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Rigorous absolute bounds for pion-pion scattering. II. Solving modified Szegö-Meiman problems

Rigorous absolute bounds for pion-pion scattering. II. Solving modified Szegö-Meiman problems

by

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ABSTRACT. — In this second paper devoted to the derivation of absolute bounds for strong interactions, we give general methods for solving further extremum problems. The main problem consists in finding at some analyticity point the extremal values of a function $F(z)$, analytic in a given domain D, knowing: *i*) the value of a certain functional

$$\int |dz| \, \rho(z) \, | F(z) |^p \qquad (p \geq 2),$$

where the integration runs over the boundary ∂D of D and $\rho(z)$ is a given positive function, *ii*) the sign of Im $F(z)$ on the boundary ∂D. The mathematical techniques involved pertain to the theory of Hardy spaces of analytic functions. Numerical applications to the calculation of absolute bounds for the pion-pion amplitude are presented.

I. INTRODUCTION

In the first paper [*1*] (hereafter denoted by (I)) of this series devoted to the derivation of rigorous absolute bounds for pion-pion scattering, we displayed the various mathematical problems there encountered, and

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solved the first of them. In this second paper we pursue our program, and tackle the two remaining extremum problems pointed out in the introduction of (I), which we first restate briefly: let the function $F(z)$ be holomorphic in some domain D (which might be the cut energy-plane when F is the amplitude at fixed transfer); we suppose that we know the value of a certain functional $\int_{\partial D} | dz | k(| F(z) |, z)$ involving $| F(z) |$ on the boundary ∂D of its analyticity domain ([3]), where $k(| F |, z)$ is a given positive function; we want to answer the two following questions:

i) assume that $F(z)$ satisfies some positivity properties (the sign of Im $F(z)$ is given on ∂D or part of it); what are the extremum values of $F(z_0)$, where the point z_0 is in D?

ii) what is the allowed domain for the couple of quantities $(F(z_0), F(z_1))$, where the points z_0 and z_1 are in D?

Similar and simpler problems have been already investigated by many authors. The question *i)*, without positivity constraints, and with a function k quadratic in $| F |$, is rather trivial, and its solution is known from a long time [2]. It has been generalized to functions $k's$ that satisfy certain growth and convexity properties [3]. Our purpose is to introduce the positivity constraints on $F(z)$. However, we shall be able to answer question *i)* (and also question *ii)*) in the only case where k is a power function in $| F |$:

$$k(| F |, z) = \rho(z) | F |^p, \qquad (p > 1). \qquad (I.1)$$

This is indeed the reason why we calculated in (I) the least upper bounds of the form (I.1) for the particular functions $k's$ relevant to the calculation of bounds ([4]).

With a functional of the form $\int_{\partial D} | dz | \rho(z) | F(z) |^p$, the natural framework for a search of extrema is the theory of Hardy spaces, namely Banach spaces of analytic functions $F(z)$ with a norm precisely defined by that functional (raised to the power $1/p$). In fact, considering such spaces is unavoidable, inasmuch as some non trivial theorems of the theory of Hardy spaces will turn out to be essential in the derivation of our results. In the next two sections, we shall state the mathematical problems directly in Hardy spaces, without attending to their precise connection with the physical problems, that will be made explicit in section IV and in the third paper of this series. Furthermore, the domain D will be taken as the unit disk $| z | < 1$ (image through a suitable conformal mapping of the cut energy-plane).

Section II deals with the above mentioned problem *i)*. We first give

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([3]) See Eqs. (I.3) and (I.3)' in (I).
([4]) See Eq. (II.17) in (I). With the notation of (I), the function k is nothing but $(| F |/v_p)^p$.

precise definitions and establish some general properties of the solution (subsection (II.a)); then, we tackle the case $p = 2$ (subsection (II.b)) by resorting to geometrical considerations; we completely solve the problem in this case, which turns out to be rather easy when $\rho(z) \equiv 1$, whereas more elaborate techniques are required when $\rho(z) \not\equiv$ const., especially if $\rho(z)$ has zeros (which happens in our applications); in the general case $p > 1$, the solution cannot be given in a closed form, and we shall afford instead a method for approaching the extremum from the correct side (subsection (II.c)). In section III, we solve the problem ii), first in the case of a quadratic functional (subsection (III.a)), next in the case where k has the more general form (I.1) (subsection (III.b)). Section IV is devoted to some applications that will show how much taking into account the positivity constraints improves previous bounds. Most of the proofs are deferred to appendices. In the first one we give, for the convenience of the reader, a résumé of the main facts of the theory of Hardy spaces which are used all along this paper.

II. EXTREMUM PROBLEMS WITH POSITIVITY CONSTRAINTS

In the theory of Hardy spaces H^p presented in the mathematical litterature, the norm is defined as above with $\rho(z) \equiv 1$. In order to state properly our extremum problems, for which $\rho(z) \not\equiv$ constant, we have first to give a precise definition of the class of functions where these extrema are to be found, and to connect it to standard Hardy spaces.

II.a. Preliminaries and statement of the problem.

From now on, we always deal with functions $f(z)$ holomorphic in the unit disk $|z| < 1$ and « real analytic » ($f(z^*) = f^*(z)$). The weight function $\rho(z)$ defined on the unit circle $|z| = 1$ will be denoted $\rho(\theta)$ and will be assumed to satisfy the following properties:

$$\rho(\theta) = \rho(-\theta) \geqslant 0 \quad (-\pi < \theta < \pi), \qquad \text{Log } \rho(\theta) \in L^1. \quad \text{(II.1)}$$

In order to refine some of our results, we shall actually need further restrictions on ρ, one of them being:

$$\rho(\theta) \in L^1. \quad \text{(II.2)}$$

Let us define the class $H^p(\rho)$ for $p > 1$ as ([5]):

$$H^p(\rho) = \{ f(z) : f \in N^+, f(e^{i\theta}) \equiv \lim_{r \nearrow 1} f(re^{i\theta}) \text{ exists a. e., } f(e^{i\theta}) \in L^p(\rho) \} \quad \text{(II.3)}$$

([5]) Note that $H^p(1)$ coincides with the standard Hardy space H^p.

where N^+ is the Smirnov class (see Appendix A. a), and $L^p(\rho)$ is the usual space of functions g defined on the unit circle, with the norm

$$\left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \mid g(\theta) \mid^p \right)^{\frac{1}{p}}.$$

Our aim is to make $H^p(\rho)$ a Banach space with the norm of $f(z)$ given by the $L^p(\rho)$-norm of its boundary value $f(e^{i\theta})$. This is the reason why we are led to introduce the technical assumption that f belongs to N^+. At this point an interesting question arises. We know indeed that $f(z)$, which is nothing but a scattering amplitude, has a boundary value in the sense of distributions. This property, together with the condition $f(e^{i\theta}) \in L^p(\rho)$, limits severely the rate of growth of $f(z)$ as well as the accumulation of zeros near the boundary $\mid z \mid = 1$. This is precisely what is controlled by the condition $f \in N^+$. So it is not impossible that such a condition be a consequence of the former properties. We were unable to answer this question. It is worth mentioning however that any function in $H^p(\rho)$ has a boundary value in the sense of distributions, at least for the weights $\rho(\theta)$ corresponding to the physical situation (see appendix B). These considerations are in fact related to a deeper problem, namely to find a precise characterization of the class of analytic functions the boundary values of which are physical amplitudes. This would require a closer examination of the interplay between the distribution-theoretic aspect of the amplitudes and the (non linear) unitarity relation (see footnote at the beginning of section II in (I)).

We now introduce a function $G(z)$ holomorphic in $\mid z \mid < 1$, such that the $L^p(\rho)$-norm of $f(e^{i\theta})$ can be written as $\left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mid f(e^{i\theta})/G(e^{i\theta}) \mid^p \right]^{1/p}$. A convenient choice is:

$$G(z) = \exp \left(- \frac{1}{p} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \text{Log } \rho(\theta) \right). \tag{II.4}$$

$G(z)$, which is an outer function (appendix A. a), has the following properties:

   i) it has no zeros in $\mid z \mid < 1$ and is real positive on the real axis,

   ii) it has radial limits $G(e^{i\theta})$ almost everywhere, and

$$\mid G(e^{i\theta}) \mid = \rho(\theta)^{-\frac{1}{p}}. \tag{II.5}$$

Then, for any $f \in H^p(\rho)$, the function

$$h(z) = \frac{f(z)}{G(z)} \tag{II.6}$$

belongs to the Hardy space H^p. More precisely, one has the

PROPOSITION 1. — *Under the hypothesis* (II.1), *the class* $H^p(\rho)$ *is a linear*

space isomorphic to H^p. *The isomorphism is isometric when* $H^p(\rho)$ *is equipped with the norm*

$$\| f \|_{p,\rho} = \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) | f(e^{i\theta}) |^p \right)^{\frac{1}{p}}. \tag{II.7}$$

Proof. — For any $f \in H^p(\rho)$, let h be defined by Eq. (II.6). Firstly h belongs to N^+ as a consequence of the factorization theorem (theorem (A.4)) because $1/G$ is an outer function and $f \in N^+$. Secondly,

$$h(e^{i\theta}) \equiv \lim_{r \nearrow 1} h(re^{i\theta}) = \frac{f(e^{i\theta})}{G(e^{i\theta})} \qquad \text{a. e.,} \tag{II.8}$$

and

$$| h(e^{i\theta}) | = \rho(\theta)^{\frac{1}{p}} | f(e^{i\theta}) | \in L^p. \tag{II.9}$$

By theorem (A.2), these two properties imply that $h \in H^p$.

Conversely, for any $h \in H^p$, let $f(z)$ be $h(z)G(z)$. A similar reasoning shows that $f \in H^p(\rho)$.

This proves in particular that the class $H^p(\rho)$ is a linear space, which was not at all obvious from its definition.

Finally, the isometry is evident.

The isomorphism established in proposition 1 enables us to make use of the whole machinery of Hardy spaces. However, in the transformation $f \to h$, the positivity constraint

$$\pm \operatorname{Im} f(e^{i\theta}) \geqslant 0, \qquad 0 < \pm \theta < \pi, \tag{II.10}$$

becomes a condition on $h(e^{i\theta})$ which mixes its real and imaginary parts, and we shall see later on that this makes the problem a difficult one.

The set of functions of $H^p(\rho)$ which fulfil Eq. (II.10) is a convex cone $\mathscr{C}$.

We are now in a position to state our extremum problem: given a real point x inside the unit circle, find the supremum M and the infimum m of $f(x)$ when f ranges over $\mathscr{C}$ with a norm $\| f \|_{p,\rho} \leqslant 1$.

The mapping $f \to f(x)$ defines a linear functional on $H^p(\rho)$ that will be denoted by Δ, with $\Delta(f) = f(x)$. Let us show that Δ is continuous. According to theorem (A.5), we can write a Cauchy formula on the unit circle for the H^p function $f(z)/G(z)$. Then, using the Hölder inequality, one gets:

$$\left| \frac{f(x)}{G(x)} \right| = \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{f(e^{i\theta})}{G(e^{i\theta})} \frac{e^{i\theta}}{e^{i\theta} - x} \right|$$

$$\leqslant \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left| \frac{f(e^{i\theta})}{G(e^{i\theta})} \right|^p \right)^{\frac{1}{p}} \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{| e^{i\theta} - x |^q} \right)^{\frac{1}{q}}, \tag{II.11}$$

so that

$$| \Delta(f) | \leqslant \text{const} \| f \|_{p,\rho} \tag{II.12}$$

$\left(\text{here and in the following, } \dfrac{1}{p} + \dfrac{1}{q} = 1 \right).$

It is useful to notice that with hypothesis (II.2), one has necessarily $M > 0$ and $m < 0$. This is readily seen by considering the functions $f_\pm(z) \equiv \pm \varepsilon$ ($\varepsilon > 0$ and small enough) which verify the required conditions $f_\pm \in \mathscr{C}$ ($\text{Im } f_\pm(e^{i\theta}) = 0$) and $\| f_\pm \|_{p,\rho} \leq 1$. Moreover, since $\text{Inf } \Delta(f) = - \text{Sup } [-\Delta(f)]$, we can restrict ourselves to the search of the supremum. Before proceeding, we have to know whether the supremum is really reached on the allowed set, and whether the solution is unique. The answer is given by the

PROPOSITION 2. — *The supremum* M *is reached for some function* $\hat{f}(z)$ *of the cone* $\mathscr{C}$. *Under the hypothesis* (II.2), *this function is unique and has a unit norm* [6] :

$$M \equiv \underset{\substack{\|f\|_{p,\rho} \leq 1 \\ f \in \mathscr{C}}}{\text{Max}} \ \Delta(f) = \Delta(\hat{f}), \qquad \| \hat{f} \|_{p,\rho} = 1 . \tag{II.13}$$

That $\hat{f}$ must have a unit norm results trivially from the fact that $M > 0$. The other parts of proposition 2 are proved in appendix C.

II.b. Geometrical solution in $H^2(\rho)$.

When $p = 2$, the space $H^2(\rho)$, isometric to H^2, becomes a Hilbert space. This allows us to use simple geometrical reasonings, at least in the first step towards the solution.

We first remark that the linear functional $f(x)$ can now be written as a scalar product (Δ, f), where $\Delta(z)$ is an element of $H^2(\rho)$.

The reader will verify by an elementary calculation that:

$$\Delta(z) = \frac{G(x)G(z)}{1 - xz} . \tag{II.14}$$

We notice that the functional Δ applied to the function $\Delta(z)$ itself gives $(\Delta, \Delta) = \Delta(x)$, so that [7]:

$$\| \Delta \| = \sqrt{\Delta(x)} = \frac{G(x)}{\sqrt{1 - x^2}} \tag{II.15}$$

($G(x) > 0$ from Eq. (II.4)).

Furthermore, when Δ belongs to the cone $\mathscr{C}$, the solution of the maximum problem is obviously given by the function $\hat{f}(z) = \Delta(z)/\| \Delta \|$, and coincides with the solution of the Szegö-Meiman problem without positivity constraint [2]. When $\Delta \notin \mathscr{C}$, $\hat{f}$ is the (normalized) element of $\mathscr{C}$ which makes the smallest angle with Δ. Its construction will come out from geometrical considerations. The idea is to decompose Δ into two orthogonal compo-

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[6] We use Max (resp. Min) instead of Sup (resp. Inf) to indicate that the corresponding extremum is reached.

[7] Here and in the following, as long as it will be unambiguous, we shall write $\| \ \ \|$ for $\| \ \ \|_{2,\rho}$.
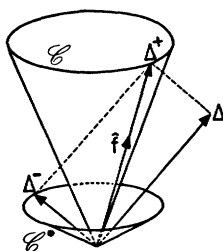
FIG. 1. — Decomposition $\Delta = \Delta^+ - \Delta^-$ of proposition 3.

nents, $\Delta = \Delta^+ - \Delta^-$, the first one being colinear to $\hat{f}$. Then, a naive picture in a three-dimensional space (fig. 1) suggests that the plane orthogonal to Δ^- is a supporting plane of $\mathscr{C}$ along Δ^+, from which it follows that Δ^- belongs to the dual cone $\mathscr{C}^*$. This is indeed true also in the infinite dimensional space $H^2(\rho)$:

PROPOSITION 3. — *Let* $\Delta^+ \equiv (\Delta, \hat{f})\hat{f}$ *and* $\Delta^- \equiv \Delta^+ - \Delta$. *Then:*

$$(\Delta^+, \Delta^-) = 0 \quad and \quad \Delta^+ \in \mathscr{C}, \quad \Delta^- \in \mathscr{C}^*.$$

Proof:

i) $(\Delta^+, \Delta^-) = \| \Delta^+ \|^2 - (\Delta^+, \Delta) = 0$ since $\| \hat{f} \| = 1$ (proposition 2).

ii) $\Delta^+ \in \mathscr{C}$ because $\hat{f} \in \mathscr{C}$ and $(\Delta, \hat{f}) = M > 0$.

iii) By definition, $\mathscr{C}^*$ is the convex cone

$$\mathscr{C}^* = \{ g(z) : g \in H^2(\rho), (g, f) \geqslant 0 \quad \forall f \in \mathscr{C} \}. \tag{II.16}$$

Assume $\Delta^- \notin \mathscr{C}^*$. Then, there exists $h \in \mathscr{C}$ of unit norm such that $(\Delta^-, h) < 0$. Let:

$$f = [1 - \varepsilon(\hat{f}, h) + O(\varepsilon^2)]\hat{f} + \varepsilon h,$$

where the term of order ε^2 is such that $\| f \| = 1$. Furthermore, ε is taken positive and sufficiently small, so that the coefficients of $\hat{f}$ and h are positive. Thus $f \in \mathscr{C}$.

Next, at the first order in ε:

$$(\Delta, f - \hat{f}) = \varepsilon[(\Delta, h) - (\Delta, \hat{f})(\hat{f}, h)] = - \varepsilon(\Delta^-, h) > 0.$$

Hence, $(\Delta, f) > (\Delta, \hat{f})$ and we get a contradiction.

Proposition 3 establishes the existence of a particular decomposition of Δ from the very existence of $\hat{f}$. Let us now show that, conversely, such a decomposition is unique and gives the solution of the maximum problem.

PROPOSITION 4. — *There exists a unique decomposition* $\Delta = \Delta^+ - \Delta^-$ *such that:*

$$(\Delta^+, \Delta^-) = 0, \tag{II.17}$$

$$\Delta^+ \in \mathscr{C}, \quad \Delta^- \in \mathscr{C}^*. \tag{II.18}$$

Moreover:

$$M \equiv \underset{\substack{\|f\| \leqslant 1 \\ f \in \mathscr{C}}}{\text{Max}} \, (\Delta, f) = \| \Delta^+ \|. \tag{II.19}$$

Proof. — Let $\Delta = \Delta^+ - \Delta^-$ be a decomposition verifying Eqs. (II.17) and (II.18). Consider the two-dimensional plane Π spanned by Δ^+ and Δ^- (fig. 2). From Eq. (II.18), the line δ (support of Δ^+) is the projection onto Π of a supporting plane of the cone $\mathscr{C}$, namely the plane orthogonal to Δ^-. Consequently, the projection of $\mathscr{C}$ onto Π lies in the half plane Π^- limited by δ and containing Δ^- (shaded area). Then, Δ^+ being both in $\mathscr{C}$ and on the border of Π^-, it is a trivial matter to show that the maximum of (Δ, f) is reached for a vector f colinear to Δ^+. So $\hat{f} = \Delta^+/\| \Delta^+ \|$, which implies Eq. (II.19). The uniqueness of Δ^+ follows from that of $\hat{f}$ (proposition 2).
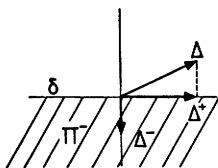

FIG. 2. — Section of H^2 by the plane Π (proof of proposition 4).

According to proposition 4, the problem amounts to constructing the above decomposition of Δ. A few remarks are now in order. The definition of the cone $\mathscr{C}$ involves only the imaginary parts of the functions $f(z)$ on the boundary $|z| = 1$. On the other hand, the scalar product in $H^2(\rho)$,

$$(g, f) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) g^*(e^{i\theta}) f(e^{i\theta}), \tag{II.20}$$

involves both the real and imaginary parts of $f(e^{i\theta})$, and we know actually that the real part is not independent of the imaginary part (Poisson formula). Then, in order to have a manageable characterization of the dual cone $\mathscr{C}^*$, it is convenient to reexpress the scalar product (g, f) in terms of $\text{Im} \, f(e^{i\theta})$. It turns out that such an operation is much easier when $\rho(\theta) \equiv 1$ than in the general case. So we shall treat separately the two cases.

