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Section A :
Physique théorique.

Covariant decompositions of symmetric tensors in the theory of gravitation

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ABSTRACT. — Covariant orthogonal decompositions of symmetric tensors have proven to be of great interest in the theory of gravitation and in characterizing spaces of Riemannian metrics. The known transverse decomposition « t » and a transverse-traceless decomposition « TT » introduced recently are described and compared. The consistency and compatibility of these two procedures are demonstrated by showing that if T^{ab} is an arbitrary symmetric tensor, then $T^{ab}_{TT} = (T^{ab}_t)_{TT} = (T^{ab}_{TT})_t$. The relationships of the various remaining « longitudinal » and trace parts of T^{ab} are exhibited. We find the result that every transverse tensor can be uniquely and orthogonally decomposed into a sum of a transverse-traceless part and another part that is transverse but has in general a non-vanishing trace. Physical interpretation of the relation between transverse and transverse-traceless tensors is provided by the canonical momentum of a gravitational field. Geometrical interpretation follows from considering the structure of the space of conformal metrics on closed manifolds.

1. INTRODUCTION

Gravitational fields can be characterized in terms of symmetric tensors on spacelike three-dimensional manifolds. It is therefore important to be able to split orthogonally these tensors into their irreducible (« spin ») parts. Gravitational degrees of freedom (spin-two) are described by the transverse-traceless part while gauge-like lower spin pieces constitute the remaining parts. The decomposition can be carried out non-covariantly in terms of an auxiliary fixed flat « background » geometry and coordinate conditions with operations modeled on their flat-space counterparts in Cartesian coördinates [*1*]. However, this procedure is coordinate-dependent and does not clearly exhibit the role of curvature of the manifold. Because they are coordinate-free, orthogonal, and explicitly account for curvature, covariant decompositions of the kind treated in this paper are preferable.

The subject of covariant decompositions has been treated by a number of authors [*2*] [*3*] [*4*]. A covariant orthogonal decomposition into a transverse part and a remainder is well known [*2*] [*3*]. This result has a clear geometrical interpretation and a definite connection to the theory of gravitation. This decomposition is reviewed in the next section. However, a finer splitting is more pertinent to gravitation, i. e., finding the simultaneously transverse *and* traceless part of a given tensor. Such a procedure has been introduced recently and was shown to be of great interest in the initial-value problem of general relativity [*4*]. The aspects of this work needed here are reviewed and elaborated in Section 3. The previous study [*4*] did not deal in detail with the important question of the relation between transverse and transverse-traceless tensors. The question of the connection between these two decompositions arises naturally and proves to be of considerable interest.

In this paper I explicitly exhibit the interrelation of the two decompositions. It is shown that if T^{ab}_{TT} is the transverse-traceless part of T^{ab} and if T^{ab}_{t} is the transverse part, each obtained respectively from the t—and TT—decompositions, then $T^{ab}_{TT} = (T^{ab}_{t})_{TT} = (T^{ab}_{TT})_{t}$. The t—and TT—operations therefore can be represented in a « commutative diagram ». Explicit results relating the various longitudinal and trace parts are also found. The decompositions hinge on two different linear, strongly elliptic, self-adjoint, second-order vector operators, whose properties are examined. A theorem relating « harmonic » functions of these operators is proved. These results are described in Section 4.

The above results are used in Section 5 to establish that every transverse tensor may itself be uniquely and orthogonally decomposed into a sum of two transverse tensors, one with a vanishing trace and the other a non-vanishing trace. This result is interpreted physically in terms of the canonical gravitational field momentum. The transverse-traceless part is a free dynamical variable, while the other (« vector ») part defines the total canonical momentum, the generator of spacelike translations.

The geometrical interpretation relates the t— decomposition to the full coordinate-free Riemannian geometry of space, an interpretation that is well known [*3*]. I have previously pointed out that the TT— decomposition is related to the conformally invariant Riemannian geometry of the manifold [4]. This interpretation is elaborated to the extent necessary to show that the space of conformal Riemannian geometries on closed manifolds ([1]) is « stratified » by the presence of conformal Killing vectors, in a manner analogous to the previously established stratification of the space of Riemannian geometries on closed manifolds by (ordinary) Killing vectors [5].

