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The free Euclidean massive vector field in the Stückelberg gauge

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ABSTRACT. — One defines a free Euclidean massive vector field in close analogy with the Stückelberg formalism. The propagator has the same high momentum behaviour as in the scalar case, and no indefinite metric occurs. One determines the most general injection of the space of Minkowski test functions into the Euclidean one that preserves the transversality condition and has the correct relations with space time translations. This injection is non local in time.

1. INTRODUCTION

The Euclidean formulation of quantum field theory was successfully used in the last years to obtain important results especially for the $P(\Phi)_2$ and Φ_3^4 theories [*1*]. In this context the probabilistic approach developed by E. Nelson led to the notion of Euclidean Markov field and to establish a Feynman-Kac formula (for a comprehensive treatment and references, see [*2*]) which gives rigorous meaning to the formal functional integration techniques.

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It is rather natural to try to extend these probabilistic methods to models involving spin one particles, in the light of the importance that functional integration always had in the quantization and renormalization of these theories.

A first step in this direction was taken by L. Gross [3] who discussed the free massive and massless vector fields. In the present paper we propose a slightly different approach to the case of the free massive vector field.

The construction of the covariant Euclidean fields and their relation to the Minkowski fields involve two steps. The first one is the choice of the relevant Euclidean field or equivalently of the scalar product in the space of Euclidean test functions, the second one consists in defining an injection of the space of Minkowski fixed time test functions into the corresponding Euclidean space. There is some arbitrariness in each of these steps connected respectively with the choice of the gauge and with the difference in the number of degrees of freedom between the Minkowski and the Euclidean world. In order to reduce this arbitrariness we shall take as a guideline the usual treatment of theories involving vector mesons coupled to conserved currents ([4], chap. 7). Such theories can be formulated in essentially two ways. In the Proca formalism the basic field is divergenceless as an operator but the free propagator does not tend to zero at high momentum, so that the theory is non renormalizable in four dimensions. In the Stückelberg formalism this defect is remedied by the introduction of an extra field (ghost field) which comes out proportional to the divergence of the vector field and turns out to be a free field with mass depending on an arbitrary parameter. The free propagator has now the same high momentum behaviour as in the scalar theory, at the price however of introducing an indefinite metric. In the Hilbert space of physical states the ghost field has average value zero and the metric is positive definite. Practical computations and renormalizations are most easily performed in the Stückelberg gauge [5].

In our version of the Euclidean theory the free Euclidean vector field will be chosen as the natural analogue of the Stückelberg field. That this is at all possible follows from the fact that the scalar product associated with the free propagator of the Stückelberg theory becomes positive definite in the Euclidean world. The Euclidean field so defined is easily seen to be Markovian. Furthermore the injection of the Minkowski test function space at fixed time in the Euclidean test function space will be chosen to have range in the subspace of the (Euclidean) divergenceless functions. This will insure that the Euclidean scalar product in the range of the injection does not depend on the mass of the ghost field.

It then turns out that these two requirements essentially determine the injection up to some unitary transformations. It is important to remark that no choice of the latter can make the injection local in time. However a free Feynman-Kac formula can be easily established. It depends only on

Wick's theorem and on the relation between the injection and time translation. It is not directly related to the Markov property of the Euclidean field.

2. THE MINKOWSKI FIELD AND THE EUCLIDEAN FIELD

We first briefly describe the basic objects of the Proca theory of the free vector field with mass $m > 0$ in $s + 1$ dimensional spacetime.

The one particle Hilbert space is constructed as follows. One considers first the space of complex vector functions $h_\mu(k)$ defined on the mass hyperboloid $k^2 = m^2$, $k^0 > 0$, with Lorentz invariant inner product

$$\langle h, h \rangle = \frac{1}{(2\pi)^s} \int \frac{d^s k}{2\omega} \overline{h_\mu(k)} \left(- g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) h_\nu(k) \tag{2.1}$$

Here $k^0 = \omega(\vec{k}) \equiv (\vec{k}^2 + m^2)^{\frac{1}{2}}$ and $g^{\mu\nu} = (1, -1, \dots -1)$. This product is not positive definite. One then takes as the one particle Hilbert space the quotient of the space defined above by the subspace of the $h_\mu(k)$ of the form $\lambda(k)k_\mu$. Each vector in the one particle space (*i. e.* each equivalence class modulo k_μ), has a unique transverse representative, *i. e.* a representative for which $k_\mu h^\mu(k) = 0$, and, for a given choice of the time axis, a unique horizontal representative, *i. e.* a representative for which $h^0(k) = 0$.

