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Section A :
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Ultraviolet and infrared singularity structure of generic Feynman amplitudes

by

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ABSTRACT. — The generic Feynman amplitude for a graph G, defined in an earlier paper by an integral over Feynman parameters, is itself a function of regularizing parameters $\underline{\lambda}$, ν. Here we work out in detail the desingularization of the integral which is needed to exhibit the meromorphic structure in these variables. All ultraviolet and infrared singularities are determined, and these are shown to correspond to certain types of sub and quotient graphs of G, respectively.

RÉSUMÉ. — L'amplitude de Feynman générique pour un graphe G, qu'on a définie dans une étude précédente comme une intégrale dans l'espace des paramètres de Feynman, est elle-même une fonction des paramètres régularisants $\underline{\lambda}$, ν. Ici nous développons en détail la désingularisation de l'intégrale qui est nécessaire pour montrer la structure méromorphe dans ces variables. Nous déterminons toutes les singularités ultra-violettes et infra-rouges, et nous démontrons que celles-ci correspondent à certains graphes quotients de G et celles-là à certains sous-graphes de G.

In an earlier paper [*1*], M. J. Westwater and I defined the *generic Feynman amplitude* of a Feynman graph as a function of complex parameters $(\lambda_1, \ldots, \lambda_N, \nu)$, where the λ_ℓ variables, one for each line of the graph, are the usual variables of analytic regularization, and ν is a complex space-

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time dimension. The problem was proposed, but not solved, of determining the precise meromorphic structure of the amplitude as a function of these parameters. That structure is determined in this paper, giving a complete description of both ultraviolet and infrared singularities of the amplitude. [It should be pointed out, however, that in discussing infrared singularities we restrict ourselves to non-singular Euclidean momenta.]

The structure problem has been solved for graphs without massless lines in [*2*]. Significant differences caused by the presence of massless lines are:

(*a*) While ultraviolet divergences still give singularities associated with various subgraphs, infrared singularities are associated with quotient (contracted) graphs;

(*b*) Singularities occur in families considerably more complicated in structure than the *s*-families of [*2*];

(*c*) The scaling transformations of the integration variables, necessary to exhibit the minimal singularities, involve blow-downs as well as blow-ups.

In § 1 we define the *s*-families involved, and state the main results concerning them. The proofs are fairly complicated, and are given in some detail in § 2; this section may be omitted at first reading. Analytic consequences (exhibition of the singularity structure) are given in § 3.

In § 4 we establish the existence of a region in parameter space in which the integral defining the generic amplitude is absolutely convergent. This eliminates the need to define the amplitude as the sum of analytic continuations of various parts, as was done in [*1*]. Thus it is easy to determine whether a physical Feynman integral is infrared divergent (again, for non-singular Euclidean momenta); we simply ask if the physical point $\lambda_\ell = 0$, $\nu = 4$ lies in the convergence region. The same ideas furnish a proof that the singularities we produce are actually present, i. e., that we have in fact found a minimum set of singularities.

We wish to make the following additional comment. It may be seen from the discussion of this paper that Theorem 3.1 of [*1*], describing the nature of the singularity of a generic Feynman amplitude when the mass of one line vanishes, is not correct. The result holds as stated for a massive graph, but not in the general case. A corrected version of this result should appear soon.

I. GRAPH-THEORETICAL CONCEPTS

Throughout this paper we take G_0 as a fixed, 2-connected Feynman graph, with a set of N lines Ω, massive lines $\Omega^M \subset \Omega$, n vertices Θ, and external vertices $\Theta^E \subset \Theta$; moreover, we assume that either $|\Omega^M| \geq 1$ or $|\Theta^E| \geq 2$. Subgraphs are defined as usual; if $H \subset G_0$ is such a subgraph, Ω_H denotes the lines of H, etc., $N(H) = |\Omega_H|$, $n(H) = |\Theta_H|$, $c(H)$ denotes the number of connected components of H, and $h(H) = N(H) - n(H) + c(H)$

the number of loops. H is *irreducible* if it is 2-connected [1] [3] or consists of a single line, and any subgraph H decomposes naturally into maximal irreducible subgraphs which we call the *pieces* of H. Finally, if G is a graph and $\chi \subset \Omega_G$, $G[\chi]$ denotes the minimal subgraph of G with $\Omega_{G[\chi]} = \chi$.

If $S \subset G_0$ has components $S_1, \ldots, S_{c(S)}$, the quotient graph $Q = G_0/S$ is obtained by collapsing each S_i to a single vertex, so that we may take $\Omega_Q = \Omega - \Omega_S$ and, denoting a vertex in Q by the corresponding subset of Θ, $\Theta_Q = (\Theta - \Theta_S) \cup \bigcup_1^{c(S)} \{ \Theta_{S_i} \}$. The mapping $\pi_Q : G_0 \to Q$ (a mapping of graphs in the sense of [3] and slightly different from the map of [1]) is then specified by $\pi_Q|_{\Omega-\Omega_S} = \text{identity}$, $\pi_Q|_{\Theta-\Theta_S} = \text{identity}$, $\pi_Q(\Theta_{S_i} \cup \Omega_{S_i}) = \Theta_{S_i}$. If Q is a contraction of the graph K, we write $Q < K$; if $Q_1 = G_0/S_1$ and $Q_2 = G_0/S_2$, $Q_1 \vee Q_2 = G_0/S_1 \cap S_2$. For $\chi \subset \Omega$, $Q[\chi] = G_0/G_0[\Omega - \chi]$.

If K is any graph, a tree T in K is a *maximal* subset $T \subset \Omega_K$ such that $h(K[T]) = 0$; a tree in G_0 is called simply a tree. Finally, G_0^∞ denotes the graph obtained by adjoining an additional vertex ∞ to Θ, connected by one line to each vertex of Θ^E.

Before proceeding formally we give an intuitive discussion of the scale transformations which will be necessary to exhibit the analytic behavior. Our definition of the generic Feynman amplitude in this paper differs by an overall factor from that of [1]. We take

$$F(\underline{s}, \underline{z}; \underline{\lambda}, \nu) = \Gamma(\mu) \int_{\mathscr{D}} \prod_{\Omega} \alpha_\ell^{\lambda_\ell} d(\underline{\alpha})^{\lambda_0} D(\underline{\alpha}, \underline{s}, \underline{z})^{-\mu} \eta \tag{1.1}$$

where $\mathscr{D} = \{ \underline{\alpha} \in \mathbb{P}^{N-1} \,|\, \alpha_\ell \geq 0,\ \ell \in \Omega \}$ and η is the fundamental projective differential form in $\mathbb{P}^{N-1}$ [14]. The $\underline{s}$ variables are the invariants constructed from the external momenta, with $s(\chi) = \left(\sum_{i\in\chi} p_i\right)^2$ for $\chi \subset \Theta^E$, the $\underline{z}$ variables $\{ z_i \,|\, i \in \Omega^M \}$ represent the squared masses, and

$$d(\underline{\alpha}) = \sum_T \prod_{\ell \notin T} \alpha_\ell, \tag{1.2}$$

$$D(\underline{\alpha}, \underline{s}, \underline{z}) = \frac{1}{2} \sum_{\chi \subset \Theta^E} s(\chi) \Big(\sum_{T_2} \prod_{\ell \notin T_2} \alpha_\ell \Big) - \Big(\sum_{\Omega^M} \alpha_\ell z_\ell \Big) d(\alpha), \tag{1.3}$$

the sums running respectively over all trees T of G_0 and all 2-trees T_2 which separate χ and $\Theta^E - \chi$. Finally

$$\mu = -\frac{1}{2} h(G_0)\nu + \sum_{\Omega} (\lambda_\ell + 1) ;$$

$$\lambda_0 = \mu - \nu/2 .$$

For further details see [1].

Consider now a point on the boundary of the integration region, where $\alpha_\ell = 0$ for all lines ℓ in some subgraph H. The d and D functions vanish at this point to order $h(\mathrm{H})$, suggesting the introduction of a scaling variable t by

$$\alpha_\ell = t\beta_\ell, \tag{1.4}$$

for $\ell \in \Omega_{\mathrm{H}}$, with the β_ℓ variables normalized (e. g. by $\beta_{\ell_0} = 1$ for some $\ell_0 \in \Omega_{\mathrm{H}}$). In fact, for massive graphs [2], it suffices for complete desingularization to make such transformations for all 2-connected graphs H; with each such H there is associated an (ultraviolet) singularity.

A new complication arises in the general case, however, if H contains all massive lines and connects all external vertices. Then $\mathrm{D}(\alpha, s)$ vanishes to order $h(\mathrm{H}) + 1$, because $(\Sigma\alpha_\ell z_\ell) = 0$, and each 2-tree in (1.3) must intersect H in at least $h(\mathrm{H}) + 1$ lines. Again a scaling is necessary but (as before) not for all subgraphs of this type. First, we may discard pieces of H which do not contribute to the additional degeneracy of D; the resulting subgraph H′ we call a *link*. Moreover, the corresponding (infrared) singularity is most closely associated with the quotient graph $\mathrm{Q} = \mathrm{G}/\mathrm{H}'$; note that because the α_ℓ are homogeneous coordinates we may rewrite (1.4) as

$$\alpha_\ell = \gamma_\ell / t \tag{1.5}$$

for $\ell \in \Omega_{\mathrm{Q}}$, where again the γ's are normalized. Paralleling the subgraph case, it then turns out to be necessary to make such scalings not for all of Q, but rather separately for the irreducible pieces of Q.

There is one further complication: in addition to the new infrared singularities, we lose certain ultraviolet singularities which would have occurred in the corresponding massive graph-those associated with subgraphs which are not *saturated*. For further comment see Remark 4.7 (*a*).

