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# Coherent states and symmetric spaces. II

by

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ABSTRACT. — Properties of system of the generalized coherent states related to representations of class I of principal series of the motion groups of symmetric spaces of arbitrary rank have been studied. It has been proved that such states are given by horospherical kernels and are the generalization of the plane waves for the case of symmetric spaces. The case of symmetric spaces of the tube type is studied in more detail.

## 0. INTRODUCTION

This paper deals with the further study, started in paper [*1*], of the systems of generalized coherent states (CS), which are not square-integrable. The generalized CS introduced in paper [*2*], as well as usual CS [*3*] [*4*], turn out to be very convenient for the solution of a number of problems possessing a dynamical symmetry.

Thus, for instance, in papers [*5*] the problems of boson and fermion pair creation in alternating homogeneous external field were solved with their aid. In paper [*6*] CS for the rotation group of three-dimensional space (previously introduced in paper [*7*]) were used to obtain estimates for the partition function of the quantum spin system. In papers [*8*] [*9*] such states were applied in the so called Dicke model describing the interaction of radiation with matter.

In the following we shall call generalized CS for short simply CS. Note that the CS system is an overcomplete and non-orthogonal system of states of Hilbert space.

Under the additional assumption on square-integrability a number of properties of such systems was considered in papers [10]-[14]. In paper [1] some CS systems which are not square-integrable, namely the systems related to the unitary irreducible representations (UIRs) of class I of the principal series of symmetric space motion groups of rank 1 were studied.

In the present paper we study the CS systems related to the symmetric spaces of arbitrary rank.

The special attention is paid to those symmetric spaces which are Hermitean spaces and can be realized in the form of tube domains.

It is known that there exist four series of domains of this type (so called classical domains) and one special domain. The classical domains of this type are the following coset spaces:

$$\begin{aligned} X_p^{I} &= SU(p, p)/SU(p) \times SU(p) \times U(1) \\ X_p^{II} &= Sp(p, \mathbb{R})/U(p) \\ X_{2p}^{III} &= SO^*(4p)/U(2p) \\ X_p^{IV} &= SO_0(p, 2)/SO(p) \times SO(2) \end{aligned}$$

All these domains may be considered as the phase spaces of special dynamical systems and their motion groups-as the dynamical symmetry groups of the corresponding Hamiltonians. Note that there is a relation between the CS method and the so called problem of quantization considered by Kirillov [15] and Kostant [16].

We shall not consider here this relation but we shall give only some examples of physical problems related to the considered groups.

1. The $SU(p, p)$ group for odd $p$ is the dynamical symmetry group of the problem of boson pair creation in alternating homogeneous external field [5].

The $SU(4p)$ group is the dynamical symmetry group of the corresponding problem of fermion pair creation [5].

2. The symplectic group $Sp(p, \mathbb{R})$ is the group of linear homogeneous canonical transformations of boson creation and annihilation operators $a_j^+$, $a_k$ ($j, k = 1, \ldots, p$) [17]. The space, which is dual, according to Cartan, to the space $X_p^{II}$, appears in the problem of distribution of eigenvalues of random Hermitean matrices considered by Dyson [18].

3. The space, which is dual, according to Cartan, to the space $X_{2p}^{III}$, appears in the consideration of linear canonical transformations of fermion creation and annihilation operators.

4. The $SO(p, 2)$ group is the group of conformal transformations of $p$-dimensional Minkowsky space, i. e. pseudo-euclidean space of signature $(p - 1, 1)$. It is also the symmetry group of spherical functions of $p$-dimensional space. For $p = 3$ the $SO(3, 2)$ group (de Sitter group) is locally isomorphic to the $Sp(2, \mathbb{R})$ group. For $p = 4$ the $SO(4, 2)$ group is locally isomorphic to the $SU(2, 2)$ group, and its Lie algebra is isomorphic to the algebra of Dirac's matrices. Since recently, the study of representa-

tions of SO(4, 2) group has been of particular interest in view of consideration of the field theories with conformal invariance. See for example review paper [*19*].

In the present paper in Sec. 1 the necessary facts are collected from the theory of induced representations and the CS system is constructed for UIRs of principal series of class I and it is proved that the kernels which determine the coherent states are constants on horospheres of corresponding symmetric space.

In Sec. 2 the horospheres in the symmetric space are studied. In Sec. 3 the properties of the classical domains are investigated. The tube domains are studied in more detail. Finally in Sec. 4 explicit formulæ for the coherent states are obtained and their properties are investigated.

## 1. COHERENT STATES RELATED TO REPRESENTATIONS OF PRINCIPAL SERIES

Let G be a connected real semisimple Lie group with finite center. It is well known [*20*] that such a group possesess a series of unitary irreducible representations (UIRs) of class I, i. e. the series of UIRs for which in the Hilbert space there is a vector $|\psi_0\rangle$ invariant under the action of maximal compact subgroup K of the group G. Let $T(g)$ be such a representation. According to [*2*] the system of generalized CS of the type $(T, |\psi_0\rangle)$ is called the set of states $\{T(g)|\psi_0\rangle\}$. It can be easily seen that the elements $g_1$ and $g_2$ which belong to one coset G on K determine the same state. Therefore CS of the type $(T, |\psi_0\rangle)$ is given by the point of the cosetspace $X = G/K$. Just choosing some element $g_x \in G$ in the coset $gK$ corresponding to the element $x \in X$ we get the CS system $\{|x\rangle\}$:

$$|x\rangle = T(g_x)|0\rangle, \quad |0\rangle = |\psi_0\rangle \tag{1.1}$$

For representations of class I of the so called principal series we may use their explicit realization as induced representations. Let us remind it. The group G has the Iwasawa decomposition: G = KAN where K is the maximal compact subgroup of group G, A is Abelian noncompact subgroup and N is the maximal nilpotent subgroup. Let M be centralizer A in K, B be the subgroup G equal MAN, $\Xi$ be coset space $B\backslash G = M\backslash K$ and $d\mu(\xi)$ be K-invariant measure on $\Xi$ which is normalized so that $\int d\mu(\xi) = 1$. The UIRs of the principal series of class I are called the representations of the group G induced by unitary characters $\chi^\lambda(b)$ of the subgroup B trivial on M (i. e. $\chi^\lambda(b) = 1$, if $b \in M$). In other words, UIRs of the principal series of class I can be realized in the space of square-integrable functions $L_2(\Xi, d\mu)$. The operator of representation $T^\lambda(g)$ is given by the formula:

$$T^\lambda(g)f(\xi) = \alpha^\lambda(\xi, g)f(\xi_g), \tag{1.2}$$

where

$$\alpha^\lambda(\xi, g) = \left[\frac{d\mu(\xi_g)}{d\mu(\xi)}\right]^{1/2} \chi^\lambda(a(\xi, g)) \tag{1.3}$$

and elements $\xi_g \in \Xi$ and $a(\xi, g) \in A$ are determined from the expansion

$$g_\xi \cdot g = \text{MAN}\, g_\eta, \quad \eta = \xi_g, \quad a = a(\xi, g) \tag{1.4}$$

Here $g_\xi$ and $g_\eta$ are the elements of cosets space B\G corresponding to elements $\xi$ and $\eta$. Note that the quantities $\frac{d\mu(\xi_g)}{d\mu(\xi)}$ and $\chi^\lambda$ are functions of the quantities $a(\xi, g)$.

For the symmetric space $X = G/K$ of the rank $r$ (1) such representations are determined by the real numbers $\lambda_1, \ldots, \lambda_r$ (2). The function $\alpha^\lambda(\xi, g)$ which determines the representation $T^\lambda(g)$ satisfies the functional equation:

$$\alpha^\lambda(\xi, g_2 g_1) = \alpha^\lambda(\xi, g_2)\alpha^\lambda(\xi_{g_2}, g_1) \tag{1.5}$$

Going to another function $\tilde{f}(\xi) = f_0(\xi) f(\xi)$, if necessary, we may assume that

$$\alpha^\lambda(\xi, k) \equiv 1 \quad \text{for } k \in K \text{ and therefore}$$
$$\alpha^\lambda(\xi, gk) = \alpha^\lambda(\xi, g) \tag{1.6}$$

It means that

$$\alpha^\lambda(\xi, g) = K_\lambda(x, \xi) \tag{1.7}$$

where $x = x(g)$ is the coset $gK$ corresponding to element $g$.

It is easy to see that the function $\Psi_0(\xi) \equiv 1$ belongs to $L_2(\Xi, d\mu)$ and it is invariant under transformations of the maximal compact subgroup K. Acting on it by the operator $T^\lambda(g)$ we obtain the expression for coherent states in the $\xi$-representation:

$$T^\lambda(g)\Psi_0(\xi) = \alpha^\lambda(\xi, g) = \Psi^\lambda_x(\xi) = \langle \xi, \lambda \mid x \rangle \tag{1.8}$$

Thus CS$\mid x \rangle$ is determined by the kernel $\langle \xi, \lambda \mid x \rangle$, where $\xi \in \Xi$ and $x \in X$.

Let us go now to the study of properties of these kernels. First of all we shall prove.

PROPOSITION 1.1. — For fixed $\xi \in \Xi$ the kernel $\langle \xi, \lambda \mid x \rangle$ is constant on orbits of the group $N_\xi$ which is conjugate to the group N present in the Iwasawa decomposition $G = KAN$ and having the fixed point $\xi$.

