_1, h_2] &= 0 & h_1, h_2 &\in H \\
[h, e_\alpha] &= (h, \alpha)e_\alpha & h &\in H \\
[e_\alpha, e_\beta] &= N_{\alpha,\beta}e_{\alpha+\beta}
\end{aligned} \tag{15}$$

where $N_{\alpha,\beta}$ are real numbers.

The relations (15) are characteristic for the algebra B_n [11] and they show that Σ is the root system of $R \approx B_n$ (we do not differ any more between R and B_n). We have furthermore that $\Sigma = \Sigma_+ \cup \Sigma_-$,

$$\begin{aligned}
\Sigma_+ &= (\quad \omega_i \pm \omega_j, \quad \omega_K \,|\, i, j, K = 1, \ldots, n \,; i < j) \\
\Sigma_- &= (- \omega_i \pm \omega_j, \ - \omega_K \,|\, i, j, K = 1, \ldots, n \,; i < j)
\end{aligned} \tag{16}$$

where $\Sigma_+(\Sigma_-)$ contains all positive (negative) roots. Denote by $X_\pm$ the root vectors corresponding to $\Sigma_\pm$. Then we have $(i = 1, \ldots, n)$

$$a_i \in X_+, \qquad b_i \in X_- \tag{17}$$

i. e., the para-Fermi annihilation (creation) operators belong to the system of positive (negative) root vectors of B_n. Thus, the problem to find all reservoir states can be formulated now in the following way.

PROBLEM. — Let π be an arbitrary finite dimensional irreducible representation of the algebra B_n, i. e., a homomorphic mapping $(\pi : g \to \hat{g}, g \in B_n, \hat{g} \in \hat{B}_n)$ of B_n onto the set $\hat{B}_n$ of linear operators in the finite dimensional space W and let Σ be the root system of B_n as defined in (13). Find the subspace

$$V = (v \,|\, v \in W, \hat{e}_{\omega_i} \cdot v = 0, i = 1, \ldots, n) \subset W \tag{18}$$

that is the set of all vectors $v \in W$ annihilated by the operators $\hat{e}_{\omega_i}$, $i = 1, \ldots, n$.

In order to solve this problem, it is convenient to choose the basis in W which consists of eigenvectors of the operators $\hat{h}$, $h \in H$. An arbitrary weight λ is defined then by its canonical co-ordinates $[\lambda_1, \ldots, \lambda_n]$ or by the co-ordinates $(l_1, \ldots, l_n)$ in the orthogonal basis $\Omega = (\omega_i \mid i = 1, \ldots, n)$ introduced by the relation (11). From the equality $\displaystyle\sum_{i=1}^{n} \lambda_i f_i = \sum_{i=1}^{n} l_i \omega_i$ it follows that

$$l_1 = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-1} + \frac{\lambda_n}{2}$$

$$l_2 = \qquad \lambda_2 + \ldots + \lambda_{n-1} + \frac{\lambda_n}{2}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

$$l_{n-1} = \qquad\qquad\qquad\qquad \lambda_{n-1} + \frac{\lambda_n}{2}$$

$$l_n = \qquad\qquad\qquad\qquad\qquad \frac{\lambda_n}{2}$$

Therefore the co-ordinates $(L_1, \ldots, L_n)$ of the highest weight Λ are either all integers or half-integers depending on whether the nth canonical co-ordinate of Λ is even or odd. Moreover, since the canonical co-ordinates of Λ are non-negative integers, we have

$$L_1 \geqslant L_2 \geqslant \ldots L_n \geqslant 0 \qquad (20)$$

A given representation π of B_n is defined up to isomorphism by its highest weight, i. e., by its co-ordinates $L_1, \ldots, L_n$. On the contrary, to every set of integer or half-integer numbers, satisfying (20), there corresponds a representation of B_n. Since all roots of B_n have integer co-ordinates in Ω, it follows from (6) that all weights have in the orthogonal basis either integer or half-integer coefficients.

Using the invariance of all weights Γ with respect to the Weyl group S, it is easy to show that the n-tuple obtained from a given weight $(l_1, \ldots, l_n)$ by arbitrary reflections $l_i \rightleftarrows -l_i$ and permutations $l_i \rightleftarrows l_j$ of its co-ordinates in the orthogonal basis Ω is also a weight. This property together with formulae (6) and (9) gives that the n-tuple $(l_1, l_2, \ldots, l_n)$ is a weight if and only if its co-ordinates satisfy the inequalities

$$| l_{i_1} | + | l_{i_2} | + \ldots + | l_{i_m} | \leqslant L_1 + L_2 + \ldots + L_m,$$
$$i_1 \neq i_2 \neq \ldots \neq i_m = 1, \ldots, n \quad (21)$$

for $m = 1, 2, \ldots, n$.