2. TRANSVERSE DECOMPOSITION

Let T^{ab} be an arbitrary symmetric tensor defined on a smooth Riemannian three-manifold M with metric g_{ab}. The transverse decomposition of T^{ab} is defined by [2] [3]

$$T^{ab} = T_t^{ab} + (KX)^{ab} \tag{2.1}$$

where $(KX)^{ab} \equiv \nabla^a X^b + \nabla^b X^a$ is the « Killing form » of X^a. Note that $(KX)_{ab} = \mathscr{L}_X g_{ab}$ is the Lie derivative of g_{ab} along X^a. The transverse part T_t^{ab} of T^{ab} is defined by $T^{ab} - (KX)^{ab}$, where

$$\nabla_b (KX)^{ab} = \nabla_b T^{ab} \tag{2.2}$$

Let us now investigate the linear second-order vector operator

$$\nabla_b (KX)^{ab} \equiv (\Delta_K X)^a = \Delta X^a + \nabla^a(\nabla_b X^b) + R_b^a X^b \tag{2.3}$$

which is the key element in this procedure (Note that the conventions being used are $\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c = X_d R^d_{\cdot cba}$ and $R_{ab} = R^d_{\cdot adb}$). In (2.3), Δ stands for the ordinary Laplacian: $\Delta = \nabla^2 = g^{ab}\nabla_a \nabla_b$. The ellipticity of an operator depends only upon its « principal part », i. e., the highest derivatives acting on the unknown quantities which it contains. To construct its « symbol », each derivative operator ∇_a occuring in its principal part is replaced by an arbitrary vector V_a. The symbol of the operator defines a linear transformation σ_v. The operator is said to be elliptic if σ_v is an isomorphism [3]. In the present case,

$$[\sigma_v(\Delta_K)]_b^a = V_b V^a + \delta_b^a V_c V^c. \tag{2.4}$$

Here, σ_v operates on vectors X^b and defines a vector-space isomorphism when the determinant of σ_v is non-vanishing for all non-vanishing V^a. That $\det \sigma_v \neq 0$ here is readily verified, for example, by choosing $V^a = (1, 0, 0)$ in a local Cartesian frame. The operator is said to be « strongly elliptic » if all the eigenvalues of σ_v are non-vanishing and have the same sign. Again, this is easily checked and Δ_K is strongly elliptic.

In contrast to ellipticity, the Hermiticity or self-adjointness of an operator depends upon its full structure. In the case of Δ_K, this property requires

$$\int \sqrt{g} V_a \nabla_b (KW)^{ab} d^3x = \int \sqrt{g} W_a \nabla_b (KV)^{ab} d^3x, \tag{2.5}$$

([1]) The term « closed manifold » refers to a compact manifold without boundary.

which follows upon integration by parts and Gauss's theorem. On a closed manifold there are no boundary terms. On an asymptotically flat topologically Euclidean space, the vectors vanish asymptotically at least as fast as $0(r^{-1})$. This requires that we assume the given tensor T^{ab} vanishes « fast enough » asymptotically, e. g., such that $\nabla_b T^{ab} \sim 0(r^{-(3+\varepsilon)})$, $\varepsilon > 0$ (2).

To discuss the existence and uniqueness of solutions X^a of (2.2), we must first find the regular « harmonic » functions of Δ_K. These are the smooth vectors which get mapped to zero everywhere, i. e., the elements of the kernel $\Delta_K^{-1}(0)$ of Δ_K. In the present case, these are at most just the isometries of g_{ab}, if any are admitted. Thus, suppose $(\Delta_K Z)^a = 0$. Then, using Gauss's theorem,

$$0 = \int \sqrt{g} Z_a \nabla_b (KZ)^{ab} d^3x = -\frac{1}{2} \int \sqrt{g} (KZ)_{ab} (KZ)^{ab} d^3x. \qquad (2.6)$$