A) CASE $\rho(\theta) \equiv 1$

Because of the « reality » condition of $f(z)$, we notice first that:

$$\begin{cases} \text{Re} \, f(e^{-i\theta}) = \text{Re} \, f(e^{i\theta}), \\ \text{Im} \, f(e^{-i\theta}) = - \text{Im} \, f(e^{i\theta}). \end{cases} \tag{II.21}$$

Then, the scalar product (II.20) reads:

$$(g, f) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [\text{Re} \, g(e^{i\theta}) \, \text{Re} \, f(e^{i\theta}) + \text{Im} \, g(e^{i\theta}) \, \text{Im} \, f(e^{i\theta})] \tag{II.22}$$

On the other hand, we can write a Cauchy formula on the unit circle for the function $f(z)g(z)$:

$$g(o)f(o) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} g(e^{i\theta}) f(e^{i\theta})$$

$$= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [\mathrm{Re}\, g(e^{i\theta})\, \mathrm{Re}\, f(e^{i\theta}) - \mathrm{Im}\, g(e^{i\theta})\, \mathrm{Im}\, f(e^{i\theta})]. \quad \text{(II.23)}$$

Indeed, according to theorem (A.2), gf belongs to H^1 because: i) g and f belong to N^+, which implies $gf \in N^+$; ii) the radial limit of $g(z)f(z)$ exists almost everywhere and belongs to L^1. This entails the validity of the above Cauchy formula (theorem (A.5)).

Subtracting Eq. (II.23) from Eq. (II.22), we get the proper expression of the scalar product:

$$(g, f) = g(o)f(o) + 2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mathrm{Im}\, g(e^{i\theta})\, \mathrm{Im}\, f(e^{i\theta}). \quad \text{(II.24)}$$

We now use the fact that, due to Poisson formula, $f(o)$ and $\mathrm{Im}\, f(e^{i\theta})$ are independent quantities: any real $f(o)$ and $\mathrm{Im}\, f(e^{i\theta}) \in L^2$ determine one (and only one) function $f(z)$ in H^2 (this results from theorem (A.6), where $f(z)$ is to be replaced by $if(z)$). The definition (II.16) is then easily translated into a new characterization of the cone $\mathscr{C}^*$, namely:

$$\mathscr{C}^* = \{ g(z) : g \in H^2, g(o) = 0, \pm \mathrm{Im}\, g(e^{i\theta}) \geqslant 0\ (0 < \pm \theta < \pi) \}. \quad \text{(II.25)}$$

The solution of the decomposition problem defined by Eqs. (II.17) and (II.18) readily follows. Let $\Delta^+(z)$ be the function defined (via Poisson formula) by:

$$\begin{cases} \Delta^+(o) = \Delta(o) \\ \mathrm{Im}\, \Delta^+(e^{i\theta}) = \begin{cases} \mathrm{Im}\, \Delta(e^{i\theta}) & \text{whenever } \mathrm{Im}\, \Delta(e^{i\theta}) \geqslant 0, \\ 0 & \text{»} \quad \mathrm{Im}\, \Delta(e^{i\theta}) < 0. \end{cases} \end{cases} (0 < \theta < \pi) \quad \text{(II.26)}$$

Obviously $\Delta^+ \in \mathscr{C}$ and $\Delta^- \equiv \Delta^+ - \Delta$ belongs to $\mathscr{C}^*$. Moreover, $\Delta^-(o) = 0$ and the intersection of the supports of $\mathrm{Im}\, \Delta^+(e^{i\theta})$ and $\mathrm{Im}\, \Delta^-(e^{i\theta})$ is of measure zero, so that $(\Delta^+, \Delta^-) = 0$. Hence Δ^+ and Δ^- realize the desired decomposition. To go further, we need the explicit form of $\mathrm{Im}\, \Delta(e^{i\theta})$, which results from Eq. (II.14) with $G(z) \equiv 1$:

$$\mathrm{Im}\, \Delta(e^{i\theta}) = \frac{x \sin \theta}{1 - 2x \cos \theta + x^2}. \quad \text{(II.27)}$$

It appears that when $x \geq 0$, $\Delta \in \mathscr{C}$ and $\Delta^+(z)$ coincides with $\Delta(z)$. When $x < 0$, Eq. (II.26) gives $\Delta^+(z) \equiv \Delta(o) = 1$. Finally, we get from Eq. (II.15) the very simple result:

$$M = \begin{cases} \dfrac{1}{\sqrt{1 - x^2}} & \text{for } x \geqslant 0, \\ 1 & \text{for } x < 0. \end{cases} \quad \text{(II.28)}$$

Similarly:

$$m = \begin{cases} -1 & \text{for } x \geqslant 0 \\ -\dfrac{1}{\sqrt{1-x^2}} & \text{for } x < 0. \end{cases} \qquad (\text{II}.29)$$

This is to be compared with the extrema obtained without positivity constraint, namely $M = -m = (1-x^2)^{-1/2}$ for all x.

B) CASE $\rho(\theta) \not\equiv$ CONST.

In this case, the trick used to eliminate the unwanted term Re g. Re f in the expression (II.20) of the scalar product non longer applies. So one has to find a substitute for the formula (II.24) expressing (g, f) in terms of $f'(0)$ and Im $f(e^{i\theta})$. If the function $\rho(\theta)$ has zeros, the construction of such a substitute crucially depends on the way $\rho(\theta)$ vanishes. We were able to cook up the required formula only when $\rho(\theta)$ has zeros of even multiplicity, more precisely when

$$\rho(\theta) = (1 - \cos \theta)^{n_0}(1 + \cos \theta)^{n_1}\prod_{i=2}^{N}(\cos \theta - \cos \theta_i)^{2n_i}\,\bar\rho(\theta), \quad (\text{II}.30)$$

where the $n_i's$ are positive integers, $n_0 + n_1$ is even, and $\bar\rho(\theta) = \bar\rho(-\theta)$ is bounded from above and below. Let us note that the $\rho's$ of the family (II.30) automatically fulfil both conditions (II.1) and (II.2). For the sake of simplicity, we shall consider here the only case of interest for our applications (see section IV), namely:

$$\rho(\theta) = |\,1 - e^{2i\theta}\,|^2\,\bar\rho(\theta), \quad 0 < b \leqslant \bar\rho(\theta) = \bar\rho(-\theta) \leqslant B \quad (-\pi \leqslant \theta \leqslant \pi), \quad (\text{II}.31)$$

corresponding to $n_0 = n_1 = 1$, $n_i = 0$ for $i \geq 2$. Then, one has the

PROPOSITION 5. — *Under the hypothesis* (II.31):
i) *any (real) continuous linear functional Γ on $H^2(\rho)$ can be written in a unique way as*

$$\Gamma(f) = \gamma_0 f(o) + \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\,\rho(\theta)\gamma(\theta)\,\text{Im}\,f(e^{i\theta}) \qquad (f \in H^2(\rho)), \quad (\text{II}.32)$$

where γ_0 and $\gamma(\theta)$ are real and $\gamma(\theta) = -\gamma(-\theta) \in L^2(\rho)$;
ii) *conversely, any expression $\Gamma(f)$ of the form* (II.32) *defines a continuous linear functional on $H^2(\rho)$, thus a function $g \in H^2(\rho)$ such that $\Gamma(f) = (g, f)$. Moreover:*

$$g(z) = G(z)\left[\gamma_0 G(o) - i\int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\,\frac{G(e^{i\theta})}{1 - ze^{i\theta}}\,\rho(\theta)\gamma(\theta)\right]. \quad (\text{II}.33)$$

This proposition is the particular case $p = 2$ of a more general proposition valid in $H^p(\rho)$, and proved in appendix D. That the representation (II.32) must hold is strongly suggested by a Poisson-type formula

in the space $H^2(\rho)$, which, for the class (II.31) of functions $\rho(\theta)$, reads as follows (see Eq. (D.2)):

$$f(z) = f(o) + \frac{2z}{1 - z^2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \sin \theta \, \mathrm{Im} f(e^{i\theta}). \qquad (II.34)$$

More precisely, any function $f(z)$ in $H^2(\rho)$ is shown to have the representation (II.34). Conversely, given $f(o)$ and $\mathrm{Im} f(e^{i\theta})$ real, with

$$\mathrm{Im} f(e^{i\theta}) = - \mathrm{Im} f(e^{-i\theta}) \in L^2(\rho),$$

the right hand side of Eq. (II.34) defines a function $f(z)$ which belongs to $H^2(\rho)$ [8]. We warn the reader against the temptation of writing here the usual Poisson representation, which is *not* valid in $H^2(\rho)$ [9].

By using proposition 5 and the independence of $f(o)$ and $\mathrm{Im} f(e^{i\theta})$, we deduce as previously the following characterization of the cone $\mathscr{C}^*$:

$$\mathscr{C}^* = \left\{ g(z) : g(z) = \frac{1}{i} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{G(z)G(e^{i\theta})}{1 - ze^{i\theta}} \rho(\theta)\gamma(\theta); \right.$$
$$\left. \gamma(\theta) = - \gamma(- \theta) \in L^2(\rho), \pm \gamma(\theta) \geqslant 0 \quad (0 < \pm \theta < \pi) \right\}. \qquad (II.35)$$

It is worth noticing that for $\rho(\theta) \equiv 1$, one has $\gamma_0 = g(o)$ and $\gamma(\theta) = 2 \, \mathrm{Im} \, g(e^{i\theta})$, so that one recovers Eq. (II.25). The complication arising when $\rho(\theta) \not\equiv \mathrm{const.}$ can be viewed in the following way. Let us consider the two sets $(g(o), \mathrm{Im} \, g(e^{i\theta}))$ and $(\gamma_0, \gamma(\theta))$ as the « contravariant » and the « covariant » components respectively of the same « vector » g (both sets of components determine univocally $g(z)$ via Eq. (II.34) with g in place of f or Eq. (II.33)). The conditions defining the cone $\mathscr{C}$ bear on the contravariant components, whereas the natural conditions defining its dual $\mathscr{C}^*$ bear on the covariant ones. Only when $\rho(\theta) \equiv \mathrm{const.}$ the two sets of components are proportional (the « metric tensor » is diagonal). In the general case, $(g(o), \mathrm{Im} \, g(e^{i\theta}))$ and $(\gamma_0, \gamma(\theta))$ are related in a « non local » way. The exact relations follow immediately from Eq. (II.33).

Combining Eq. (II.35) and proposition 4, we are now led to a theorem which gives the solution of our problem:

THEOREM 1. — Let $\delta^-(\theta)$ be a function satisfying the following properties:

$$\begin{cases} \delta^-(\theta) = - \delta^-(- \theta), & (II.36a) \\ \delta^-(\theta) \geqslant 0 \quad (0 < \theta < \pi), & (II.36b) \\ \mathrm{Im} \, \Delta^-(e^{i\theta}) + \mathrm{Im} \, \Delta(e^{i\theta}) \geqslant 0 \quad (0 < \theta < \pi), & (II.36c) \\ \text{Measure of } \{ \, \mathrm{Supp} \, \delta^- \cap \mathrm{Supp} \, (\mathrm{Im} \, \Delta^- + \mathrm{Im} \, \Delta) \, \} = 0 & (II.36d) \end{cases}$$

---

[8] In other words $H^2(\rho)$ is isomorphic to the Hilbert space $\mathbb{R} \oplus L^2(\rho)$. This isomorphism is indeed homeomorphic, and the right-hand side of Eq. (II.32) is the general form of the continuous linear functionals on $\mathbb{R} \oplus L^2(\rho)$.

[9] In fact, Eq. (II.34) is nothing but the twice-subtracted dispersion relation with only one subtraction constant, satisfied by the symmetrical scattering amplitude.

where:

$$\text{Im } \Delta^-(e^{i\theta}) = \lim_{r \uparrow 1} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \text{Re} \left[\frac{G(e^{i\theta})G^*(e^{i\varphi})}{1 - re^{i(\theta - \varphi)}} \right] \rho(\varphi)\delta^-(\varphi). \quad \text{(II.37)}$$

Under hypothesis (II.30), *the set of equations* (II.36) *has one and only one solution* $\delta^-(\theta)$ *in* $L^2(\rho)$, *and:*

$$M = \left[\frac{G^2(x)}{1 - x^2} + G(x) \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\delta^-(\theta) \text{ Im } \frac{G(e^{i\theta})}{1 - xe^{i\theta}} \right]^{\frac{1}{2}} \quad \text{(II.38)}$$

Proof :

i) Let $\Delta^+(z)$ and $\Delta^-(z) = \Delta^+(z) - \Delta(z)$ be the two functions defined by Eqs. (II.17) and (II.18). Then, according to the first part of proposition 5, there exist δ_0 real and $\delta^-(\theta) = -\delta^-(-\theta) \in L^2(\rho)$ such that:

$$(\Delta^-, f) = \delta_0 f(o) + \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\delta^-(\theta) \text{ Im } f(e^{i\theta}) \quad \text{for all} \quad f \in H^2(\rho). \quad \text{(II.39)}$$

Moreover, from Eq. (II.33):

$$\Delta^-(z) = G(z) \left[\gamma_0 G(o) - i \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{G(e^{i\theta})}{1 - ze^{i\theta}} \rho(\theta)\delta^-(\theta) \right]. \quad \text{(II.40)}$$

But $\Delta^- \in \mathscr{C}^*$ implies $\delta_0 = 0$ and $\delta^-(\theta) \geq 0$ $(0 < \theta < \pi)$. Thus $\delta^-(\theta)$ verifies the properties (II.36a and b).

Also Eq. (II.37) results from Eq. (II.40), and the property (II.36c) from the fact that $\Delta^+ \in \mathscr{C}$.

Finally, Eq. (II.39) gives:

$$(\Delta^-, \Delta^+) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\delta^-(\theta) \text{ Im } [\Delta^-(e^{i\theta}) + \Delta(e^{i\theta})], \quad \text{(II.41)}$$

so that Eqs. (II.36b and c) entail Eq. (II.36d), since $(\Delta^-, \Delta^+) = 0$. This shows that the set of Eqs. (II.36) has a solution.

ii) Conversely, let $\delta^-(\theta)$ be a solution of Eqs. (II.36) in $L^2(\rho)$. Let us define $\Delta^-(z)$ by Eq. (II.40) with $\delta_0 = 0$, and $\Delta^+(z)$ by $\Delta^+ = \Delta^- + \Delta$. Then Δ^- (and also Δ^+) belongs to $H^2(\rho)$ according to the second part of proposition 5, and $\Delta^-(z)$ satisfies Eq. (II.37). Furthermore, $\Delta^- \in \mathscr{C}^*$ and $\Delta^+ \in \mathscr{C}$ as a consequence of $\delta_0 = 0$ and Eqs. (II.36b and c).

Next, (Δ^-, Δ^+) is given again by the right hand side of Eq. (II.41), so that the condition (II.36d) implies $(\Delta^-, \Delta^+) = 0$.

Hence, the functions $\Delta^-(z)$ and $\Delta^+(z)$ just constructed realize the decomposition of proposition 4. Since this decomposition is unique, and since the correspondence between $\delta^-(\theta) \in L^2(\rho)$ and $\Delta^-(z) \in H^2(\rho)$ is one-to-one, we conclude that the solution of Eqs. (II.36) is unique.

Finally, according to Eqs. (II.19) and (II.17),

$$M = (\|\Delta\|^2 - \|\Delta^-\|^2)^{1/2} = (\|\Delta\|^2 + (\Delta^-, \Delta))^{1/2}.$$

Inserting here the form (II.14) of $\Delta(z)$ and using Eq. (II.39), we get the expression (II.38) for the maximum.

The above proof is valid under the hypothesis (II.31). However, as we have already mentioned, theorem 1 still holds for all functions $\rho(\theta)$ of the form (II.30). Such a generalization is straightforward after suitable (and rather obvious) changes are made in the Poisson-type formula (II.34) and in proposition 5.

The expression (II.38) for the maximum M has been written in a form which directly exhibits the improvement obtained by taking into account the positivity condition. As a last remark, let us indicate that the equations (II.36) and (II.37) furnish a real-variable algorithm for computing $\delta^-(\theta)$ from Im $\Delta(e^{i\theta})$. In the course of such a calculation, it is required however to invert Eq. (II.37). The form of this equation given above is not very convenient for doing that inversion, since it involves a limiting process. Therefore, at the price of a slight additional assumption on the weight $\rho(\theta)$, it is useful to recast Eq. (II.37) in a more tractable form. It turns out that the function $\bar{\rho}(\theta)$ (as defined in Eq. (II.31)) which appears in our calculation of bounds, can be written as (see subsection IV.a)):

$$\bar{\rho}(\theta) = \rho_0 \sqrt{\cos \frac{\theta}{2}} + \bar{\bar{\rho}}(\theta), \qquad (\text{II}.42)$$

where $\bar{\bar{\rho}}(\theta)$ is Lipschitz continuous of order $\mu > \dfrac{1}{2}$ ([10]). Then one has the

THEOREM 2. — *Under hypotheses* (II.31) *and* (II.42), *Eq.* (II.37) *is equivalent to:*

$$2\sqrt{\rho(\theta)} \text{ Im } \Delta^-(e^{i\theta}) = \sqrt{\rho(\theta)}\delta^-(\theta)$$

$$+ \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \frac{\sin\left[\dfrac{\varphi - \theta}{2} - \Phi(\varphi) + \Phi(\theta)\right]}{\sin \dfrac{\varphi - \theta}{2}} \sqrt{\rho(\varphi)}\delta^-(\varphi), \quad (\text{II}.43)$$

where $\Phi(\theta) = \text{Arg } G(e^{i\theta})$.

Moreover, considered as an integral equation for the function $\sqrt{\rho(\theta)}\delta^-(\theta) \in L^2$, *Eq.* (II.43) *is a Fredholm equation with a Hilbert-Schmidt kernel.*

The proof is given in appendix E. Eqs. (II.36) and (II.43) provide us with a modified algorithm which is now perfectly suited for numerical investigations. We refer to subsection (IV.b) for the practical use of theorems 1 and 2 in computing absolute bounds on the pion-pion amplitude. The case $\rho(\theta) = |1 - e^{2i\theta}|^2$, which is not very far from the realistic situation, is completely soluble; it is treated in appendix F.

([10]) Actually, $\bar{\bar{\rho}}(\theta)$ is Lipschitz continuous of order μ, for any $\mu < 1$.

II.c. Solution in $H^p(\rho)(p > 1)$.

The geometrical considerations of subsection (II.*b*) no longer apply when $p \neq 2$, since in that case $H^p(\rho)$ is not a Hilbert space. We shall thus resort to a completely different method. Our purpose is to set up an approximation scheme allowing us to approach the maximum M arbitrarily well. A first idea would be to restrict the space $H^p(\rho)$ to a finite subspace of dimension N for which the problem is tractable, and let N ultimately increase. Unfortunately, the approximate maxima obtained in these finite-dimensional spaces are smaller than M (they are maxima on smaller sets), and consequently they are not upper bounds for $f(x)$. Upper bounds come out however when, instead of restricting the set where the maximum is looked for, we enlarge it. A way to do this is to approximate the cone $\mathscr{C}$ by the convex cone limited by a finite number of supporting planes of $\mathscr{C}$. Such will be precisely the method we shall use in the following, the essential tool being the important « duality relation » which connects our maximum problem in H^p to a minimum problem in the dual space $(H^p)^*$.