We are now ready to define these ideas formally.

DEFINITION 1.1. — If $\mathrm{H} \subset \mathrm{G}_0$, let $\mathrm{G}_0^\infty/\mathrm{H}$ have pieces $\mathrm{Q}_1, \ldots, \mathrm{Q}_k$, numbered so that $\infty \notin \Theta_{\mathrm{Q}_i}$ and $\Omega_{\mathrm{Q}_i}^{\mathrm{M}} = \emptyset$ for $i > i_0$. Then

$$\bar{\mathrm{H}} = \mathrm{H} \cup \pi_{\mathrm{G}_0^\infty/\mathrm{H}}^{-1}(\mathrm{Q}_{i_0+1} \cup \ldots \cup \mathrm{Q}_k)$$

is called the *saturation* of H; H is *saturated* if $\bar{\mathrm{H}} = \mathrm{H}$.

DEFINITION 1.2. — A subgraph $\mathrm{S} \subset \mathrm{G}_0$ is a *link* if (*a*) $\bar{\mathrm{S}} = \mathrm{G}_0$ and (*b*) the removal of any piece of S destroys property (*a*). [Note that (*a*) is equivalent to $\Omega^{\mathrm{M}} \subset \Omega_{\mathrm{S}}$ and $\Theta_{\mathrm{S}_1} \supset \Theta^{\mathrm{E}}$ for some component S_1 of S].

Let $\mathscr{H} = \{ \mathrm{H} \subset \mathrm{G}_0 \mid \mathrm{H} = \bar{\mathrm{H}}$, and H is irreducible $\}$, $\mathscr{Q} = \{ \mathrm{Q} = \mathrm{G}_0/\mathrm{S} \mid \mathrm{S}$ is a link, and Q is irreducible $\}$. We will see that ultraviolet and infrared singularities correspond to graphs in $\mathscr{H}$ and $\mathscr{Q}$, respectively.

DÉFINITION 1.3. — If $\mathrm{H} \in \mathscr{H}$ with $\mathrm{H} \neq \mathrm{G}_0$, a *link in H* is an irreducible subgraph $\mathrm{H}_1 \subset \mathrm{H}$ with $\bar{\mathrm{H}}_1 = \mathrm{H}$. If $\mathrm{Q} \in \mathscr{Q}$, with $\mathrm{Q} = \mathrm{G}_0/\mathrm{S}$, a subgraph $\mathrm{S}_1 \subset \mathrm{Q}$ is a *link in Q* if $\mathrm{S} \cup \mathrm{S}_1$ is a link. For unity of terminology we sometimes refer to a link as a *link in G_0*.

EXAMPLE 1.4. — In the graph of figure 1 (where dashed lines are massless, and wavy lines indicate external vertices), $G_0[1, 2]$ and $G_0[1, 2, 3, 4]$ are links, but $G_0[1, 2, 3]$ is not. $G_0[6]$ is not in $\mathscr{H}$, since $\overline{G_0[6]} = G_0[6, 7]$. Typical elements of $\mathscr{Q}$ are $Q[3, 4, 5]$, $Q[7]$.

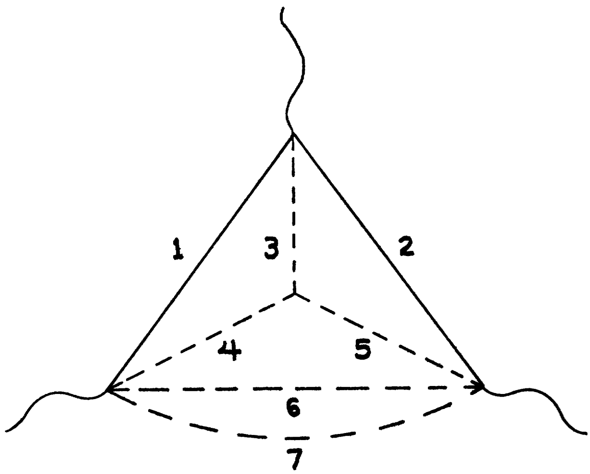

FIG. 1. — Feynman graph.

We now define the s-families associated with singularities of F. They are allied to the labelled s-families of [2], and a direct generalization of the s-families used in [5]. For any $\mathscr{E} \subset \mathscr{Q} \cup \mathscr{H}$ we write $\mathscr{E}_q = \mathscr{E} \cap \mathscr{Q}$, $\mathscr{E}_h = \mathscr{E} \cap \mathscr{H}$, $\mathscr{E}^0 = \{ K \in \mathscr{E} \mid \Omega_K \text{ is maximal} \}$ and, for $K \in \mathscr{E}$, $\mathscr{E}(K) = \{ K' \in \mathscr{E} \mid \Omega_{K'} \subsetneqq \Omega_K \}$. Similarly, $\mathscr{E}_q^0(K) = ((\mathscr{E}(K))^0)_q$, etc.

DEFINITION 1.5. — An *s-family* $\mathscr{E} \subset \mathscr{Q} \cup \mathscr{H}$ is a maximal family satisfying:

1) $G_0 \in \mathscr{E}$;

2) The sets Ω_K, $K \in \mathscr{E}$, are non-overlapping, i. e., for $K_1, K_2 \in \mathscr{E}$, either $\Omega_{K_1} \subset \Omega_{K_2}$, $\Omega_{K_2} \subset \Omega_{K_1}$, or $\Omega_{K_1} \cap \Omega_{K_2} = \emptyset$.

3) If $K \in \mathscr{E}$, then $\bigvee_{\mathscr{E}_q^0(K)} Q = K/S$, where (*a*) S is a link in K, and (*b*) the pieces of K/S are precisely the elements of $\mathscr{E}_q^0(K)$.

4) If $K \in \mathscr{E}$, (*a*) the pieces of $\bigcup_{\mathscr{E}_h^0(K)} H = H_1$ are precisely the elements of $\mathscr{E}_h^0(K)$, (*b*) if $K = G_0$, $\bar{H}_1 \neq G_0$, (*c*) if $K = G_0/S \in \mathscr{Q}$, the pieces of $H_1 \cup S$ are precisely the pieces of H_1 together with the pieces of S.

A family $\mathscr{E} \subset \mathscr{Q} \cup \mathscr{H}$ satisfying (1) – (4) but not necessarily maximal will be called a *weak* s-family.

EXAMPLE 1.6. — In the graph of figure 1, typical s-families are $\{ G_0, G_0[3], G_0[4], G_0[5], G_0[4, 5, 6, 7], G_0[1], Q[7] \}$ and $\{ G_0, G_0[1], G_0[3], Q[6], Q[7], Q[3, 4, 5], Q[5] \}$. (An algorithm for generating such families will be given shortly.) However, $\{ G_0, Q[3], Q[5] \}$ violates (3a), $\{ G_0, G_0[1], G_0[3], G_0[4] \}$ violates (4a), $\{ G_0, G_0[1], G_0[2] \}$ (4b), and $\{ G_0, G_0[3], G_0[4], Q[3, 4, 5] \}$ (4c). In the graph of figure 2 $\{ G_0, Q[5, 6, 7, 8, 9], Q[10, 11, 12, 13, 14] \}$ violates (3b); thus the given conditions are independent.

We now give the principle results on the nature of s-families, to be proved in § 2. See Remark 1.11 for a discussion of their significance.

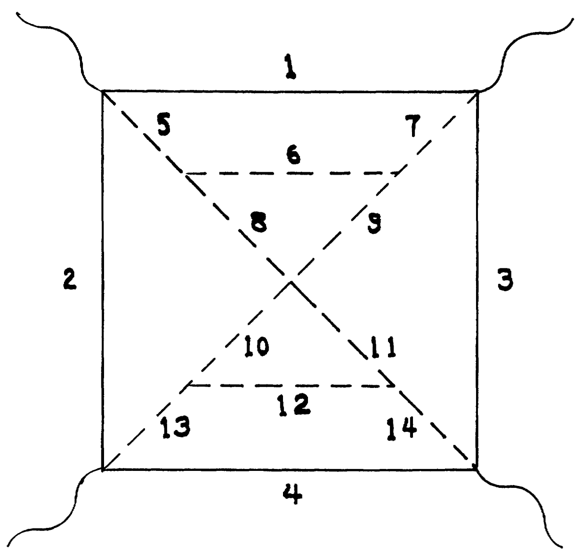

FIG. 2. — Feynman graph.

THEOREM 1.7. — Given a weak s-family $\mathscr{E}$ and a line $\ell \in \Omega$, let $K \in \mathscr{E}$ be the minimal element for which $\ell \in \Omega_K$. Define

$$G = K[\ell] \cup \bigcup_{H \in \mathscr{F}} H,$$

where $\mathscr{F} \subset \mathscr{E}_h^0(K)$ is the maximal subset such that (a) G is a link in K, if any such $\mathscr{F}$ exists, or (b) G is irreducible, otherwise. Let $\tilde{G} = K$ in case (a), $\tilde{G} = \bar{G}$ in case (b), and let $Q_1, \ldots, Q_j$ be the pieces of $\tilde{G}/G$. Then $\mathscr{E}' = \mathscr{E} \cup \{ \tilde{G} \} \cup \{ Q_1, \ldots, Q_j \}$ is a weak s-family.

THEOREM 1.8. — If $\mathscr{E}$ is an s-family and $K \in \mathscr{E}$ there is precisely one line, denoted $\sigma(K)$, in $\left[\Omega_K - \bigcup_{K' \in \mathscr{E}^0(K)} \Omega_{K'} \right]$. In particular, this implies $| \mathscr{E} | = | \Omega | = N$.