Now we can prove the following theorem.

THEOREM 1. — The root vector x_l is a reservoir state if and only if its weight $l = (l_1, \ldots, l_n)$ satisfies the condition

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} L_i \tag{22}$$

Proof. — Let the weight of x_l satisfy (22). Then the vector $x_{l'}' = \hat{e}_{\omega_i} x_l$ is (a) either zero, or (b) it has a weight $l' = (l_1', \ldots, l_n')$ with $l_i' = l_i + 1$, $l_j' = l_j$, $i \neq j$, $j = 1, \ldots, n$. Such weight, however, cannot exist since it violates the conditions (21)

$$\sum_{j=1}^{n} l_j' = \sum_{j=1}^{n} l_j + 1 > \sum_{j=1}^{n} L_j$$

Therefore $e_{\omega_i} x_l = 0$ for all $i = 1, \ldots, n$. The proof of the sufficient part of the theorem is based on the following two statements.

1) *Irreducibility.* — Given a Lie algebra R with basis $e_1, \ldots, e_N$. Consider a representation of R in the space W and let $x \in W$. Then W is linearly spanned by all elements

$$\hat{e}_{i_1} \hat{e}_{i_2} \ldots \hat{e}_{i_m} x \qquad m = 1, 2, \ldots \tag{23}$$

2) *Poincaré-Birkhoff-Witt theorem* [11]. — All ordered monomials $e_1^{j_1} e_2^{j_2}, \ldots, e_N^{j_N}$ constitute a basis of the universal enveloping algebra $\mathcal{U}$ of R.

Consider the associative algebra $\hat{\mathcal{U}}$ generated by all operators $\hat{e}_{i_1}, \ldots, \hat{e}_{i_m}$. Since $\hat{\mathcal{U}}$ is an (associative) homomorphic image of U, it follows from 2) that $\hat{\mathcal{U}}$ is a linear envelope of the operators $\hat{e}_1^{j_1} \hat{e}_2^{j_2}, \ldots, \hat{e}_N^{j_N}$. Therefore, an arbitrary element $z \in W$ is a finite linear combination of elements

$$\hat{e}_1^{j_1} \hat{e}_2^{j_2} \ldots \hat{e}_N^{j_N} . x \tag{24}$$

Let $l = (l_1, \ldots, l_n)$ be such a weight that

$$\sum_{i=1}^{n} l_i < \sum_{i=1}^{n} L_i \tag{25}$$

and suppose there exists a weight vector x_l corresponding to l and annihilated by all operators $\hat{e}_{\omega_i}$, i. e.,

$$\hat{e}_{\omega_i} x_l = 0 \qquad i = 1, \ldots, n \tag{26}$$

Divide the basis elements of B_n into three groups $\{ e_\alpha \}$, $\{ e_\beta \}$, $\{ \omega_K \mid K = 1, \ldots, n \}$, where

$$\alpha \in \{ -\omega_i \pm \omega_j, \omega_i - \omega_j, -\omega_K \mid i < j, i, j, K = 1, \ldots, n \}$$
$$\beta \in \{ \quad \omega_i + \omega_j, \omega_K \mid i < j, i, j, K = 1, \ldots, n \}$$

and order them within each group in an arbitrary way. According to (24), the representation space W is spanned on the elements

$$\hat{e}^{i_1}_{\alpha_1} \ \ldots \ \hat{e}^{i_p}_{\alpha_p} \hat{e}^{j_1}_{\beta_1} \ \ldots \ \hat{e}^{j_q}_{\beta_q} \hat{\omega}^{K_1}_1 \ \ldots \ \hat{\omega}^{K_n}_n . x_l \tag{27}$$

Since x_l is an eigenvector of all ω_i from (26) we have that W is a linear envelope of all vectors

$$\hat{e}^{i_1}_{\alpha_1} \ \ldots \ \hat{e}^{i_p}_{\alpha_p} . x_l \tag{28}$$

As it can be easily checked, all weights $(l'_1, \ldots, l'_n)$ corresponding to the vectors of the above type (28) have the property

$$\sum_{i=1}^{n} l'_i \leqslant \sum_{i=1}^{n} l_i < \sum_{i=1}^{n} L_i \tag{29}$$

This, however, is impossible, since it means that W does not contain, for instance, the highest weight vector. We run into contradiction. Hence, Eq. (26) with a weight satisfying (25) is impossible. This completes the proof. The following corollary is an immediate consequence of the above theorem and the property that the vectors corresponding to different weights are linearly independent.