The time zero vector field has three independent components for which we can and shall take its space components ([6], chap. 3). The relevant test functions at time zero are real vector functions $\widehat{h}(\vec{x})$ of the space variable $\vec{x}$. They are horizontal ($h^0 = 0$) since we define the field by its space components. In momentum space they satisfy the reality condition

$$h(\vec{k}) = \overline{h(-\vec{k})}. \tag{2.2}$$

The two point function of the field is then

$$\langle \Omega, \mathrm{V}(h)\mathrm{V}(h)\Omega \rangle = \frac{1}{(2\pi)^s} \int \frac{d^s k}{2\omega} \overline{h_i(\vec{k})} \left(\delta_{il} + \frac{k_i k_l}{m^2} \right) h_l(\vec{k}) \equiv \langle h, h \rangle_{\mathrm{M}} \tag{2.3}$$

where Ω is the free vacuum. This scalar product makes the space of test functions into a Hilbert space M, the complexification of which yields the one particle space. Here $h(\vec{k})$ is identified with the horizontal representative of the corresponding equivalence class, taken at the point $k = (\omega, \vec{k})$.

We now define our version of the free Euclidean vector field. Let E be the Hilbert space of the $s + 1$ vector valued function of the $s + 1$ Euclidean vector k satisfying the reality condition

$$f(k) = \overline{f(-k)} \tag{2.4}$$

with scalar product

$$\langle f, f \rangle_{\mathrm{E}} = \frac{1}{(2\pi)^{s+1}} \int d^{s+1}k \sum_{\mu,\nu=0}^{s} \overline{f_\mu(k)} \mathrm{S}_{\mu\nu}(k) f_\nu(k). \tag{2.5}$$

Here

$$\mathrm{S}_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2 + m^2} + \frac{k_\mu k_\nu}{m^2}\left(\frac{1}{k^2 + m^2} - \frac{1}{k^2 + \sigma m^2}\right). \tag{2.6}$$

$k^2 = \sum_{\mu=0}^{s} k_\mu^2$ and σ is an arbitrary positive number. The quantity σm^2 is the mass squared of the ghost field ([4], chap. 7). The scalar product (2.5) is positive definite for any positive σ. The operator $\mathrm{S}_{\mu\nu}(k)$ is easily inverted:

$$(\mathrm{S}^{-1})_{\mu\nu} = (k^2 + m^2)\delta_{\mu\nu} + \left(\frac{1}{\sigma} - 1\right) k_\mu k_\nu. \tag{2.7}$$

In particular S^{-1} is a differential operator in configuration space. The Euclidean field $\mathrm{A}(f)$ is now defined as the Gaussian process over E with mean value zero and covariance given by the scalar product (2.5) ([2], chap. 1). Since S^{-1} is a differential operator the field $\mathrm{A}(f)$ is Markovian, the proof being the same as in the scalar case ([2], chap. 3).

3. RELATION BETWEEN THE MINKOWSKI AND THE EUCLIDEAN FIELDS

In the whole section the greek indices run from 0 to s, the latin indices from 1 to s; for any vector l in the Euclidean space $\vec{l}$ will denote its space part and

$$l^2 \equiv \sum_{\mu=0}^{s} l_\mu^2, \qquad \vec{l}^{\,2} \equiv \sum_{\mu=1}^{s} l_\mu^2.$$

We now look for an injection j of M into E with the following properties.