*Proof.* — Let us fix the point $\xi \in \Xi$ and consider the function $f_\xi(g) = T^\lambda_g \Psi_0(\xi) = \alpha^\lambda(\xi, g)$ as function of the variable $g$. Let us denote

(1) Let us remind that the rank of symmetric space $G \mid K$ is called the number of independent metric invariants of a pair of its points. This number equals to the dimension of the subgroup A of the group G [*21*].

(2) As is shown in paper [*22*] all representations of principal series are irreducible. In this paper it is also proved that for the spaces of rank I each UIR of class I belongs either to principal or to complementary series which is obtained from the principal series by means of an analytic continuation in $\lambda$.

by $H_\xi$ the stability subgroup of the point $\xi$. Putting $\eta = \xi$ in (1.4) we obtain

$$H_\xi = g_\xi^{-1} B g_\xi \tag{1.9}$$

i. e. the group $H_\xi$ is conjugated to the group $B = MAN$. Correspondingly, the nilpotent part $N_\xi$ of the group $H_\xi$ is

$$N_\xi = g_\xi^{-1} N g_\xi \tag{1.10}$$

and, as it is easy to see, it keeps the point $\xi$ to be fixed.

Let $h$ be an element of $N_\xi : h = g_\xi^{-1} n g_\xi$, $n \in N$. Then the function $\alpha^\lambda(\xi, h)$ is completely determined by the quantities $a(\xi, g)$ which enter in the expansion (1.4). Furthermore

$$g_\xi h = g_\xi g_\xi^{-1} n g_\xi = n g_\xi \tag{1.11}$$

Consequently, $a(\xi, h) = e$ ($e$ is the identity element of the group G).

So we have proven that for $h \in N_\xi$

$$\alpha^\lambda(\xi, h) = 1 \tag{1.12}$$

From this it follows also that for $h \in M_\xi$, $M_\xi = g_\xi^{-1} M g_\xi$

$$\alpha^\lambda(\xi, h) = 1 \tag{1.13}$$

Finally, for $h \in A_\xi = g_\xi^{-1} A g_\xi$, $h = g_\xi^{-1} a g_\xi$

$$\alpha^\lambda(\xi, h) = f_\xi(h) = f_\xi(a) \tag{1.14}$$

Let us consider now an arbitrary element of group G. Then $g = g_x k$ and therefore

$$\alpha^\lambda(\xi, g) = \alpha^\lambda(\xi, g_x k) = \alpha^\lambda(\xi, g_x) \tag{1.15}$$

Let us denote $y = x_h$, $h \in N_\xi$. It means that

$$h^{-1} g_x = g_y k_1, \qquad g_x = h g_y k_1, \qquad k_1 \in K \tag{1.16}$$

Now in view of the functional equation (1.5)

$$\alpha^\lambda(\xi, g_x) = \alpha^\lambda(\xi, h g_y k_1) = \alpha^\lambda(\xi, h g_y) = \alpha^\lambda(\xi, g_y) \tag{1.17}$$

Therefore

$$\Psi_x^\lambda(\xi) = \Psi_y^\lambda(\xi), \qquad y = x_h, \qquad h \in N_\xi \tag{1.18}$$

Note that when $h$ goes over the whole group $N_\xi$, $x_h$ goes over the orbit of this group in the space X. These orbits are called the horospheres of maximal dimension of the symmetric space X or the horocycles. Hence the proposition 1.1 can be formulated in the following equivalent form.

PROPOSITION 1.1′. — The kernel $\langle \xi, \lambda \mid x \rangle$ describing coherent state $\mid x \rangle$ is constant on horocycles of group $N_\xi$.

It is naturally to call these kernels by horospherical kernels. This relates the coherent state method for the considered case to the horosphere method developed in paper by Gel'fand and Graev [*23*] and considered in detail for the case of symmetric space in paper by Helgason [*24*].

Let us consider the connection of coherent states with the horocycles of symmetric space in some more detail.

## 2. HOROCYCLES IN SYMMETRIC SPACE

Note once more, that horospheres of maximal dimension in the space X or horocycles are called the orbits of subgroups conjugated to the subgroup N. In this conception one usually includes also horospheres of lesser dimension—the horospheres of non-general position, but here we shall not consider those. Let us consider the properties of horocycles.

PROPOSITION 2.1. — Let $\Omega = \{ \omega \}$ be the set of horocycles. Then $\Omega = G/MN$.

The proof is given in paper [*24*].

PROPOSITION 2.2.

1) $K \backslash G/K = A/W$
2) $MN \backslash G/MN = A \times W$

where W is the Weyl group of the symmetric space: $W = N(A)/M$, $N(A)$ is normalizer A in K.

Every horocycle may be represent in the form

$$\omega = (ka)^{-1}\omega_0, \qquad \omega_0 = N \cdot 0$$

(0 is the origin of the space X), while the elements $k$ and $mk$ determine the same horocycle, and the element $a$ is unique. Hence the quantities $\xi$ and $a$ determine the horocycle $\omega$ unambiguously. This gives the possibility to introduce the horospherical system of coordinats: the element $\xi \in \Xi$ is called the normal to horosphere $\omega$, and element $a$ is called the complex distance from the horocycle $\omega_0$.

The important special case of symmetric space is the case of Hermitean symmetric space, i. e. the case of symmetric space possessing a complex structure. It is known that these spaces can be realized in the form of bounded domains in $n$-dimensional complex space—the space $\mathbb{C}^n$.

In this case we can obtain complete description of boundary of symmetric spaces and its horospheres by using the conception of boundary component introduced by Pyatetsky-Shapiro in [*25*].

DÉFINITION 2.3. — Let X be the Hermitean symmetric space which is realized in the form of a bounded domain $\mathcal{D}$ in the space $\mathbb{C}^n$, $\partial\mathcal{D}$ be the boundary of $\mathcal{D}$. The subset $\mathcal{F} \subset \partial\mathcal{D}$ is called by boundary component, if every analytic curve $\varphi(t)$ $(t \in \mathbb{C}, |t| < \varepsilon)$ which as a whole belongs to $\partial\mathcal{D}$ and cross $\mathcal{F}$, as a whole belongs to $\mathcal{F}$.

As an arbitrary Hermitean symmetric space is a direct product of irreducible Hermitean symmetric spaces then, in order to prove the following statements it is sufficient to consider only irreducible spaces. Let $\mathcal{D}$ be an irreducible Hermitean symmetric space of rank $r$ realized in the form of bounded symmetric domain in $\mathbb{C}^n$, $\partial\mathcal{D}$ be the boundary of this domain. In this case the following proposition is valid.

PROPOSITION 2.4. — The number of nonequivalent components $\partial\mathcal{D}$ is equal to $r$.

*Proof.* — This statement follows from the explicit enumeration of these components for the case of classical domains [*25*]. The proof which does not use the enumeration see in [*26*]. From this proposition it is easy to obtain the description of orbits in $\partial\mathcal{D}$.

LEMMA 2.5. — The number of nonequivalent G-orbits in $\partial\mathcal{D}$ is equal to $r$.

*Proof.* — From the proposition (2.4) the existence follows of $r$ analytically nonequivalent components $\mathcal{F}^{(i)}$ $(i = 1, \ldots, r)$. Let us consider the set $F_i = \bigcup_{g \in G} \mathcal{F}^{(i)}_g$. It is easy to see that $F_i \cap F_j = \emptyset$, if $i \neq j$, and that any $\mathcal{F}^{(i)}$ can be transfer into $\tilde{\mathcal{F}}^{(i)}$ with the help of analytical automorphisms of domains $\mathcal{D}$. So we obtain $r$ G-nonequivalent orbits.

As well known, among the subsets of boundary there exist Bergman-Shilov boundary (BS-boundary). Let us describe this boundary in the terms of boundary components.

LEMMA 2.6. — The BS-boundary consists of zerodimensional components.

*Proof.* — Let $\varphi(t)$ be analytical curve $\varphi = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_1(t), \ldots, \varphi_n(t)$ are functions which are analytic in the disk $|t| < \varepsilon$. Let us denote $M = \sup(|\varphi_1|^2 + \ldots + |\varphi_n|^2)$. Then

$$|\varphi_1(0)|^2 + \ldots + |\varphi_n|(0)|^2 \leqslant M$$

and equality is valid if and only if all $\varphi_i(t)$ are constants. From the definition of BS-boundary S follows that for each analytical function $\psi(z)$, $z \in \mathcal{D}$ there exist the point of BS-boundary $s_0 \in S$ in which the quantity $|\psi(z)|$ achives maximal value. Then it follows from the above discussion and from the definition 2.3 that the boundary component which contains the point $s_0$ consists only from this point. Let us apply the construction of lemma 2.5 to the point $s_0$. The orbit $Gs_0$ coincides with the BS-boundary.

For the following study of the boundary and horospheres it is appropriate to use another though equivalent definition horospheres which was introduced by Gel'fand and Graev [*23*].

Let us consider a subgroup $\mathrm{P} \subset \mathrm{G}$ so that

$$\lim_{t \to \infty} g(-t)\mathrm{P}g(t) = e \tag{2.1}$$

Here $g(t)$ is the one-parameter subgroup in G such that $g(t) \in \mathrm{A}$. Let us denote by $\mathrm{N}_i$ the nilpotent part of group P.