DEFINITION 1.9. — Given an s-family $\mathscr{E}$, the domain $\mathscr{D}(\mathscr{E}) \subset \mathscr{D}$ (see (1.1)) consists of all $\underline{\alpha}$ satisfying

$$\sup_{H \in \mathscr{E}_h^0(K)} \alpha_{\sigma(H)} \leq \alpha_{\sigma(K)} \leq \inf_{Q \in \mathscr{E}_q^0(K)} \alpha_{\sigma(Q)} \tag{1.6}$$

THEOREM 1.10. — (a) For any $\underline{\alpha} \in \mathscr{D}$, there is an s-family $\mathscr{E}$ with $\alpha \in \mathscr{D}(\mathscr{E})$; ($b$) If $\mathscr{E}_1 \neq \mathscr{E}_2$, $\mathscr{D}(\mathscr{E}_1) \cap \mathscr{D}(\mathscr{E}_2)$ has Lebesque measure zero.

REMARK 1.11. — Theorem 1.7 is the basic algorithm for the construction of s-families: starting with some weak s-family, e. g. $\{ G_0 \}$, we make the construction of Theorem 1.7 successively for each line of Ω; this produces an s-family. [A given application of Theorem 1.7 does not necessarily produce a new s-family; in fact, it fails to do so precisely when

$$\left(\Omega_K - \bigcup_{\mathscr{E}^0(K)} \Omega(K')\right) = \{ \ell \} .$$

See the proof of Theorem 1.8.] If $\ell_1 < \ldots < \ell_N$ is a fixed ordering of Ω, and we apply Theorem 1.7, starting with $\{ G_0 \}$ and using successively $\ell = \ell_1$, $\ell = \ell_2$, etc., we obtain an s-family $\mathscr{E}$ for which $\mathscr{D}(\mathscr{E})$ contains the region $\alpha_{\ell_1} \leq \ldots \leq \alpha_{\ell_N}$ (see proof of Theorem 1.10). The reader is strongly urged to generate the two s-families of Example 1.6 in this way, using for example the orderings $3<4<5<6<1<7<2$ and $1<3<2<6<4<7<5$, respectively.

The final result of this chapter will be needed to discuss the behavior of the functions d and D in the domain $\mathscr{D}(\mathscr{E})$.

THEOREM 1.12. — Let $\mathscr{E}$ be an s-family, let $\mathscr{E}^* \subset \mathscr{E}_h$ consist of all graphs H such that $\sigma(H)$ is a piece of $G_0[\sigma(H)] \cup \bigcup_{\mathscr{E}_h^0(H)} H'$. Let $\sigma(\mathscr{E}^*) = T$. Then $T \cap \Omega_K$ is a (spanning) tree in K for each K in $\mathscr{E}$, moreover, either (a) $\sigma(G_0) \in \Omega^M$, (b) $\sigma(G_0) \in T$, and the 2-tree $T - \sigma(G_0)$ separates Θ^E into two nonempty subsets ψ, $\Theta^E - \psi$, or (c) both.

II. PROOFS OF GRAPH-THEORETICAL RESULTS

We recall the following characterization [3]: a graph K is 2-connected if and only if, given lines $\ell_1, \ell_2 \in \Omega_K$, there is a circuit C in K containing both ℓ_1 and ℓ_2.

LEMMA 2.1. — Let G be 2-connected, $S \subset G$ have pieces $S_1, \ldots, S_k$ and let $K \subset G/S$ have pieces $K_1, \ldots, K_m$. Then any piece of $\pi^{-1}(K)$ ($\pi = \pi_{G/S}$) is either an S_i or has the form

$$H = \bigcup_{p \in \psi} G[\Omega_{K_p}] \cup \bigcup_{i \in \chi} S_i , \tag{2.1}$$

with ψ a non-empty subset of $\{1,\ldots,m\}$. The ψ's corresponding to the various pieces are disjoint and exhaust $\{1,\ldots,m\}$, and the various χ's are disjoint. If K_p is a piece of G/S, then the piece (2.1) for which $p \in \psi$ has $\chi \neq \emptyset$.

Proof. — We first observe that any two lines $\ell_1, \ell_2 \in \Omega_{K_p}$ must lie in the same piece of $\pi^{-1}(K_p)$. For although given a circuit C in K_p containing ℓ_1 and ℓ_2, the lines Ω_C will not necessarily form a circuit in G, we can produce a circuit C_1 in $\pi^{-1}(K_p)$ such that $\pi(C_1) = C$, by adjoining arcs in the various components of S to $G[\Omega_C]$. Since $\ell_1, \ell_2 \in \Omega_{C_1}$, they lie in the same piece of $\pi^{-1}(K_p)$. Now the piece structure as described is clear.

Finally, if K_p is a piece of G/S, let C_2 be a circuit in G intersecting both Ω_K and $\Omega - \Omega_K$. Let C be a circuit contained in $\pi(C_2) \cap \Omega_{Kp}$, and take $C_1 \in \pi^{-1}(K_p)$ with $\pi(C_1) = C$ as above. Since $\Omega_{C_1} \cap \Omega_{Kp} \subset \Omega_C \cap \Omega_{Kp}$, and this latter set of lines contains no circuit in G, $C_1 \not\subset G[\Omega_K]$, i. e., $\Omega_{C_1} \cap \Omega_S \neq \emptyset$, thus $\chi \neq \emptyset$.

We remark that any piece S_i of S with $S_i \not\subset \pi^{-1}(K)$ is also a piece of $S \cup \pi^{-1}(K)$.

DÉFINITION 2.2. — Suppose that R, $S \subset G_0$ have pieces $R_1, \ldots, R_k$ and $S_1, \ldots, S_j$, respectively. R and S are *piecewise isomorphic* if $k = j$ and (possibly with renumbering) $R_i \approx S_i$. We write $R \,||\, S$ if $R \cup S$ has $k+j$ pieces, equivalently, if $R \cup S$ is piecewise isomorphic to the disjoint union of copies of R and S. (Note $R \,||\, S$ implies $\Omega_R \cap \Omega_S = \emptyset$.) If $Q = G_0/S$ and $H \subset Q$, we write $H \subset_0 Q$ if H is piecewise isomorphic to $G_0[\Omega_H]$.

Using Lemma 2.1 it is easy to see that, for R, $S \subset G_0$, $R \,||\, S$ iff R is piecewise isomorphic to $\pi_{G_0/S}(R)$. From this, $H \subset_0 G_0/S$ iff $G_0[\Omega_H] \,||\, S$. If H is also irreducible (and in other cases when no confusion can arise) we do not then distinguish between H and $G_0[\Omega_H]$.

LEMMA 2.3. — Suppose $H_1, H_2, R_0 \subset G_0$, with $H_1 \cup R_0$ a link and $H_1 \,||\, (R_0 \cup H_2)$. Then $\overline{H_2 \cup R_0} \neq G_0$.

Proof. — If $\Omega^M \cap \Omega_{H_1} \neq \emptyset$ the result is immediate, since then $\Omega^M \not\subset \Omega_{R_0 \cup H_2}$. If $\bar{R}_0 = G_0$, then $H_1 \cup R_0$ a link and $H_1 \,||\, R_0$ would contradict (*b*) of Def. 1.2. We therefore assume $\bar{R}_0 \neq G_0$ and $\Omega^M \cap \Omega_{H_1} = \emptyset$, and suppose that $\overline{H_2 \cup R_0} = G_0$. Let $G_i \subset H_i \cup R_0$ $(i = 1, 2)$ be minimal subgraphs which connect all vertices of Θ^E. Since $\bar{R}_0 \neq G_0$, $G_1 \not\subset R_0$, and some pair of external vertices must be joined by different paths in G_1 and G_2; $G_1 \cup G_2$ must therefore contain a circuit C. C must intersect both H_1 and $R_0 \cup H_2$, contradicting $H_1 \,||\, R_0 \cup H_2$.

LEMMA 2.4. — Take $H_0 \in \mathscr{H}$, S_0 a link in H_0, and $K_0 \subset H_0/S_0$ with $K_i \not\subset_0 H_0/S_0$ for any piece K_i of K_0. Then $R = S_0 \cup \pi^{-1}_{H_0/S_0}(K_0)$ is a link in H_0.

Proof. — Certainly $\bar{R} = H_0$. If $H_0 \neq G_0$, we must show that R is irreducible; this follows from Lemma 2.1 and the irreducibility of S_0. If $H_0 = G_0$, it suffices by an induction argument to treat the case in which K_0 is irreducible; by Lemma 2.1 R has one piece H of the form (2.1) and some pieces $\{ S_i \mid i \notin \chi \}$ which are also pieces of S_0. Now

$$\overline{R - H} = \bigcup_{i \notin \chi} S_i \neq G_0,$$

because S_0 is a link, while for $i \notin \chi$, $\overline{R - S_i} \neq G_0$ by Lemma 2.3 (take $R_0 = S_0 - S_i$, $H_1 = S_i$, $H_2 = H$). Thus R is a link.

LEMMA 2.5. — Suppose that $K \in \mathscr{Q} \cup \mathscr{H}$, that S is a link in K, and that $G \subset S$ is an irreducible subgraph of K; if $K = G_0$ we also admit the possibility that G is a link. Then:

(*a*) If $K \in \mathscr{Q}$, G is a link in K or $G \subset_0 K$. In the first case, set $\tilde{G} = K$; otherwise, or if $K \in \mathscr{H}$, set $\tilde{G} = \bar{G}$.

(*b*) Let K/G have pieces $Q_1, \ldots, Q_m$. Up to renumbering,

$$\tilde{G} = \pi_{K/G}^{-1}(Q_1 \cup \ldots \cup Q_j) \cup G$$

for some $j \leq m$. Moreover, if $K \in \mathscr{Q}$ and $G \subset_0 K$, $\tilde{G} \subset_0 K$.