(1) j is linear,

(2) for any $h \in \mathrm{M}$, $f = jh$ satisfies $k_\mu f_\mu(k) = 0$,

(3) for any $\vec{x} \in \mathrm{R}^s$, $t \in \mathrm{R}$ and for any $h \in \mathrm{M}$, $f = jh$ satisfies the equality

$$\langle f, e^{i(\vec{k}\cdot\vec{x} + k^0 t)} f \rangle_{\mathrm{E}} = \langle h, e^{i\vec{k}\cdot\vec{x} - \omega|t|} h \rangle_{\mathrm{M}}.$$

Condition (3) expresses the fact that j is isometric, commutes with space translations, and intertwines the Euclidean time translations with the semi-group generated by the free hamiltonian.

We then prove

PROPOSITION 1. — The most general injection j satisfying conditions (1), (2), (3) is given by:

$$\vec{jh} = Q\beta P\vec{h}$$

$$(jh)_0 = \frac{i\vec{k}}{\sqrt{\vec{k}^2}} \cdot \beta P\vec{h} \equiv \frac{i}{\sqrt{\vec{k}^2}} \sum_{l=1}^{s} k_l(\beta P\vec{h})_l \tag{3.1}$$

where P, Q and β are k-dependent $s \times s$ matrices,

$$P = \mathbb{1} - \frac{\vec{k} \otimes \vec{k}}{\vec{k}^2} + \frac{\omega}{m} \frac{\vec{k} \otimes \vec{k}}{\vec{k}^2}, \tag{3.2}$$

$$Q = \mathbb{1} - \frac{\vec{k} \otimes \vec{k}}{\vec{k}^2} - i\frac{k_0}{\sqrt{\vec{k}^2}} \frac{\vec{k} \otimes \vec{k}}{\vec{k}^2}, \tag{3.3}$$

and β is unitary, satisfies the reality condition

$$\beta(k) = \overline{\beta(-k)} \tag{3.4}$$

but is otherwise arbitrary.

Proof. — Let $h \in M$ and $f \equiv jh$. Then conditions (2) and (3) and the identity

$$\int \frac{dk_0}{k_0^2 + \omega^2} e^{ik_0 t} = \frac{\pi}{\omega} e^{-|t|\omega} \tag{3.5}$$

yield

$$\sum_{\mu=0}^{s} \overline{f_\mu(k)} f_\mu(k) = \sum_{i,l=1}^{s} \overline{h_i(\vec{k})}\left(\delta_{il} + \frac{k_i k_l}{m^2}\right) h_l(\vec{k}). \tag{3.6}$$

Eq. (3.6) and condition (1) imply that f can be written as

$$f_\mu(k) = \sum_{i=1}^{3} \alpha_{\mu i}(k) h_i(\vec{k}) \tag{3.7}$$

and that

$$\sum_{\mu=0}^{s} \overline{\alpha_{\mu i}(k)} \alpha_{\mu l}(k) = \delta_{il} + \frac{k_i k_l}{m^2}. \tag{3.8}$$

Then condition (2) becomes

$$\sum_{\mu=0}^{s} k_\mu \alpha_{\mu i}(k) = 0. \tag{3.9}$$

Using eq. (3.9) we can eliminate the α_{0i} from eq. (3.8) and obtain the following equation for the $s \times s$ matrix $\alpha = (\alpha_{il}(k))$:

$$\alpha(k)^+\left(\mathbb{1} + \frac{\vec{k} \otimes \vec{k}}{k_0^2}\right)\alpha(k) = \mathbb{1} + \frac{\vec{k} \otimes \vec{k}}{m^2}. \tag{3.10}$$

Let $\alpha = Q\beta P$ with P and Q defined respectively by Eq. (3.2) and Eq. (3.3). Since

$$P^2 = \mathbb{1} + \frac{\vec{k} \otimes \vec{k}}{m^2} \qquad \text{and} \qquad Q^+Q = \mathbb{1} + \frac{\vec{k} \otimes \vec{k}}{k_0^2}$$

Eq. (3.10) means that $\beta(k)$ is a unitary matrix. The reality condition (see Eqs (2.2) and (2.4)) implies Eq. (3.4). Collecting now all the information, we obtain for j the form described in Proposition 1.

Conversely such a j obviously satisfies conditions (1), (2), and (3). Q. E. D.

Because of the $\sqrt{k^2}$ denominator in Eqs (3.1) and (3.3) the injection is not local in time. One can easily verify that no choice of β can make it local. Obviously Proposition 1 and the last remark do not depend on the coefficient function of $k_\mu k_\nu$ in the propagator $S_{\mu\nu}$ (see Eq. (2.6)) because of condition (2).