COROLLARY. — The subspace $V \subset W$ of all reservoir states is the linear envelope of all weight vectors with weights $(l_1, \ldots, l_n)$ satisfying the Eq. (22).

The representation space W is linearly spanned on all elements

$$x_{\alpha_1 \ldots \alpha_m} = e_{\alpha_1} e_{\alpha_2} \ \ldots \ e_{\alpha_m} . x_\Lambda \qquad \alpha_q \in \Sigma_-, \quad q = 1, \ldots, m \tag{30}$$

If some of the roots α_q are equal to $- \omega_K$ or $- \omega_i - \omega_j$ $(i, j, K = 1, \ldots, n)$, then the weight of $x_{\alpha_1, \ldots, \alpha_n}$ does not satisfy the condition (22) and hence $x_{\alpha_1, \ldots, \alpha_n} \notin V$. Therefore an arbitrary vector $x \in V$ can be represented as

$$x = P(e_{-\omega_i + \omega_j}) x_\Lambda \tag{31}$$

with P being a polynomial of the operator $\hat{e}_\alpha$,

Let
$$\alpha \in \Sigma'_- = (- \omega_i + \omega_j | i < j ; j, i = 1, \ldots, n) \tag{32}$$

$$\Sigma'_+ = (\omega_i - \omega_j | i < j ; i, j = 1, \ldots, n) \tag{33}$$

Then for any $x \in V$ and $\alpha \in \Sigma' = \Sigma'_- \cup \Sigma'_+$ we have

$$\hat{e}_\alpha x \in V \tag{34}$$

Denote
$$\Omega' = \{ \omega'_i | \omega'_i = \omega_i - \omega_{i+1}, i = 1, \ldots, n - 1 \} \tag{35}$$

The vectors ω'_i, $i = 1, \ldots, n$, e_α, $\alpha \in \Sigma'$ constitute a basis of a subalgebra of B_n isomorphic to A_{n-1} [real form SU(n)]. Ω' can be chosen as a basis

of the Cartan subalgebra H' of A_{n-1} and then Σ' is its root system. With respect to the initial basis Ω the set $\Sigma'_+(\Sigma'_-)$ is the positive (negative) root system of A_{n-1} and the elements of Ω' are the simple roots.

Returning to (34), we observe that the reservoir subspace $V \subset W$ is invariant with respect to the algebra A_{n-1}. With arguments similar to those used in the proof of Theorem 1, it is possible to show that the highest weight vector x_Λ of the representation π of B_n is the only vector belonging to V and annihilated by the operators $\hat{e}_\alpha$, $\alpha \in \Sigma'_+$. This indicates that the representation π' of A_{n-1} in V is an irreducible one. The vector x_Λ is the highest weight vector for both the representation π of B_n in W and π' of A_{n-1} in V. The representation π' is characterized by its canonical co-ordinates $[\Lambda'_1, \ldots, \Lambda'_{n-1}]$,

$$\Lambda'_i = \frac{2(\Lambda, \omega'_i)}{(\omega'_i, \omega'_i)} \qquad i = 1, \ldots, n \qquad (36)$$

Using the co-ordinates of Λ in the orthogonal basis Ω, i. e., $\Lambda = \sum_{i=1}^{n} L_i \omega_i$ we obtain from (12) and (35) that $\Lambda'_i = L_i - L_{i+1}$. If now the representation π has canonical co-ordinates $[\Lambda_1, \ldots, \Lambda_n]$, then from (19) we derive that π' is characterized by the canonical co-ordinates

$$[\Lambda_1, \ldots, \Lambda_{n-1}] \qquad (37)$$

We collect the above-proved results in a theorem.

THEOREM 2. — Any irreducible representation $[\Lambda_1, \ldots, \Lambda_n]$ of the algebra B_n defines an irreducible representation of n pairs of para-Fermi operators a_i, b_i, $i = 1, \ldots, n$. The reservoir states from the representation space W span A_{n-1} invariant subspace V, in which the irreducible representation $[\Lambda_1, \ldots, \Lambda_{n-1}]$ of the algebra A_{n-1} is realized.