For convenience, we choose to work in the space H^p instead of $H^p(\rho)$, by making use of the isomorphism defined by Eq. (II.6). The norm in H^p will be denoted $\| \quad \|_p$. The image $\mathscr{X}$ of the cone $\mathscr{C}$ through the isomorphism is the convex cone:

$$\mathscr{X} = \{ h(z) : h \in H^p, \; \pm \, \mathrm{Im} \, [G(e^{i\theta})h(e^{i\theta})] \geqslant 0 \qquad (0 < \pm \theta < \pi) \} \quad (\mathrm{II}.44)$$

Just as was done in the case $p = 2$, we restrict ourselves to the particular class of functions $\rho(\theta)$ which is of interest for us (see section IV), namely:

$$\rho(\theta) = |\, 1 - e^{2i\theta} \,|^p \, \bar{\rho}(\theta), \quad 0 < b \leqslant \bar{\rho}(\theta) = \bar{\rho}(-\theta) \leqslant B \quad (-\pi \leqslant \theta \leqslant \pi). \quad (\mathrm{II}.45)$$

Then, we have the following result, which parallels proposition 5:

PROPOSITION 6. — *Under the hypothesis* (II.45), *any real continuous linear functional* Γ *on* H^p *can be written in a unique way as:*

$$\Gamma(h) = \gamma_0 h(o) + \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\gamma(\theta) \, \mathrm{Im} \, [G(e^{i\theta})h(e^{i\theta})], \qquad (h \in H^p), \quad (\mathrm{II}.46)$$

where γ_0 *and* $\gamma(\theta)$ *are real and* $\gamma(\theta) = -\gamma(-\theta) \in L^q(\rho)$, $\left(\dfrac{1}{p} + \dfrac{1}{q} = 1 \right)$.

This result is an immediate consequence, through the isomorphism (II.6), of the proposition proved in appendix D ([11]). Besides, Eq. (II.34) is still valid for any f in $H^p(\rho)$, which implies the independence of $h(o)$ and $\mathrm{Im} \, [G(e^{i\theta})h(e^{i\theta})]$. So, by a similar argument, the cone $\mathscr{X}^*$, dual of $\mathscr{X}$, is given by:

$$\mathscr{X}^* = \{ \Gamma \in (H^p)^* : \gamma_0 = 0, \; \pm \gamma(\theta) \geqslant 0 \; (0 < \pm \theta < \pi) \}. \quad (\mathrm{II}.47)$$

([11]) Notice that γ_0 has been redefined so as to include an extra factor $G(o)$.

Now, from Eq. (II.13), the maximum M we are looking for can be written as:

$$M = G(x) \max_{\substack{||h||_p \leqslant 1 \\ h \in \mathscr{K}}} \Delta(h),$$ (II.48)

where Δ is the continuous linear functional ([12]) on H^p:

$$\Delta(h) \equiv h(x) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta}}{e^{i\theta} - x} h(e^{i\theta}).$$ (II.49)

Any linear functional Γ in $\mathscr{K}^*$ defines a supporting plane of $\mathscr{K}$, $\Gamma(h) = 0$, and the half-space $\Gamma(h) \geq 0$ in H^p is a convex cone which contains $\mathscr{K}$. Thus:

$$\sup_{\substack{||h||_p \leqslant 1 \\ \Gamma(h) \geqslant 0}} \Delta(h) \geqslant \max_{\substack{||h||_p \leqslant 1 \\ h \in \mathscr{K}}} \Delta(h).$$ (II.50)

Moreover, according to proposition 2, still valid when $\mathscr{C}$ is replaced by the cone $\Gamma(h) \geq 0$, the supremum over this cone is reached (it will thus be denoted by Max). Finally, letting Γ range over $\mathscr{K}^*$, one obviously gets from Eqs. (II.48) and (II.50):

$$M \leqslant G(x) \inf_{\Gamma \in \mathscr{K}^*} \max_{\substack{||h||_p \leqslant 1 \\ \Gamma(h) \geqslant 0}} \Delta(h).$$ (II.51)

The search of the Max in Eq. (II.51) requires the knowledge of the (unique) solution h_0 of the Szegö-Meiman problem in H^p without positivity constraint:

$$\max_{||h||_p \leqslant 1} \Delta(h) = \Delta(h_0).$$ (II.52)

It is a well known fact in the theory of Hardy spaces that the solution of this auxiliary problem when $p \neq 2$ is not given by simply applying Hölder inequality to Eq. (II.49). Indeed, it is not possible to saturate this inequality with a function h in H^p, since the necessary and sufficient conditions for saturation fix both the modulus (within a constant), and the phase of $h(e^{i\theta})$:

$$\begin{cases} |h(e^{i\theta})|^p = \text{const} \left| \dfrac{e^{i\theta}}{e^{i\theta} - x} \right|^q, \\[4mm] \text{Arg } h(e^{i\theta}) + \text{Arg } \dfrac{e^{i\theta}}{e^{i\theta} - x} = 0 \pmod{2\pi}. \end{cases} \qquad (-\pi < \theta < \pi) \quad \text{(II.53)}$$

It has to be noticed however that the kernel associated with the linear functional Δ is not uniquely defined. Actually, in the right-hand side of Eq. (II.49) one can add to the kernel $e^{i\theta}/(e^{i\theta} - x)$ a term of the form $e^{i\theta}u(e^{i\theta})$, where $u(z)$ is an analytic function in H^q, without changing the value of the integral. It turns out that there exists a (unique) function $u(z)$ such that the

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([12]) We use the same symbol Δ to denote the functional which sends a function onto its value at x, whatever may be the space ($H^p(\rho)$ of H^p) the function belongs to.

saturation conditions of the Hölder inequality for the new kernel are fulfilled with a function $h_0(z)$ in H^p. The equivalent kernel and the function $h_0(z)$ constructed in that way are called respectively the extremal kernel and the extremal function associated with Δ. We give in appendix A.e the algorithm, as presented by Duren [4], that leads both to the extremal kernel and the extremal function. In the present case, one finds:

$$h_0(z) = \frac{(1 - x^2)^{\frac{1}{p}}}{(1 - xz)^{\frac{2}{p}}}. \tag{II.54}$$

By assumption, h_0 does not belong to $\mathscr{K}$ (otherwise, it would obviously give the solution of our initial problem, and $M = G(x)/(1 - x^2)^{1/p}$). Therefore, there are Γ's in $\mathscr{K}^*$ such that $\Gamma(h_0) < 0$. As we shall see in a moment, it happens that the Inf in Eq. (II.51) is reached precisely for a Γ satisfying the inequality $\Gamma(h_0) < 0$. Thus nothing is lost by limiting the range of Γ to the subset of $\mathscr{K}^*$ defined by $\Gamma(h_0) < 0$. Anyhow, restricting the range of Γ does not spoil the inequality (II.51):

$$M \leqslant G(x) \inf_{\substack{\Gamma \in \mathscr{K}^* \\ \Gamma(h_0) < 0}} \max_{\substack{\|h\|_p \leqslant 1 \\ \Gamma(h) \geqslant 0}} \Delta(h). \tag{II.55}$$

Furthermore, when $\Gamma(h_0) < 0$, the maximum of $\Delta(h)$ in the half-space $\Gamma(h) \geq 0$ is reached for a function h on the boundary of this half-space, namely $\Gamma(h) = 0$:

$$\max_{\substack{\|h\|_p \leqslant 1 \\ \Gamma(h) \geqslant 0}} \Delta(h) = \max_{\substack{\|h\|_p \leqslant 1 \\ \Gamma(h) = 0}} \Delta(h). \tag{II.56}$$

Proof. — First of all, according to proposition 2 (where $\mathscr{C}$ is to be replaced by the cone $\Gamma(h) \geq 0$), the maximum is reached for a function $h_1(z)$ which is normalized, $\| h_1 \|_p = 1$. Suppose that $\Gamma(h_1) > 0$. Define $h_\lambda(z)$ by (see Fig. 3):

$$h_\lambda(z) = (1 - \lambda)h_0(z) + \lambda h_1(z), \qquad (0 \leqslant \lambda \leqslant 1). \tag{II.57}$$

Then $\|h_\lambda\|_p \leq 1$ because of the convexity of the unit ball in H^p. Moreover:

$$\Gamma(h_\lambda) = \lambda[\Gamma(h_1) - \Gamma(h_0)] + \Gamma(h_0) \geqslant 0 \quad \text{for} \quad \lambda \geqslant \bar{\lambda} \equiv \frac{-\Gamma(h_0)}{\Gamma(h_1) - \Gamma(h_0)}. \tag{II.58}$$
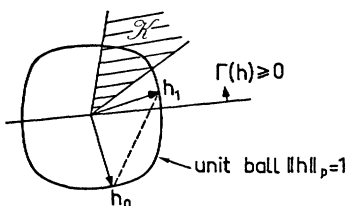

FIG. 3. — Landscape in H^p.

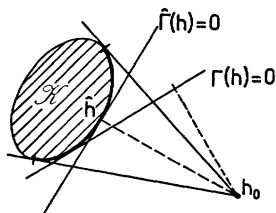Obviously $0 < \bar{\lambda} < 1$, and for $\bar{\lambda} \le \lambda < 1$:

$$\Delta(h_\lambda) = (1 - \lambda)\Delta(h_0) + \lambda\Delta(h_1) > \Delta(h_1). \qquad (II.59)$$

since $\Delta(h_0) > \Delta(h_1)$ (uniqueness of h_0).

This is in contradiction with the fact that $\text{Max } \Delta(h)$ is reached in h_1. So $\Gamma(h_1) = 0$.

The virtue of Eq. (II.56) is that it reduces an extremum problem with inequality constraints, $\Gamma(h) \ge 0$, to a simpler problem with equality cons-traints, $\Gamma(h) = 0$, for which one can make use of the duality relation. Note that Eq. (II.56) is valid only for those Γ's in $\mathcal{X}^*$ such that $\Gamma(h_0) < 0$, as it can be viewed on a naïve picture where H^p is replaced by a three-dimen-sional space (see Fig. 4).

FIG. 4. — The supporting plane $\hat{\Gamma}(h) = 0$ separates h_0 and the cone $\mathcal{X}$ (the picture represents the section of H^p by the unit sphere).

Inserting Eq. (II.56) into Eq. (II.55) provides us with an inequality which actually is an equality. The proof is deferred to appendix G. We establish there the existence of a supporting plane $\hat{\Gamma}(h) = 0$ which, firstly, contains the function $\hat{h}(z)$ where the maximum of $\Delta(h)$ is reached ($\hat{\Gamma}(\hat{h}) = 0$) and secondly, is such that $\hat{\Gamma}(h_0) < 0$ (see Fig. 4). It then follows that:

$$M = G(x) \mathop{\text{Min}}_{\substack{\Gamma \in \mathcal{X}^* \\ \Gamma(h_0) < 0}} \mathop{\text{Max}}_{\substack{||h||_p \le 1 \\ \Gamma(h) = 0}} \Delta(h). \qquad (II.60)$$

We are now ready to apply the duality relation. The general formulation of this relation, a consequence of the Hahn-Banach theorem, goes as follows [5]. Let B be a Banach space, S a (closed) subspace of B. For any continuous linear functional Δ on B, one has ([13]):

$$\mathop{\text{Sup}}_{\substack{||h||_B \le 1 \\ h \in S}} |\Delta(h)| = \mathop{\text{Min}}_{\Psi \in S^\perp} ||\Delta + \Psi||_{B^*}, \qquad (II.61)$$

where the minimum has to be taken over all the linear functionals Ψ that belong to the subspace $S^\perp$ of the dual B^* of B. $S^\perp$, the annihilator of S, is the set of elements Ψ of B^* such that $\Psi(h) = 0$ for all h in S.

In order to apply the duality relation (II.61) to Eq. (II.60), one is tempted to identify B with H^p and S with the plane $\Gamma(h) = 0$. This would lead us to a formula involving norms of functionals in the dual space $(H^p)^*$, norms

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([13]) Eq. (II.61) has a very simple geometrical interpretation when B is a Hilbert space. A picture in a three-dimensional space is particularly illuminating.

which are not tractable for computational purposes. We shall instead imbed ([14]) H^p in the space L^p, which we identify with B (so that $B^* = L^q$), and define S as:

$$S = \{ h(e^{i\theta}) : h(z) \in H^p, \Gamma(h) = 0 \} . \qquad (II.62)$$

Now, from Eqs. (II.46) and (II.47), and using the symmetry properties in θ, one can write for any Γ in $\mathscr{K}^*$:

$$\Gamma(h) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{i} \rho(\theta)\gamma(\theta)G(e^{i\theta})h(e^{i\theta}) . \qquad (II.63)$$

Furthermore, according to Cauchy theorem in H^1 (theorem A.5), one has:

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta} . u(e^{i\theta})h(e^{i\theta}) = 0 \qquad \text{for all} \quad u(z) \in H^q . \qquad (II.64)$$

As a consequence, the annihilator of S is found to be:

$$S^{\perp} = \left\{ \psi(\theta) : \psi(\theta) = \frac{\lambda}{i} \rho(\theta)\gamma(\theta)G(e^{i\theta}) + e^{i\theta}u(e^{i\theta}), \lambda \in \mathbb{R}, u \in H^q \right\} \qquad (II.65)$$

Obviously, any function $\psi(\theta)$ of the form (II.65) belongs to $S^{\perp}$. That the reciprocal is true follows easily from the discussion in appendix A.d. Taking into account Eqs. (II.49), (II.61) and (II.65), we obtain:

$$\max_{\substack{\|h\|_p \leqslant 1 \\ \Gamma(h)=0}} \Delta(h) = \min_{\substack{\lambda \in \mathbb{R} \\ u \in H^q}} \left\| \frac{e^{i\theta}}{e^{i\theta} - x} + \frac{\lambda}{i} \rho(\theta)\gamma(\theta)G(e^{i\theta}) + e^{i\theta}u(e^{i\theta}) \right\|_{L^q} \qquad (II.66)$$

We remark that the maximum of $\Delta(h)$ without the constraint $\Gamma(h) = 0$ is not given by the right-hand side of Eq. (II.66) with $\lambda = 0$ and $u(e^{i\theta}) \equiv 0$. It is then convenient to translate the functions $u(z)$ in order to make the extremal kernel associated with Δ appear explicitly. Using the algorithm mentioned above (see appendix A.e), one is led to:

$$u(z) = - \frac{1}{z - x} + \left(\frac{1 - xz}{1 - x^2} \right)^{1 - \frac{2}{q}} \frac{1}{z - x} + v(z), \qquad v \in H^q . \qquad (II.67)$$

Finally, collecting all these results, we obtain the:

THEOREM 3. — *Under hypothesis (II.45), the solution of the maximum problem (II.13) is given by:*

$$M = G(x) \min_{\substack{\gamma(\theta) \in \mathscr{Q} \cup (-\mathscr{Q}) \\ v \in H^q}}$$

$$\left\| \left(\frac{1 - xe^{i\theta}}{1 - x^2} \right)^{1 - \frac{2}{q}} \frac{e^{i\theta}}{e^{i\theta} - x} + \frac{1}{i} \rho(\theta)\gamma(\theta)G(e^{i\theta}) + e^{i\theta}v(e^{i\theta}) \right\|_{L^q} , \qquad (II.68)$$

([14]) More precisely, we identify the analytic functions $h(z)$ with their restrictions $h(e^{i\theta})$ to the unit circle, which belong to L^p; L^p is the space of functions defined on the unit circle with the usual L^p-norm.

where $\mathscr{D}$ is the convex cone in $L^q(\rho)$:

$$\mathscr{D} = \left\{ \gamma(\theta) : \gamma(\theta) = - \gamma(-\theta) \in L^q(\rho), \ \pm \gamma(\theta) \geqslant 0 \ (0 < \pm \theta < \pi), \right.$$
$$\left. \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\gamma(\theta) \ \mathrm{Im} \ \frac{G(e^{i\theta})}{(1 - xe^{i\theta})^{\frac{2}{p}}} \leqslant 0 \right\}. \quad \text{(II.69)}$$

The intuitive content of this theorem can be most easily visualized on a figure in $\mathbb{R}^2$ which displays the various cones involved, $\mathscr{K}$, $\mathscr{K}^*$ and $\mathscr{D}$ (Fig. 5). Note that $\gamma(\theta)$ in Eq. (II.68) has to range over the union of the two cones $\mathscr{D}$ and $-\mathscr{D}$. This is due to our ignorance of the sign of λ in Eq. (II.66). It turns out that for $p \neq 2$, the minimum in Eq. (II.68) may be reached in either $\mathscr{D}$ or $(-\mathscr{D})$.
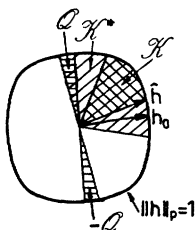

FIG. 5. — Configuration of the various cones involved in theorem 3.

The interesting feature of the formula (II.68) is that, when one restricts the ranges of γ and v to (possibly finite-dimensional) subsets of the allowed sets, its right-hand side provides us with upper bounds for M. Since the least upper bound without positivity constraint is recovered by putting $\gamma(\theta) = v(z) \equiv 0$, some improvement over this bound is practically insured by using any subset of trial functions γ and v containing the null functions. Numerical applications of formulas (II.68) and (II.69) to the calculation of pion-pion bounds will be carried out in section IV.c.

III. COUPLED EXTREMA PROBLEM

We now turn our attention to the second problem mentioned in the introduction.

Let $f(z)$ belong to $H^p(\rho)$ ($p > 1$) and let x and y be real inside the unit circle; find in the two-dimensional plane the range of the couple $(f(x), f(y))$ when $\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) | f(e^{i\theta}) |^p \leqslant 1$. The weight function $\rho(\theta)$ is assumed to fulfil condition (II.1).

First of all, making use of the isomorphism (II.6), we can immediately

reformulate the problem in the space H^p as follows: let Λ be the continuous linear mapping $H^p \to \mathbb{R}^2$:

$$\Lambda(h) = (h(x), h(y)), \qquad h \in H^p ; \qquad (III.1)$$

find the image $\mathscr{D} \equiv \Lambda(\| h \|_p \leq 1)$ of the unit ball ($\mathscr{D}$ is obviously convex). Here again, it turns out that the case $p = 2$ can be dealt with more easily than the general case.

III.a. Solution in H^2.