The most natural β seems to be the identity matrix.

We can now establish a Feynman-Kac formula for the free field. This is nothing but the second quantized form of the equality

$$j^+ e^{ik_0 t} j = e^{-|t|\omega} \tag{3.11}$$

which is part of condition (3). We will need the following definition

$$j_t = e^{ik_0 t} j. \tag{3.12}$$

PROPOSITION 2. — Let $h_i \in M$, $1 \leq i \leq n$, let $t_1 \leq t_2 \leq \ldots \leq t_n$, let H_0 be the free hamiltonian in Fock space ; then

$$\langle \Omega, V(h_1) e^{-(t_2 - t_1)H_0} \ldots e^{-(t_n - t_{n-1})H_0} V(h_n) \Omega \rangle = \left\langle \prod_{i=1}^{n} A(j_{t_i} h_i) \right\rangle. \tag{3.13}$$

The average on the RHS of Eq. (3.13) is taken with the gaussian measure.

Proof. — The proof relies on the following lemma.

LEMMA. — Let $h_i \in M (1 \leq i \leq n)$, let $e_i (1 \leq i \leq n)$ be bounded operators in M, let $\Gamma(e_i)$ be their exponential in the sense of the symmetric tensor product ([2], chap. 1), then the Wick expansion of $\Gamma(e_1)V(h_1)\Gamma(e_2) \ldots \Gamma(e_n) V(h_n)$ is obtained from that of $V(h_1) \ldots V(h_n)$ by the following replacements:

1) replace $< h_i, h_l >_M$, $i < l$ by $< h_i, e_{i+1} \ldots e_l h_l >_M$,
2) in each normal product replace $V(h_i)$ by $V(e_1 e_2 \ldots e_i h_i)$.

Proof of the lemma. — By induction.

Since $e^{-tH_0} = \Gamma(e^{-t\omega})$, we can compute the LHS of Eq. (3.13) using the lemma with $e_i = e^{-(t_i - t_{i-1})\omega}$. On the other hand from Eqs. (3.11) and (3.12) it follows that

$$j_s^+ j_t = e^{-|t-s|\omega}. \tag{3.14}$$

The proposition is now a consequence of the equality

$$j_{t_i}^{+} j_{t_l} = e^{-(t_l - t_i)\omega} = e_{i+1} \dots e_l \qquad \text{for} \qquad i < l$$

and of Wick's theorem.

Q. E. D.

4. CONCLUDING REMARKS

In the Stückelberg gauge the physical states are supposed to be obtained by applying field operators with divergenceless test functions to the vacuum. This led us to impose the condition (2) on the injection j. Actually one can easily check that the subspace $\sum_{\mu=0}^{s} k_\mu f_\mu = 0$ of E is the closed linear span of the ranges of the j_t for all t. This means that the appropriate space to describe the physics of the free fields is the Euclidean Fock space constructed over the one particle subspace with $\sum_{\mu=0}^{s} k_\mu f_\mu = 0$.

When turning to interacting theories, the first natural attempt would be to couple the vector field to a conserved current. On a formal level one would expect the average value of $\partial_\mu A_\mu$ to vanish on the physical states. However the need of regularizing the theory by ultraviolet and space cutoffs usually destroys this property. Then it is necessary to have the whole Euclidean space at one's disposal and it is very useful, at least in perturbation theory, that the free propagator should have a good behaviour at high momentum as given by Eq. (2.6). Of course one expects that the physical quantities will not depend on the mass of the ghost after the removal of the cutoffs.

A consequence of our choice is that the injection is not local in time. This is to be contrasted with the approach of L. Gross [3] who takes an injection that is local in time, in close analogy with the scalar case. As a consequence he cannot impose the condition $k.f = 0$ (see the remarks after Proposition 1). Then the reduction from four to three degrees of freedom depends critically on the choice of a Euclidean propagator that does not tend to zero at high momentum. For instance this reduction does not occur naturally with the propagator (2.6).

It will be interesting to test the preceding ideas on interacting theories.

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