It turn out to be that to different one-parameter subgroups corresponds different horospheres, which are orbits of subgroups, conjugated to the subgroup $\mathrm{N}_i$.

In the book [*25*] it is proved that in any boundary component there exist one and only one point $\delta$ which may be connected by geodesic with the fixed point $x_0$ of domain $\mathscr{D}$. This geodesic is, of course, not unique.

PROPOSITION 2.7. — The number of nonequivalent horospheres equal to $r$. Indeed, let us fix the point $x_0$ (for example the origin of $\mathscr{D}$). The geodesic $x(t)$ has the form $g(t)x_0$, where $g(t)$ is one-parameter subgroup such as $g(t) \in \mathrm{A}$. The number of different nonequivalent oneparameter subgroups equal to $r$. The geodesic $x(t)$ determine the point $\delta = \lim_{t \to \infty} x(t)$ of boundary $\partial\mathscr{D}$. It is easy to see that to different boundary components corresponds different type of geodesic.

Let $\mathrm{R}_i$ be the subgroup of all transformations of group G, keeping each point of $\mathscr{F}^{(i)}$ to be fixed in the following sence: for each geodesic $x(t)$ such that $\lim_{t \to \infty} x(t) = \delta \in \mathscr{F}^{(i)}$ follows that $\rho(x(t), gx(t)) = 0$ for $g \in \mathrm{R}_i$. Here $\rho(x, y)$ is distance between the points $x$ and $y$. It is evident that $g(t)\delta = \delta$. Then if $g \in \mathrm{R}_i$, so $\lim_{t \to \infty} \rho(x(t), g(x(t)) = 0$. Let us denote $x(t) = g(t)x_0$, then we get $\lim_{t \to \infty} (g(t)x_0, gg(t)x_0) = 0$. Hence $\lim_{t \to \infty} g(-t)gg(t) = e$ and the nilpotent subgroups of group $\mathrm{R}_i$ coincides up to conjugation with nilpotent parts of subgroup P in (2.1).

The following result will be presented without a proof.

The stability subgroup G of point $s_0 \in \mathrm{S}$ of BS-boundary is isomorphic to the group $\mathrm{K}_0\mathrm{AN}$, where $\mathrm{K}_0$ is compact subgroup which contains the centralizer M. From this follows that $\dim \Xi \geqslant \dim \mathrm{S}$ (The proof see e. g. in [*26*]).

In conclusion this section we show that there exist the natural equivariant mapping of space $\Xi$ into space S. Let us remind the definition of equivariant mapping.

DÉFINITION 2.8. — Let X and Y are G-invariant spaces. The mapping $f : \mathrm{X} \to \mathrm{Y}$ is called equivariant mapping if the diagramm

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{f} & \mathrm{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{X} & \xrightarrow[f]{} & \mathrm{Y} \end{array}$$

is commutative.

Each normal $\xi \in M/K$ determine the unique horocycle containing the origin. This horocycle determine the point BS-boundary. It is obvious that this mapping commute with the action of group G.

Note that the group K acts transitively on the space $\Xi$ and hence from equivariance of mapping $\Xi \to S$ follows that K acts transitively also and on BS-boundary.

Let us consider in more detail the classical domains.

## 3. CLASSICAL DOMAINS

As is known (see e. g. [25] [27]) there exist four types of classical domains

$$\mathcal{D}^{\mathrm{I}}_{p,q}, \quad p \geqslant q, \quad G = SU(p, q), \quad K = SU(p) \times SU(q) \times U(1) \quad (3.1)$$

$$\mathcal{D}^{\mathrm{II}}_{p}, \quad G = Sp(p, R), \quad K = U(p) \quad (3.2)$$

$$\mathcal{D}^{\mathrm{III}}_{p}, \quad G = SO^*(2p), \quad K = U(p) \quad (3.3)$$

Here $SO^*(2p)$ be subgroup of group $SO(2p, \mathbb{C})$ which leave invariant the form $z_1\bar{z}_{p+1} - z_{p+1}\bar{z}_1 + \ldots$

$$\mathcal{D}^{\mathrm{IV}}_{p}, \quad G = SO_0(p, 2), \quad K = SO(p) \times SO(2) \quad (3.4)$$

where $SO_0(p, 2)$ denotes the connected component of the identity transformation of the group $SO(p, 2)$.

All these domains are irreducible Hermitean symmetric spaces except the domain type IV for $p = 2$. Let us give the table of the principal characteristics of classical domains. Note that dimensions the space X and $\Xi$ connected by the formula $\dim \Xi = \dim X - r$, which follows from the formula $\dim G/K = \dim G/MN$.

Let us denote by $K_0$ the maximal compact subgroup of stability subgroup $G_0$ of the point $s_0 \in S$. As the group K acts transitively on S, then the group $K_0$ is stability subgroup of the point $s_0$ relative to the action of the group K. Let us remind that $G_0 = K_0AN$.

TABLE 1

| $\mathcal{D}$ | $\dim \mathcal{D}$ | $\dim \Xi$ | $\dim S$ | $\dim K_0$ |
|---|---|---|---|---|
| $\mathcal{D}^{\mathrm{I}}$ | $2pq$ | $2pq - q$ | $q(2p-q)$ | $(p-q)^2 + q^2 - 1$ |
| $\mathcal{D}^{\mathrm{II}}$ | $p^2 + p$ | $p^2$ | $\frac{p(p+1)}{2}$ | $\frac{p(p-1)}{2}$ |
| $\mathcal{D}^{\mathrm{III}}$ | $p^2 - p$ | $p^2 - p - \left[\frac{p}{2}\right]$ | $\frac{p(p-1)}{2} + \frac{(1+(-1)^{p+1})}{2}(p-1)$ | $\frac{p(p+1)}{2} - \frac{(1+(-1)^{p+1})}{2}(p-1)$ |
| $\mathcal{D}^{\mathrm{IV}}$ | $2p$ | $2p - 2$ | $p$ | $\frac{(p-1)(p-2)}{2}$ |

The compact subgroup $K_0$ may be described as follows

| | | |
|---|---|---|
| I. | $K_0 = SU(q) \times U(p-q)$ | |
| II. | $K_0 = SO(p)$ | |
| III. | $K_0 = Sp(p/2)$ | for even $p$ |
| | $K_0 = Sp((p-1)/2) \times SO(2)$ | for odd $p$ |
| IV. | $K_0 = SO(p-1)$ | |

As it is known there exist two realizations of classical domains: bounded realization (of the type of unit disk) and unbounded realization (of the type of upper halfplane).

The bounded realization is described in detail e. g. in [27]. In this paper is given also description of BS-boundary. To reach our aims it is convenient to use unbounded realization. We restricted to the consideration of the most important class of domains namely the tube domains.

Let us remind that the homogeneous domain $\mathscr{D}$ is called the tube domain, if it may be represented in the form: $\mathscr{D} = \{ Z : Z = X + iY, X \in \mathbb{R}^n, Y \in V^n \}$, where $V^n \subset \mathbb{R}^n$ is convex selfadjoint homogeneous cone.

The tube domains belong to the four classical series and there is one exceptional domain.

| | |
|---|---|
| I. | $\mathscr{D}^{I}_{pp} = SU(p, p)/SU(p) \times SU(p) \times U(1)$ |
| II. | $\mathscr{D}^{II}_{p} = Sp(p, R)/U(p)$ |
| III. | $\mathscr{D}^{III}_{2p} = SO^*(4p)/U(2p)$ |
| IV. | $\mathscr{D}^{IV}_{p} = SO_0(p, 2)/SO(p) \times SO(2)$ |
| V. | $\mathscr{D}^{V} = E^{R}_{7}/E^{C}_{6} \times SO(2)$ |

Here $E^R_7$ is certain real form of the exceptional group $E_7$, $E^C_6$ is the compact form of the exceptional group $E_6$, dim $\mathscr{D}^V = 54$ and the rank the domain $\mathscr{D}^V$ equal to 3.

It is known that the possibility of representation of domain $\mathscr{D}$ in the form of tube domain is connected with the structure of its root diagram. The domain $\mathscr{D}$ has a form of tube domain if its root diagram has the type $C_p$ and cannot be realized as a tube if the root diagram has the type $BC_p$ (see e. g. [26]).

In order to obtain explicit formulae for the kernels describing CS one should find the character $\chi^\lambda(a)$ of the representation $T^\lambda$. Note that geometrically the element $a \in A$ characterizes the complex distance between parallel horocycles.

PROPOSITION 3.1. — Let $o$ be origin in $\mathscr{D}$, $\omega_0 = N \cdot 0$ be the horocycle parallel to $\omega$ and $x \in \omega$. Then the element $a \in A$ defining the complex distance between the horocycle $\omega_0$ and $\omega$ may be found from the formula

$$x = a(x)n(x) \cdot 0, \qquad x \in \omega \tag{3.6}$$

The paper [23] contains the proof of this statement for the case of coset space of a complex group. This proof can be also applied to our case because the family of horocycles is transitive relative to the action of group G.

Let us go to the explicit calculation of the complex distance and some necessary subgroups for the four series of the classical tube domains. Some general results will clear up the structure of these calculations.

PROPOSITION 3.2. — Any analytic automorphism of tube domain which fixes infinity point has the form

$$Z \rightarrow AZ + B \tag{3.7}$$

where A is an affine transformation of the cone V on itself and B is a real vector.