Because $(KZ)_{ab}(KZ)^{ab} \geqslant 0$, (2.6) can hold if and only if $(KZ)_{ab} = 0$ everywhere, which are just the familiar Killing equations satisfied by isometries of g_{ab} (3). In general, there will be none. Suppose, however, that such a $Z^a \neq 0$ does exist. On an asymptotically flat manifold Z^a must approach asymptotically one of the familiar translation or rotation Killing vectors of Euclidean space. However, these do not vanish at infinity and must therefore be ruled out here. The operator Δ_K has no regular harmonic functions *vanishing at infinity* on a smooth, topologically Euclidean, asymptotically flat space. It follows that for a given « source » $\nabla_b T^{ab}$ in (2.2), the solution of (2.2) must be unique. However, for linear elliptic equations of the present kind, uniqueness implies existence as well [7] (4).

For a *closed* manifold with S linearly independent Killing vectors, the kernel of Δ_K is not empty and has dimension S. Nevertheless, the source $\nabla_b T^{ab}$ will always be orthogonal to a Killing vector, as seen from

$$\int \sqrt{g} Z_a \nabla_b T^{ab} d^3x = -\frac{1}{2} \int \sqrt{g} T^{ab} (KZ)_{ab} d^3x = 0, \qquad (2.7)$$

as $(KZ)_{ab} = 0$ here. In this case we appeal to the result that a self-adjoint operator such as Δ_K can be inverted on a closed manifold if its source is

(2) Here « r » refers to a « radial » coordinate. We assume, for example, that outside a bounded region we can introduce « spherical polar » coordinates (r, θ, ϕ) such that $g^{ab} r_{,a} r_{,b} = 1$, $g^{ab} r_{,a} \phi_{,b} = g^{ab} r_{,a} \theta_{,b} = g^{ab} \theta_{,a} \phi_{,b} = 0$; $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$. In these coordinates the metric takes its usual flat-space form plus corrections that vanish as $r \to \infty$.

(3) For closed manifolds, $(\Delta_K Z)^a = 0$ iff $\mathcal{L}_Z g_{ab} = 0$ was established by Yano and Bochner [6].

(4) In Ref. [7], this result is established for compact manifolds with Dirichlet-type boundary data. We assume that it also holds when the boundary is pushed away to « infinity » by postulating appropriate asymptotic fall-off of the metric and of the « source » terms.

globally orthogonal to its harmonic functions [7] ([5]). Thus Δ_K can still be inverted and solved for X^a. A similar argument shows that the solution X^a is unique up to isometries. However, Killing vectors cannot affect the « longitudinal » part $T_l^{ab} = (KX)^{ab}$, so the decomposition itself has a unique result. As a corollary, one notes that a Killing vector can never be expressed globally in the form $\nabla_b T^{ab}$ for any choice of T^{ab} whatsoever.

It is clear that the transverse and longitudinal tensors as defined in this section are orthogonal:

$$\int \sqrt{g}(KV)_{ab} T_t^{ab} d^3x = -\frac{1}{2}\int \sqrt{g} V_a \nabla_b T_t^{ab} d^3x = 0 \tag{2.8}$$

for any T_t^{ab} and $(KV)^{ab}$.

Before passing to the TT— decomposition, observe that the trace-free part of T_t^{ab} i. e., $T_t^{ab} - \frac{1}{3} g^{ab} T_t$, $T_t = g_{cd} T_t^{cd}$, is no longer transverse since $\nabla_b T_t = \nabla_b (T - 2\nabla_c X^c) \neq 0$ in general. On the other hand, consider the t— part of $T^{ab} - \frac{1}{3} g^{ab} T$. We have

$$T^{ab} - \frac{1}{3} g^{ab} T = \left(T^{ab} - \frac{1}{3} T g^{ab}\right)_t + (KS)^{ab} \tag{2.9}$$

for some S^a. Therefore $\left(T^{ab} - \frac{1}{3} T g^{ab}\right)_t$ is transverse, but is no longer trace-free, because

$$g_{ab}\left(T^{ab} - \frac{1}{3} T g^{ab}\right)_t = 2\nabla_c S^c \tag{2.10}$$

which cannot be expected to vanish in general. These two elementary facts indicate why a different procedure is needed to construct the transverse-traceless part of T^{ab}.