(*c*) For any disjoint subsets χ, $\psi \subset \{ 1, \ldots, j \}$,

$$R = K\left[(\Omega_S - \Omega_{\tilde{G}}) \cup \Omega_G \cup \bigcup_{\chi} \Omega_{Q_i} \cup \bigcup_{\psi} (\Omega_{Q_i} \cap \Omega_S) \right]$$

is a link in K.

(*d*) Each Q_i, $1 \leq i \leq j$, is in $\mathscr{Q}$, and $Q_i[\Omega_S \cap \Omega_{Q_i}]$ is a link in Q_i.

Proof. — (*a*) Suppose $K = G_0/T$. If $G \not\subset_0 K$, then $\pi_K^{-1}(G) \cup T$ is a link by Lemma 2.4, and hence G is a link in K.

(*b*) The special case $K \in \mathscr{Q}$, $G \not\subset_0 K$, $K = \tilde{G}$ is trivial, so we may assume $\tilde{G} = \bar{G}$. Let the pieces of G_0^∞/G be $Q'_1, \ldots, Q'_S$, with $\infty \notin Q'_i$ and $\Omega^M \cap \Omega_{Q'_i} = \emptyset$ when $i < j$, so that

$$\tilde{G} = \bar{G} = \pi_{G_0^\infty/G}^{-1}(Q'_1 \cup \ldots \cup Q'_j) \cup G. \tag{2.2}$$

If $K \in \mathscr{H}$, $K = \bar{K} \supset \bar{G}$ implies $\Omega_{Q'_i} \subset \Omega_K$ for $i \leq j$, so that Q'_i, $i \leq j$, is a piece of K/G. With possible renumbering, $Q'_i = Q_i$ ($i \leq j$), completing the proof in this case.

In the case $K = G_0/T \in \mathscr{Q}$, $G \subset_0 K$, we claim

$$T \subset_0 Q'_{j+1} \cup \ldots \cup Q'_S \tag{2.3}$$

Since $T \parallel G$, $\subset_0$ in (2.3) follows from $\subset$. Suppose the contrary; then, using again $T \parallel G$, some piece T_1 of T satisfies $T_1 \subset_0 Q'_i$, $i \leq j$. But an application of Lemma 2.3 (taking $R_0 = T - T_1$, $H_1 = T_1$, $H_2 = G$) shows that this is impossible, proving (2.3). But now from (2.2) and (2.3), $T \parallel \tilde{G}$, so $\tilde{G} \subset_0 G_0/T = K$. Moreover, $Q'_1, \ldots, Q'_j$ are pieces of $G_0/T \cup G = K/G$, which completes the proof.

(*c*) Suppose first that $K \in \mathscr{H}$ and that $\chi = \psi = \emptyset$. The pieces of K/R will consist of (*i*) the pieces of $K/S \cup \tilde{G}$ and (*ii*) the pieces of $\tilde{G}/G$; since $\bar{G} = \tilde{G}$ and $\bar{S} = K$, certainly $\bar{R} = K$. If $\bar{G} = \tilde{G} = K$, then $R = G$, so that R is a link in K. Otherwise, we may assume that G is irreducible. Let S have pieces $S_1, \ldots, S_p$, with $G \subset S_1$, and S_1/G have pieces

$$\{ Q_{ik} \mid i = 1, \ldots, m, \ k = 1, \ldots, r(i) \},$$

with $Q_{ik} \subset Q_i$. Then R has pieces

$$S'_1 = G \cup \pi^{-1}_{S_1/G}\Bigg(\bigcup_{\substack{j<i\leq m \\ 1\leq k\leq r(i)}} Q_{ik} \Bigg) \subset S_1, \ S_2, \ldots, S_k$$

(note that Lemma 2.1 and the irreducibility of G imply that S'_1 is irreducible), and the omission of any of these pieces would destroy the property $\bar{R} = K$, by the corresponding fact for S. Hence R is a link.

If χ or ψ is not empty, the result follows from Lemma 2.4, taking $H_0 = K$ and $S_0 = K[(\Omega_S - \Omega_{\tilde{G}}) \cup \Omega_G]$. It is necessary to verify that $K[\Omega_{Q_i}]$ and $K[\Omega_{Q_i} \cap \Omega_S]$ are not $||$ to S_0; in the first case this follows from Lemma 2.1, in the second, from the fact that S is a link. Finally, for $K \in \mathscr{Q}$, let $K = G_0/T$; applying the case already proved to $S \cup T$ in G_0 shows that $R \cup T$ is a link in G_0, i. e., R is a link in K.

(*d*) We first reduce to the case $K = G_0$ as follows: if $K = G_0/T \in \mathscr{Q}$, let $K' = G_0$ and $S' = S \cup T$; if $K \in \mathscr{H}$, let $K' = G_0$, $S' = G_0[(\Omega - \Omega_K) \cup \Omega_S]$. K', S', and G satisfy the hypotheses of the Lemma (an application of (*c*) shows that S' is a link in the non-trivial case $K \in \mathscr{H}$), and Q_i, $Q_i[\Omega_S \cap \Omega_{Q_i}]$ $(1 \leq i \leq j)$ are not changed. Now apply (*c*) with $\chi = \{ 1 \ldots \hat{\imath} \ldots j \}$, taking $\psi = \varnothing$ to show $Q_i \in \mathscr{Q}$, $\psi = \{ i \}$ to show $Q_i[\Omega_S \cap \Omega_{Q_i}]$ a link in Q_i.

Proof of Theorem 1.7. — We must verify the conditions of Def. 1.5 for the family $\mathscr{E}'$. Let S be defined by $\bigvee_{\mathscr{E}^0_q(K)} Q = K/S$; S is a link in K by Def. 1.5. Thus we are in precisely the situation of Lemma 2.5, and most of our results follow from that Lemma. Note first that $\tilde{G} \in \mathscr{H} \cup \mathscr{Q}$ (in case (*a*), because $\tilde{G} = K$, in case (*b*), from Lemma 2.1) and $Q_1, \ldots, Q_j \in \mathscr{Q}$ (from Lemma 2.5 (*d*)). Condition (1) is also immediate, since $G_0 \in \mathscr{E}$ implies $G_0 \in \mathscr{E}'$.

It is clear that $\tilde{G}$, $Q_1, \ldots, Q_j$ do not overlap each other; to verify (2), we must check that they do not overlap any $K' \in \mathscr{E}(K)$. Suppose first that $\Omega_{K'} \subset \Omega_H$, for some $H \in \mathscr{E}^0_h(K)$. If $H \in \mathscr{F}$, $\Omega_H \subset \Omega_{\tilde{G}}$ and $\Omega_H \cap \Omega_{Q_i} = \varnothing$. If $H \notin \mathscr{F}$, then $H || G$ (otherwise $\mathscr{F}$ would not be maximal), and hence $\Omega_H \subset \Omega_Q$, for some piece Q of K/G; Lemma 2.5 (*c*) then implies that either $\Omega_H \subset \Omega_{Q_i} \subset \Omega_{\tilde{G}}$ for some *i*, or $\Omega_H \cap \Omega_{\tilde{G}} = \varnothing$, for all *i* $(1 \leq i \leq j)$. Hence K' is non-overlapping. A slight modification of the last part of the argument also does the case $\Omega_{K'} \subset \Omega_{Q'}$, $Q' \in \mathscr{E}^0_q(K)$.

To check (3) we must look at $Q_a = \bigvee_{\mathscr{E}_q'^0(\tilde{G})} Q = \tilde{G}/G$, $Q_{bi} = \bigvee_{\mathscr{E}_q'^0(Q_i)} Q = Q_i/S_{bi}$, and (if $K \neq \tilde{G}$) $Q_c = \bigvee_{\mathscr{E}_q'^0(K)} Q = K/S_c$. That G is a link in $\tilde{G}$ and $Q_1, \ldots, Q_j$ are precisely the pieces of Q_a follows from the definitions in Theorem 1.7. If $Q_i \in \mathscr{E}_q^0(K)$, there is nothing new to check, otherwise, $S_{bi} = Q_i[\Omega_S \cap \Omega_{Q_i}]$ is a link in Q_i by Lemma 2.5 (*d*); S_c is a link in K by Lemma 2.5 (*c*) (take $\chi = \{1, \ldots, j\}$). In these cases the piece structure of Q_{bi} and Q_c follows from the known piece structure of K/S.

We check (4) similarly. $\mathscr{E}_h'^0(\tilde{G}) = \mathscr{F} \subset \mathscr{E}_h^0(K)$, so $\bigcup_{\mathscr{E}_h'^0(\tilde{G})} H$ satisfies the conditions. Again, either $Q_i \in \mathscr{E}_q^0(K)$ (a trivial case) or $\mathscr{E}_h'^0(Q_i) \subset \mathscr{E}_h^0(K)$, and if $K \neq \tilde{G}$, $\mathscr{E}_h'^0(K) \subset [\mathscr{E}_h^0(K) - \mathscr{F}] \cup \{\tilde{G}\}$; in both these cases (4) follows from (4) for $\mathscr{E}$ and from the maximality of $\mathscr{F}$.

Proof of Theorem 1.8. — Note that in the construction of Theorem 1.7.

$$\Omega_{\tilde{G}} - \bigcup_{\underline{K} \in \mathscr{E}'(\tilde{G})} \Omega_{\underline{K}} = \{ \ell \} \tag{2.4}$$

Now let $\mathscr{E}$ be an *s*-family and suppose that for $K \in \mathscr{E}$ there are distinct lines $\ell_1, \ell_2 \in \Omega_K - \bigcup_{K' \in \mathscr{E}(K)} \Omega_{K'}$. Apply the construction, taking $\ell = \ell_1$. Since in $\mathscr{E}'$ there is a graph $\tilde{G}$ satisfying (2.4), $\mathscr{E}' \neq \mathscr{E}$, and $\mathscr{E}$ is not maximal. As as corollary we note that if $\mathscr{E}$, $\mathscr{E}'$ are as in Theorem 1.7, and $\mathscr{E}'' \supset \mathscr{E}'$ is an *s*-family, then $\sigma(\tilde{G}) = \ell$ for $\mathscr{E}''$.