PROPOSITION 3.3. — The BS-boundary S is a part of the boundary $\partial\mathcal{D}$ which consists of the points $\{ Z = X, Y = 0 \}$. BS-boundary is invariant under the automorphisms of the domain $\mathcal{D}$, and $\dim_{R} S = \dim_{C} \mathcal{D}$.

It follows from the proposition 3.2 that all transformations of the BS-boundary are of the form (3.7). Consider all transformations of the group G which keep the point $s$ of the BS-boundary S to be fixed. Choose $s$ as the infinity point. The stability group of this point is $G_0 = K_0AN$. If $G_0$ acts on $\mathcal{D}$ as the group of fractional linear transformations

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \rightarrow (AZ + B)(CZ + D)^{-1} \qquad \text{then } C = 0$$

Let us describe the nilpotent subgroup of the group G for the case of tube domains.

PROPOSITION 3.4. — Let N be the nilpotent subgroup present in the Iwasawa decomposition of G which acts in $\mathcal{D}$. Then N is a semi-direct product $N = N_1 \ltimes N_2$ where $N_1$ is a nilpotent subgroup of the group of affine transformations of the cone and $N_2$ is the commutative invariant subgroup—the translation group of the BS-boundary.

The group $G_1$ of affine transformations of the cone V acts as follows

$$g : Y \rightarrow Y' = AYA^{+}, \quad Y, Y' \in V, \quad g \in G_1, \tag{3.8}$$

where A belongs, correspondingly, to (notations see in [21])

$$\begin{array}{lll} 1) & SL(p, C) \times R^{+} & \text{for } \mathcal{D}^{I} \\ 2) & SL(p, R) \times R^{+} & \text{for } \mathcal{D}^{II} \\ 3) & SU^{*}(2p, R) \times R^{+} & \text{for } \mathcal{D}^{III} \end{array} \tag{3.9}$$

where $SU^{*}(2p, R) \times R^{+}$ is the group of real quaternionic matrices, $R^{+}$ is a multiplicative group of positive real numbers,

$$4) \qquad SO(p-1, 1) \times R^{+} \qquad \text{for } \mathcal{D}^{IV}$$

Note that the nilpotent subgroups $N_1$ are the maximal nilpotent subgroups of the above groups $G_1$ and any element $Y \in V$ can be represented in the form

$$Y = AY_0A^+ \tag{3.9'}$$

where $A \in N_1$, $Y_0$ is the diagonal matrix with positive elements.

Note that the cones connected with the domains of the types $\mathscr{D}^{I}$, $\mathscr{D}^{II}$, $\mathscr{D}^{III}$, $\mathscr{D}^{V}$ can be described as:

I) The cone of positive definite complex hermitian matrices in the case of $\mathscr{D}^{I}$.

II) The cone of positive definite real symmetric matrices in the case of $\mathscr{D}^{II}$.

III) The cone of positive definite hermitian-quaternion matrices in the case of $\mathscr{D}^{III}$.

IV) The cone of positive definite « hermitian » $3 \times 3$ matrices over the Cayley numbers (octonions) in the case of $\mathscr{D}^{V}$.

It follows from the above considerations that in the root diagram $C_p$ multiplicity $m$ of the roots $\pm (e_i \pm e_j)$ in the case of $\mathscr{D}^{I}$ is equal two, $m = 2$, in the case of $\mathscr{D}^{II}$, $m = 1$, in the case of $\mathscr{D}^{III}$ $m = 4$ and in the case of $\mathscr{D}^{V} m = 8$.

Finally, we note that for the domain of the type $\mathscr{D}^{IV}$ the cone consists of a set of vectors $\{ y = (y_1, \ldots, y_p) ; y_1^2 - y_2^2 - \ldots - y_p^2 > 0, y_1 > 0 \}$.

Let us consider the classical tube domains in more detail. We shall use the following notations: $A^+$—the matrix hermitian conjugated to the matrix A, $A > 0$ where A is a hermitian matrix means the positivity of all eigenvalues of the matrix A, $A^{(p,q)}$—the matrix with $p$ lines and $q$ columns, $A^{(p)}$—the matrix of order $p$.

### 3A. **The tube domains of the type I.**

Let us remind that this domain is defined as

$$\mathscr{D}_p^{I} = SU(p, p)/SU(p) \times SU(p) \times U(1)$$

or

$$\mathscr{D}_p^{I} = \{ Z : Z = X + iY, X^+ = X, Y^+ = Y, Y > 0 \}$$

where $G = SU(p, p) = \{ g \}$ acts on $\mathscr{D}$ as the group of fractional linear transformations.

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \to Z' = (AZ + B)(CZ + D)^{-1}, \det g = 1 \tag{3.10}$$

where A, B, C, D are the matrices of order $p$.

The BS-boundary consists of the hermitian matrices $Z = X$, $Y = 0$.

From the invariance of the BS-boundary it follows that

$$A^+C = C^+A, \; B^+D = D^+B, \; A^+D - C^+B = \lambda I, \; D^+A - B^+C = \lambda I \tag{3.11}$$

(where $\lambda$ is a real number. Let us put $\lambda = 1$. Then it is easy to verify that the condition (3.11) is equivalent to the condition

$$g^+ \mathrm{E} g = \mathrm{E} \tag{3.12}$$

where $\mathrm{E} = \begin{pmatrix} 0 & -i\mathrm{I} \\ i\mathrm{I} & 0 \end{pmatrix}$ and I is the unit matrix.

It means that $\mathrm{G} = \mathrm{SU}(p, p)$.

Note that in the case of the bounded realization the matrix $g$ satisfies the condition

$$g^+ \mathrm{E}_1 g = \mathrm{E}_1 \qquad \text{where} \qquad \mathrm{E}_1 = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} \tag{3.13}$$

From relation (3.12) we can find that the Lie algebra of the group G consists of the following matrices:

$$\begin{pmatrix} \tilde{\mathrm{A}} & \tilde{\mathrm{B}} \\ \tilde{\mathrm{C}} & -\tilde{\mathrm{A}}^+ \end{pmatrix} \qquad \text{where} \qquad \tilde{\mathrm{B}}^+ = \tilde{\mathrm{B}},\ \tilde{\mathrm{C}}^+ = \tilde{\mathrm{C}},\ \mathrm{sp}\,\tilde{\mathrm{A}} = \mathrm{sp}\,\tilde{\mathrm{A}}^+ \tag{3.14}$$

Let us give some useful information on the structure of the group G and its Lie algebra $\mathscr{G}$.

1) The maximal compact subgroup $\mathrm{K} = \{ k \}$ is isomorphic to the group $\mathrm{SU}(p) \times \mathrm{SU}(p) \times \mathrm{U}(1)$. Its Lie algebra $\mathscr{K}$ consists of the matrices:

$$\begin{pmatrix} \tilde{\mathrm{A}} & \tilde{\mathrm{B}} \\ -\tilde{\mathrm{B}} & \tilde{\mathrm{A}} \end{pmatrix} \qquad \text{where} \qquad \tilde{\mathrm{A}}^+ = -\tilde{\mathrm{A}},\ \tilde{\mathrm{B}}^+ = \tilde{\mathrm{B}},\ \mathrm{sp}\,\tilde{\mathrm{A}} = 0 \tag{3.15}$$

2) Exponenting the matrix (3.15) we obtain the expression for a matrix $k \in \mathrm{K}$

$$k = \frac{1}{2}\begin{pmatrix} \mathrm{A} + \mathrm{D}, & \pm i(\mathrm{A} - \mathrm{D}) \\ \mp i(\mathrm{A} - \mathrm{D}), & \mathrm{A} + \mathrm{D} \end{pmatrix},$$
$$\text{where} \quad \mathrm{A}^+\mathrm{A} = \mathrm{D}^+\mathrm{D} = 1,\ \det \mathrm{A} \cdot \det \mathrm{D} = 1 \tag{3.16}$$

3) The Cartan decomposition $\mathscr{G} = \mathscr{K} + \mathscr{P}$ is as follows

$$p = \begin{pmatrix} \tilde{\mathrm{A}} & \tilde{\mathrm{B}} \\ \tilde{\mathrm{B}} & -\tilde{\mathrm{A}} \end{pmatrix}, \qquad \tilde{\mathrm{A}}^+ = \tilde{\mathrm{A}},\ \tilde{\mathrm{B}}^+ = \tilde{\mathrm{B}}, \qquad p \in \mathscr{P} \tag{3.17}$$

and $k$ is obtained by the formula (3.15).

4) The maximal Abelian subalgebra $\mathscr{A} \subset \mathscr{P}$ can be chosen in the form $\mathscr{A} = \{ a \}$

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \tag{3.18}$$

$\alpha$ is a real diagonal matrix. Correspondingly, the group A consists of the matrices

$$\begin{pmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{A}^{-1} \end{pmatrix} \tag{3.19}$$

where A is a diagonal matrix with non-negative elements.

5) The subgroup M (the centralizer A in K) consists of the following matrices:

$$m = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \tag{3.20}$$

where T is a diagonal unitary matrix with det T $= \pm 1$. Correspondingly, $M = U(1) \times \ldots \times U(1)$ ($p - 1$ times).