The dimension of the reservoir subspace V is given therefore by the formula for the dimension of the irreducible representations of A_{n-1}. In terms of the canonical co-ordinates $[\Lambda_1, \ldots, \Lambda_{n-1}]$ it reads as follows:

$$\dim V = \prod_{j=0}^{l-2} \frac{1}{(j+1)!} \prod_{K=1}^{l-j-1} (\Lambda_K + \ldots + \Lambda_{K+j} + j + 1) \qquad (38)$$

Since $\Lambda_K + \ldots + \Lambda_{K+j} = L_K - L_{K+j+1}$, where $(L_1, \ldots, L_n)$ are the co-ordinates of the highest weight Λ in the orthogonal basis Ω, we have also

$$\dim V = \prod_{i=1}^{l-1} \frac{1}{i!} \prod_{K=1}^{l-i} (L_K - L_{K+i} + i) \qquad (39)$$

It follows from (39) that the representation space contains a single vacuum

state x_Λ if and only if all co-ordinates of Λ are equal, i. e., $\Lambda = (L, \ldots, L)$. Since L is non-negative integer or half integer, all single valued representations can be labelled by the integer $N = 2L$. From (12) and (15), it is easy to show that in this case

$$a_i b_j x_\Lambda = 2L\delta_{ij} x_\Lambda \qquad (40)$$

As it is known [3] the relations (40) define canonical representations of the para-Fermi operators of order $2L = N$. Finally, we discuss the multiplicity properties of the weights Γ' corresponding to the basic reservoir states, i. e., the ones which are also weight vectors. Two weights λ_1, λ_2 belong to the same equivalence class with respect to the Weyl group S of B_n if there exists $s \in S$ such that $\lambda_1 = s\lambda_2$. All weights within one equivalence class have the same multiplicity. Since the element $s \in S$ acting on a weight λ changes some of the signs or permutes its orthogonal co-ordinates, Γ' is not S invariant. Indeed, if $(\lambda_1, \ldots, \lambda_n) \in \Gamma'$ then, for instance, $s.(\lambda_1, \ldots, \lambda_n) = (-\lambda_1, \ldots, -\lambda_n) \notin \Gamma'$ and $s \in S$. The system Γ' is invariant with respect to a subgroup $S' \subset S$, which only permits the orthogonal co-ordinates and S' is nothing but the Weyl group of A_{n-1}.

On the other hand Γ' is not contained in the equivalence class of the highest weight Λ, which would have proved the simplicity of its elements. More than that, from Theorem 2 we observe that Γ' is the collection of all weights in a representation of A_{n-1} which can be an arbitrary one depending on the choice of the representation for PFO. Since in the general case the multiplicity of these weights is more than one, the weights corresponding to the basic reservoir states are not necessarily simple. In fact all weights from Γ' are simple only in the case if the orthogonal co-ordinates of the highest weight Λ are less or equal to $\dfrac{3}{2}$, i. e., if the highest weight is either of the form $\left(\dfrac{3}{2} \ldots \dfrac{3}{2} \dfrac{1}{2} \ldots \dfrac{1}{2} \right)$ or $(1 \ldots 0\ 0 \ldots 0)$. Indeed in this case the operators $\hat{e}_\alpha$, $\alpha \in \Sigma'$, corresponding to the generators of A_{n-1} of A_{n-1} can only permute the co-ordinates of Λ and hence they belong to the Weyl group. Therefore Γ' is contained in one equivalence class together with the highest weight.

CONCLUSIONS

We have studied the set of all finite dimensional representations of a given number of para-Fermi operators. In fact, this is the set of all those representations for which the Hermitian conjugate of a_i equals b_i, i. e., $a_i^+ = b_i$. Indeed one can easily verify that the above condition requires that the representation of the compact form $SO(2n + 1)$ of B_n is anti-Hermitian and therefore (if the operators are bounded) finite dimensional.

The main purpose of the present paper was to analyze the properties of the vacuum-like states within a given irreducible representation of the

PFO. Bracken and Green have considered the same problem [1] and they have established some necessary properties to be satisfied by the vacuum-like states. The former authors have shown that any irreducible representation of U(n) containing a reservoir state can occur at most once within a given irreducible representation of SO($2n + 1$). Moreover, all such representations are contained in one irreducible representation of SO($2n$). As it now follows from Theorem 2, this is really the case since there exists one irreducible representation of SU(n) containing reservoir and only reservoir states. We have also shown that, in the general case, some of the weights corresponding to the basic reservoir states are not simple and hence are not contained in the equivalence class of the highest weight. This result disagrees with those obtained in [1]. If, however, the generalized para-statistic introduced by Bracken and Green is of order $p \leq 3$, then all weights of the basic reservoir states have multiplicity ones. Therefore the results in [1] concerning the applications of the generalized para-statistics of order 3 to a new, modified quark model, remain unaltered.

Finally, we wish to point out that further information about the vacuum-like states can be obtained by studying the multiplicity properties of the weights in the representations of A$_n$. This is, however, a rather hard problem, since apart from some trivial cases, explicit formulae for the multiplicities do not exist.

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