3. TRANSVERSE-TRACELESS DECOMPOSITION

Consider T^{ab} and g_{ab} as above. The transverse-traceless decomposition (TT) is defined by (4):

$$T^{ab} = T_{TT}^{ab} + (LY)^{ab} + \frac{1}{3} T g^{ab} \tag{3.1}$$

where $T = g_{cd} T^{cd}$ and

$$(LY)^{ab} \equiv \nabla^a Y^b + \nabla^b Y^a - \frac{2}{3} g^{ab} \nabla_c Y^c = (KY)^{ab} - \frac{2}{3} g^{ab} \nabla_c Y^c.$$

Here we have a different « longitudinal » part $T_L^{ab} = (LY)^{ab}$ which is trace-

([5]) In fact, it can be shown (Ref. [9]) for strongly elliptic operators such as Δ_K that on closed manifolds, the operator has a complete countable set of differentiable eigenfunctions with real eigenvalues whose only accumulation point is $+\infty$. See also Matzner [10], who discussed in another context some properties of Δ_K on closed manifolds.

free and a trace part $T^{ab}_{TR} = \frac{1}{3} g^{ab} g_{cd} T^{cd}$. The TT— part is defined by (3.1), where

$$\nabla_b (LY)^{ab} = \nabla_b \left(T^{ab} - \frac{1}{3} T g^{ab} \right). \tag{3.2}$$

We shall briefly repeat a discussion of the type given in Section 2 with the necessary changes. We first note that $(LY)^{ab}$ may be called the « conformal Killing form » of Y^a for the following reason. Just as $\mathcal{L}_X g_{ab} = (KX)_{ab}$, here we recall the definition of the conformal metric $\tilde{g}_{ab} \equiv g^{-1/3} g_{ab}$ (independent of arbitrary overall scale changes in g_{ab}) and observe that

$$\mathcal{L}_Y \tilde{g}_{ab} = g^{-1/3} (LY)_{ab}.$$

This describes the action of « infinitesimal coordinate transformations » on the conformal metric.

The relevant linear second order vector operator is [4]

$$\nabla_b (LY)^{ab} \equiv (\Delta_L Y)^a = \Delta Y^a + \frac{1}{3} \nabla^a (\nabla_b Y^b) + R^a_b Y^b. \tag{3.3}$$

The symbol of Δ_L is given by

$$[\sigma_v(\Delta_R)]^a_b = \frac{1}{3} V^a V_b + \delta^a_b V_c V^c, \tag{3.4}$$

which shows that Δ_L is both elliptic and strongly elliptic. Self-adjointness is shown by the computation

$$\int \sqrt{g} V_a \nabla_b (LW)^{ab} d^3x = \int \sqrt{g} W_a \nabla_b (LV)^{ab} d^3x \tag{3.5}$$

The kernel of Δ_L consists at most of « conformal Killing vectors » C^a satisfying $(LC)_{ab} = 0$. These are just the « isometries » of $\tilde{g}_{ab}$ if any are admitted, or, equivalently, the vectors leaving g_{ab} invariant up to an overall scaling factor. These vectors will not vanish at infinity in an asymptotically flat space and so may be ignored ([6]). In a closed space we have orthogonality between C^a and $\nabla_b \left(T^{ab} - \frac{1}{3} T g^{ab} \right)$:

$$0 = \int \sqrt{g} C_a \nabla_b \left(T^{ab} - \frac{1}{3} T g^{ab} \right) d^3x$$
$$= -\frac{1}{2} \int \sqrt{g} \left(T^{ab} - \frac{1}{3} T g^{ab} \right) (LC)_{ab} d^3x \tag{3.6}$$

([6]) However, if $(\Delta_L \xi)^a = 0$ is considered subject to the boundary condition that ξ^a asymptotically approaches a *conformal Killing vector* C^a, rather than asymptotically approaching zero, the equation has a *unique* solution ξ^a such that asymptotically $\xi^a \to C^a + O(r^{-1})$. The key point is that $(\Delta_L \xi)^a = 0$ and $(L\xi)_{ab} = 0$ are equivalent equations on closed manifolds or whenever the boundary terms that occur in Gauss's theorem may be ignored. On asymptotically flat spaces, these boundary terms *are* ignorable if the vector vanishes asymptotically, but are *not* ignorable in the present case, where ξ^a does not vanish at infinity. Similar remarks hold for $(\Delta_K \xi)^a$ and $(K\xi)_{ab}$. This is very useful in defining the total momentum of a gravitational field, as described in Section 5 below. See also Ref. [14].