6) Choose the point $s$ of the BS-boundary S as the infinity point. Then as follows from (3.10), the stability group $G_0$ of this point consists of the matrices

$$g_0 = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \qquad B^+D = D^+B,\ A^+D = I \tag{3.21}$$

The group $G_0$ acts on $\mathscr{D}$ as:

$$g_0 : Z \to Z' = AZA^+ + BA^+ \tag{3.22}$$

or

$$g_0 : X \to X' = AXA^+ + BA^+,\ Y \to Y' = AYA^+ \tag{3.23}$$

The group $G_0$ acts transitively but not simply transitively on $\mathscr{D}$. For example, the stability subgroup of the point $Z_0 = iI$ is isomorphic to the group SU($p$) (see table I) and in this realization it is a group of matrices:

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \right\} \qquad U^+U = 1,\ \det U = \pm 1.$$

7) Now we shall find the maximal nilpotent subalgebra of the algebra $\mathscr{G}$ or, what is the same, of the algebra $\mathscr{G}_0$. To do this we shall find the root subspaces of the algebra $\mathscr{G}_0$ which correspond to the positive roots with respect to the algebra $\mathscr{A}$. We have

$$\left[ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} A & B \\ 0 & -A^+ \end{pmatrix} \right] = \lambda \begin{pmatrix} A & B \\ 0 & -A^+ \end{pmatrix}, \qquad B^+ = B \tag{3.24}$$

Calculating the commutator we get the following conditions:

$$[\alpha, A] = \lambda A, \qquad \alpha B + B\alpha = \lambda B \tag{3.25}$$

Now we introduce the short notations: $e_{ij}$ — the matrix with elements $[e_{ij}]_{ke} = \delta_{ik}\delta_{je}$ (where $\delta_{ij}$ is the Kroneker symbol). Then $\alpha = \sum_{i=1}^{p} \alpha_i e_{ii}$.

Now we will investigate the cases $B = 0$ and $A = 0$ separately:

*a*) $B = 0$, $A = e_{ij}$ or $A = ie_{ij}$,

$$[e_{ii}, A] = A, \qquad [e_{jj}, A] = -A$$

and other commutators are equal to zero.

Thus the matrix of the form

$$\begin{pmatrix} A & B \\ 0 & -A^+ \end{pmatrix}$$

where $A = e_{ij}$ or $A = ie_{ij}$ and $B = 0$ corresponds to the root $(e_i - e_j)$. The multiplicity of this root is equal to 2.

*b)* $A = 0$, $B = e_{ij} + e_{ji}$ or $B = i(e_{ij} - e_{ji})$, $i < j$. These matrices correspond to the root $e_i + e_j$, $i < j$ and the multiplicity of this root is equal to 2 also.

*c)* $A = 0$, $B = e_{ii}$, $e_{ii}B + Be_{ii} = 2B$. This is the case of the root $2e_i$ and the multiplicity is equal to one.

The nilpotent subalgebra $\mathcal{N}$ corresponds to the set of all positive roots. So $\mathcal{N}$ consists of the matrices such as

$$n = \begin{pmatrix} A & B \\ 0 & -A^+ \end{pmatrix} \tag{3.26}$$

where A is the upper triangular matrix with diagonal zero, and B is hermitian matrix.

8) The subalgebra $\mathcal{N}$ is the semidirect sum of two subalgebras $\mathcal{N} = \mathcal{N}_1 \boxed{\sim} \mathcal{N}_2$

$$\mathcal{N}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^+ \end{pmatrix} \right\}, \quad \mathcal{N}_2 = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}$$

where matrices A, B satisfy the conditions of point 7 and $\mathcal{N}_2$ is the ideal in $\mathcal{N}$.

9) We can find the group N exponenting the algebra $\mathcal{N}$

$$N = \left\{ \begin{pmatrix} A & AB_1 \\ 0 & (A^+)^{-1} \end{pmatrix} \right\} \tag{3.27}$$

where A is the upper triangular matrix with unities on the diagonal, $B_1$ is hermitian matrix, N is the semidirect product of two groups $N_1$ and $N_2$, where

$$N_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \right\}, \quad N_2 = \left\{ \begin{pmatrix} 1 & B_1 \\ 0 & 1 \end{pmatrix} \right\}$$

The group $N_1$ naturally acts on the cone V

$$n_1 : Y \to Y' = AYA^+. \tag{3.28}$$

Its action on the cone is not transitive. The orbits of this group may be naturally called as the horocycles of the cone. The group $N_2$ acts as the translation group.

So the horocycle $\omega$ of the whole symmetric space can be represented by

$$\omega = \{ X + iY,\ Y \in \omega_c,\ X \in S \}$$

10) The group AN in this realization consists of the following matrices:

$$\begin{pmatrix} A & AB_1 \\ 0 & (A^+)^{-1} \end{pmatrix} \tag{3.29}$$

Here A is the upper triangular matrix with positive numbers on the diagonal. The group AN acts simply transitively on $\mathscr{D}$ and $AN_1$ acts on the cone V.

11) Let $Z = X + iY$ be a general point of the domain $\mathscr{D}$. It is easy to see that we can transfer the point Z into the point $iY$ by the transformation of the group $N_2$ and then by the transformation of the group $AN_1$ transfer it into the point $Z_0 = iI$. Therefore $Z = X + iAA^+$, $Y = AA^+$. The matrix Y is positive definite and it can be represented unambiguously in the form $\tilde{A}Y_0\tilde{A}^+$ where $\tilde{A}$ is the complex matrix with unities on the diagonal, $Y_0$ is a diagonal matrix with $y_{ii} > 0$. It follows from *7a* that $\tilde{A}$ corresponds to the set of roots $(e_i - e_j)$, $i < j$. The elements of the matrix $Y_0$ define the complex distance between matrix Y and the standard horocycle passing through the point $Z_0 = iI$. It is not difficult to obtain the following expression for the elements $y_{ii}$

$$y_{ii} = \frac{\Delta_{p-i+1}(Y)}{\Delta_{p-i}(Y)} \tag{3.30}$$

where $\Delta_i(Y)$ is the principal lower angle minor of $i$-th order, $\Delta_0(Y) = I$. The values $y_{ii}$ are positive according to Sylvester cryterion of positive definition of the matrices Y and the cone V is defined by the inequalitites:

$$y_{11} > 0, \ \ldots, \ y_{pp} > 0 \tag{3.31}$$

It is easy to find the half sum of the positive roots $\rho = (\rho_1, \ldots, \rho_p)$

$$\rho = \frac{1}{2}\Sigma(2e_i) + \frac{1}{2}\,.\,2\Sigma(e_i - e_j) + \frac{1}{2}2\Sigma(e_i + e_j)$$

Thus we obtain

$$\rho_i = (2p - 2i + 1) \tag{3.32}$$

Now we shall consider the domains of the second type.

### 3B. The tube domains of the type II.

The considerations are the same as in the case of domains of the first type with some evident modifications. So we only describe the matrix realization of the domain $\mathscr{D}^{\mathrm{II}}$ and formulate the final results.

1) $$\mathscr{D}_p^{\mathrm{II}} = Sp(p, \mathbb{R})/U(p)$$

and in the tube realization

$$\mathscr{D}_p^{\mathrm{II}} = \{ Z : Z = X + iY \}$$

where X and Y are real symmetric matrices of the order $p$ and Y is positive definite. The group $G = Sp(p, \mathbb{R})$ acts as the group of the fractional linear transformations (3.10). And matrices A, B, C and D satisfy the following conditions

$$A'C = C'A, \quad B'D = D'B, \quad A'D - C'B = I \tag{3.33}$$

2) Let $Z_0 = iI$, $Z = X + iY$. As in 11) of the previous section we obtain $Y = AA'$ (here A is the real upper triangle matrix with positive elements on the diagonal). Let us represent the matrix Y as $\tilde{A}Y_0\tilde{A}'$ where $\tilde{A}$ is the upper triangle matrix with unities on the diagonal and $Y_0$ is the diagonal matrix with elements $y_{11}, \ldots, y_{pp}$. We obtain

$$y_{ii} = \frac{\Delta_{p-i+1}(Y)}{\Delta_{p-i}(Y)}, \qquad \Delta_0(Y) = 1$$

The halfsum of the positive roots $\rho = (\rho_1, \ldots, \rho_p)$ in this case is represented by

$$\rho_i = (p - i + 1) \tag{3.34}$$

Now we are going to domains of the third type.

## 3C. The tube domains of the type III.

The domain $\mathscr{D}_p^{\mathrm{III}}$ of the type III is a tube if $p$ is even. In this realization

$$\mathscr{D}_{2p}^{\mathrm{III}} = \{ Z : Z = X + iY, X^+ = X, Y^+ = Y, Y > 0 \}$$

where X and Y are the matrices of order $2p$ satisfying the additional condition:

$$ZJ = J'Z \tag{3.35}$$

where

$$J = \begin{pmatrix} j & 0 & & \\ 0 & j & & \\ & & \ddots & \\ & & & j \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.36}$$

It is convenient to break all matrices of order $2p$ in $p^2$ blocks of matrices of order 2 considered as quaternions

$$q = q^0\tau_0 + q^1\tau_1 + q^2\tau_2 + q^3\tau_3 \tag{3.37}$$

where

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{3.38}$$

$$\tau_k^+ = -\tau_k .$$

As it is known elements $\tau_0$, $\tau_k$ $(k = 1, 2, 3)$ are the basis of the noncommutative but associative algebra with the division over the field of the real numbers.