Let Δ_x and Δ_y be the elements of H^2 such that $h(x) = (\Delta_x, h)$ and $h(y) = (\Delta_y, h)$. For any h in H^2, the (real symmetric) matrix

$$W = \begin{pmatrix} (h, h) & (h, \Delta_x) & (h, \Delta_y) \\ (\Delta_x, h) & (\Delta_x, \Delta_x) & (\Delta_x, \Delta_y) \\ (\Delta_y, h) & (\Delta_y, \Delta_x) & (\Delta_y, \Delta_y) \end{pmatrix} \qquad (III.2)$$

is positive semi-definite. Thus $\det W \geqslant 0$, that is to say:

$$h(x)^2 \| \Delta_y \|^2 - 2h(x)h(y)(\Delta_x, \Delta_y) + h(y)^2 \| \Delta_x \|^2$$
$$\leqslant [\| \Delta_x \|^2 \| \Delta_y \|^2 - (\Delta_x, \Delta_y)^2] \| h \|^2 . \quad (III.3)$$

When h is in the unit ball, this inequality implies

$$\frac{h(x)^2}{1 - y^2} - 2\frac{h(x)h(y)}{1 - xy} + \frac{h(y)^2}{1 - x^2} \leqslant \frac{(x - y)^2}{(1 - x^2)(1 - y^2)(1 - xy)^2} . \quad (III.4)$$

Here, we have used $\| \Delta_x \| = (1 - x^2)^{-\frac{1}{2}}$, $\| \Delta_y \| = (1 - y^2)^{-\frac{1}{2}}$, and $(\Delta_x, \Delta_y) = (1 - xy)^{-1}$ (see Eqs. (II.14) and (II.15)). The region $\mathscr{D}$ we are looking for is contained in the ellipse (III.4). In fact $\mathscr{D}$ coincides with the elliptic disk (III.4): given $h(x)$ and $h(y)$ satisfying Eq. (III.4), let h be the unique vector in the plane spanned by Δ_x and Δ_y such that $(\Delta_x, h) = h(x)$, $(\Delta_y, h) = h(y)$; then obviously h satisfies Eq. (III.3) with an equality sign, which, together with Eq. (III.4), shows that $\| h \| \leqslant 1$.

The generalization to the case where one deals with more than two points (x and y) is straightforward [6].

III.b. Solution in $H^p(p > 1)$.

In this case, it is not possible to characterize the region $\mathscr{D}$ in a closed form similar to Eq. (III.4). One can however describe the boundary of $\mathscr{D}$ as the envelope of a family of straight lines. Let us define the family of continuous linear functionals on H^p:

$$\Psi_\tau(h) = \cos\tau \, h(y) - \sin\tau \, h(x), \qquad -\frac{\pi}{2} < \tau \leqslant \frac{\pi}{2}, \qquad (III.5)$$

and let $M(\tau)$ be the least upper bound of $| \Psi_\tau(h) |$ on the unit ball:

$$M(\tau) = \underset{\|h\|_p \leqslant 1}{\text{Max}} | \Psi_\tau(h) | . \qquad (III.6)$$

Then, $\mathscr{D} = \Lambda(\|h\|_p \leq 1)$ is the intersection of the following family of strips:

$$\mathscr{D} = \bigcap_{-\frac{\pi}{2} < \tau \leq \frac{\pi}{2}} \{ (h(x), h(y)) : |\cos\tau\, h(y) - \sin\tau\, h(x)| \leq M(\tau) \} \quad \text{(III.7)}$$

Proof. — Let E_τ be the strip $|\cos\tau\, h(y) - \sin\tau\, h(x)| \leq M(\tau)$. $\mathscr{D}$ is obviously contained in the intersection of the $E_\tau's \left(-\dfrac{\pi}{2} < \tau \leq \dfrac{\pi}{2} \right)$. In order to show that the converse is also true, assume the existence of a point (α, β) such that $(\alpha, \beta) \in \bigcap_\tau E_\tau$ and $(\alpha, \beta) \notin \mathscr{D}$. Then, since $\mathscr{D}$ is convex, there exists τ_0 $\left(-\dfrac{\pi}{2} < \tau_0 \leq \dfrac{\pi}{2} \right)$ such that the straight line d with slope $\operatorname{tg} \tau_0$ passing through (α, β) does not intersect $\mathscr{D}$. As $d \subset E_{\tau_0}$, there is an element h_0 of H^p, $\|h_0\|_p \leq 1$, such that the point $(h_0(x), h_0(y))$ lies on d. Moreover $(h_0(x), h_0(y)) \in \mathscr{D}$ because $\|h_0\|_p \leq 1$, and we get a contradiction.

The problem now amounts to calculating $M(\tau)$, that is to say, constructing the extremal kernel and the extremal function associated with the linear functional Ψ_τ. This is done in appendix H, according to the recipe given in appendix A . *e*. The final result is most conveniently expressed in terms of a new parameter σ, in place of τ. The region $\mathscr{D}$ eventually appears as the intersection of two families of strips,

$$\mathscr{D} = \bigcap_{-1 \leq \sigma \leq 1} (E'_\sigma \cap E''_\sigma). \quad \text{(III.8)}$$

The strip E'_σ is defined by the inequality:

$$\left| (x - \sigma)\left(\frac{1 - x^2}{1 - \sigma x}\right)^{\frac{1}{p} - \frac{1}{q}} h(x) - (y - \sigma)\left(\frac{1 - y^2}{1 - \sigma y}\right)^{\frac{1}{p} - \frac{1}{q}} h(y) \right|$$
$$\leq \frac{|y - x|}{(1 - xy)^{\frac{1}{p}}} \left[\frac{(1 + \sigma^2)(1 + xy) - 2\sigma(x + y)}{(1 - x^2)(1 - y^2)} \right]^{\frac{1}{q}}, \quad \text{(III.9)}$$

whereas E''_σ is defined by:

$$\left| (1 - \sigma x)\left(\frac{1 - x^2}{1 - \sigma x}\right)^{\frac{1}{p} - \frac{1}{q}} h(x) - (1 - \sigma y)\left(\frac{1 - y^2}{1 - \sigma y}\right)^{\frac{1}{p} - \frac{1}{q}} h(y) \right|$$
$$\leq \frac{|y - x|}{(1 - xy)^{\frac{1}{p}}} \left[\frac{(1 + \sigma^2)(1 + xy) - 2\sigma(x + y)}{(1 - x^2)(1 - y^2)} \right]^{\frac{1}{q}}, \quad \text{(III.10)}$$

These formulas form an essential tool in the calculation of bounds that will be reported in the third paper of this series.

IV. EFFECTIVE CALCULATION OF BOUNDS

We now turn to the application of the method developed in section II. It is interesting to calculate, at the same points $s = t = 2$ and $s = 3, t = 2$ that were considered in (I), the new bounds (incorporating the positivity constraints) on the modulus of the (twice-subtracted) $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude. The comparison with former bounds will give us an idea of the improvement that can be expected in general by taking into account the positivity condition. The first extremum problem, solved in (I), provided us with various functions, $\lambda_p A^{1/p}$, $\mu_p A^{1/p}$, $\nu_p A^{1/p}$ ($p \geq 2$), $F_{LM}(A)$ and $F(A)$, each of them leading to the bounds for $|F(2, 2)|$ and $|F(3, 2)|$ listed in Table 1 (see (I), p. 334) [15]. Of course, only the bounds derived from the power functions const. $A^{1/p}$ are open to improvements with the techniques of the present paper. To apply these techniques, one has to construct the corresponding weight functions $\rho(\theta)$ and to check that they fulfil the required properties. This is simpler with the function $\lambda_p A^{1/p}$, and we shall restrict ourselves to that case.

IV.a. Construction of the functions $\rho(\theta)$.

In accordance with the introduction of (I), we first write a twice-subtracted dispersion relation at fixed transfert ($0 < t < 4$):

$$F(s, t) - F(4 - t, t)$$
$$= \frac{s(s+t-4)}{\pi} \int_4^\infty ds' \frac{2s'+t-4}{s'(s'+t-4)(s'-4)(s'+t+s-4)} A(s', t), \quad (IV.1)$$

where we shall take eventually $s = 3, t = 2$. With the proper normalization, Eqs. (II.9) and (II.10) of (I) give:

$$A(s', t) \geq \left[\frac{\sqrt{\dfrac{s'-4}{s'}}}{2\Sigma_{\frac{1}{p-1}}\left(1 + \dfrac{2t}{s'-4}\right)} \right]^{p-1} |F(s', 0)|^p, \qquad (p \geq 2), \quad (IV.2)$$

where:

$$\Sigma_\omega(\xi) = \sum_{l \text{ even}} \frac{2l+1}{[P_l(\xi)]^\omega}, \qquad (\omega > 0, \xi > 1). \qquad (IV.3)$$

---

[15] An unfortunate error crept into this table: the first bound in the column $\lambda_2\sqrt{A}$ should read 878 instead of 845.

Eqs. (IV.1) and (IV.2) imply:

$$F(s, t) - F(4 - t, t) \geqslant \frac{1}{\pi} \int_4^\infty ds' w(s') \mid F(s', 0) \mid^p, \qquad \text{(IV.4)}$$

where:

$$w(s') = \frac{s(s + t - 4)(2s' + t - 4)}{s'(s' + t - 4)(s' - 4)(s' + t + s - 4)} \left[\frac{\sqrt{\dfrac{s' - 4}{s'}}}{2\Sigma_{\frac{1}{p-1}}\left(1 + \dfrac{2t}{s' - 4}\right)} \right]^{p-1} . \qquad \text{(IV.5)}$$

Next we map the cut s'-plane at zero transfer onto the unit disk $\mid z \mid < 1$ in such a way that the image of the point $(4 - t, 0)$ is $z = 0$:

$$z = \frac{i\sqrt{t(4 - t)} - \sqrt{s'(s' - 4)}}{i\sqrt{t(4 - t)} + \sqrt{s'(s' - 4)}} . \qquad \text{(IV.6)}$$

Then, with the notation $f(z) = F(s', 0)$, one obtains:

$$F(s, t) - F(4 - t, t) \geqslant \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \mid f(e^{i\theta}) \mid^p, \qquad \text{(IV.7)}$$

where:

$$\rho(\theta) = \frac{1}{4} t(4 - t) \frac{\left| \sin \dfrac{\theta}{2} \right|}{\cos^3 \dfrac{\theta}{2} \sqrt{1 + \dfrac{t(4 - t)}{4} \tan^2 \dfrac{\theta}{2}}} w(s'), \qquad \text{(IV.8)}$$

$$s' = 2 + \sqrt{4 + t(4 - t) \tan^2 \frac{\theta}{2}} . \qquad \text{(IV.9)}$$

Eqs. (IV.3), (IV.5), (IV.8) and (IV.9) completely define the weight function $\rho(\theta)$ ($= \rho(- \theta)$). It remains to show that this function satisfies Eq. (II.45) with a $\bar\rho(\theta)$ which for $p = 2$ has the form (II.42). The argument relies on the following formula, proved in appendix J:

$$\Sigma_\omega(\xi) = \frac{1}{2(\xi - 1)} \int_0^\infty \frac{x dx}{[I_0(x)]^\omega}$$

$$+ \frac{1}{2\sqrt{2(\xi - 1)}} \int_0^\infty \frac{dx}{[I_0(x)]^\omega} + R_\omega(\xi), \qquad (\xi > 1), \quad \text{(IV.10)}$$

where $R_\omega(\xi)$ is bounded for $\xi > 1$, and $(\xi - 1)R_1(\xi)$ is Lipschitz continuous of order μ on $]1, \infty[$ for all μ ($0 < \mu < 1$).

Then, it is a trivial matter to check that: i) when $\theta \to 0, \rho(\theta) \simeq \text{const.} \mid \theta \mid^p$; ii) when $\theta \to \pm \pi$, $\rho(\theta) \simeq \text{const.} (\pi \mp \theta)^p$; iii) when $\theta \neq 0$ or $\pm \pi$, $0 < \rho(\theta) < \infty$; so that $\bar\rho(\theta) = \rho(\theta)/\mid 2 \sin \theta \mid^p$ satisfies inequality (II.45).

As for the decomposition (II.42) of $\bar\rho(\theta)$ when $p = 2$, we first notice that $\bar\rho(\theta)$ is analytic in $]-\pi, \pi[$. Furthermore, when $\theta \to + \pi$, $(\xi - 1) \simeq$ const. $(\pi - \theta)$ and one easily finds that $\bar\rho(\theta) = \tau(\xi)/(\xi - 1)\Sigma_1(\xi)$ where $\tau(\xi)$ is analytic in the neighbourhood of $\xi = 1$. Then, using Eq. (IV.10) and the Lipschitz continuity of $(\xi - 1)R_1(\xi)$, one deduces that $\bar\rho(\theta)$ can be written as a sum of const. $(\sqrt{\pi - \theta} + \sqrt{\pi + \theta})$ with a Lipschitz continuous function of order μ for all $\mu < 1$. This is equivalent to Eq. (II.42).

IV.b. Computation of M and m in $H^2(\rho)$.

We are now ready to calculate $M = \text{Max} \, [f(o)/\| f \|_{2,\rho}]$ by solving Eqs. (II.36) and (II.43). For numerical purposes, we replace the integral in Eq. (II.43) by a discrete sum over N points, say $\theta_1, \ldots, \theta_N$. As we do not know the support of the unknown function $\delta^-(\theta)$ we have to examine all the possible supports. Let S_α be such a support: S_α is a non empty proper subset of $(\theta_1, \ldots, \theta_N)$ containing N_α points, $1 \le N_\alpha < N$. If S_α is the correct ansatz, then, due to Eq. (II.36d):

$$\text{Im} \, \Delta^-(e^{i\theta}) = - \, \text{Im} \, \Delta(e^{i\theta}) \qquad \text{for} \quad \theta \in S_\alpha . \qquad (IV.11)$$

Thus the inhomogeneous term in Eq. (II.43) is known on the support of δ^-, and the set of N_α linear equations, discrete version of the Fredholm equation (II.43) on S_α, can be solved. Let $\delta_\alpha^-(\theta)$ be the (odd) solution. If $\delta_\alpha^-(\theta)$ violates the inequalities (II.36b), the ansatz S_α for the support of δ^- has to be rejected. Otherwise, it remains to check that inequalities (II.36c) are satisfied, which requires the knowledge of Im $\Delta^-(e^{i\theta})$. This function is given by Eq. (II.43) for $\theta \notin S_\alpha$. According to the uniqueness property of theorem 1, these ultimate conditions (II.36c) are met for one and only one support, say S_β. The corresponding function $\delta_\beta^-(\theta)$ provides us with the solution of Eqs. (II.36) and (II.43). Finally, formula (II.38) with $x = 0$ gives the desired maximum M. Using the phase $\bar\Phi(\theta)$ as defined in appendix E (see Eq. (E.1)), this formula can be written as:

$$\frac{M}{G(o)} = \left[1 + \frac{1}{G(o)} \int_0^\pi \frac{d\theta}{\pi} \sqrt{\rho(\theta)}\delta^-(\theta) \cos(\bar\Phi(\theta) - \theta) \right]^{\frac{1}{2}} \qquad (IV.12)$$

The quantity $M/G(o)$ precisely measures the improvement due to the positivity constraints.

The minimum m is obtained in the same way.

Our calculations have been performed on a computer IBM 360. In each case (M and m), the uniqueness of the solution came out quite well. Moreover, we have observed that, when increasing N, the solution becomes stable very rapidly. Figure 6 exhibits the functions $- \sqrt{\rho(\theta)}\delta^-(\theta)$ and
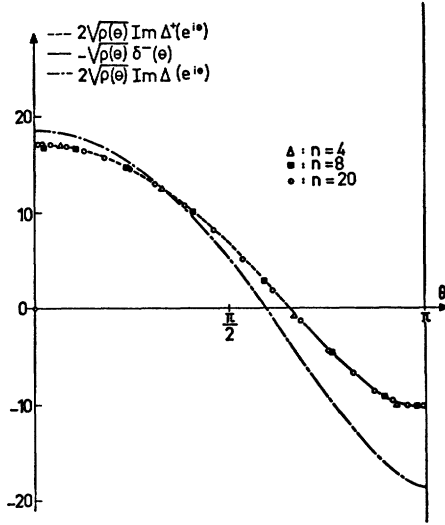
FIG. 6. — Solution of the maximum problem in $H^2(\rho)$. Graphs of the functions $-\sqrt{\rho(\theta)}\delta^-(\theta)$ and $2\sqrt{\rho(\theta)}$ Im $(\Delta + \Delta^-)(e^{i\theta})$ calculated with n $(= 4, 8, 20)$ integration points on $[0, \pi]$. One sees that the points for $n = 4$ fall remarkably close to the curve drawn with $n = 20$. The function $2\sqrt{\rho(\theta)}$ Im $\Delta(e^{i\theta})$ is also plotted.

$2\sqrt{\rho(\theta)}$ Im $(\Delta + \Delta^-)(e^{i\theta})$ corresponding to the maximum M. The values of $M/G(o)$ and $m/G(o)$ are found to be, for $s = 3$ and $t = 2$:

$$\frac{M}{G(o)} = 0.9006, \qquad \frac{m}{G(o)} = -0.8214. \qquad (IV.13)$$

The calculation of the resulting bounds on $|F(2, 2)|$ and $|F(3, 2)|$ is deferred to the next subsection.

IV.c.   Computation of bounds in $H^p(\rho)(p \geq 2)$.

For $x = 0$, Eqs. (II.68) and (II.69) can be simplified. Putting:

$$\zeta(\theta) = \pm \rho(\theta)^{\frac{1}{q}}\gamma(\theta), \qquad 0 < \pm \theta < \pi, \qquad (IV.14)$$

one obtains:

$$\frac{M}{G(o)} = \operatorname*{Min}_{\substack{\zeta \in \mathscr{T} \cup (-\mathscr{T}) \\ v \in H^q}} \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | 1 + e^{i(\bar{\Phi}(\theta) - \theta)}\zeta(\theta) + e^{i\theta}v(e^{i\theta}) |^q \right\}^{\frac{1}{q}}, \quad (IV.15)$$

where $\mathscr{T}$ is the cone:

$$\mathscr{T} = \left\{ \zeta(\theta) : \zeta(\theta) = \zeta(-\theta) \in L^q, \ \zeta(\theta) \geqslant 0, \right.$$
$$\left. \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \zeta(\theta) \cos [\bar{\Phi}(\theta) - \theta] \leqslant 0 \right\}, \quad (IV.16)$$

and where, as previously, $\bar{\Phi}(\theta)$ is the phase of $\bar{G}(e^{i\theta})$ defined in appendix D (see Eq. (D.3)). Similarly, the minimum m of $f(o)$ is given by:

$$\frac{m}{G(o)} = - \underset{\substack{\zeta \in \mathcal{T}' \cup (-\mathcal{T}') \\ v \in H^q}}{\text{Min}} \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |1 + e^{i(\bar{\Phi}(\theta) - \theta)} \zeta(\theta) + e^{i\theta} v(e^{i\theta})|^q \right\}^{\frac{1}{q}}, \quad (IV.17)$$

where

$$\mathcal{T}' = \left\{ \zeta(\theta) : \zeta(\theta) = \zeta(-\theta) \in L^q, \zeta(\theta) \geqslant 0, \right.$$
$$\left. \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \zeta(\theta) \cos [\bar{\Phi}(\theta) - \theta] \geqslant 0 \right\}. \quad (IV.18)$$

Here again, the integrals in Eqs. (IV.15) to (IV.18) are replaced by discrete sums over N points $\theta_1, \ldots, \theta_N$. A first set of parameters to be varied in the minimization process is the set of values $\zeta(\theta_1), \ldots, \zeta(\theta_N)$. As for the function $v(e^{i\theta})$, it is restricted to range over the subspace of polynomials of a given order (< 10). The (real) coefficients of these polynomials constitute the second set of parameters to be varied. To perform the minimization, use has been made of the CERN program MINUIT. The results are given in Table 1 for the values 4.3 and 5 of p which gave in (I) the best bounds on $|F(2, 2)|$ and $|F(3, 2)|$ respectively.