REMARK 2.6. — Suppose that K is an element of some *s*-family $\mathscr{E}$, and $\ell = \sigma(K)$. Then:

(*a*) If we apply the construction of Theorem 1.7, we will find $\mathscr{F} = \mathscr{E}_h^0(K)$ and, if $K \in \mathscr{H}$,

$$K = \tilde{G} = \overline{K[\ell] \cup \bigcup_{\mathscr{F}} H}. \tag{2.5}$$

(*b*) Take $\mathscr{E}^* \subset \mathscr{E}_h$ such that $\mathscr{E}_h^0(K) \subset \mathscr{E}^*$, $\ell \notin \Omega_H$ for $H \in \mathscr{E}^*$, and $\mathscr{E}^* \subset \mathscr{E}_h(K)$ if $K \in \mathscr{Q}$. We modify the construction of Theorem 1.7 to require $\mathscr{F}$ to be the maximal subfamily *of* $\mathscr{E}^*$ such that (*a*) G is a link in G_0, if any such $\mathscr{F}$ exists, or (*b*) G is irreducible, otherwise. Then we claim that

$$\mathscr{F} = \mathscr{F}^* \equiv \mathscr{E}_h^0(K) \cup \bigcup_{H \in \mathscr{E}_h^0(K)} \mathscr{E}^*(H) \,;$$

in particular, $\mathscr{E}_h^0$ is characterized as the set of maximal elements of $\mathscr{F}$, and (2.5) still holds if $K \in \mathscr{H}$.

To show this note that the $K \in \mathscr{Q}$ case is trivial. If $K \in \mathscr{H}$, $\mathscr{F} \supset \mathscr{F}^*$ by maximality. Let K′ be the minimal element of $\mathscr{E}$ containing G. By (4*c*) of Definition 1.5, a non-empty intersection of G with any $Q \in \mathscr{E}^0_q(K')$ is a piece of G; (*a*) and (*b*) then imply $G \subset Q$, contradicting the choice of K′. If $K \neq K'$, G cannot contain $\sigma(K')$ since $\ell \in \Omega_{K'}$, and hence

$$G \subset \bigcup_{\mathscr{E}^0_h(K)} H ;$$

(4*a*, *b*) of Definition 1.5 then imply $G \subset H \in \mathscr{E}^0_h(K)$, again contradicting the choice of K′. Thus $K = K'$ and $\mathscr{F} = \mathscr{F}^*$.

Proof of Theorem 1.10. — (*a*) If $\underline{\alpha} \in \mathscr{D}$ with $\alpha_{\ell_1} \leq \ldots \leq \alpha_{\ell_N}$, apply the construction of Theorem 1.7 repeatedly, starting with $\mathscr{E} = \mathscr{E}_1 = \{ G_0 \}$ and taking $\ell = \ell_1, \ell_2, \ldots$ in turn. This generates a sequence

$$G_i = K_i[\ell_i] \cup \bigcup_{\mathscr{F}_i} H, \qquad \mathscr{E}_{i+1} = \mathscr{E}_i \cup \tilde{G}_i \cup \{ Q_{i1}, \ldots, Q_{ik(i)} \}, \quad \text{etc.,}$$

with $\mathscr{E} = \mathscr{E}_{N+1}$ an *s*-family; we will show that $\underline{\alpha} \in \mathscr{D}(\mathscr{E})$. Now clearly $\sigma(\tilde{G}_i) = \ell_i$, so that for any $K \in \mathscr{E}$, $K = \tilde{G}_i$ for some *i*; moreover, $\mathscr{E}^0_h(K) = \mathscr{F}_i$, and $\mathscr{E}^0_q(K) = \{ Q_{i1}, \ldots, Q_{ik(i)} \}$. If $H \in \mathscr{F}_i$, necessarily $H = \tilde{G}_j$ for some $j < i$, so that $\sigma(H) = \ell_j$ and

$$\alpha_{\sigma(H)} \leq \alpha_{\sigma(\tilde{G}_i)} . \tag{2.6}$$

Now note that if $Q_{j,r} = Q_{j',r'}$, we must have $j = j'$ and $r = r'$ [for if, say, $\tilde{G}_j \supset \tilde{G}_{j'}$, then by (2) of Def. 1.5 either $j = j'$, $\Omega_{\tilde{G}_{j'}} \subset \Omega_{Q_{js}}$ for some *s*, or $\Omega_{\tilde{G}_j} \subset \Omega_H$ for some $H \in \mathscr{F}_j$, and each of the latter two cases precludes $Q_{jr} = Q_{j'r'}$]. Thus for any Q_{ir}, $Q_{ir} \notin \mathscr{E}_i$, so $Q_{ir} = \tilde{G}_j$ with $j > i$, and

$$\alpha_{\sigma(Q_{ir})} = \alpha_{\ell_j} \geq \alpha_{\ell_i} = \alpha_{\sigma(K)} . \tag{2.7}$$

(2.6) and (2.7) imply $\underline{\alpha} \in \mathscr{D}(\mathscr{E})$.

(*b*) Let $\mathscr{E}_1$, $\mathscr{E}_2$ be distinct *s*-families, with $\sigma_a : \mathscr{E}_a \to \Omega$ $(a = 1, 2)$ the corresponding maps; we assert that $\underline{\alpha} \in \mathscr{D}(\mathscr{E}_1) \cap \mathscr{D}(\mathscr{E}_2)$ must satisfy $\alpha_\ell = \alpha'_\ell$ for some $\ell \neq \ell'$. Specifically, we will prove by induction on *i* that if

$$\alpha_{\ell_1} < \ldots < \alpha_{\ell_N} \tag{2.8}$$

then

$$\sigma_1^{-1}(\ell_i) = \sigma_2^{-1}(\ell_i) \tag{2.9}$$

for all *i*, and hence $\mathscr{E}_1 = \mathscr{E}_2$.

For $i = 1$, let $K_a = \sigma_a^{-1}(\ell_1)$ $(a = 1, 2)$. $K_a \notin \mathscr{Q}$ since, if $\tilde{K}_a \in \mathscr{E}_a$ were the minimal element with $K_a < \tilde{K}_a$, then by (1.6) $\alpha_{\sigma_a(\tilde{K}_a)} \leq \alpha_{\sigma_a(K_a)} = \alpha_{\ell_1}$ for $\underline{\alpha} \in \mathscr{D}(\mathscr{E}_a)$, contradicting (2.8). Hence $K_a \in \mathscr{H}$, and by similar reasoning $\mathscr{E}^0_{ah}(K_a) = \varnothing$; using (2.5), $K_a = \overline{G_0[\ell_1]}$, $a = 1, 2$.

Now assume that (2.9) holds for $i < j$, so that $\mathscr{E}' = \bigcup_{i<j} \sigma_a^{-1}(\ell_i)$ is independent of *a*, and let $K_a = \sigma_a^{-1}(\ell_j)$. Suppose first that (say) $K_1 \in \mathscr{Q}$, and that $\tilde{K} \in \mathscr{E}_1$ is the minimal element with $K_1 < \tilde{K}$. As above, $\underline{\alpha} \in \mathscr{D}(\mathscr{E}_1)$ implies $\tilde{K} \in \mathscr{E}'$ (and by the induction assumption $\sigma_1(\tilde{K}) = \sigma_2(\tilde{K})$), similarly $\mathscr{E}^0_{ah}(\tilde{K}) \subset \mathscr{E}'$, $a = 1, 2$. If we apply Remark 2.6 to $\tilde{K}$, taking

$\mathscr{E}^* = \{ H \in \mathscr{E}'_h \mid H \subset \tilde{K} \}$ and $\mathscr{E} = \mathscr{E}_1$ or $\mathscr{E} = \mathscr{E}_2$, we see that $\mathscr{E}^0_{1h}(\tilde{K}) = \mathscr{E}^0_{2h}(\tilde{K})$ [both being characterized as the maximal elements of $\mathscr{F}$]. Since K_1 is a piece of $\tilde{K} \Big/ \Big[\tilde{K}[\sigma_1(\tilde{K})] \cup \bigcup_{\mathscr{E}^0_{1h}(\tilde{K})} H \Big]$, $K_1 \in \mathscr{E}_2$ also, and since $\ell_j \in \Omega_{K_1}$ we must have $K_2 < K_1$ or $K_2 \subset K_1$. If $K_2 \in \mathscr{Q}$, reversing the argument gives $K_1 < K_2$, so $K_1 = K_2$. But $K_2 \in \mathscr{H}$ is impossible, for since (again from $\underline{\alpha} \in \mathscr{D}(\mathscr{E}_1)$) $\mathscr{E}^0_{1h}(K_1) \subset \mathscr{E}'$ and hence $\mathscr{E}^0_{1h}(K_1) \subset \mathscr{E}_{2h}(K_1)$, this would imply that a union of elements of $\mathscr{E}_{2h}(K_1)$, namely $K_2 \cup \bigcup_{\mathscr{E}^0_{1h}(K_1)} H$, would contain a link $K_1[\ell_j] \cup \bigcup_{\mathscr{E}^0_{1h}(K_1)} H$ in K_1, and this can be seen to contradict (4) of Def. 1.5.