As usual we can define the conjugate quaternion $\bar{q} = q^0\tau_0 - q^k\tau_k$ and the norm $N(q) = \bar{q}q = q\bar{q} = (q^0)^2 + q^k q_k$. As it is easy to verify

$$jq'j^{-1} = \bar{q} \tag{3.39}$$

Let us define the complex-conjugate quaternion by the formula $q_c = q_c^0 + q_c^k\tau_k$ where $q_c^k$ is a number complex conjugate to $q^k$. Then $q^+ = q_c^0\tau_0 - q_c^k\tau_k$.

Let $X = (x_{ke})$ be the quaternionic hermitian matrix of order $p$ satisfying the condition (3.35).

From (3.39) we obtain

$$\tilde{X} = JX'J^{-1} = (\bar{x}_{ek}) \tag{3.40}$$

But as $X^+ = X$, then $x_{ek}^+ = \bar{x}_{ek} = x_{ke}$ and consequently matrix X is real quaternionic and hermitian as the matrix Y too.

Positive definite matrices Y ($Y > 0$) form a cone V in $p(2p - 1)$-dimensional real linear space. The affine transformations of the cone V have the form

$$Y \to Y_1 = AYA^+ \tag{3.41}$$

From condition the matrix Y to belong to the cone V it follows the condition

$$AYA^+ = JY_1'J^{-1} = J(AYA^+)'J^{-1} = J\bar{A}Y'A'J^{-1} = J\bar{A}J^{-1}YJA'J^{-1}$$

So A must satisfy the condition

$$JAJ^{-1} = \tilde{A} = A^+ \tag{3.42}$$

i. e. be real quaternionic but not necessary hermitian. The real quaternionic matrices form a group $SU^*(2p) \times R^+$. This is the group of the affine transformations of the cone V.

We shall not enumerate the properties of the third type domains because they are analogous to those of the domains of the type I and II.

We note only that the root diagram is as in the previous cases of the type $C_p$. It contains the roots $\pm(e_i \pm e_j)$, $i < j$ with multiplicity 4 and the roots $\pm 2e_i$ with multiplicity 1. It follows from this that $\rho = (\rho_1, \ldots, \rho_p)$ has the form

$$\rho_i = (4p - 4i + 1) \tag{3.43}$$

### 3D. **The exceptional domain $D^V$.**

Beside the « classical » tube domains an exceptional domain exists in $\mathbb{C}^{27}$. It is $\mathscr{D}^V = E_7^R/E_6^C \times SO(2)$ where $E_7^R$ is the real form of exceptional simple group $E_7$. This space can be represented as

$$\mathscr{D}^V = \{ Z : Z = X + iY \}$$

where X and Y are hermitian matrices for order $3 \times 3$ over Cayley numbers and Y is positive definite matrix ($Y > 0$). The matrices Y form a convex selfconjugated cone in the space $\mathbb{R}^{27}$. The rank $\mathscr{D}^{\mathrm{V}}$ equals to 3. The root diagram is of the type $C_3$. The multiplicities of the roots $\pm(e_1 \pm e_2)$, $\pm(e_2 \pm e_3)$, $\pm(e_3 \pm e_1)$ are equal to 8 and of the roots $\pm 2e_1$, $\pm 2e_2$, $\pm 2e_3$ are 1. The vector of the halfsum of the positive roots is represented by

$$\rho = (17, 9, 1) \tag{3.44}$$

### 3E. **The domains of the type IV.**

All of these domains are tubes.

This domain can be realized in the form:

$$\mathscr{D}_p^{\mathrm{IV}} = \{ Z : Z = X + iY, X \in \mathbb{R}^p, Y \in V \}$$

where the cone V is defined by inequalitites

$$y_1^2 - y_2^2 - \dots - y_p^2 > 0, y_1 > 0 \tag{3.45}$$

or in the other metrics:

$$2y_1 y_p - y_2^2 - \dots - y_{p-1}^2 > 0, y_1 > 0 \tag{3.45'}$$

The affine transformations of this cone can be represented by

$$y \to Ay \tag{3.46}$$

where A is a real matrix of order $p$ such as $A'EA = \lambda E$ ($\lambda$ is some positive number) and the matrix E has the form

$$E = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & -1 & & \\ & & \ddots & \\ & 0 & & -1 \end{pmatrix}$$

for the cone (3.45) and

$$E = \begin{pmatrix} 0 & & & 1 \\ & -1 & & \\ & & \ddots & \\ & & & -1 & \\ 1 & & & 0 \end{pmatrix} \tag{3.47'}$$

for the cone (3.45').

It follows from this that the group of affine transformations of the cone is $SO(p-1, 1) \times R^+$. The dimension of the group A is equal to 2. It contains the dilatation $y_i \to \lambda y_i$ and the Lorentz transformation in the plane $y_1$, $y_p$,

$$y_1^1 = y_1 \operatorname{ch} \tau + y_p \operatorname{sh} \tau,$$
$$y_p^1 = y_1 \operatorname{sh} \tau + y_p \operatorname{ch} \tau$$

Now we shall find the root spaces following our general scheme. Choose the metric matrix in the form ($q = p - 2$)

$$E = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_q & 0 \\ I_2 & 0 & 0 \end{pmatrix} \tag{3.48}$$

In this metrics

$$x^2 = 2(x_1 x_{p+1} + x_2 x_{p+2}) + x_3^2 + \dots + x_p^2$$

The group A consists of matrices

$$a = \left(\begin{array}{cc|c|cc} \lambda_1 & 0 & & & \\ 0 & \lambda_2 & 0 & 0 & \\ \hline 0 & & I_q & 0 & \\ \hline & & & \lambda_1^{-1} & 0 \\ 0 & & 0 & 0 & \lambda_2^{-1} \end{array}\right) \tag{3.49}$$

and acts as

$$x_1 \to \lambda_1 x_1, \qquad x_{p+1} \to \lambda_1^{-1} x_{p+1}$$
$$x_2 \to \lambda_2 x_2, \qquad x_{p+2} \to \lambda_2^{-1} x_{p+2}$$

The Lie algebra $\mathscr{A} = \{ a \}$ is represented by

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0_q & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \tag{3.50}$$

where $\alpha$ is the real diagonal matrix of the second order.

Let us find the Lie algebra $\mathscr{G}$ of the group G using the equation for its elements:

$$A'E + EA = 0 \tag{3.51}$$

It is convenient to represent A in the block form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \underbrace{A_{31}}_{2} & \underbrace{A_{32}}_{q} & \underbrace{A_{33}}_{2} \end{pmatrix} \begin{matrix} \}2 \\ \}q \\ \}2 \end{matrix} \tag{3.52}$$

So the condition (3.51) is equivalent to the following conditions

$$A'_{31} = -A_{31}, \qquad A'_{22} = -A_{22}, \qquad A'_{13} = -A_{13}$$
$$A_{33} = -A'_{11}, \qquad A_{32} = -A'_{21}, \qquad A_{23} = -A'_{12}$$

The root spaces are found by the condition

$$[a, A] = \lambda A$$

Calculating the commutator we obtain the following relations

$$[\alpha, A_{11}] = \lambda A_{11}, \qquad \alpha A_{12} = \lambda A_{12}, \qquad \alpha A_{13} + A_{13}\alpha = \lambda A_{13}$$
$$A_{21}\alpha = -\lambda A_{21}, \qquad \alpha A_{13} + A_{31}\alpha = -\lambda A_{31}$$

From this it is easy to obtain

*a)* $A_{11} = e_{12}$ corresponds to the root $e_1 - e_2$ with the multiplicity 1.

$b)$ $A_{13} = e_{12} + e_{21} \rightarrow e_1 + e_2$ with the multiplicity 1. (3.53)

$c)$ $A_{12} = e_{1i} \rightarrow e_1$ with the multiplicity $p - 2$.

$d)$ $A_{12} = e_{2i} \rightarrow e_2$ with the multiplicity $p - 2$.

If we make the transformation in the root space

$$e_1 - e_2 = 2f_2, \qquad e_1 + e_2 = 2f_1 \tag{3.54}$$

we obtain the root diagram of the type $C_2$

$$\{ \pm 2f_i, \pm (f_i \pm f_j) \}, \qquad i, j = 1, 2$$

Here $2f_1$ and $2f_2$ have the multiplicity 1 and $(f_1 + f_2)$, $(f_1 - f_2)$ have the multiplicity $p - 2$.

The halfsum of positive roots has the form

$$\rho = (p - 1)f_1 + f_2, \qquad \rho = (p - 1, 1) \tag{3.55}$$

The metrics (3.45′) is obtained by means of the transformation $A \rightarrow UAU^{-1}$ where

$$U = \frac{1}{\sqrt{2}} \left( \begin{array}{cc|c|cc} 1 & 0 & \multirow{2}{*}{0} & 1 & 0 \\ 0 & 0 & & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_q & \multicolumn{2}{c}{0} \\ \hline 1 & 0 & \multirow{2}{*}{0} & 1 & 0 \\ 0 & 0 & & 0 & 0 \end{array} \right)$$

and the elements of matrix $Y_0$ are given as

$$Y_0^1 = y_1 + y_2, \; Y_0^2 = \frac{y_1^2 - \ldots - y_p^2}{y_1 + y_2} \tag{3.56}$$

## 4. PROPERTIES OF THE COHERENT STATES

Now we will obtain the explicit formulæ for the system of coherent states and study their properties.