Table 1

	$p = 2$	$p = 4.3$	$p = 5$
$M/G(o) \leq$	0.9034	0.8078	0.7920
$- m/G(o) \leq$	0.8221	0.8810	0.8967
$G(o)$	9.262	22.34	28.61

Due to the fact that the trial functions $\zeta(\theta)$ and $v(e^{i\theta})$ have been restricted to proper subsets of $\mathcal{T} \cup (-\mathcal{T})$ and H^q respectively, the figures appearing in the first two lines of Table 1 are only upper bounds for $M/G(o)$ and $- m/G(o)$. In the case $p = 2$, one observes however that these bounds are quite close above the exact extrema (IV.13), and one may expect that roughly the same accuracy occurs for higher values of p. One also remarks that when p increases, the factor $M/G(o)$ on the upper bound gets better, whereas the factor $- m/G(o)$ on the lower bound gets worse. Unfortunately, it is this last factor which is relevant for the calculation of absolute bounds on the amplitude.

Choosing now $s = 3$, $t = 2$ in Eq. (IV.7) and using the bounds:

$$m \| f \|_{p,\rho} \leqslant f(o) \equiv F(2, 2) \leqslant M \| f \|_{p,\rho}, \quad (IV.19)$$

one gets:

$$m[F(3, 2) - F(2, 2)]^{\frac{1}{p}} \leqslant F(2, 2) \leqslant M[F(3, 2) - F(2, 2)]^{\frac{1}{p}}. \quad (IV.20)$$

On the other hand, as explained in the introduction of (I), the inequality (I.3)″ of that paper results in a bound of the form:

$$| F(3, 2) | \leqslant M'[F(3, 2) - F(2, 2)]^{\frac{1}{p}}. \quad (IV.21)$$

From Eqs. (IV.20) and (IV.21), one obtains for $p = 2$:

$$| F(2, 2) | \leqslant 709, \qquad | F(3, 2) | \leqslant 8010, \quad (IV.22)$$

instead of 878 and 8154 respectively when ignoring the positivity constraints. For $p = 4.3$, the bound 98 on $| F(2, 2) |$ becomes:

$$| F(2, 2) | \leqslant 86.3, \quad (IV.23)$$

whereas for $p = 5$, the bound 460 on $| F(3, 2) |$ is practically unchanged.

The above figures show that the absolute bounds are not very sensitive to the inclusion of the positivity condition. In view of these rather disappointing results, it has seemed to us not worth trying to improve the bounds derived in (I) with the function $v_p A^{1/p}$, bounds which are actually far from the best ones, obtained with the function F(A) (see Table 1 in (I)). However, it will be seen in the next paper that the use of the solution of the coupled extrema problem given in section III turns out to be more rewarding.

Let us remark that we could have got out of elaborating a particular method when p is equal to 2, since the general method is valid for any $p > 1$. However, in other physical contexts (for example when dealing with vertex functions involving electromagnetic or weak hadronic currents), one is led to extremum problems similar to the ones considered in this paper, and the only value of p which naturally appears is $p = 2$. In such problems, the obvious thing to do is to use the complete solution provided by the Hilbert space techniques ($p = 2$). This is one of the reasons for which here we have developed in details the Hilbert space approach. An application to the calculation of bounds on the K_{l3} form factor will be presented in a forthcoming publication.

As a conclusion, beyond the poor numerical results obtained in the particular problem we have solved, we would like to emphasize the role played here by the theory of Hardy spaces, and to stress how powerful and manageable this theory may be in the solution of physical problems.

APPENDIX A

This appendix is intended to provide the reader with a short introduction to the theory of Hardy spaces, giving in a compact form the main definitions and results relevant for this paper. Needless to say, it is far from being exhaustive, and many important topics of the theory are ignored. Also the proofs are omitted. The interested reader is referred to the excellent textbook of Duren [4]. See also ref. [7].

a) Definitions and basic properties.

For any function $f(z)$ holomorphic in $|z| < 1$, let $M_p(r, f)$, $(0 \leq p \leq \infty)$, be defined as follows:

$$
M_p(r, f) = \begin{cases}
\displaystyle \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \operatorname{Log}^+ | f(re^{i\theta}) |, & p = 0, \\[12pt]
\displaystyle \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | f(re^{i\theta}) |^p \right]^{\frac{1}{p}}, & 0 < p < \infty, \\[12pt]
\displaystyle \operatorname*{Max}_{-\pi < \theta \leqslant \pi} | f(re^{i\theta}) |, & p = \infty,
\end{cases}
\tag{A.1}
$$

where:

$$
\operatorname{Log}^+ x = \begin{cases}
\operatorname{Log} x & x > 1, \\
0 & x \leqslant 1.
\end{cases}
$$

DEFINITIONS. — The *Hardy space* H^p $(0 < p \leq \infty)$ is the class of analytic functions $f(z)$ such that $M_p(r, f)$ is bounded for $0 \leq r < 1$. The *Nevanlinna class* N is the class of analytic functions $f(z)$ such that $M_0(r, f)$ is bounded for $0 \leq r < 1$.

The H^p's, as well as N, are linear spaces. For $1 \leq p \leq \infty$, H^p is shown to be a Banach space with the norm $\| f \|_p = \lim_{r \uparrow 1} M_p(r, f)$. (For $0 < p < 1$, $\| f \|_p$ is not a norm, but H^p is a complete metric space with the distance $\| f - g \|_p^p$).

Obviously $H^{p_1} \subset H^{p_2} \subset N$ for all $0 < p_2 < p_1 \leq \infty$.

One of the first important results concerns the existence and properties of boundary values on the unit circle. In the following, we shall denote by $f(e^{i\theta})$ the radial limit, when it exists, of $f(z) : f(e^{i\theta}) = \lim_{r \uparrow 1} f(re^{i\theta})$.

THEOREM A.1. — i) *If $f \in N$, then $f(e^{i\theta})$ exists almost everywhere and $\operatorname{Log} | f(e^{i\theta}) | \in L^1$* (unless $f(z) \equiv 0$).

ii) *If $f \in H^p$ $(0 < p < \infty)$, then $f(e^{i\theta}) \in L^p$, and:*

$$
\| f \|_p = \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | f(e^{i\theta}) |^p \right]^{\frac{1}{p}},
\tag{A.2}
$$

$$
\lim_{r \nearrow 1} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | f(re^{i\theta}) - f(e^{i\theta}) |^p = 0.
\tag{A.3}
$$

iii) *If $f \in H^\infty$, then $f(e^{i\theta}) \in L^\infty$, and:*

$$
\| f \|_\infty = \operatorname*{Ess\ sup}_{-\pi < \theta < \pi} | f(e^{i\theta}) |.
\tag{A.4}
$$

Remarks:

1) For $1 \leq p < \infty$, Eq. (A.3) amounts to saying that $f(re^{i\theta})$ converges strongly to $f(e^{i\theta})$ when $r \uparrow 1$.

2) For $p = 2$, H^2 is a (separable) Hilbert space with the scalar product:

$$(g, f) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} g^*(e^{i\theta}) f(e^{i\theta}) .$$

3) The analog of Eq. (A.3) in H^∞, namely $\lim_{r\uparrow 1} \operatorname{Ess\,sup}_{-\pi < \theta < \pi} | f(re^{i\theta}) - f(e^{i\theta}) | = 0$, is wrong $\left(\text{consider for example the function } f(z) = \exp\left(-\dfrac{1 + z}{1 - z}\right)\right).$

4) The analogs of both Eqs. (A.2) and (A.3) in N are wrong. In particular, there exist functions $f \in N$ for which the equation:

$$\lim_{r\nearrow 1} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \operatorname{Log}^+ | f(re^{i\theta}) | = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \operatorname{Log}^+ | f(e^{i\theta}) | \tag{A.5}$$

is violated $\left(\text{consider for example the function } f(z) = \exp\dfrac{1 + z}{1 - z}\right).$

This last remark shows that N cannot be viewed as the « natural limit » of the H^{p}'s when $p \to 0$. It is then useful to introduce the *Smirnov class* N^+, which is defined as the subclass of N for which Eq. (A.5) holds.

Notice that $H^p \subset N^+$ for all $0 < p \le \infty$.

According to theorem (A.1), any function in H^p has a radial limit in L^p. The converse is not true. However one has the

THEOREM A.2. — *If* $f(z) \in N^+$ *and* $f(e^{i\theta}) \in L^p$ *for some* $0 < p \le \infty$, *then* $f(z) \in H^p$.

Remark. — The assumption $f \in N^+$ cannot be replaced by the weaker one $f \in N$ as it can be seen by considering again the function $f(z) = \exp \dfrac{1 + z}{1 - z} .$

b) Representation theorems in N, N^+ and H^p.

The important factorization theorem given below analyses the structure of H^p functions, and provides us with information about their distribution of zeros as well as their boundary behaviour. Before stating the theorem, some definitions are needed:

An *inner function* $f(z)$ is a function holomorphic in $| z | < 1$, such that $| f(z) | \le 1$ in $| z | < 1$, and $| f(e^{i\theta}) | = 1$ a. e.

It can be shown that any inner function $f(z)$ can be factorized in a unique way as $f(z) = B(z)S(z)$, where:

i) B(z) is a *Blaschke product*:

$$B(z) = z^m \prod_n \frac{| a_n |}{a_n} \cdot \frac{a_n - z}{1 - a_n^* z} , \tag{A.6}$$

with m a non negative integer, $| a_n | < 1$ and $\displaystyle\sum_n (1 - | a_n |) < \infty.$

ii) S(z) is a *singular inner function*:

$$S(z) = \exp\left[-\int_{-\pi}^{\pi} d\mu(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} \right], \tag{A.7}$$

with $\mu(\theta)$ a bounded non decreasing function such that $\mu'(\theta) = 0$ a. e. (Note that the above

mentioned function $\exp\left(-\dfrac{1+z}{1-z}\right)$ is nothing but a singular inner function with the measure $d\mu(\theta)$ concentrated at $\theta = 0$).

An *outer function* $Q(z)$ in $|z| < 1$ is a function of the form:

$$Q(z) = e^{i\gamma} \exp \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Log} \chi(\theta), \qquad (A.8)$$

where γ is real, $\chi(\theta) \geq 0$ and $\operatorname{Log} \chi(\theta) \in L^1$.

Both singular inner functions and outer functions have no zeros in $|z| < 1$. Some interesting properties of outer functions are given by the

THEOREM A.3. — *Let $Q(z)$ be an outer function as in Eq. (A.8). Then:*

i) $Q \in N^+$,

ii) $\lim\limits_{r \nearrow 1} |Q(re^{i\theta})| = \chi(\theta)$ a. e.,

iii) $Q \in H^p$ $(0 < p \leqslant \infty)$ *if and only if* $\chi(\theta) \in L^p$; *then* $\|Q\|_p = \|\chi\|_{L^p}$.

Now comes the canonical factorization theorem:

THEOREM A.4 :

i) *A function $f(z) \not\equiv 0$ belongs to N if and only if it has the form:*

$$f(z) = B(z) \frac{S_1(z)}{S_2(z)} Q(z), \qquad (A.9)$$

where $B(z)$ is a Blaschke product, $S_1(z)$ and $S_2(z)$ are singular inner functions, and $Q(z)$ is an outer function.

ii) *A function $f(z) \not\equiv 0$ belongs to N^+ if and only if it has the form:*

$$f(z) = B(z)S_1(z)Q(z), \qquad (A.10)$$

where $B(z)$, $S_1(z)$ and $Q(z)$ are as in i).

iii) *A function $f(z) \not\equiv 0$ belongs to H^p $(0 < p \leq \infty)$ if and only if it has the form (A.10) with $Q \in H^p$.*

That N is a class too large for being the « natural limit » of the H^p's when $p \to 0$ clearly appears in this theorem.

A second kind of representation theorems concerns the problem of recovering a function $f(z)$ from its radial limit $f(e^{i\theta})$. The relevant formulas are of course Cauchy and Poisson formulas. Establishing their validity however requires some care, since the integration path is pushed on to the boundary of the analyticity domain.

THEOREM A.5. — *If a function $f(z)$ defined in $|z| < 1$ satisfies one of the three following properties, it satisfies the two other ones:*

i) $f \in H^1$.

ii) $f(z)$ *is the Cauchy integral on the unit circle of a function $\varphi \in L^1$, satisfying the conditions:*

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{in\theta} \varphi(\theta) = 0, \qquad (n = 1, 2, \ldots). \qquad (A.11)$$

Namely:

$$f(z) = \frac{1}{2\pi i} \int_{z' = e^{i\theta}} dz' \frac{\varphi(\theta)}{z' - z}. \qquad (A.12)$$

iii) $f(z)$ *is the Poisson integral on the unit circle of a function $\psi \in L^1$, satisfying conditions (A.11):*

$$f(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{dt}{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \psi(t). \qquad (A.13)$$

Moreover:

$$\varphi(\theta) = \psi(\theta) = f(e^{i\theta}) \qquad \text{a. e.} \tag{A.14}$$

Remark. — Obviously the representations (A.12) and (A.13), valid in H^1, are *a fortiori* valid in H^p for $1 < p \leq \infty$.

It is expected that for reconstructing an analytic function (up to an imaginary constant), the knowledge of the only real part of its radial limit is sufficient. This is indeed true when $1 \leq p \leq \infty$. More precisely, one has the

THEOREM A.6:

i) *Any function $f(z)$ of the form:*

$$f(z) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(\theta), \tag{A.15}$$

where $u \in L^p$ $(1 < p < \infty)$, belongs to H^p.

ii) *Any function $f(z) \in H^p$ $(1 \leq p \leq \infty)$ has the representation (A.15) with*

$$u(\theta) = i \operatorname{Im} f(0) + \operatorname{Re} f(e^{i\theta}) \qquad \text{a. e.}$$

Remark. — The first part of the theorem is wrong both for $p = 1$ and $p = \infty$. Counter-examples can be explicitly constructed. For $p = 1$, take $u(\theta) = \sum_{n=2}^{\infty} \frac{\cos n\theta}{\operatorname{Log} n}$ (see ref. [4], p. 63); for $p = \infty$, take $u(\theta)$ as the characteristic function of any arc $\theta_0 \leq \theta \leq \theta_1$.

c) Conjugate functions.

So far, only analytic functions have been considered. Obviously, the definition (A.1) can be extended to real functions $u(z)$ harmonic in $|z| < 1$, and the quantities $M_p(r, u)$ lead to the definition of spaces h^p of harmonic functions analogous to the H^p's. Among the many results involving the spaces h^p, of particular interest for us is the remarkable theorem of M. Riesz about harmonic conjugate functions.

THEOREM A.7. — *Let $u(z)$ be a real harmonic function in $|z| < 1$, and let $v(z)$ be its harmonic conjugate, normalized so that $v(o) = 0$. If $u \in h^p$ $(1 < p < \infty)$, then $v \in h^p$. Furthermore:*

$$M_p(r, v) \leqslant \left(\frac{p}{p-1}\right)^{\frac{1}{p}} M_p(r, u), \qquad (0 \leqslant r < 1). \tag{A.16}$$

Remark. — The failure of this theorem when $p = 1$ and ∞ is the reason for the failure of the first part of theorem (A.6) under the same conditions. The counter-examples there exhibited work as well here.

d) Representation of linear functionals.

If one identifies a function $f(z) \in H^p$ with its radial limit $f(e^{i\theta})$, which according to theorem (A.1) belongs to L^p, H^p appears as a subspace of L^p. Now, according to Hahn-Banach theorem, any continuous linear functional Γ on H^p can be extended to a continuous linear functional on L^p with the same norm. Then, knowing that the dual of L^p is $L^q \left(\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p < \infty\right)$, there exists $\gamma(\theta) \in L^q$ such that $\Gamma(f) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \gamma^*(\theta) f(e^{i\theta})$ for all $f \in H^p$. When $1 < p < \infty$, only that part $\gamma^+(\theta)$ of the Fourier series of $\gamma(\theta)$ which contains the non negative frequencies contributes to the above integral. Furthermore, it can be

shown by using theorem (A.7) that $\gamma^+(\theta)$ is the radial limit of a function $g(z)$ in H^q, so that finally:

$$\Gamma(f) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} g^*(e^{i\theta}) f(e^{i\theta}). \qquad (A.17)$$

The precise statement can be formulated as follows:

THEOREM A.8. — *Any continuous linear functional Γ on H^p ($1 < p < \infty$) is representable in the form (A.17) by a unique function $g \in H^q \left(\dfrac{1}{p} + \dfrac{1}{q} = 1 \right)$. For $p = 1$, the same representation (A.17) holds with $g(e^{i\theta})$ replaced by a (non unique) function $\gamma(\theta) \in L^\infty$.*

The Hölder inequality applied to Eq. (A.17), $|\Gamma(f)| \le \| g \|_q \cdot \| f \|_p$, shows that the norm $\| \Gamma \| = \underset{\|f\|_p \le 1}{\text{Sup}} \cdot |\Gamma(f)|$ of the functional Γ is bounded by:

$$\| \Gamma \| \le \| g \|_q. \qquad (A.18)$$

Except for the case $p = 2$, Eq. (A.18) is in general a strict inequality, because Hölder inequality cannot be saturated with g and f both analytic functions, as explained in section (II.c) (Cf. discussion following Eq. (II.52)). The actual calculation of $\| \Gamma \|$ amounts to finding an extremal function $f_0(z)$ or an extremal kernel $K_0(e^{i\theta})$ associated with Γ, which by definition satisfy $\| \Gamma \| = \Gamma(f_0) = \| K_0 \|_{L^q}$. This can be a very difficult problem. However, in the case where $g(z)$ is a rational function, an algorithm has been devised which leads to the solution.

e) Algorithm for constructing extremal kernels and functions.

Assuming that $g(z)$ is a rational function (with all its poles located in $|z| > 1$), one can obviously rewrite Eq. (A.17) as:

$$\Gamma(f) = \frac{1}{2\pi i} \int_{|z|=1} dz K(z) f(z), \qquad (A.19)$$

where $K(z)$ is a rational function. The kernel $K(z)$ is thus analytic in $|z| \le 1$, except for poles at $\beta_1, \beta_2, \ldots, \beta_n$ ($|\beta_k| < 1$), each pole being repeated according to its multiplicity.

Then, the extremal functions $f_0(z)$ and the extremal kernels $K_0(z)$ have the form:

$$f_0(z) = A \prod_{i=\lambda+1}^{\kappa} \frac{z - \sigma_i}{1 - \sigma_i^* z} \prod_{i=1}^{n-1} (1 - \sigma_i^* z)^{\frac{2}{p}} \prod_{i=1}^{n} (1 - \beta_i^* z)^{-\frac{2}{p}}, \qquad (A.20)$$

$$K_0(z) = B \prod_{i=1}^{\lambda} \frac{z - \sigma_i}{1 - \sigma_i^* z} \prod_{i=1}^{n-1} (1 - \sigma_i^* z)^{\frac{2}{q}} \prod_{i=1}^{n} \frac{(1 - \beta_i^* z)^{1 - \frac{2}{q}}}{z - \beta_i}, \qquad (A.21)$$

where A and B are complex numbers, λ and κ two integers satisfying

$$0 \le \lambda \le \kappa \le n - 1, \qquad (A.22)$$

and the $\sigma_i's$ ($i = 1, \ldots, n - 1$) are complex numbers such that:

$$\begin{cases} |\sigma_i| < 1 & \text{for} & 1 \le i \le \kappa. \\ |\sigma_i| = 1 & \text{for} & \kappa + 1 \le i \le n - 1. \end{cases} \qquad (A.23)$$

In order to determine completely $K_0(z)$, one has to find B, λ, κ and σ_i ($i = 1, \ldots, n - 1$) subjected to the restrictions (A.22) and (A.23), such that the kernel $K_0(z)$ be equivalent to the given kernel $K(z)$ (that is to say, $K_0(z)$ and $K(z)$ must have the same principal part at each of the poles β_i). The solution of this algebraic problem exists and is unique for

all $1 \leq p \leq \infty$, and the function $K_0(z)$ so constructed is *the* external kernel associated with the functional (A.19).