Finally, if K_1 and K_2 are in $\mathscr{H}$, apply Remark 2.6 with

$$\mathscr{E}^* = \{ H \in \mathscr{E}' \mid \ell_j \notin \Omega_H \} ;$$

K_1 and K_2 are given by (2.5) and hence are equal.

Proof of Theorem 1.12. — Note first that, if $Q = K/S$ and T_1, T_2 are trees in Q and S, respectively, then $T_1 \cup T_2$ is a tree in K; if a graph G has pieces $G_1, \ldots, G_k$, and T_i is a tree in G_i, then $\bigcup_1^k T_i$ is a tree in G. We will use this observation to show that for $K \in \mathscr{E}$, $T \cap \Omega_K$ is a tree in K. The proof is by induction on $|\Omega_K|$. If $|\Omega_K| = 1$, the result is trivial, since a tree in $Q[\ell]$ is by definition the null set, and if $K = G_0[\ell]$, then $\ell \in T$.

For $|\Omega_K| > 1$, we write $\bigvee_{\mathscr{E}^0_q(K)} Q = K/S$, with $S = G_0[\sigma(K)] \cup S_1$, $S_1 = \bigcup_{\mathscr{E}^0_h(K)} H$.
By the induction assumption $T \cap \Omega_Q$ is a tree in Q (for $Q \in \mathscr{E}^0_q(K)$) and by (3) of Def. 1.5, the pieces of K/S are precisely the elements of $\mathscr{E}^0_q(K)$; thus it suffices to show that $T \cap \Omega_S$ is a tree in S. Again, by the induction assumption and (4) of Def. 1.5, $T \cap \Omega_{S_1}$ is a tree in S_1. Then $T \cap \Omega_{S_1}$ (respectively $(T \cap \Omega_{S_1}) \cup \sigma(K)$) is a tree in S precisely when $G_0[\sigma(K)]$ is not (respectively is) a piece of S, i. e., precisely when $\sigma(K) \notin T$ (respectively $\sigma(K) \in T$), completing the induction step. [The last equivalence above is true by definition of T if $K \in \mathscr{H}$; if $K \in \mathscr{Q}$, S must be 2-connected or be at the form $Q[\sigma(K)]$, so $G_0[\sigma(K)]$ is not a piece of S.]

To show that (*a*), (*b*), or (*c*) must hold, suppose that $\sigma(G_0) \notin \Omega^M$, and let $S_1 = \bigcup_{\mathscr{E}^0_h(G_0)} H$. Since $\overline{S_1 \cup \sigma(G_0)} = G_0$, $\Omega_{S_1} \supset \Omega^M$, but we cannot have $\bar{S}_1 = G_0$, since this would contradict (4) of Def. 1.5. Thus S_1 must fail to connect all external vertices; this is possible only if $\sigma(G_0) \in T$ (since otherwise $T \subset S_1$) and if $T - \sigma(G_0)$ separates external vertices into two disjoint sets, q. e. d.

III. SINGULARITIES OF THE GENERIC AMPLITUDE

In this section we decompose the generic amplitude (1.1) as a sum of terms corresponding to s-families, and determine the singularity structure of each term. For any $\ell \in \Omega$, we set

$$\nu_\ell = -\lambda_\ell + \frac{\nu}{2} - 1, \tag{3.1}$$

and for any sub or quotient graph K of G_0

$$\pi_K = \sum_{\ell \in \Omega_K} \nu_\ell - [n(K) - c(K)]\frac{\nu}{2}. \tag{3.2}$$

(The set of variables $\{ \nu_\ell \}$, ν, is sometimes more convenient than $\{ \lambda_\ell \}$, ν.) In particular, note that $\pi_{G_0[\ell]} = -(\lambda_\ell + 1)$, $\pi_{Q[\ell]} = \nu_\ell$, and $\pi_{G_0} = -\mu$.

THEOREM 3.1. — Write

$$F = \sum_{\mathscr{E}} F_{\mathscr{E}}, \tag{3.3}$$

the sum taken over all s-families for G_0, where

$$F_{\mathscr{E}} = \Gamma(\mu) \int_{\mathscr{D}(\mathscr{E})} \Pi \alpha_\ell^{\lambda_\ell} d(\underline{\alpha})^{\lambda_0} D(\underline{\alpha}, \underline{s}, \underline{z})^{-\mu} \eta \tag{3.4}$$

(compare (1.1)). Then for $\underline{s}$, $\underline{z}$ in the Symanzik region $s(\chi) > 0$, $z_\ell < 0$, the integral converges when $\pi_Q > 0$, $Q \in \mathscr{E}_q$, and $\pi_H < 0$, $H \in \mathscr{E}_h$; moreover, it has a meromorphic extension to all of $\mathbb{C}^{N+1}$ with (possible) simple poles on the varieties

$$\begin{aligned} &\pi_Q = 0, -1, -2, \ldots, && Q \in \mathscr{E}_q; \\ &\pi_H = 0, 1, 2, \ldots, && H \in \mathscr{E}_h. \end{aligned}$$

REMARK 3.2. — (a) In [1] an equation similar to (3.3) was used to define F. In § 4, however, we will prove the existence of a region of parameter space in which the original integral (1.1) is convergent. Then (3.3) is an identity valid in this region, by Theorem 1.10, and Theorem 3.1 allows us to continue F to $\mathbb{C}^{N+1}$.

(b) It is easy to verify that the convergence region for (3.4) is a nonempty subset of $\mathbb{C}^{N+1}$, but since this follows from the more general results of § 4, we omit the proof.

Proof of 3.1. — For $\underline{\alpha} \in \mathscr{D}(\mathscr{E})$ we define new variables $\{ t_K \mid K \in \mathscr{E} \}$ by

$$\alpha_\ell = \prod_{\substack{\ell \in \Omega_H \\ H \in \mathscr{E}_h}} t_H \prod_{\substack{\ell \in \Omega_Q \\ Q \in \mathscr{E}_q}} t_Q^{-1} \tag{3.5}$$

Normalizing the homogeneous coordinates in $\mathbb{P}^{N-1}$ by $\alpha_{\sigma(G_0)} = t_{G_0} = 1$, we have [4]

$$\eta = \prod_{\ell \neq \sigma(G_0)} d\alpha_\ell = \prod_{\mathscr{E}_h(G_0)} t_H^{N(H)-1} dt_H \prod_{\mathscr{E}_q} t_Q^{-N(H)-1} dt_Q, \tag{3.6}$$

and the domain $\mathscr{D}(\mathscr{E})$ reduces to the cube

$$0 \leq t_K \leq 1, \qquad K \in \mathscr{E}(G_0).$$

Now any tree or 2-tree in G_0 must intersect each subgraph $H \in \mathscr{E}_h(G_0)$ in at most $n(H) - 1$ lines, and each quotient $Q \in \mathscr{E}_q$ in at least $n(Q) - 1$ lines. Moreover, from Theorem 1.12, these numbers are exact for the tree T and, in case (*b*) or (*c*) of that theorem, for the 2-tree $T - \sigma(G_0)$. Thus (1.2) becomes

$$d(\underline{\alpha}) = \prod_{\mathscr{E}_h(G_0)} t_H^{h(H)} \prod_{\mathscr{E}_q} t_Q^{-h(Q)}(1 + e_1(\underline{t})), \tag{3.7}$$

with e_1 a polynomial having positive coefficients. The factorization of $D(\underline{\alpha}, \underline{s}, \underline{z})$ is similar, but depends on the various cases of Theorem 1.12 :

$$D(\alpha, s, z) = \prod_{\mathscr{E}_h(G_0)} t_H^{h(H)} \prod_{\mathscr{E}_q} t_Q^{-h(Q)}(\zeta_{\mathscr{E}} + e_2(\underline{t}, \underline{s}, \underline{z})), \tag{3.8}$$

where

$$\zeta_{\mathscr{E}} = \begin{cases} -z_{\sigma(G)}, & \text{case } (a)\,; \\ s(\psi), & \text{case } (b)\,; \\ s(\psi) - z_{\sigma(G)}, & \text{case } (c)\,; \end{cases}$$

and e_2 is non-negative for $t_K \geq 0$ and $(\underline{s}, \underline{z})$ in the Symanzik region. Inserting (3.5)-(3.8) into (3.4), and recalling $\lambda_0 = \mu - \nu/2$, we have

$$F_{\mathscr{E}} = \Gamma(-\pi_G) \int_0^1 \cdots \int_0^1 \prod_{\mathscr{E}_h(G_0)} t_H^{-\pi_H - 1} dt_H \prod_{\mathscr{E}_q} t_Q^{\pi_Q - 1} dt_Q$$
$$\times (1 + e_1)^{\lambda_0} (\zeta_{\mathscr{E}} + e_2)^{-\mu}. \tag{3.9}$$

Since $\zeta_{\mathscr{E}} > 0$ in the Symanzik region, the theorem follows by considering $t_K^{\pm \pi_K - 1}$ as a distribution [6].