Note that studying the tube domains we have used unbounded realization of the space $X = G/K$. Therefore it will be convenient for us to give the formulæ for the horospherical kernels $\langle \xi, \lambda \mid x \rangle = \Psi_x^\lambda(\xi)$ in this realization.

Let $\mathscr{D}$ be a tube domain. There exists an involutive automorphism $\theta$ of this domain $\theta : \mathscr{D} \rightarrow \mathscr{D}$, $\theta^2 = I$ with the fixed point $Z_0 \in \mathscr{D}$. After this transformation the point $s = \infty$ of BS-boundary transforms at the point $s = 0$.

The map $\theta$ has the following form:

1) For the domains $\mathscr{D}^{\mathrm{I}}$, $\mathscr{D}^{\mathrm{II}}$, $\mathscr{D}^{\mathrm{III}}$

$$\theta : Z \to -\frac{1}{Z}, \qquad Z_0 = i\mathrm{I} \tag{4.1}$$

2) For the domain $\mathscr{D}^{\mathrm{IV}}$

$$\theta : z \to z' = (-z_2, -z_1, z_3, \ldots, z_p)\frac{1}{\lambda(z)}, \tag{4.2}$$

where

$$\lambda(z) = z_1 z_2 - z_3^2 - \ldots - z_p^2, \qquad z_0 = (i, i, 0, \ldots, 0)$$

Therefore if the nilpotent subgroup of stability group of the point $s = \infty$ is the group N then the stability group of the point $s = 0$ will be $\bar{\mathrm{N}} = \theta \mathrm{N} \theta^{-1}$. As it follows from Bruhat decomposition (see e. g. [*24*]) the diffeomorphism $\bar{\mathrm{N}}$ on the open everywhere dense set in $\Xi = \mathrm{M}\backslash\mathrm{K}$ exists. So we can take that $\xi \in \bar{\mathrm{N}}$. We shall use the scheme of sec. 1 to obtain the explicit formulæ for the horospherical kernels (with some evident modifications).

Let $\Psi_0^\lambda(\xi)$ be a K-invariant function on $\Xi$. The function $\Psi_0^\lambda(\xi)$ in unbounded realization is found from the condition

$$\mathrm{T}(k)\Psi_0^\lambda(\xi) = \alpha^\lambda(\xi, k)\Psi_0^\lambda(\xi_k) = \Psi_0^\lambda(\xi)$$

Let us denote $f(\xi) = \Psi_0^\lambda(\xi)\tilde{f}(\xi)$ then

$$\mathrm{T}(g)\Psi_0^\lambda(\xi)\tilde{f}(\xi) = \alpha^\lambda(\xi, g)\Psi_0^\lambda(\xi_g)\tilde{f}(\xi_g)$$

So

$$\mathrm{T}(g)\tilde{f}(\xi) = \tilde{\alpha}^\lambda(\xi, g)\tilde{f}(\xi_g) \tag{4.3}$$

where

$$\tilde{\alpha}^\lambda(\xi, g) = \Psi_0^\lambda(\xi_g)[\Psi_0^\lambda(\xi)]^{-1}\alpha(\xi, g)$$

So in order to find $\Psi_Z^\lambda(\xi)$ it is sufficient to calculate the multiplicator

$$\tilde{\alpha}^\lambda(\mathrm{Z}, \xi) = \left[\frac{d\mu(\xi_g)}{d\mu(\xi)}\right]^{1/2} \chi^\lambda(a)$$

Find at first $\tilde{\alpha}^\lambda(\mathrm{Z}, \xi_0)$ where $\xi_0$ corresponds to the point $s \in \mathrm{S}$, $s = \infty$. As it is known, $\tilde{\alpha}^\lambda(\mathrm{Z}, \xi_0)$ depends only on the function $a(\mathrm{Z}, \xi_0)$. We have obtained the explicit expression for the distance $a(\mathrm{Z}, \xi_0)$ in sec. 3. And hence to obtain the final expression we can use the known formula for the Jacobian of the transformation

$$\left[\frac{d\mu(\xi_g)}{d\mu(\xi)}\right] = e^{-2(\rho a(\mathrm{Z}, \xi))} \tag{4.4}$$

Using the explicit expression for $\rho : \rho = (\rho_1, \ldots, \rho_p)$ we finally obtain:

PROPOSITION 4.1.

$$\langle \xi_0, \lambda \mid \mathrm{Z} \rangle = \Psi_Z^\lambda(\xi_0) = \prod_{j=1}^{p} y_j^{-\rho_j + i\lambda_j}\Psi_0^\lambda(\xi_0) \tag{4.5}$$

Here $p$ is the rank of the domain $\mathscr{D}$, $y_i$ are the diagonal elements of the matrix $Y_0$ (see (3.9')) and $\rho_i$ are obtained by (3.32) in the first type case, (3.34) for $\mathscr{D}^{\text{II}}$, (3.43) for $\mathscr{D}^{\text{III}}$, (3.44) for $\mathscr{D}^{\text{V}}$ and (3.55) for $\mathscr{D}^{\text{IV}}$.

Using the involutive automorphism $\theta$ and the group properties of the multiplicator $\tilde{\alpha}^\lambda(Z, \xi)$ we obtain the general formula. As the group $\bar{N}$ is the stability subgroup of the point $s = 0, s \in S$ and the group N acts transitively on $\Xi$ we shall obtain

$$\tilde{\alpha}^\lambda(\xi^0, \bar{n}.g_Z) = \tilde{\alpha}^\lambda(\xi^0, \bar{n})\tilde{\alpha}^\lambda(\xi^0_{\bar{n}}, g_Z)$$

hence

$$\Psi^\lambda_Z(\xi) = \frac{\tilde{\alpha}^\lambda(Z_{\bar{n}^{-1}}, \xi^0)}{\tilde{\alpha}^\lambda(Z^{(0)}_{\bar{n}^{-1}}, \xi^0)}\Psi^\lambda_0(\xi) \tag{4.6}$$

here $\xi = \xi^0_{\bar{n}}$, and $\Psi^\lambda_{Z^0}(\xi) = \Psi^\lambda_0(\xi)$.

So $\Psi^\lambda_{Z_g}(\xi) = \Psi^\lambda_Z(\xi)$, $g \in N_\xi$ as it follows from the Proposition 1.1 (Here $\bar{n}_\xi$ is an element of the group $\bar{N}$ transforming $\xi_0 \to \xi$ and $N_\xi = \bar{n}_\xi^{-1} N \bar{n}_\xi$ is the stability group of the point $\xi$).

Note that

$$|\Psi^\lambda_Z(\xi)|^2 = e^{-2(\rho(a(Z,\xi)))}|\Psi^\lambda_0(\xi)|^2 \tag{4.7}$$

It coincides with the well known Poisson kernels formula (see e. g. [*24*]).

The system of CS is overcomplete and nonorthogonal and possesses a number of remarkable properties some of which are enumerated below (We omit the proofs of these propositions).

1. The system $\{|x\rangle\}$ is complete. It follows from irreducibility of the $T^\lambda(g)$ representation.

2. Coherent states are normalized to unity

$$\langle x | x \rangle = ||\Psi^\lambda_x(\xi)||^2 = \int |\Psi^\lambda_x(\xi)|^2 d\mu(\xi) = 1 \tag{4.8}$$

It follows from the unitarity of the $T^\lambda(g)$ representation.

3. Operator $T^\lambda(g)$ acts transitively on the set of CS $\{|x\rangle\}$

$$T^\lambda(g)|x\rangle = |x'\rangle \qquad \text{where} \qquad x' = x_{g^{-1}} \tag{4.9}$$

4. Horospherical kernels $\Psi^\lambda_x(\xi)$ are the eigenfunctions of the Laplace-Beltrami operator of the space $X = G/K$

$$\Delta_x \Psi^\lambda_x(\xi) = -(\lambda^2 + \rho^2)\Psi^\lambda_x(\xi) \tag{4.10}$$

It follows from combined results by Harish Chandra [*24*] [*28*] and from the fact that the space of representations of class I consists of zonal spherical functions on X.

5. Coherent states are nonorthogonal to each other

$$\langle x | y \rangle = \Phi_\lambda(\tau) \tag{4.11}$$

where $\tau = \tau(x, y)$ is the distance between the points $x, y \in X$, $\Phi_\lambda(\tau)$ is defined by formula $\Phi_\lambda(\tau) = \langle 0 | T^\lambda(g) | 0 \rangle$ or

$$\Phi_\lambda(\tau) = \langle 0 | x \rangle = \int \Psi_x^\lambda(\xi) d\mu(\xi) \tag{4.12}$$

Here $\tau = \tau(x_0, gx_0)$, $x_0 = 0$, $\Phi_\lambda(\tau)$ is the so called zonal spherical function.