As for the extremal functions, one has to distinguish two cases:

1) $1 < p \leq \infty$: by inserting the values of the $\sigma_i's$ calculated above into Eq. (A.20) and computing A by requiring that $\| f_0 \|_p = 1$, $\Gamma(f_0) > 0$, one obtains the (unique) normalized extremal function $f_0(z)$.

2) $p = 1$: in that case, the $\sigma_i's$ for $\lambda + 1 \leq i \leq n - 1$ are not determined by the above algorithm, since $K_0(z)$ takes the form:

$$K_0(z) = B \prod_{i=1}^{\lambda} \frac{z - \sigma_i}{1 - \sigma_i^* z} \prod_{i=1}^{n} \frac{1 - \beta_i^* z}{z - \beta_i} . \tag{A.24}$$

Then, if it turns out that $\lambda = n - 1$, the normalized extremal function is unique and given by Eq. (A.20) with the first product missing. Otherwise ($0 \leq \lambda < n - 1$), the remaining $\sigma_i's$ can be chosen arbitrarily in the disk $| \sigma_i | \leq 1$, and Eq. (A.20) characterizes the family of extremal functions.

APPENDIX B

In order to ensure that the analytic functions in our spaces $H^p(\rho)$ have boundary values in the sense of distributions, the condition (II.1) on the weight function $\rho(\theta)$ is not sufficient. One needs to restrict further the class of ρ's. A sufficient condition is:

$$\frac{1}{\rho(\theta)} \in L^\omega \quad \text{for some} \quad \omega > 0, \tag{B.1}$$

as is shown by the:

PROPOSITION. — *Under hypothesis (II.1) and (B.1), any function in $H^p(\rho)$ ($p > 1$) has a boundary value on the circle $|z| = 1$ in the sense of distributions.*

Proof. — According to proposition 1, any f in $H^p(\rho)$, can be written as $f(z) = G(z)h(z)$, where $h \in H^p$ and G is the outer function defined by Eq. (II.4). Actually $G \in H^{p\omega}$. This results from theorem (A.3) after noticing that under hypothesis (B.1) the radial limit $|G(e^{i\theta})| = \rho(\theta)^{-1/p}$ belongs to $L^{p\omega}$. Furthermore:

$$\| G \|_{p\omega} = \left\| \rho^{-\frac{1}{p}} \right\|_{p\omega} = \left\| \rho^{-1} \right\|_\omega^{\frac{1}{p}}. \tag{B.2}$$

Besides, any function $h(z)$ in H^p is bounded by:

$$| h(z) | = | \Delta_z(h) | \leqslant \| \Delta_z \|_q \| h \|_p, \tag{B.3}$$

where the linear functional Δ_z is defined as in Eq. (II.49). The calculation of $\| \Delta_z \|_q$ proceeds through the determination of the extremal kernel associated with Δ_z, and gives (by using the algorithm of appendix A.e):

$$\| \Delta_z \|_q = \frac{1}{\left(1 - |z|^2\right)^{\frac{1}{p}}}. \tag{B.4}$$

Hence:

$$| h(re^{i\theta}) | \leqslant \frac{\| h \|_p}{\left(1 - r^2\right)^{\frac{1}{p}}}, \tag{B.5}$$

and similarly for G in $H^{p\omega}$:

$$| G(re^{i\theta}) | \leqslant \frac{\| G \|_{p\omega}}{\left(1 - r^2\right)^{\frac{1}{p\omega}}}. \tag{B.6}$$

Finally, we get:

$$| f(re^{i\theta}) | \leqslant \frac{\| f \|_{p,\rho} \left\| \rho^{-1} \right\|_\omega^{\frac{1}{p}}}{\left(1 - r^2\right)^{\frac{1}{p} + \frac{1}{p\omega}}}. \tag{B.7}$$

This bound (inverse power of the distance of z to the boundary) implies, according to a well-known theorem, the existence of a boundary value of $f(z)$ in the sense of distributions.

Remarks:

i) The two hypotheses (II.1) and (B.1) are truly independent. For instance, the function $\rho(\theta) = e^{1/\theta^2}$ fulfils condition (B.1) but not (II.1); conversely $\rho(\theta) = e^{-1/\sqrt{|\theta|}}$ fulfils (II.1) but not (B.1). Furthermore, with this last function, one can show that condition (II.1) is not sufficient to ensure the validity of the above proposition.

ii) Functions ρ's of the form (II.30), (II.31) and (II.45) obviously satisfy Eq. (B.1).

iii) It is worth noticing that the property « existence of a boundary value in the sense of distribution » is conserved in any conformal mapping of the cut energy-plane onto the unit disk, provided that the boundary value in the energy-plane is a tempered distribution.

APPENDIX C

Proof of proposition 2.

i) The supremum M is reached:

We first show that the cone $\mathscr{C}$ is weakly closed in $H^p(\rho)$. Let $\{g_i\}$ be a sequence in $\mathscr{C}$ which converges weakly to $g \in H^p(\rho)$. Assume $g \notin \mathscr{C}$. This implies that there exists a subset e of the unit circle, with a non zero measure, where Im $g(e^{i\theta}) < 0$. Let $\chi(\theta)$ be the characteristic function of e, and define the linear functional Φ on $H^p(\rho)$:

$$\Phi(f) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \chi(\theta) \rho(\theta)^{\frac{1}{p}} \text{ Im } f(e^{i\theta}), \qquad f \in H^p(\rho). \tag{C.1}$$

Φ is continuous:

$$|\Phi(f)| \leq \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \chi(\theta) \right)^{\frac{1}{q}} \cdot \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) | \text{ Im} f(e^{i\theta}) |^p \right)^{\frac{1}{p}}$$

$$\leq (\text{measure of } e)^{\frac{1}{q}} \| f \|_{p,\rho} \tag{C.2}$$

Moreover $\Phi(g_i)$ is obviously non negative, since Im $g_i(e^{i\theta}) \geq 0$ a. e., and, by construction, $\Phi(g) < 0$. As $\Phi(g_i)$ converges to $\Phi(g)$ we get a contradiction. Thus $g \in \mathscr{C}$, and $\mathscr{C}$ is weakly closed.

Besides, the space H^p, and then also $H^p(\rho)$, being reflexive, the closed unit ball $\| f \|_{p,\rho} \leq 1$ is weakly compact [8]. Then its intersection with the weakly closed set $\mathscr{C}$ is weakly compact. Therefore the supremum M of the continuous functional Δ on this intersection is reached.

ii) The function $\hat{f}(z)$ where Δ reaches its maximum is unique:

Assume there are two such functions $\hat{f}_1(z)$ and $\hat{f}_2(z)$. First $\| \hat{f}_1 \|_{p,\rho} = \| \hat{f}_2 \|_{p,\rho} = 1$. Let $g = (\hat{f}_1 + \hat{f}_2)/\| \hat{f}_1 + \hat{f}_2 \|_{p,\rho}$. Then $g \in \mathscr{C}$, $\| g \|_{p,\rho} = 1$, and:

$$\Delta(g) = 2M/\| \hat{f}_1 + \hat{f}_2 \|_{p,\rho} \tag{C.3}$$

which is strictly larger than M, unless $\hat{f}_1 = \hat{f}_2$ (the Minkowski inequality

$$\| \hat{f}_1 + \hat{f}_2 \|_{p,\rho} \leq \| \hat{f}_1 \|_{p,\rho} + \| \hat{f}_2 \|_{p,\rho}$$

is saturated only if $\hat{f}_1 = \hat{f}_2$).

APPENDIX D

This appendix is devoted to the proof of a representation theorem for the continuous linear functionals on $H^p(\rho)$ ($p > 1$). We first need a Poisson-type formula in the space $H^p(\rho)$, valid in the particular case where the weight function $\rho(\theta)$ has the form (II.45):

LEMMA. — *Under hypothesis* (II.45), *any function* $f(z)$ *in* $H^p(\rho)$ *has the representation:*

$$f(z) = f(o) + \frac{2z}{1 - z^2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \sin \theta \, \text{Im} \, f(e^{i\theta}) . \qquad (D.1)$$

Conversely, given φ_0 *and* $\varphi(\theta)$ *real with* $\varphi(\theta) = -\varphi(-\theta) \in L^p(\rho)$, *the function:*

$$f(z) = \varphi_0 + \frac{2z}{1 - z^2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \sin \theta \, \varphi(\theta) \qquad (D.2)$$

belongs to $H^p(\rho)$. *Furthermore,* $f(o) = \varphi_0$ *and* $\text{Im} \, f(e^{i\theta}) = \varphi(\theta)$.

Proof. — The factorization (II.45) of $\rho(\theta)$ entails a corresponding factorization of the function $G(z)$:

$$G(z) = \frac{1}{1 - z^2} \bar{G}(z) , \qquad (D.3)$$

where $\bar{G}(z)$ is given in terms of $\bar{\rho}(\theta)$ as $G(z)$ in terms of $\rho(\theta)$ in Eq. (II.4). The lower and upper bounds b and B on $\bar{\rho}(\theta)$ are readily transferred to $|\bar{G}(z)|$:

$$B^{-\frac{1}{p}} \leqslant |\bar{G}(z)| \leqslant b^{-\frac{1}{p}}, \qquad |z| \leqslant 1 . \qquad (D.4)$$

Now, for any $f(z)$ in $H^p(\rho)$, it is easily shown that $[f(z) - f(o)]/z$ still belongs to $H^p(\rho)$. Next, because of proposition 1, the function:

$$\frac{1}{G(z)} \frac{f(z) - f(o)}{z} = \frac{1}{\bar{G}(z)} \frac{1 - z^2}{z} [f(z) - f(o)] \qquad (D.5)$$

belongs to H^p and, because of Eq. (D.4), the same is true for the function

$$k(z) \equiv (1 - z^2) z^{-1} [f(z) - f(o)] .$$

Finally, according to theorem A.6, $k(z)$ can be represented as the Poisson integral:

$$k(z) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, \text{Re} \, k(e^{i\theta}) , \qquad (D.6)$$

from which one immediately deduces Eq. (D.1).

The reciprocal is proved by simply reversing the argument. This is left to the reader.

According to this lemma, the knowledge of any function f in $H^p(\rho)$ is equivalent to the knowledge of $f(o) \in \mathbb{R}$ and $\text{Im} \, f(e^{i\theta}) \in L^p(\rho)$. In other words, the correspondence $f \to (f(o), \text{Im} \, f(e^{i\theta}))$ establishes an isomorphism between $H^p(\rho)$ and the direct sum $\mathbb{R} \oplus L^p(\rho)$. We equip the space $\mathbb{R} \oplus L^p(\rho)$ with the topology of the direct sum for Banach spaces, given by the norm:

$$|f(o)| + \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) |\text{Im} \, f(e^{i\theta})|^p \right]^{\frac{1}{p}} . \qquad (D.7)$$

It will be proved in a moment that this norm is in fact equivalent to the norm $\| f \|_{p,\rho}$

in $H^p(\rho)$. As a consequence, any continuous linear functional on one of the two spaces $H^p(\rho)$ or $\mathbb{R} + L^p(\rho)$ defines, through the above isomorphism, a continuous linear functional on the other. More precisely, one has the

PROPOSITION. — *Under hypothesis* (II.45), *any (real) continuous linear functional* Γ *on* $H^p(\rho)$ ($p > 1$) *can be written in a unique way as:*

$$\Gamma(f) = \gamma_0 f(o) + \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\gamma(\theta) \text{ Im } f(e^{i\theta}), \qquad f \in H^p(\rho), \qquad (D.8)$$

where γ_0 *and* $\gamma(\theta)$ *are real, and* $\gamma(\theta) = -\gamma(-\theta) \in L^q(\rho)$.

Conversely, any expression $\Gamma(f)$ *of the form* (D.8) *defines a continuous linear functional on* $H^p(\rho)$; *in the case* $p = 2$, *it is associated with an element* $g \in H^2(\rho)$ *such that* $\Gamma(f) = (g, f)$, *which is given by:*

$$g(z) = G(z)\left[\gamma_0 G(o) - i \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{G(e^{i\theta})}{1 - ze^{i\theta}} \rho(\theta)\gamma(\theta)\right]. \qquad (D.9)$$

Proof:
i) Equivalence of the two norms, $\| f \|_{p,\rho}$ and (D.7):
Using the expression of $f(z)$ in terms of the function $k(z)$ defined in the previous lemma, the Minkowski inequality, the decomposition (II.45) of $\rho(\theta)$, and the bound $\bar{\rho}(\theta) \leq B$, we successively get:

$$\| f \|_{p,\rho} \leq \left\| f(o) + \frac{z}{1 - z^2} k(z) \right\|_{p,\rho}$$

$$\leq \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \right)^{\frac{1}{p}} | f(o) | + B \| k \|_p$$

$$\leq \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \right)^{\frac{1}{p}} | f(o) | + B(\| \text{Re } k \|_{L^p} + \| \text{Im } k \|_{L^p}). \qquad (D.10)$$

Now, according to theorem (A.7), $\| \text{Im } k \|_{L^p} \leq \left(\dfrac{p}{p-1} \right)^{1/p} \| \text{Re } k \|_{L^p}$, so that:

$$\| f \|_{p,\rho} \leq \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \right)^{\frac{1}{p}} | f(o) | + B\left[1 + \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \right] \| \text{Re } k \|_{L^p}. \qquad (D.11)$$

Expressing k in terms of f gives $\text{Re } k(e^{i\theta}) = 2 \sin \theta \text{ Im } f(e^{i\theta})$, and:

$$\| \text{Re } k \|_{L^p} = \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | 2 \sin \theta |^p | \text{Im } f(e^{i\theta}) |^p \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{b} \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) | \text{Im } f(e^{i\theta}) |^p \right)^{\frac{1}{p}}. \qquad (D.12)$$

Furthermore, one trivially has:

$$\begin{cases} f(o) \leq G(o) \| f \|_{p,\rho}, \\ \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) | \text{Im } f(e^{i\theta}) |^p \right)^{\frac{1}{p}} \leq \| f \|_{p,\rho}. \end{cases} \qquad (D.13)$$

Finally, from Eqs. (D.11), (D.12) and (D.13), there are two positive constants C' and C'' such that:

$$C' \| f \|_{p,\rho} \leq | f(o) | + \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) | \text{Im } f(e^{i\theta}) |^p \right)^{\frac{1}{p}} \leq C'' \| f \|_{p,\rho}. \qquad (D.14)$$

ii) It follows from these inequalities that any linear functional Γ which is continuous in the topology of the norm (D.7) (resp. $\| f \|_{p,\rho}$) is also continuous in the topology of the norm $\| f \|_{p,\rho}$ (resp. the norm (D.7)). To obtain Eq. (D.8), it only remains to remark that its right hand side is the characteristic representation of the continuous linear functionals on $\mathbb{R} \oplus L^p(\rho)$. This results, firstly, from the fact that the dual of a direct sum of Banach spaces is the direct sum of the duals, secondly, from the Riesz representation theorem in $L^p(\rho)$. As for the uniqueness of the representation (D.8), it is an immediate consequence of the independence of $f(o)$ and Im $f(e^{i\theta})$ implied by the previous lemma.

iii) Case $p = 2$:

In order to determine the function $g(z) \in H^2(\rho)$ associated with the linear functional Γ, let us consider the function

$$\Delta_z(z') = \frac{G(z)^* G(z')}{1 - z^* z'}, \tag{D.15}$$

which does not belong to $H^2(\rho)$ (the elements of $H^p(\rho)$ are « *real* » analytic functions) but is a linear combination with complex coefficients of two functions in $H^2(\rho)$: $\Delta_z^R(z') + i\Delta_z^I(z')$. Now, we notice that in Eq. (D.8), we can replace Im $f(e^{i\theta})$ by $- if(e^{i\theta})$ since Re $f(e^{i\theta})$ is odd and gives no contribution. Thus:

$$\Gamma(\Delta_z) = \Gamma(\Delta_z^R) + i\Gamma(\Delta_z^I) = \gamma_0 G^*(z)G(o) - i \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta)\gamma(\theta) \frac{G^*(z)G(e^{i\theta})}{1 - z^* e^{i\theta}} . \tag{D.16}$$

On the other hand $\Gamma(\Delta_z) = (g, \Delta_z^R) + i(g, \Delta_z^I)$, and an elementary calculation gives $\Gamma(\Delta_z) = g(z)^*$. Eq. (D.9) then follows.

APPENDIX E

Proof of theorem 2.

The function $\bar{\rho}(\theta)$ of Eq. (II.42) is Lipschitz continuous of order 1/2. Let us remark that theorem 2 is not valid for any function $\bar{\rho}$ of Lipschitz order 1/2. In Eq. (II.42), we have extracted the most singular part (of order 1/2) of $\bar{\rho}$, which turns out to be soft enough not to spoil the Hilbert-Schmidt character of the kernel in Eq. (II.43).