REMARK 3.3. — It can be shown [8] that (3.5) is the local form of a suitable birational transformation, i. e., that there exists a closed non-singular abstract variety V and birational map $Z \subset \mathbb{P}^{N-1} \times V$, such that (*a*) V may be covered with coordinate charts $\{ U_{\mathscr{E}} \}$ indexed by the set of all *s*-families, with $\{ t_K \mid K \in \mathscr{E}, K \neq G_0 \}$ local coordinates in $U_{\mathscr{E}}$, and (*b*) if $V' \subset V$ denotes the Zariski open set given by $V' \cap U_{\mathscr{E}} = \{ \underline{t} \mid t_K \neq 0, \forall K \}$, then Z is given by (3.5) on $\{ \underline{\alpha} \mid \alpha_\ell \neq 0, \forall \ell \} \times (V' \cap U_{\mathscr{E}})$. Moreover the results of [8] may be used to show that V is a projective variety. According to [8], it suffices to find an integer n_K for each $K \in \mathscr{H} \cup \mathscr{Q}$ so that the half spaces $P_K = \left\{ \underline{x} \in \mathbb{R}^N \mid a_K \left[\sum_{\ell \in K} x_\ell - n_K \right] \geq 0 \right\}$, where $a_K = 1$ or -1 for $K \in \mathscr{H}$ or $K \in \mathscr{Q}$, respectively, intersect in a non-empty convex set C; in addition, for each s-family $\mathscr{E}$, $\underline{x}^{\mathscr{E}} = \bigcap_{K \in \mathscr{E}} P_K$ must be a distinct vertex of C. This condition gives a set of linear inequalities which the $\{ n_K \}$ must satisfy, and a solution may be shown to exist by use of « theorems of the alternative » [9]. We omit details.

IV. CONVERGENCE REGION AND EXISTENCE OF SINGULARITIES

In this section we discuss the convergence of the integral (1.1) defining F, and prove that the singularities described in Theorem 3.1 actually occur in F. We need a few graph-theoretical preliminaries. If K is a sub or quotient graph of G_0, let $\mathscr{T}_K$ denote the set of trees T in G_0 such that $T \cap \Omega_K$ is a tree in K. We will use the fact that every subset $\chi \subset \Omega_G$ forming no loops can be enlarged to a tree in G.

LEMMA 4.1. — Let H and S be subgraphs of G_0, with H having pieces $H_1, \ldots, H_k$. Then there is a tree T in G_0 such that $T \in \mathscr{T}_H$, $T \notin \mathscr{T}_S$, unless there exists $\chi \subset \{ 1, \ldots, k \}$ with $\Omega_H \cap \Omega_S = \bigcup_{i \in \chi} \Omega_{H_i}$, and with $S \Big/ \left(\bigcup_{\chi} H_i \right)$ a union of pieces of $G_0 \Big/ \left(\bigcup_{\chi} H_i \right)$.

Proof. — Case 1, $\Omega_H = \emptyset$. The condition reduces to $\Omega_S = \emptyset$ or $\Omega_S = \Omega$. Suppose otherwise, choose $\ell \in \Omega_S$, $\ell' \in \Omega_{G_0} - \Omega_S$, and let C be a circuit in G_0 containing ℓ and ℓ'. Then if $C - G_0[\ell]$ is extended to a tree T, $T \notin \mathscr{T}_S$ [and $T \in \mathscr{T}_H$ since $\Omega_H = T \cap \Omega_H = \emptyset$].

Case 2, $\Omega_H \cap \Omega_S = \emptyset$. Again we suppose $\emptyset \neq \Omega_S \neq \Omega_{G_0}$. If in some piece Q of G_0/H, $\emptyset \neq \Omega_S \cap \Omega_Q \neq \Omega_Q$, we may apply Case 1 to $\pi_{G_0/H}(S) \cap Q$ to produce a tree T_1 in Q with $T_1 \cap \Omega_S$ not a tree in $\pi_{G_0/H}(S) \cap Q$; the union of T_1 with arbitrary trees in H and in all other pieces of G_0/H then lies in $\mathscr{T}_H - \mathscr{T}_S$. Hence we may assume $\Omega_S \cap \Omega_{Q_1} = \Omega_{Q_1}$, $\Omega_S \cap \Omega_{Q_2} = \emptyset$ for some pieces Q_1, Q_2 of G_0/H. But then the extension of any tree in H to a tree in G_0 is in $\mathscr{T}_H - \mathscr{T}_S$.

Apply Case 1 to each H_i, taking $G_0 = H_i$, $S = S \cap H_i$, $H = \emptyset$, to conclude that the desired tree can be found unless $\Omega_S \cap \Omega_{H_i} = \emptyset$ or $\Omega_S \cap \Omega_{H_i} = \Omega_{H_i}$ for each i, let $\chi = \{ i \mid \Omega_S \cap \Omega_{H_i} = \Omega_{H_i} \}$, and apply Case 2 in each piece of $G_0 \Big/ \left(\bigcup_{\chi} H_i \right)$ to reach the desired conclusion. [In each case a tree found in H_i or a piece of $G_0 \Big/ \Big(\bigcup_{\chi} H_i \Big)$ is easily extended to a tree $T \in \mathscr{T}_H - \mathscr{T}_S$.]

DÉFINITION 4.2. — The *convergence region* $\Lambda \subset \mathbb{C}^{N+1}$ for the generic amplitude F_{G_0} is the region $\{ (\underline{\lambda}, \nu) \mid \operatorname{Re} \pi_H < 0,\ \text{all } H \in \mathscr{H};\ \operatorname{Re} \pi_Q > 0,\ \text{all } Q \in \mathscr{Q} \}$.

REMARK 4.3. — (*a*) We will shortly show $\Lambda \neq \varnothing$. A comparison with Theorem 3.1 shows that, for $(\underline{\lambda}, \nu) \in \Lambda$, all integrals (3.4) defining the $F_{\mathscr{E}}$ converge. Thus the integral (1.1) for F converges and (3.3) is valid in Λ (using Theorem 1.10).

(*b*) Definition 4.2 includes the condition $\operatorname{Re} \mu = -\operatorname{Re} \pi_{G_0} > 0$, not actually necessary for convergence of the integral in (1.1). Omission of this condition would not change our conclusions, but it seems most natural to include it since we avoid treating G_0 on a different footing from other elements of $\mathscr{H}$. Moreover, the factor $\Gamma(\mu)$ in (1.1) may be viewed as arising from an integration over t_{G_0} [2] [5], and this integration is convergent for $\operatorname{Re} \mu > 0$.

LEMMA 4.4. — Choose $\nu_0 \in \mathbb{C}$ with $\operatorname{Re} \nu_0 > 0$, and for any $H \subset G_0$, define $\underline{\lambda}^H \in \mathbb{C}^L$ by (see (3.1))

$$\lambda_\ell^H = \frac{\nu_0}{2} - \nu_\ell^H - 1 \,;$$

$$\nu_\ell^H = \frac{| \mathscr{T}_H \cap \mathscr{T}_{G_0[\ell]} |}{| \mathscr{T}_H |} \frac{\nu_0}{2} . \tag{4.1}$$

Then for $S \subset G_0$ and $Q = G_0/S$,

$$\operatorname{Re} \pi_S(\underline{\lambda}^H, \nu_0) \geq 0 , \tag{4.3}$$

$$\operatorname{Re} \pi_Q(\underline{\lambda}^H, \nu_0) \leq 0 , \tag{4.3}$$

with equality only if $\mathscr{T}_H \subset \mathscr{T}_S$, i. e., only if H and S satisfy the conditions of Lemma 4.1.

Proof. — From (3.2),

$$\pi_S(\underline{\lambda}^H, \nu_0) = \left[\sum_{\ell \in \Omega_S} \frac{| \mathscr{T}_H \cap \mathscr{T}_{G_0[\ell]} |}{| \mathscr{T}_H |} - (n(S) - c(S)) \right] \frac{\nu_0}{2}$$

$$= \left[\frac{1}{| \mathscr{T}_H |} \sum_{T \in \mathscr{T}_H} | T \cap \Omega_S | - (n(S) - c(S)) \right] \frac{\nu_0}{2} .$$

But $| T \cap \Omega_S | \leq n(S) - c(S)$, with equality only if $T \in \mathscr{T}_S$, proving (4.2). The proof of (4.3) is similar.

THEOREM 4.5. — (*a*) The convergence region Λ for F_{G_0} is not empty:

(*b*) For each $K \in \mathscr{H} \cup \mathscr{Q}$, the boundary $\partial\Lambda$ of Λ intersects the hyperplane $\{ \pi_K^0 = 0 \}$ in a nonempty open set in the hyperplane.

(*c*) For each $K \in \mathscr{H} \cup \mathscr{Q}$, F_{G_0} has a simple pole on the variety $\{ \pi_K = 0 \}$, i. e., the leading pole in each series of Theorem 3.1 is actually present in F_{G_0}.

Proof. — (*a*) From Lemma 4.4, $(\underline{\lambda}^{G_0}, \nu_0)$ satisfies $\pi_{G_0}(\underline{\lambda}^{G_0}, \nu_0) = 0$, $\operatorname{Re} \pi_H(\underline{\lambda}^{G_0}, \nu_0) > 0$, $\operatorname{Re} \pi_Q(\underline{\lambda}^{G_0}, \nu_0) < 0$, for $H \in \mathscr{H}$ $(H \neq G_0)$ and $Q \in \mathscr{Q}$. Thus for ε satisfying $1 \gg \operatorname{Re} \varepsilon > 0$, the point $\nu_\ell = \nu_\ell^{G_0} - \varepsilon$, $\nu = \nu_0$ lies in Λ.

(*b*) Consider first the case $K = H \in \mathscr{H}$. The point $(\underline{\lambda}^H, \nu_0)$ lies on $\{ \pi_H = 0 \}$; we claim that if $(\underline{\lambda}^H, \nu_0) \in (\pi_{K_1} = 0 \}$ for $K_1 \in \mathscr{H} \cup \mathscr{Q}$, necessarily $K_1 \in \mathscr{H}$ and $K_1 \supset H$. For certainly if $K_1 \in \mathscr{H}$, then by Lemma 4.1 and the irreducibility of H we must have $K_1 \supset H$. If $K_1 = G_0/S \in \mathscr{Q}$, S cannot be disjoint from H (Lemma 4.1 would imply $\Omega_S \neq \emptyset$ or $\Omega_S = \Omega$), so $S \supset H$ and $\pi_{G_0/H}(S)$ is a union of pieces of G_0/H. Since H is saturated, every piece of G_0^∞/H contains ∞ or intersects Ω_M; contraction of S cannot change this property unless $S = G_0$, corresponding to a trivial quotient, and therefore $K_1 \notin \mathscr{Q}$. Now the point $(\underline{\lambda}, \nu)$ with $\nu = \nu_0$, $\nu_\ell = \nu_\ell^H$ for $\ell \in \Omega_H$, and $\nu_\ell = \nu_\ell^H - \varepsilon$, $\ell \notin \Omega_H$ $(0 < \operatorname{Re} \varepsilon \ll 1)$ is on $(\partial\Lambda) \cap \{\pi_H = 0 \}$ and on no other $\{ \pi_{K_1} = 0 \}$; this proves (*b*) for $K \in \mathscr{H}$.