A corollary of (3.4) is that a conformal Killing vector can never be expressed globally in the form $\nabla_b\left(T^{ab} - \frac{1}{3}Tg^{ab}\right)$. Therefore, Δ_L can be inverted for a unique solution up to conformal Killing vectors. However, $T_L^{ab} = (LY)^{ab}$ is insensitive to these vectors, so the decomposition gives a unique result. That TR—, TT—, and L— tensors are all mutually orthogonal is readily verified.

4. RELATION BETWEEN THE TWO DECOMPOSITIONS

We first establish the fact that $(T_{TT}^{ab})_t = T_{TT}^{ab}$.

Thus, suppose we put

$$T_{TT}^{ab} = (T_{TT}^{ab})_t + (KV)^{ab} \tag{4.1}$$

where $(KV)^{ab} = (T_{TT}^{ab})_l$. We find that $\nabla_b(KV)^{ab} = 0$ which implies $(KV)^{ab} = 0$ and $(T_{TT}^{ab})_t = T_{TT}^{ab}$. Next we show that $(T_t^{ab})_{TT} = T_{TT}^{ab}$. Suppose

$$T^{ab} = T_t^{ab} + (KX)^{ab} \tag{4.2}$$

and

$$T^{ab} = T_{TT}^{ab} + (LY)^{ab} + \frac{1}{3}Tg^{ab}. \tag{4.3}$$

On the other hand, one has

$$T_t^{ab} = (T_t^{ab})_{TT} + (LM)^{ab} + \frac{1}{3}T_t g^{ab} \tag{4.4}$$

for some M^a. Each of the tensors $(KX)^{ab}$, $(LY)^{ab}$ and $(LM)^{ab}$ is unique. Substituting (4.4) into (4.2) and noting that $T_t = T - 2\nabla_c X^c$, we obtain

$$T^{ab} = (T_t^{ab})_{TT} + (LM)^{ab} + (LX)^{ab} + \frac{1}{3}Tg^{ab}, \tag{4.5}$$

where we also used $(LX)^{ab} = (KX)^{ab} - \frac{2}{3}g^{ab}\nabla_c X^c$.

Equating (4.5) and (4.3) we find

$$T_{TT}^{ab} - (T_t^{ab})_{TT} = [L(M + X - Y)]^{ab} \tag{4.6}$$

since the operator « L » is linear. The left side of (4.6) is a TT— tensor while the right side is an L— tensor. Using the orthogonality relation, the only way that this can hold everywhere is when both sides vanish. Therefore, $T_{TT}^{ab} = (T_t^{ab})_{TT}$. Moreover, this also establishes a relation between the several longitudinal parts:

$$(LM)^{ab} + (LX)^{ab} = (LY)^{ab}, \tag{4.7}$$

or

$$T_L^{ab} = (T_t^{ab})_L + T_l^{ab} - 2g^{ab}\nabla_c X^c. \tag{4.8}$$

Hence, the t— and TT— decompositions of an arbitrary symmetric tensor can be displayed in a commutative diagram defined by the result

$$(TT) \circ (t)T^{ab} = (t) \circ (TT)T^{ab} = (TT)T^{ab},$$

where $\circ$ denotes the composition of these operations as defined by equations (4.2), (4.3) and (4.4).