As in appendix D, we use the factorisation $G(z) = \bar{G}(z)/(1 - z^2)$ (see Eq. (D.3)). Accordingly:

$$\Phi(\theta) = \bar{\Phi}(\theta) \pm \frac{\pi}{2} - \theta, \qquad 0 < \pm \theta < \pi, \tag{E.1}$$

where $\Phi(\theta)$ and $\bar{\Phi}(\theta)$ are respectively the phases of $G(e^{i\theta})$ and $\bar{G}(e^{i\theta})$.

i) *Lipschitz continuity of* $\bar{\Phi}(\theta)$:
It is convenient to rewrite Eq. (II.42) as:

$$\bar{\rho}(\theta) = \tilde{\rho}(\theta) e^{\lambda \cos \frac{\theta}{4} \sqrt{\cos \frac{\theta}{2}}}, \tag{E.2}$$

where λ can be adjusted in such a way that $\tilde{\rho}(\theta)$ is Lipschitz continuous of the same order μ as $\bar{\rho}(\theta)$. This entails the following factorization of $\bar{G}(z)$:

$$\bar{G}(z) = \tilde{G}(z) \exp\left(-\frac{\lambda}{2} \sqrt{\frac{1+z}{2}} \right), \tag{E.3}$$

and consequently the following decomposition of $\bar{\Phi}(\theta)$:

$$\bar{\Phi}(\theta) = \tilde{\Phi}(\theta) - \frac{\lambda}{2} \sin \frac{\theta}{4} \sqrt{\cos \frac{\theta}{2}}. \tag{E.4}$$

Here, as in Eq. (II.4), one has:

$$\tilde{G}(z) = \exp\left[-\frac{1}{2} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \text{Log } \tilde{\rho}(\alpha) \right], \tag{E.5}$$

so that $\tilde{\Phi}(\theta)$, the phase of $\tilde{G}(e^{i\theta})$, is given by:

$$\tilde{\Phi}(\theta) = -\frac{1}{2} \lim_{r \uparrow 1} \text{Im} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \frac{e^{i\alpha} + re^{i\theta}}{e^{i\alpha} - re^{i\theta}} \text{Log } \tilde{\rho}(\alpha). \tag{E.6}$$

Now, $\tilde{\rho}(\theta)$ is obviously bounded from above and below: $0 < b' \le \tilde{\rho}(\theta) \le B'$. Using the fact that the function Log is analytic on $[b', B']$, it is a trivial matter to show that Log $\tilde{\rho}(\theta)$ is Lipschitz continuous of order μ. Then, according to Plemelj-Privalov theorem about Hilbert transforms [9], the Lipschitz property of Log $\tilde{\rho}(\theta)$ survives the integral transform in Eq. (E.6). As a result, one can write:

$$|\tilde{\Phi}(\theta) - \tilde{\Phi}(\varphi)| < C_1 \left| \sin \frac{\theta - \varphi}{2} \right|^\mu, \qquad (-\pi \le \theta, \varphi \le \pi). \tag{E.7}$$

ii) *Extraction of the singular part of the kernel in* Eq. (II.37):
We first remark that the singular part of the kernel in Eq. (II.37) is nothing but the

Poisson kernel times $\mathrm{Re}\,[G(e^{i\theta})G^*(e^{i\varphi})]/2$. This leads us to rewrite Eq. (II.37) as follows:

$$
\mathrm{Im}\,\Delta^-(e^{i\theta}) = \frac{1}{2}\lim_{r\uparrow 1}\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}
$$

$$
\left\{2\,\mathrm{Re}\,\frac{G(e^{i\theta})G^*(e^{i\varphi})}{1-re^{i(\theta-\varphi)}} - \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,\mathrm{Re}\,[G(e^{i\theta})G^*(e^{i\varphi})]\right\}\cdot\rho(\varphi)\delta^-(\varphi)
$$

$$
+ \frac{1}{2}\lim_{r\uparrow 1}\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,\mathrm{Re}\,[G(e^{i\theta})G^*(e^{i\varphi})]\rho(\varphi)\delta^-(\varphi). \quad (E.8)
$$

The last integral in this equation is the Poisson transform of

$$
\frac{1}{2}\,\mathrm{Re}\,[G(e^{i\theta})G^*(e^{i\varphi})]\rho(\varphi)\delta^-(\varphi) = \frac{1}{2}\cos(\Phi(\theta)-\Phi(\varphi))\sqrt{\frac{\rho(\varphi)}{\rho(\theta)}}\,\delta^-(\varphi), \quad (E.9)
$$

which, as a function of φ, obviously belongs to L^2 ($\delta^- \in L^2(\rho)$ from theorem 1). Then the last term in Eq. (E.8), which is the radial limit of the Poisson integral of the function (E.9), exists almost everywhere and is equal to $\frac{1}{2}\delta^-(\theta)$ [10].

Hence, Eq. (E.8) becomes:

$$
2\sqrt{\rho(\theta)}\,\mathrm{Im}\,\Delta^-(e^{i\theta}) = \sqrt{\rho(\theta)}\delta^-(\theta) + \lim_{r\uparrow 1}\int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}K_r(\theta,\varphi)\sqrt{\rho(\varphi)}\delta^-(\varphi), \quad (E.10)
$$

where:

$$
K_r(\theta,\varphi) = \frac{(1+r^2)\cos[\Phi(\theta)-\Phi(\varphi)] - 2r\cos[\Phi(\theta)-\Phi(\varphi)-\theta+\varphi]}{1-2r\cos(\theta-\varphi)+r^2}. \quad (E.11)
$$

iii) *Interchange of limit and integration in* Eq. (E.10):
Expressing the kernel K_1 in terms of $\bar{\Phi}$, one gets:

$$
K_1(\theta,\varphi) = \pm\frac{\sin\left[\frac{3}{2}(\theta-\varphi)-\bar{\Phi}(\theta)+\bar{\Phi}(\varphi)\right]}{\sin\left(\frac{\theta-\varphi}{2}\right)}, \quad (\sin\theta\sin\varphi\gtrless 0). \quad (E.12)
$$

Let us first prove that $K_1(\theta,\varphi)$ is a Hilbert-Schmidt kernel, that is to say that the integral:

$$
\|K_1\|_{\mathrm{H.S.}}^2 = \iint_{-\pi}^{\pi}\frac{d\theta}{2\pi}\frac{d\varphi}{2\pi}\frac{\sin^2\left[\frac{3}{2}(\theta-\varphi)-\bar{\Phi}(\theta)+\bar{\Phi}(\varphi)\right]}{\sin^2\left(\frac{\theta-\varphi}{2}\right)} \quad (E.13)
$$

is finite. From Eqs. (E.4) and (E.7), we have:

$$
\sin^2\left[\frac{3}{2}(\theta-\varphi)-\bar{\Phi}(\theta)+\bar{\Phi}(\varphi)\right]
$$

$$
= \sin^2\left[\frac{3}{2}(\theta-\varphi)-\tilde{\Phi}(\theta)+\tilde{\Phi}(\varphi)+\frac{\lambda}{2}\left(\sin\frac{\theta}{4}\sqrt{\cos\frac{\theta}{2}}-\sin\frac{\varphi}{4}\sqrt{\cos\frac{\varphi}{2}}\right)\right]
$$

$$
\leqslant 2\sin^2\left[\frac{3}{2}(\theta-\varphi)-\tilde{\Phi}(\theta)+\tilde{\Phi}(\varphi)\right] + 2\sin^2\left[\frac{\lambda}{2}\left(\sin\frac{\theta}{4}\sqrt{\cos\frac{\theta}{2}}-\sin\frac{\varphi}{4}\sqrt{\cos\frac{\varphi}{2}}\right)\right]
$$

$$
\leqslant C_2\left|\sin\frac{\theta-\varphi}{2}\right|^{2\mu} + \frac{\lambda^2}{2}\left(\sin\frac{\theta}{4}\sqrt{\cos\frac{\theta}{2}}-\sin\frac{\varphi}{4}\sqrt{\cos\frac{\varphi}{2}}\right)^2. \quad (E.14)
$$

Thus:

$$\| K_1 \|_{\text{H.S.}}^2 \leqslant C_2 \iint_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{d\varphi}{2\pi} \left| \sin \frac{\theta - \varphi}{2} \right|^{2\mu - 2}$$

$$+ \lambda^2 \iint_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{d\varphi}{2\pi} \left(\frac{\sin \dfrac{\theta}{4} \sqrt{\cos \dfrac{\theta}{2}} - \sin \dfrac{\varphi}{4} \sqrt{\cos \dfrac{\varphi}{2}}}{\sin \dfrac{\theta - \varphi}{2}} \right)^2 . \quad (E.15)$$

The first term in the right hand side of this inequality is finite since $\mu > \dfrac{1}{2}$. As for the second term, it is not immediately obvious that it is also finite. Actually, the integrand is singular when $\theta \to \pm \pi, \varphi \to \mp \pi$, and this is the only place where a singularity occurs. A closer examination shows that this singularity is in fact integrable. Hence $\| K_1 \|_{\text{H.S.}} < \infty$.

Now, let us call $\psi(\varphi)$ the function $\sqrt{\rho(\varphi)} \delta^-(\varphi) \in L^2$, and denote by $(K_r \psi)(\theta)$ the integral in the right hand side of Eq. (E.10). Then, by Schwarz inequality:

$$| (K_r \psi)(\theta) - (K_1 \psi)(\theta) |^2 \leqslant \left(\int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} | K_r(\theta, \varphi) - K_1(\theta, \varphi) |^2 \right) . \| \psi \|_{L^2}^2 . \quad (E.16)$$

When $\theta \neq \pm \pi$, straightforward calculations give:

$$\int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} | K_r(\theta, \varphi) - K_1(\theta, \varphi) |^2 = (1-r)^4 \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \frac{\sin^2 [\bar{\Phi}(\theta) - \bar{\Phi}(\varphi) - \theta + \varphi]}{\left[(1-r)^2 + 4r \sin^2 \left(\dfrac{\theta - \varphi}{2} \right) \right]^2 \text{tg}^2 \left(\dfrac{\theta - \varphi}{2} \right)}$$

$$< C_3 (1-r)^4 \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \frac{\left| \sin \dfrac{\theta - \varphi}{2} \right|^{2\mu}}{\left[(1-r)^2 + 4r \sin^2 \left(\dfrac{\theta - \varphi}{2} \right) \right]^2 \text{tg}^2 \left(\dfrac{\theta - \varphi}{2} \right)}$$

$$< C_4 (1-r)^4 \int_0^1 dx \frac{(\text{Arc sin } x)^{2\mu}}{[(1-r)^2 + 4rx^2]^2 x^2}$$

$$< C_5 (1-r)^4 \int_0^1 dx \frac{x^{2\mu - 2}}{[(1-r)^2 + 4rx^2]^2}$$

$$< C_6 (1-r)^{2\mu - 1}. \quad (E.17)$$

Thus, since $\mu > \dfrac{1}{2}$, $(K_r \psi)(\theta)$ converges to $(K_1 \psi)(\theta)$ for all θ except $\theta = \pm \pi$. Eq. (II.42) then follows, and this, together with the finiteness of $\| K_1 \|_{\text{H.S.}}$ completes the proof.

APPENDIX F

A completely soluble case : $\rho(\theta) = 4 \sin^2 \theta$.

We propose here to solve explicitly Eqs. (II.36) and (II.43) in the case where $\bar{\rho}(\theta) \equiv 1$. This problem, which is not very far from the realistic one (see sect. (IV.a)), is interesting inasmuch as it provides us with a solution which is close to the solution with the true function $\bar{\rho}(\theta)$. In particular one can expect the support of the true function $\delta^-(\theta)$ to be similar to the support of the function $\delta^-(\theta)$ for $\bar{\rho}(\theta) \equiv 1$.

According to Eq. (D.3), $G(z) = (1 - z^2)^{-1}$, so that, from Eq. (E.12):

$$K_1(\theta, \varphi) = \pm [1 + 2 \cos (\theta - \varphi)], \qquad (\sin \theta \sin \varphi \gtrless 0). \tag{F.1}$$

Taking into account the symmetry property of the function $\omega(\theta) = \sin \theta \, \delta^-(\theta)$ (see Eq. (II.36a)), we can write the integral equation (II.43) as follows:

$$\omega(\theta) + \int_0^\pi \frac{d\varphi}{\pi} (1 + 2 \cos \theta \cos \varphi)\omega(\varphi) = 2 \sin \theta \, \text{Im} \, \Delta^-(e^{i\theta}), \qquad (0 < \theta < \pi). \tag{F.2}$$

When $\theta \in \text{Supp.} \, \delta^-$, $\text{Im} \, \Delta^-(e^{i\theta})$ coincides with $-\text{Im} \, \Delta(e^{i\theta})$ (see Eq. (II.36d)). Now, for the sake of simplicity, we choose $x = 0$, so that, from Eq. (II.14), $\Delta(z) = (1 - z^2)^{-1}$, and

$$2 \sin \theta \, \text{Im} \, \Delta(e^{i\theta}) = \cos \theta. \tag{F.3}$$

The support of the function $\delta^-(\theta)$ is unknown. Let us make the following ansatz:

$$\text{Supp } \delta^- = [\xi, \pi] \cup [-\pi, -\xi], \qquad (0 < \xi < \pi), \tag{F.4}$$

where ξ is to be determined. Then,

$$\omega(\theta) + \int_\xi^\pi \frac{d\varphi}{\pi} (1 + 2 \cos \theta \cos \varphi)\omega(\varphi) = -\cos \theta, \qquad (\xi < \theta < \pi), \tag{F.5}$$

$$\int_\xi^\pi \frac{d\varphi}{\pi} (1 + 2 \cos \theta \cos \varphi)\omega(\varphi) = 2 \sin \theta \, \text{Im} \, \Delta^-(e^{i\theta}), \qquad (0 < \theta < \xi). \tag{F.6}$$

By mere inspection of Eq. (F.5), one remarks that $\omega(\theta)$ is a first order polynomial in $\cos \theta$, say $\omega(\theta) = \omega_0 + \omega_1 \cos \theta$, the coefficients of which satisfy the equations:

$$\begin{cases} \omega_0(2\pi - \xi) - \omega_1 \sin \xi = 0, \\ -2\omega_0 \sin \xi + \omega_1(2\pi - \xi - \sin \xi \cos \xi) = -\pi. \end{cases} \tag{F.7}$$

Furthermore, Eq. (F.6) can be rewritten as:

$$-\cos \theta - (\omega_0 + \omega_1 \cos \theta) = 2 \sin \theta \, \text{Im} \, \Delta^-(e^{i\theta}), \qquad (0 < \theta < \xi). \tag{F.8}$$

Then, using Eq. (F.3), we get:

$$2 \sin \theta \, \text{Im} \, [\Delta(e^{i\theta}) + \Delta^-(e^{i\theta})] = -(\omega_0 + \omega_1 \cos \theta), \qquad (0 < \theta < \xi), \tag{F.9}$$
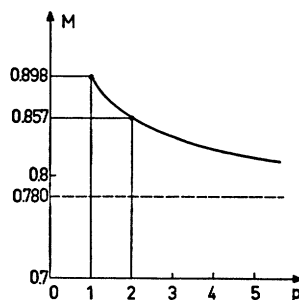
which, with inequality (II.36c), implies:

$$-(\omega_0 + \omega_1 \cos \theta) \geq 0, \qquad (0 < \theta < \xi). \tag{F.10}$$

On the other hand, inequality (II.36b) implies:

$$\omega_0 + \omega_1 \cos \theta \geq 0, \qquad (\xi < \theta < \pi). \tag{F.11}$$

FIG. 7. — The maximum M as a function of p (> 1) in the case $\rho(\theta) = |\,2\sin\theta\,|^p$. The asymptotic value of M for $p \to \infty$ is M $= 0.780$.

By comparison of the last two equations, we obtain:

$$\omega_0 + \omega_1 \cos\xi = 0. \tag{F.12}$$

Solving Eqs. (F.7) and (F.12) for ξ, ω_0 and ω_1 gives:

$$2\pi - \xi + \tan\xi = 0, \tag{F.13}$$

$$\omega_0 = -\pi\frac{\cos^2\xi}{\sin^3\xi}, \tag{F.14a}$$

$$\omega_1 = \pi\frac{\cos\xi}{\sin^3\xi}. \tag{F.14b}$$

The unique solution of Eq. (F.13) in $]0, \pi[$ is $\xi = 1.7898$ rad., and $\omega_0 = -0.15940$, $\omega_1 = -0.73378$. Now one verifies that inequalities (F.10) and (F.11) are satisfied. This means, due to the uniqueness of the solution of Eqs. (II.36) and (II.43), that our ansatz (F.4) is correct, and has provided us with *the* solution.

Finally, according to Eq. (II.38):

$$M^2 = 1 + 2\int_\xi^\pi \frac{d\theta}{\pi}(\omega_0 + \omega_1 \cos\theta)\cos\theta = -\pi\frac{\cos\xi}{\sin^3\xi}. \tag{F.15}$$

Hence M $= 0.85661$, which is to be compared with the maximum M $= 1$ obtained when discarding the positivity constraints.

This result has been used to check our computer program calculating the minimum (II.68). At the same time, with this program, we have computed M still for $x = 0$ but for other values of p (between 1 and ∞), and weight functions ρ of the form $|\,2\sin\theta\,|^p$. Note that with this family of ρ's, the maximum of $f(o)$ without positivity constraints is 1 for any p. M, as a function of p, is given in Figure 7. One observes that the improvement due to the positivity constraints increases with p.

APPENDIX G

Proof of equality (II.60).

In this appendix we show that the inequality

$$\text{M} \leqslant \text{G}(x) \underset{\substack{\Gamma \in \mathscr{X}^* \\ \Gamma(h_0) < 0}}{\text{Min}} \underset{\substack{||h||_p \leqslant 1 \\ \Gamma(h) = 0}}{\text{Max}} \Delta(h) \qquad (\text{G}.1)$$

is actually an equality. To do that we construct a supporting plane $\hat{\Gamma}(h) = 0$ of the cone $\mathscr{X}$, which contains the solution $\hat{h}$ (the existence of which is ensured by proposition 2), and separates h_0 and $\mathscr{X}$. A picture in the intersection of H^p with the two-dimensional plane spanned by h_0 and $\hat{h}$ (Fig. 8) suggests that $\hat{\Gamma}(h) = 0$ is the plane passing through the origin, and belonging to the pencil of planes defined by:

1) the tangent plane T to the unit ball at the point $\hat{h}$,
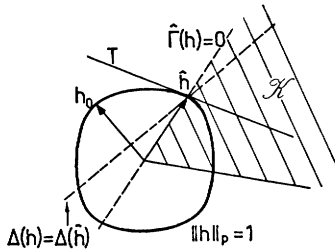2) the plane $\Delta(h) = \Delta(\hat{h})$.

FIG. 8. — Section of H^p by the plane spanned by h_0 and $\hat{h}$ (proof of equality (II.60)).