If $K = G_0/S \in \mathscr{Q}$, then $(\underline{\lambda}^S, \nu_0)$ lies on π_K. Here we claim that if $(\underline{\lambda}^S, \nu_0) \in \{ \pi_{K_1} = 0 \}$, $K_1 \in \mathscr{Q} \cup \mathscr{H}$, then necessarily $K_1 \in \mathscr{H}$ and hence by Lemma 4.1, $\Omega_{K_1} \cap \Omega_S \neq \emptyset$. For if $K_1 = G_0/R \in \mathscr{Q}$, then by Lemma 4.1 $R \cap S$ is the union of certain pieces $S_1, \ldots, S_j$ of S. Now in $G_0^\infty/(S_1 \cup \ldots \cup S_j)$, any piece Q which contains ∞ and intersects Ω, or which intersects Ω^M, must intersect both R and S, contradicting Lemma 4.1. Thus no such pieces can exist, and $S_1 \cup \ldots \cup S_j$ satisfies property (*a*) of Def. 1.2; by (*b*) of Def. 1.2, $S = S_1 \cup \ldots \cup S_j$. Since K is irreducible, G_0/S has only one piece, and again by Lemma 4.1, $R = S$ or $R = G_0$. This proves the claim. Now the point $\nu = \nu_0$, $\nu_\ell = \nu_\ell^S$, $\ell \notin \Omega_S$; $\nu_\ell = \nu_\ell^S - \varepsilon$, $\ell \in \Omega_S$ $(0 < \operatorname{Re} \varepsilon \ll 1)$, lies in $\partial\Lambda \cap \{ \pi_K = 0 \}$ and no other $\{ \pi_{K_1} = 0 \}$.

The proof of (*c*) is now immediate. For any $K \in \mathscr{H} \cup \mathscr{Q}$ there is at least one *s*-family $\mathscr{E}$ with $K \in \mathscr{E}$ (proof: apply Theorem 1.7, as in Remark 1.11, to the weak *s*-family $\{ G_0, K \}$). The residue of F on $\{ \pi_K = 0 \}$ is

$$\operatorname{res}_K F = \sum_{K \in \mathscr{E}} \operatorname{res}_K F_{\mathscr{E}}. \tag{4.4}$$

The calculation of the residue of $F_{\mathscr{E}}$ at an interior point of $\partial\Lambda \cap \{ \pi_K = 0 \}$ is easy-simply replace the factor $t_K^{\pm \pi_K - 1}$ in (3.9) by $\delta(t_K)$. *No analytic continuation in the other parameters is necessary.* However, if we take ν and all ν_ℓ real, the residue of $F_{\mathscr{E}}$ is strictly positive; hence (4.4) cannot vanish. This completes the proof.

EXAMPLE 4.6. — In Figure 3 the convergence regions in the ν_1, ν_2 plane are indicated for the bubble graph with zero, one, and two massive lines.

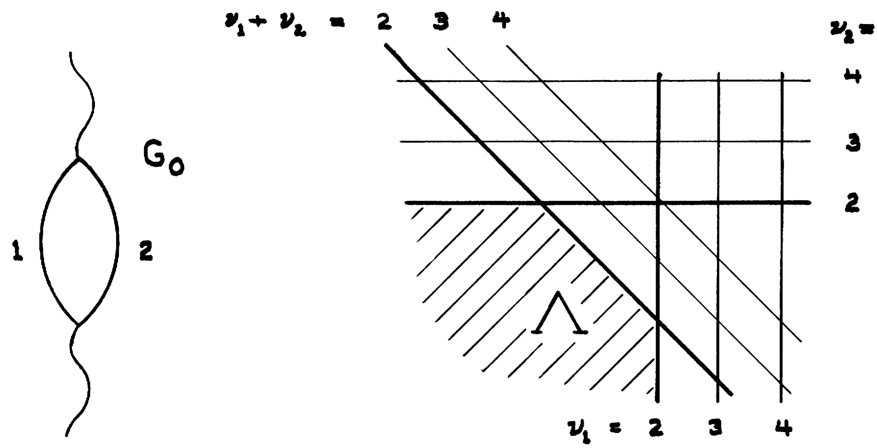

FIG. 3 *a.* — Convergence regions for bubble graphs.

In each case we have taken $\nu = 4$, and shown the first few poles of each series. Poles associated with G_0 have the form $\{ \nu_1 + \nu_2 = 2 + k \}$ $(k \geq 0)$, with $G_0[\ell]$ the form $\{ \nu_\ell = 2 + k \}$ $(k \geq 0)$, and with $Q[\ell]$ the form $\{ \nu_\ell = -k \}$ $(k \geq 0)$. Note that in each case part of the boundary of the convergence region Λ is formed by the leading pole of each series.

REMARK 4.7. — *(a)* The irreducible subgraph G[2], which corresponds to a singularity in Figure 3*a*, is not saturated, and hence does not correspond to a singularity, in Figure 3*b*. This disappearance can be partially motivated by noticing that, were this singularity ($\{ \nu_2 = 2 \}$) to appear in Figure 3*b*, it would not form part of the boundary of Λ, being hidden behind the sin-

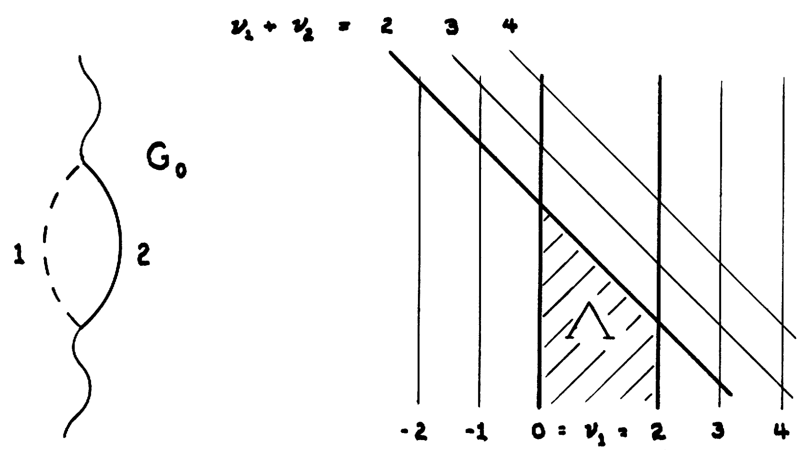

FIG. 3 *b.*

gularities $\{ \nu_1 = 0 \}$ and $\{ \nu_1 + \nu_2 = 2 \}$. Thus an attempt to prove its existence by the method of Theorem 4.4 (*c*) would fail. It can be seen that this is a general phenomena – if H is an irreducible subgraph of G_0, but $\bar{H} \neq H$, the « singularity » $\pi_H = 0$ does not form a non-trivial part of $\partial\Lambda$, being hidden behind $\{ \pi_{\bar{H}} = 0 \}$ and $\{ \{ \pi_Q = 0 \} \mid Q$ a piece of $\bar{H}/H \}$.

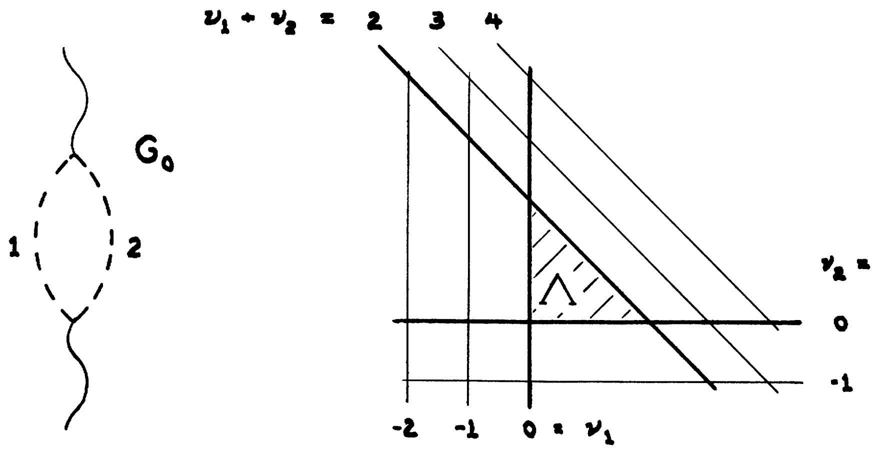

FIG. 3 *c*.

(*b*) This remark suggests a possibly helpful conjecture for the study of integrals such as (1.1)—that the leading pole of each series should, in general, form a part of the boundary of the convergence region. If a certain scaling of the integration variables produces an apparent pole which fails to satisfy this condition, a new scaling can be sought. Indeed, the non-existence of poles of (1.1) corresponding to non-saturated graphs was discovered in this way. The extraneous poles of a massive graph produced by scaling separately in all sectors $\alpha_{\ell_1} < \alpha_{\ell_2} < \dots \alpha_{\ell_N}$ [7] can be eliminated by the same criterion.

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