An alternative proof of the theorem follows from the fact that $T_l^{ab} = (KX)^{ab}$ has no TT part. To see this, suppose we write

$$T_l^{ab} = (KX)^{ab} = (T_l^{ab})_{TT} + (LZ)^{ab} + \frac{1}{3} T_l g^{ab} \tag{4.9}$$

for some Z^a. Noting that $T_l = 2\nabla_c X^c$, we have

$$(KX)^{ab} = (T_l^{ab})_{TT} + (LZ)^{ab} + \frac{2}{3} g^{ab}\nabla_c X^c \tag{4.10}$$

or

$$(T_l^{ab})_{TT} = [L(X - Z)]^{ab}, \tag{4.11}$$

which implies, using orthogonality, that

$$(T_l^{ab})_{TT} = 0. \tag{4.12}$$

As the last item of this section, I will prove a theorem relating regular harmonic functions of Δ_K and Δ_L on closed manifolds. First, consider a metric g_{ab} on a closed manifold admitting a Killing vector X^a, which will therefore be a harmonic function of Δ_K. It is obvious that X^a will be conformal Killing vector of any metric $\bar{g}_{ab}$ conformally related to g_{ab}, i. e.,

$$\bar{g}_{ab} = \phi^4 g_{ab}, \ \phi > 0.$$

Thus X^a will be a harmonic function of $\bar{\Delta}_L$. This follows from the fact that

$$\mathscr{L}_X \bar{g}_{ab} = g_{ab} \mathscr{L}_X \phi^4 \tag{4.13}$$

since $\mathscr{L}_X g_{ab} = 0$. The usual form of the conformal Killing equations for $\bar{g}_{ab}$ is $(\bar{X}_a \equiv \bar{g}_{ab}X^b ; \bar{X}^a \equiv X^a)$

$$\mathscr{L}_X \bar{g}_{ab} \equiv \bar{\nabla}_a \bar{X}_b + \bar{\nabla}_b \bar{X}_a = \frac{2}{3}\, \bar{g}_{ab} \bar{\nabla}_c X^c. \tag{4.14}$$

From $\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + 2(\delta^a_b \nabla_c \ln \phi + \delta^a_c \nabla_b \ln \phi - g_{bc}\nabla^a \ln \phi)$
we have

$$\bar{\nabla}_c X^c = \nabla_c X^c + 6X^c\nabla_c \ln \phi, \tag{4.15}$$

and it follows that

$$g_{ab} \mathscr{L}_X \phi^4 = \frac{2}{3}\, \bar{g}_{ab} \bar{\nabla}_c X^c,$$

so (4.13) becomes identical to (4.14).

A converse theorem is perhaps more interesting. Let Y^a denote a conformal Killing vector, i. e., a harmonic function of Δ_L:

$$\mathscr{L}_Y g_{ab} = -\frac{2}{3}\, g_{ab} \nabla_c Y^c. \tag{4.16}$$

Consider a metric $\bar{g}_{ab}$ in proper conformal relation to g_{ab}:

$$\bar{g}_{ab} = \phi^4 g_{ab}, \ 0 < \phi < \infty.$$

We have $(\bar{Y}_a = \bar{g}_{ab} Y^b)$

$$\mathscr{L}_Y \bar{g}_{ab} = \bar{g}_{ab}\left[\mathscr{L}_Y \phi^4 + \frac{2}{3}\phi^4 \nabla_c Y^c\right] = \bar{\nabla}_a \bar{Y}_b + \bar{\nabla}_b \bar{Y}_a. \tag{4.17}$$

Now assume that we make the special choice

$$\phi^4 = (g_{ab}Y^aY^b)^{-1} \equiv |\vec{Y}|^{-2}, \tag{4.18}$$

where we assume that the conformal Killing vector used here has nowhere vanishing norm on the manifold, i. e., the transformation generated by has no fixed points [7]. We have

$$\mathscr{L}_Y\phi^4 = \mathscr{L}_Y(g_{ab}Y^aY^b)^{-1} = -|\vec{Y}|^{-4}Y^aY^b.\frac{2}{3}g^{ab}\nabla_cY^c \tag{4.19}$$

because of (4.16) and $\mathscr{L}_YY^a = 0$. Therefore,

$$\mathscr{L}_Y\phi^4 = -\frac{2}{3}|\vec{Y}|^{-2}\nabla_cY^c \tag{4.20}$$

and the right-hand side of (4.17) vanishes. Thus Y^a has been converted into a Killing vector of $\bar{g}_{ab}$, a harmonic function of $\bar{\Delta}_K$. We have proved the *Theorem*: Let Y^a be a harmonic function of Δ_L with nowhere vanishing norm on a closed manifold M. Then there always exists a manifold $\bar{M}$ conformally related to M for which Y^a is a harmonic function of $\bar{\Delta}_K$.