In the following such a plane will be constructed, and will be shown to enjoy the required properties.

i) The linear functional

$$\Theta(h) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\hat{h}(e^{i\theta})|^{p-2} \hat{h}^*(e^{i\theta}) h(e^{i\theta}), \qquad h \in H^p, \qquad (\text{G}.2)$$

is continuous on H^p ($p > 1$). Indeed, by Hölder inequality:

$$|\Theta(h)| \leqslant \left(\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\hat{h}(e^{i\theta})|^p \right)^{\frac{1}{q}} ||h||_p = ||h||_p . \qquad (\text{G}.3)$$

Eq. (G.3) also shows that the plane $\Theta(h) = 1$ (which obviously contains $\hat{h}$) is a supporting plane of the unit ball at the point $\hat{h}$ (that is to say, $\Theta(h) \leq 1$ for any h in the unit ball). It is actually the above mentioned tangent plane T. More precisely, let us prove that:

$$||\hat{h} + \varepsilon k||_p = 1 + \varepsilon \Theta(k) + O(\varepsilon^2 + \varepsilon^p), \qquad \forall k \in H^p \ (p > 1). \qquad (\text{G}.4)$$

As a matter of fact:

$$||\hat{h} + \varepsilon k||_p^p = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\hat{h}(e^{i\theta})|^p \left| 1 + \varepsilon \frac{k(e^{i\theta})}{\hat{h}(e^{i\theta})} \right|^p$$

$$= 1 + p\varepsilon \Theta(k) + \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\hat{h}(e^{i\theta})|^p \left[\left| 1 + \varepsilon \frac{k(e^{i\theta})}{\hat{h}(e^{i\theta})} \right|^p - \left(1 + p\varepsilon \operatorname{Re} \frac{k(e^{i\theta})}{\hat{h}(e^{i\theta})} \right) \right]. \quad (\text{G}.5)$$

But there exists a constant C_p such that, for any complex z:

$$| 1 + z |^p - (1 + p \text{ Re } z) < \begin{cases} C_p | z |^p & , & 1 < p < 2, \\ C_p(| z |^2 + | z |^p), & p \geqslant 2 \end{cases} . \tag{G.6}$$

This trivially results from:

$$| 1 + z |^p - (1 + p \text{ Re } z) = \begin{cases} | z |^p + O(| z |^{p-1}) & \text{when } | z | \to \infty, \\ \dfrac{p}{2} | z |^2 + \dfrac{p(p-2)}{2} (\text{Re } z)^2 + O(| z |^3) & \text{when } | z | \to 0. \end{cases} \tag{G.7}$$

Then, using Eq. (G.6) in Eq. (G.5), one gets, in the case $p \geq 2$:

$$\| \hat{h} + \varepsilon k \|_p^p - 1 - p\varepsilon\Theta(k) < C_p\varepsilon^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [| \hat{h}(e^{i\theta}) |^{p-2} | k(e^{i\theta}) |^2 + \varepsilon^{p-2} | k(e^{i\theta}) |^p] < O(\varepsilon^2) . \tag{G.8}$$

In the case $p < 2$, the only term which remains is of order ε^p. Eq. (G.4) then follows.

ii) The equation of any plane of the previously defined pencil can be written as $\Gamma_\lambda(h) = 0$, where:

$$\Gamma_\lambda(h) = \lambda[\Theta(h) - 1] - [\Delta(h) - \Delta(\hat{h})] . \tag{G.9}$$

The plane of this pencil which passes through the origin corresponds to $\lambda = \Delta(\hat{h})$. Thus:

$$\hat{\Gamma}(h) = \Delta(\hat{h})\Theta(h) - \Delta(h) . \tag{G.10}$$

Now, it follows from $\Theta(\hat{h}) = 1$ that:

$$\hat{\Gamma}(\hat{h}) = 0 . \tag{G.11}$$

Furthermore, since $\| h_0 \|_p = 1$, and because of the uniqueness of h_0:

$$\hat{\Gamma}(h_0) \leqslant \Delta(\hat{h}) - \Delta(h_0) < 0 . \tag{G.12}$$

Next, let us prove that for any $h \in \mathcal{K}$, $\hat{\Gamma}(h) \geq 0$. The cone $\mathcal{K}$ being convex, $\hat{h} + \varepsilon(h - \hat{h}) \in \mathcal{K}$ for $0 \leq \varepsilon \leq 1$. Moreover, using Eq. (G.4):

$$\Delta\left(\frac{\hat{h} + \varepsilon(h - \hat{h})}{\| \hat{h} + \varepsilon(h - \hat{h}) \|}\right) = \frac{\Delta(\hat{h}) + \varepsilon\Delta(h - \hat{h})}{1 + \varepsilon\Theta(h - \hat{h}) + O(\varepsilon^2 + \varepsilon^p)}$$
$$= \Delta(\hat{h}) + \varepsilon[\Delta(h - \hat{h}) - \Delta(\hat{h})\Theta(h - \hat{h})] + O(\varepsilon^2 + \varepsilon^p) = \Delta(\hat{h}) - \varepsilon\hat{\Gamma}(h) + O(\varepsilon^2 + \varepsilon^p) . \tag{G.13}$$

Since this quantity must be less than or equal to $\Delta(\hat{h})$ for $0 \leq \varepsilon \leq 1$, $\hat{\Gamma}(h)$ has to be non negative for all $h \in \mathcal{K}$, that is to say:

$$\hat{\Gamma} \in \mathcal{K}^* . \tag{G.14}$$

iii) Finally, since $\Delta(\hat{h}) > 0$, $\hat{\Gamma}(h) = 0$ implies:

$$\Delta(h) = \Delta(\hat{h})\Theta(h) \leqslant \Delta(\hat{h}), \tag{G.15}$$

for any h in the unit ball. Then:

$$\underset{\substack{\|h\|_p \leqslant 1 \\ \hat{\Gamma}(h) = 0}}{\text{Max}} \Delta(h) \leqslant \Delta(\hat{h}), \tag{G.16}$$

and:

$$\underset{\substack{\Gamma \in \mathcal{K}^* \\ \Gamma(h_0) < 0}}{\text{Min}} \underset{\substack{\|h\|_p \leqslant 1 \\ \Gamma(h) = 0}}{\text{Max}} \Delta(h) \leqslant \Delta(\hat{h}) = \text{M} . \tag{G.17}$$

Comparison of Eqs. (G.1) and (G.17) gives the desired result.

APPENDIX H

Calculation of the maximum (III.13).

In this appendix, we apply the algorithm of appendix A.e to the calculation of the extremal kernel $K_\tau(z)$ and the extremal function $h_\tau(z)$ associated with the functional $\Psi_\tau(h)$ defined by Eq. (III.5). We rewrite $\Psi_\tau(h)$ as follows:

$$\Psi_\tau(h) = \int_{|z|=1} \frac{dz}{2\pi i} \left(\frac{\cos \tau}{z-y} - \frac{\sin \tau}{z-x} \right) h(z), \qquad \left(-\frac{\pi}{2} < \tau \leqslant \frac{\pi}{2} \right). \tag{H.1}$$

Keeping to the notation of appendix A.e, $n = 2$, $\beta_1 = x$, $\beta_2 = y$. Let us call $\sigma(-1 \leq \sigma \leq 1)$ the unique zero of $K_\tau(z)h_\tau(z)$ in $|z| \leq 1$. Three cases have to be considered:

1) $\lambda = \kappa = 1$.
Then the structural formulae (A.20) and (A.21) give:

$$h_\tau(z) = A_\sigma^{(1)} \left[\frac{1-\sigma z}{(1-xz)(1-yz)} \right]^{\frac{2}{p}}, \tag{H.2}$$

$$K_\tau(z) = B_\sigma^{(1)} \frac{z-\sigma}{(z-x)(z-y)} \left[\frac{1-\sigma z}{(1-xz)(1-yz)} \right]^{1-\frac{2}{p}}. \tag{H.3}$$

Equating the residues at $z = x$ and $z = y$ of $K_\tau(z)$ with $-\sin \tau$ and $\cos \tau$ respectively, one obtains:

$$\begin{cases} -\sin \tau = B_\sigma^{(1)} \dfrac{x-\sigma}{x-y} \left[\dfrac{1-\sigma x}{(1-x^2)(1-xy)} \right]^{1-\frac{2}{p}}, \\[4mm] \cos \tau = B_\sigma^{(1)} \dfrac{y-\sigma}{y-x} \left[\dfrac{1-\sigma y}{(1-y^2)(1-xy)} \right]^{1-\frac{2}{p}}. \end{cases} \tag{H.4}$$

The elimination of $B_\sigma^{(1)}$ gives us the following relation between τ and σ:

$$\tan \tau = \frac{x-\sigma}{y-\sigma} \left[\frac{(1-\sigma x)(1-y^2)}{(1-\sigma y)(1-x^2)} \right]^{1-\frac{2}{p}}. \tag{H.5}$$

Besides, the (real) constant $A_\sigma^{(1)}$ is fixed by the condition $\| h_\tau \|_p = 1$. After an elementary calculation one gets:

$$A_\sigma^{(1)} = \pm \left[\frac{(1-x^2)(1-y^2)(1-xy)}{(1+\sigma^2)(1+xy) - 2\sigma(x+y)} \right]^{\frac{1}{p}}. \tag{H.6}$$

2) $\lambda = 0$, $\kappa = 1$.
We have similarly:

$$h_\tau(z) = A_\sigma^{(2)} \frac{z-\sigma}{1-\sigma z} \left[\frac{1-\sigma z}{(1-xz)(1-yz)} \right]^{\frac{2}{p}}, \tag{H.7}$$

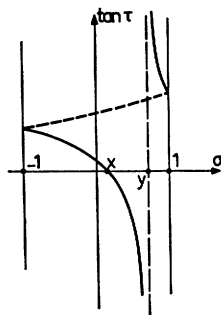$$K_\tau(z) = B_\sigma^{(2)} \frac{1-\sigma z}{(z-x)(z-y)} \left[\frac{1-\sigma z}{(1-xz)(1-yz)} \right]^{1-\frac{2}{p}}. \tag{H.8}$$

The relation between τ and σ is now:

$$\tan \tau = \frac{1-\sigma x}{1-\sigma y} \left[\frac{(1-\sigma x)(1-y^2)}{(1-\sigma y)(1-x^2)} \right]^{1-\frac{2}{p}}, \tag{H.9}$$

and it turns out that $A_\sigma^{(2)} = A_\sigma^{(1)}$.

FIG. 9. — Tan τ as a function of σ. The solid line corresponds to case 1) (Eq. (H.5)), the broken line to case 2) (Eq. (H.9)).

3) $\lambda = \kappa = 0$.

This case is the limit of the two previous ones when $\sigma \to \pm 1$, and may be included in them.

The graph of the functions (H.5) and (H.9) has been drawn on Figure 9. It shows that to a given value of $\tau \in \left] -\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$ corresponds one and only one value of σ in $[-1, 1]$. This is in accordance with the existence and uniqueness properties of the extremal kernel.

It remains to calculate $M(\tau) = |\Psi_\tau(h_\tau)| = |\cos\tau \, h_\tau(y) - \sin\tau \, h_\tau(x)|$. In both cases one finds:

$$M(\tau) = |B_\sigma^{(i)}| \left[\frac{(1 + \sigma^2)(1 + xy) - 2\sigma(x + y)}{(1 - x^2)(1 - y^2)(1 - xy)} \right]^{\frac{1}{q}}. \tag{H.10}$$

Eqs. (III.9) and (III.10) then readily follow.

APPENDIX J

Proof of Eq. (IV.10).

When $\xi \notin [-1, 1]$, the Legendre polynomials $P_l(\xi)$ behave like const. $l^{-1/2}(\xi + \sqrt{\xi^2 - 1})^l$ for large l. It follows that the series (IV.3) converges uniformly on any compact outside $[-1, 1]$, and defines a function $\Sigma_\omega(\xi)$ holomorphic in the complex ξ-plane cut along $[-1, 1]$. The subject of this appendix is to study the singularity of $\Sigma_\omega(\xi)$ at $\xi = 1$.

1) *Expansion of $\Sigma_\omega(\xi)$ for real $\xi > 1$.*

Let us first replace the series in Eq. (IV.3) by an integral. We make use of the standard summation formula:

$$\sum_{l\,\text{even}} u(l) = \frac{1}{2} u(o) + \frac{1}{2} \int_0^\infty dy\, u(y) + \frac{1}{4} \sum_{l\,\text{even}} \int_l^{l+2} dy(l + 2 - y)(y - l) u''(y), \qquad (J.1)$$

from which we deduce:

$$\left| \sum_{l\,\text{even}} u(l) - \left(\frac{1}{2} u(o) + \int_0^\infty \frac{dy}{2} u(y) \right) \right| \leqslant \frac{1}{4} \int_0^\infty dy\, |\, u''(y)\,| \,. \qquad (J.2)$$

Here, $u(y) = (2y + 1)/P_y(\xi)^\omega$. In order to evaluate the integral of $u(y)$, we use the following formula due to Watson [11]:

$$P_l (\cosh \tau) = \sqrt{\frac{\tau}{\sinh \tau}} I_0(l\tau) + c\, \frac{\tau \sinh l\tau}{l}, \qquad (J.3)$$

where $|\, c\,| < \dfrac{2}{5.6^{3/2}}$ for *all* real τ and l. The merit of this formula is to give a sharp estimate (valid both for $l \to \infty$ or $\tau \to 0$) of the error made when replacing $P_l (\cosh \tau)$ by the well known asymptotic limit $I_0(l\tau)$. Hence, one has:

$$\frac{1}{P_{\frac{x}{\tau}}(\cosh \tau)^\omega} = \left(\frac{\sinh \tau}{\tau} \right)^{\frac{\omega}{2}} \frac{1}{I_0(x)^\omega} \left[1 + c\sqrt{\frac{\tau}{\sinh \tau}}\, \frac{\tau^2 \sinh x}{x I_0(x)} \right]^{-\omega} = \frac{1}{I_0(x)^\omega} [1 + O(\tau^2)], \quad (J.4)$$

where $O(\tau^2)$ is uniform in x ($0 < x < \infty$). Then:

$$\int_0^\infty \frac{dy}{2} \frac{2y + 1}{P_y (\cosh \tau)^\omega} = \frac{1}{2\tau^2} \int_0^\infty dx\, \frac{2x + \tau}{I_0(x)^\omega} [1 + O(\tau^2)]$$

$$= \frac{1}{\tau^2} \int_0^\infty \frac{x dx}{I_0(x)^\omega} + \frac{1}{2\tau} \int_0^\infty \frac{dx}{I_0(x)^\omega} + O(1). \quad (J.5)$$

On the other hand, the right-hand side of Eq. (J.2), which is nothing but the total variation of the function $u'(y)$ between 0 and ∞ (within a factor $1/4$), turns out to be uniformly bounded in τ ($-\infty < \tau < \infty$). Actually, using again the asymptotic formula (J.3), one can show that:

i) the function $u'(y) = \dfrac{d}{dy} [(2y + 1)/P_y (\cosh \tau)^\omega]$ has only one extremum in $[0, \infty[$, say at $y = y_0$, so that the total variation of $u'(y)$ is $|\, u'(o) - u'(y_0)\,| + |\, u'(y_0)\,|$,

ii) $|\, u'(y_0)\,|$ and $|\, u'(o)\,|$ are bounded functions of τ in the neighbourhood of $\tau = 0$.

Hence, coming back to the variable $\xi = \cosh \tau$, one gets from Eqs. (J.2) and (J.5):

$$\Sigma_\omega(\xi) = \frac{1}{2(\xi - 1)} \int_0^\infty \frac{x \, dx}{I_0(x)^\omega} + \frac{1}{2\sqrt{2(\xi - 1)}} \int_0^\infty \frac{dx}{I_0(x)^\omega} + R_\omega(\xi), \qquad (J.6)$$

where $R_\omega(\xi)$ is a bounded function of ξ in $]1, \infty[$.

It remains to establish the Lipschitz continuity of the function $(\xi - 1)R_1(\xi)$. To this end, we first extend Eq. (J.6) to complex ξ.

2) *Extension to complex ξ for $\omega = 1$.*

The point at infinity is a regular point for $\Sigma_1(\xi)$, and the coefficients of its power expansion in $1/\xi$ are all positive. Indeed, the same property holds for $1/P_l(\xi)$ itself:

$$\frac{1}{P_l(\xi)} = \text{const.} \prod_{i=1}^{l/2} \frac{1}{\xi^2 - \xi_{li}^2} \qquad (l \text{ even})$$

$$= \frac{1}{\xi^l} \sum_{p=0}^\infty \frac{c_{lp}}{\xi^{2p}}, \qquad c_{lp} > 0. \qquad (J.7)$$

As a result, on a circle $|\xi| = \xi_0 > 1$, $|\Sigma_1(\xi)|$ reaches its maximum at $\xi = \xi_0$. When $|\text{Arg}(\xi - 1)| < \frac{\pi}{2}$, $|\xi - 1| = O(\xi_0 - 1)$ and:

$$|\Sigma_1(\xi)| \leqslant \Sigma_1(\xi_0)$$

$$= O\left(\frac{1}{\xi_0 - 1}\right) \qquad \text{(from Eq. (J.6))}$$

$$= O\left(\frac{1}{|\xi - 1|}\right). \qquad (J.8)$$

Obviously, $R_1(\xi)$ enjoys the same property:

$$R_1(\xi) = O\left(\frac{1}{|\xi - 1|}\right), \qquad |\text{Arg}(\xi - 1)| < \frac{\pi}{2}. \qquad (J.9)$$

This, together with the boundedness of $R_1(\xi)$ for ξ real > 1, allows us to prove that for any φ_0, $0 < \varphi_0 < \frac{\pi}{2}$:

$$R_1(1 + \rho e^{i\varphi}) = O\left(\rho^{-\frac{|\varphi|}{\varphi_0}}\right), \qquad |\varphi| \leqslant \varphi_0. \qquad (J.10)$$

This result is an immediate consequence of the Phragmén-Lindelöf theorem [12] applied to the function $r(\xi) \equiv R_1(\xi)/\exp\left[\frac{i}{2\varphi_0} \text{Log}^2(\xi - 1)\right]$. As a matter of fact, $r(1 + \rho e^{i\varphi})$ is bounded on the rays $\varphi = 0$ and $\varphi = \varphi_0$, and is holomorphic and bounded by $\rho^{-1 + \varphi/\varphi_0}$ in the angular region $0 < \varphi < \varphi_0$. Thus $r(\xi)$ is bounded in the neighbourhood of $\xi = 1$ for $0 \leqslant \text{Arg}(\xi - 1) \leqslant \varphi_0$. Eq. (J.10) then follows.

3) *Lipschitz continuity of $(\xi - 1)R_1(\xi)$.*

Let us apply the Cauchy formula to the function $T(\xi) \equiv (\xi - 1)R_1(\xi)$ in the sector S: $\rho_1 \leq |\xi - 1| \leq \rho_2$, $|\text{Arg}(\xi - 1)| \leq \varphi_1$ $(\varphi_1 < \varphi_0)$. $T(\xi)$ is holomorphic on S. Furthermore, from Eq. (J.10):

$$|T(\xi)| = O\left(|\xi - 1|^{1 - \frac{\varphi_1}{\varphi_0}}\right), \qquad \text{Arg}(\xi - 1) = \pm \varphi_1 \qquad (J.11)$$

Then, for $\xi \in S$:

$$T'(\xi) = \frac{1}{2i\pi} \int_{\partial S} d\xi' \frac{T(\xi')}{(\xi' - \xi)^2}. \qquad (J.12)$$

In this formula, one can safely let $\rho_1 \rightarrow 0$, so that:

$$T'(\xi) = \frac{1}{2i\pi}\left[-\int_0^{\rho_2} d\rho e^{i\varphi_1} \frac{T(1 + \rho e^{i\varphi_1})}{(1 + \rho e^{i\varphi_1} - \xi)^2} + \int_0^{\rho_2} d\rho e^{-i\varphi_1} \frac{T(1 + \rho e^{-i\varphi_1})}{(1 + \rho e^{-i\varphi_1} - \xi)^2}\right] + \bar{T}(\xi), \quad (J.13)$$

where $\bar{T}(\xi)$, the contribution of the arc of radius ρ_2, remains bounded when $\xi \rightarrow 1$ in S. When ξ is real, Eq. (J.13) together with the bound (J.11) give:

$$|T'(\xi)| < \text{const.} \int_0^{\rho_2} d\rho \frac{\rho^{1 - \frac{\varphi_1}{\varphi_0}}}{\rho^2 - 2\rho \cos \varphi_1 + (\xi - 1)^2} + |\bar{T}(\xi)|$$

$$< \text{const.} \int_0^\infty d\rho \frac{\rho^{1 - \frac{\varphi_1}{\varphi_0}}}{\rho^2 - 2\rho \cos \varphi_1 + (\xi - 1)^2} + |\bar{T}(\xi)|$$

$$= O((\xi - 1)^{-\frac{\varphi_1}{\varphi_0}}). \qquad (J.14)$$

Finally, for any ξ_1 and ξ_2, $\xi_2 > \xi_1 > 1$:

$$|T(\xi_2) - T(\xi_1)| \leqslant \int_{\xi_1}^{\xi_2} d\xi |T'(\xi)|$$

$$\leqslant \text{const.}\left((\xi_2 - 1)^{1 - \frac{\varphi_1}{\varphi_0}} - (\xi_1 - 1)^{1 - \frac{\varphi_1}{\varphi_0}}\right)$$

$$\leqslant \text{const.}(\xi_2 - \xi_1)^{1 - \frac{\varphi_1}{\varphi_0}}. \qquad (J.15)$$

Since φ_1 can be chosen arbitrarily small, we conclude that the function $(\xi - 1)R_1(\xi)$ is Lipschitz continuous of order μ on $]1, \infty[$ for all μ, $0 < \mu < 1$.

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