6. The expression by hypergeometrical function in the case of rank I is known [*28*] for this function. But only its asymptotic behaviour in general case is known

$$\Phi_\lambda(\tau) \underset{\tau \to \infty}{\sim} \sum_{s \in W} C(s\lambda) e^{(is\lambda - \rho)(\tau)} \tag{4.13}$$

where $W = N(A)/M$ is the Weyl group of the space, $C(\lambda)$ is defined by

$$C(\lambda) = \int e^{-(\rho + i\lambda)H(\bar{n})} d\mu(\bar{n}) \tag{4.14}$$

and the measure $d\mu(\bar{n})$ is normalized so that

$$\int e^{-2\rho(H(\bar{n}))} d\mu(\bar{n}) = 1, \qquad \text{i. e. } C(-i\rho) = 1 \tag{4.15}$$

(Let $g = k(g) \exp H(g) n(g)$ ; $g \in G$. It follows from Bruhat decomposition [*24*] that $\bar{N}MAN$ is dense in G and for $g = \bar{n}amn$, $\bar{n}$ is unambiguously defined by $g$). It was obtained [*30*] the explicit expression for $C(\lambda)$

$$C(\lambda) = \frac{I(i\lambda)}{I(\rho)}, \qquad I(\lambda) = \prod_{\alpha \in \Sigma^+} B\left(\frac{m_\alpha}{2}, \frac{m_{\alpha/2}}{4} + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right), \tag{4.16}$$

Here $B(\alpha, \beta)$ is Beta function, $\Sigma^+$ is the set of positive roots, $m_\alpha$, $m_{\alpha/2}$ are the multiplicities of roots, $\langle \alpha, \beta \rangle$ is the scalar product in the space of roots.

7. The system of functions describing the CS satisfies the following relations of completeness and orthogonality

$$\int \Psi_x^\lambda(\xi) \Psi_x^{\lambda'}(\xi') d\mu(x) = N\delta(\lambda - \lambda')\delta(\xi, \xi') \tag{4.17}$$

$$\int \bar{\Psi}_x^\lambda(\xi) \Psi_{x'}^\lambda(\xi) d\mu(\xi) d\mu(\lambda) = N_1 \delta(x, x') \tag{4.18}$$

Here the functions $\delta(x, x')$, $\delta(\xi, \xi')$ normalized so that

$$\int \delta(x, x') f(x') d\mu(x') = f(x), \qquad \int \delta(\xi, \xi') f(\xi') d\mu(\xi') = f(\xi)$$

$$d\mu(\lambda) = |C(\lambda)|^{-2} d\lambda, \qquad N_1 = |C(\lambda)|^{-2} N$$

and $C(\lambda)$ is defined in (4.16).

The proof of relation (4.17) follows from generalization of Schur lemma (see, e. g. [*15*]). The relation (4.18) follows from the completeness of the system of spherical functions.

8. Let us point out an interesting connection of considered problem with those by Furstenberg [*29*] and some problems related to generalization of Fatou theorem on symmetric spaces (see, e. g. [*24*]).

As was shown in [*29*] it may be introduced the concept of the boundary group and the boundary of the group.

Let $U = G/P$ where P is the closed subgroup in G, $\mu$ be the Borel measure on U and $g \in G$; $g$ acts on $\mu$ as $g.\mu(E) = \mu(g^{-1}E)$, E be the subset in U.

DÉFINITION 4.2. — The space U is called the Furstenberg boundary and P is the boundary group if U is compact and for every probability measure $\mu$ on U can be found the sequence $g_n$ so that $g_n\mu \to \delta(\xi)$ ($\delta(\xi)$ is the Dirac $\delta$-function, the convergence is considered in the weak topology). The maximal boundary $\tilde{U}$ exists. As it follows from [*26*] for symmetric spaces of noncompact type $\tilde{U} = \Xi = B\backslash G$.

The representations of class I are characterized by horospherical kernels $\Psi_Z^\lambda(\xi)$ which satisfy the equation (4.10). The following generalization of Fatou theorem for eigenfunctions of Laplacian is valid:

PROPOSITION 4.3. — Let

$$F_\lambda(x) = \int P_\lambda(x, \xi) f(\xi) d\mu(\xi) \tag{4.19}$$

where

$$P_\lambda(x, \xi) = e^{-(\rho + i\lambda)H(a(x,\xi))}\left[\int \Psi_x^\lambda(\xi) d\mu(\xi)\right]^{-1} \tag{4.20}$$

Note that for $\lambda = -i\rho$, $P_\lambda(x, \xi) = e^{-2\rho H(a)}$ is the Poisson kernel. Then

$$\underset{t\to\infty}{F_\lambda(x(t))} \to f(\xi), \qquad \underset{t\to\infty}{x(t)} \to \xi$$

Here $x(t)$ is the geodesic in the space X. We can consider the boundary values $f(\xi)$ of the function $F_\lambda(x)$ as distributions and hyperfunctions. The proof uses the formulæ (4.17), (4.18) but is more complicated.

Some class of representations of the group G on every Furstenberg boundary can be constructed. For nonmaximal boundaries these representations correspond to representations of degenerate series. And so analogous systems of CS can be constructed.

As was shown in [*14*], CS for representations of discrete series are given by generalized Bergmann kernels. For representations of principal series we obtain the connection with generalized Poisson kernels.

9. Now we consider the decomposition of representation $T^\lambda(g)$ into irreducible components at restriction by a maximal compact subgroup K. It follows from the Frobenius reciprocity theorem (see, e. g. [*20*]) that the

decomposition contains those and only those representations of the group K which when being restricted by the subgroup M contains the identity representation. Thereby the problem is reduced to that for the compact groups.

In conclusion let us note that both the results and the calculation methods for classical domains of the tube type are also valid for other Hermitian domains.

## REFERENCES

[1] A. M. PERELOMOV, preprint ITEP, n° 17, 1974 ; *Commun. Math. Phys.*, 1975 (in press).

[2] A. M. PERELOMOV, *Commun. Math. Phys.*, t. **26**, 1972, p. 222.

[3] R. J. GLAUBER, *Phys. Rev.*, t. **130**, p. 2529 ; t. **131**, 1963, p. 2766.

[4] J. R. KLAUDER, E. C. G. SUDARSHAN, *Fundamentals of Quantum Optics*, New York, Benjamin, 1968.

[5] A. M. PERELOMOV, *Phys. Lett.*, t. **A 39**, 1972, p. 165, 353 ; *Teor. Mat. Fiz.*, t. **16**, 1973, p. 303 ; t. **19**, 1974, p. 83.

[6] E. H. LIEB, *Commun. Math. Phys.*, t. **31**, 1973, p. 327.

[7] J. M. RADCLIFFE, *Jour. Phys.*, t. **A 4**, 1971, p. 313.

[8] F. T. ARECCHI, E. COURTENS, R. GILMORE, H. THOMAS, *Phys. Rev.*, t. **A 6**, 1972, p. 2211.

[9] K. HEPP, E. H. LIEB, *Phys. Rev.*, t. **A 8**, 1973, p. 2517.

[10] A. M. PERELOMOV, *Teor. Mat. Fiz.*, t. **6**, 1971, p. 213.

[11] V. BARGMANN, P. BUTERA, L. GIRARDELLO, J. R. KLAUDER, *Reps. Math. Phys.*, t. **2**, 1971, p. 221.

[12] A. M. PERELOMOV, *Funct. Anal. Appl.*, t. **6**, n° 4, 1972, p. 47.

[13] A. M. PERELOMOV, *Funct. Anal. Appl.*, t. **7**, n° 3, 1973, p. 57.

[14] M. I. MONASTYRSKY, A. M. PERELOMOV, *Dokl. Akad. Nauk SSSR*, t. **207**, 1972, p. 1303; *Reps. Math. Phys.*, t. **6**, 1974, p. 14.

[15] A. A. KIRILLOV, *Elementi teorii predstavlenii* (in Russian), Moscow, Nauka, 1973.

[16] B. KOSTANT, *Lecture Notes in Math.*, t. **170**, 1970, p. 87.

[17] F. A. BEREZIN, *Method of second quantization*. London, Academic Press, 1966.

[18] F. DYSON, *Commun. Math. Phys.*, t. **19**, 1970, p. 235.

[19] A. F. GRILLO, *Revista N. C.*, t. **3**, 1973, p. 146.

[20] G. W. MACKEY, *Bull. Amer. Math. Soc.*, t. **69**, 1963, p. 628.

[21] S. HELGASON, *Differential geometry and Symmetric Spaces*, New York, Academic Press, 1962.

[22] B. KOSTANT, *Bull. Amer. Math. Soc.*, t. **75**, 1969, p. 627.

[23] I. M. GELFAND, M. I. GRAEV, *Amer. Math. Soc. Transl.*, t. **37**, 1964, p. 351.

[24] S. HELGASON, *Adv. in Math.*, t. **5**, n° 1, 1970, p. 1.

[25] I. I. PYATETSKII-SHAPIRO, *Automorphic Functions and the Geometry of Classical Domains*, New York, Gordon and Breach, 1969.

[26] C. MOORE, *Amer. J. Math.*, t. **86**, 1964, p. 201, 358.

[27] L. K. HUA, Harmonic Analysis in the Classical Domains, *Amer. Math. Soc.*, 1963.

[28] HARISH-CHANDRA, *Amer. J. Math.*, t. **80**, 1958, p. 241.

[29] H. FURSTENBERG, *Ann. Math.*, t. **77**, 1963, p. 335.

[30] S. G. GINDIKIN, F. I. KARPELEVIČ, *Sov. Math.*, t. **3**, 1962, p. 962.

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