5. DECOMPOSITION OF TRANSVERSE TENSORS AND PHYSICAL INTERPRETATION

A simple and important deduction that can be made from the findings of the previous section is the following result concerning transverse tensors: Every *transverse* symmetric tensor on a Riemannian manifold can be split uniquely and orthogonally into a sum of a transverse tensor with vanishing trace and a transverse tensor with non-vanishing trace.

The proof of this theorem follows from (4.4) and the fact that $(T_t^{ab})_{TT} = T_{TT}^{ab}$. Let us write (4.4) in the form

$$p^{ab} = s^{ab} + m^{ab} \tag{5.1}$$

where $\nabla_b p^{ab} = 0$, s^{ab} is TT, and m^{ab} is a transverse tensor of the form

$$m^{ab} = (LM)^{ab} + \frac{1}{3}pg^{ab}, \tag{5.2}$$

where [8]

$$\nabla_b(LM)^{ab} = \nabla_b\left(p^{ab} - \frac{1}{3}pg^{ab}\right) = -\frac{1}{3}\nabla^a p. \tag{5.3}$$

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(7) If there is a fixed point, we have $\vec{Y} = 0$ there and (4.18) shows that ϕ^4 is infinite at that point. The fixed point could be the « origin » of an asymptotically flat space which is therefore conformally mapped to infinity. For example, if $\vec{Y}$ is a flat-space « dilatation », then one can show that $\phi^4 = (x^2 + y^2 + z^2)^{-1}$.

(8) From (5.3), it follows that the gradient of the trace of a transverse tensor is always globally orthogonal to conformal Killing vectors on closed manifolds. This is important in the investigation of the linearization stability of the initial-value equations on closed manifolds [*14*].

The trace of m^{ab} is given by

$$m = g_{ab}\left[(\mathrm{LM})^{ab} + \frac{1}{3}pg^{ab}\right] = p. \tag{5.4}$$

The orthogonality of s^{ab} and m^{ab} is apparent.

The meaning of this result in the context of general relativity can be made clear by the following observations. Four of the ten Einstein vacuum field equations refer to the initial data on a spacelike hypersurface. Three of these four equations have the form

$$\nabla_b p^{ab} = 0 \tag{5.5}$$

where $\pi^{ab} = \sqrt{g}p^{ab}$ is the canonical momentum of the gravitational field [*1*]. The momentum tensor p^{ab}, as a result of the above theorem, is composed of two independent parts. First, s^{ab} is an arbitrary TT tensor as described in Section 3. Second, m^{ab} is defined in (5.2) and satisfies (5.3). In (5.3), one sees that $(\mathrm{LM})^{ab}$ is determined by p. Therefore $(\mathrm{LM})^{ab}$ itself contains nothing arbitrary; the arbitrariness can be considered to reside in p. It is known, however, that p may be regarded as an essentially kinematical quantity defining the chosen rate of volume expansion of an « initial » surface relative to local proper time [*4*] [*11*]. Hence, all solutions of (5.5) are determined independently by one kinematical function and a purely gravitational spin-two transverse-traceless tensor S^{ab}. One can regard S^{ab} as defining the « wave » part of the momentum and p as defining an essentially arbitrary « gauge » degree of freedom.

The situation is similar when there are present three-vector currents S^a describing the flow of external matter or other fields. In place of (4.13) we have

$$\nabla_b p^{ab} = -8\pi S^a \tag{5.6}$$

In this case we again make an orthogonal splitting of p^{ab} into s^{ab} plus an m^{ab} of the form (4.10). Now, however, m^{ab} is no longer transverse. Hence s^{ab} remains « sourceless » just as before but m^{ab} is related to the external current by means of

$$\nabla_b m^{ab} = -8\pi S^a \tag{5.7}$$

Therefore,

$$\nabla_b(\mathrm{LM})^{ab} = -\frac{1}{3}\nabla^a p - 8\pi S^a \tag{5.8}$$

In this case M^a is determined by a « gauge » function p and three « source » functions S^a.

Physical interpretation of the decomposition of momentum may be completed by the observation that in an asymptotically flat space, the vector part M^a of p^{ab} completely and uniquely determines the total momentum of a gravitational field-matter configuration [9]. The vector M^a plays

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(9) The total momentum of a *closed* space is physically meaningless.

the role of a potential in the elliptic « Poisson-type » equation (5.8). The $0(r^{-1})$ and $0(r^{-2})$ parts ([10]) of M^a determine the total momentum by means of the integral

$$16\pi P_\xi = -2\oint_\infty \xi^a(LM)_{ab}\sqrt{g}dS^b = \int\sqrt{g}\left(16\pi S_a + \frac{2}{3}\nabla_a p\right)\xi^a d^3x, \quad (5.9)$$

where ξ^a is an « almost symmetry vector ». I define ([11]) ξ^a as the unique solution of the equation $(\Delta_L\xi)^a = 0$ such that ξ^a approaches a standard flat-space Killing vector at infinity. Thus, if ξ^a approaches a translation vector, (5.9) defines linear momentum. If ξ^a approaches a rotation vector, (5.9) defines angular momentum. The present formulation of total momentum is considered further in [*13*].

6. THE SPACE OF CONFORMAL GEOMETRIES

« Conformal Superspace » $\tilde{\mathcal{S}}$ may be defined for the purposes of this section as the space of conformal geometries on closed manifolds, and may be identified in a natural way with the space of conformal metrics *modulo* diffeomorphisms, or, equivalently, with the space of Riemannian metrics *modulo* diffeomorphisms and conformal transformations of the form $g_{ab} \to \phi^4 g_{ab}$ where $\phi(x)$ is an arbitrary strictly positive function. The introduction of $\tilde{\mathcal{S}}$ is motivated in a manner analogous to the introduction of ordinary « superspace » $\mathcal{S}$ in the description of gravitational dynamics. That is, in the present view [*4*] [*11*], gravitational dynamics is regarded as the time evolution of *conformal* three-geometry because the Riemannian geometry up to a scaling function ϕ^4 may be regarded as free of constraints at any instant [*12*]. The momentum conjugate to the conformal geometry is transverse-traceless ([12]). In the Hamilton-Jacobi representation, therefore, one has [*4*]

$$\tilde{\sigma}^{ab} = \tilde{g}^{5/6}s^{ab} = \frac{\delta S}{\delta\tilde{g}_{ab}} \quad (6.1)$$

([10]) The solution M^a of (5.8) in general has an $0(r^{-1})$ part on an asymptotically flat space if the right-hand side doesn't vanish. If the right-hand side vanishes, the only $M^a \to 0$ is $M^a = 0$ on a topologically Euclidean space. However, if the space has a « hole » instead of sources, M^a can still have $0(r^{-1})$ and $0(r^{-2})$ parts which define the momentum and angular momentum of the « hole ». In this case, $(LM)^{ab}$ is transverse and traceless (outside the hole) but is not the same as the TT part of p^{ab}, of course.

([11]) This definition of « symmetry » is apparently the weakest one allowing passage from (5.8) to (5.9) using Gauss's theorem [*13*]. There is a unique solution for every independent solution of the flat-space conformal Killing equations, given physically natural conditions on the asymptotic approach of the metric to flatness. If there is an *exact* conformal Killing vector in the metric, then ξ^a turns out to be that vector. Compare footnote 6.

([12]) An « almost-Hamiltonian » theory of gravity can be based on $\tilde{g}_{ij}$ and $\tilde{\sigma}^{ij}_{TT}$ as the « true » (i. e., unconstrained) dynamical variables. The constraints are eliminated by using the initial-value equations. See [*15*].

where the transverse-tracelessness of $\tilde{\sigma}^{ab}$ is equivalent to the fact that S is a functional of the coordinate-free conformal three-geometry. This follows from the fact that changes of $\tilde{g}_{ab}$ of the form

$$\delta\tilde{g}_{ab} = \varepsilon g^{-1/3}(\mathrm{LY})_{ab} = \varepsilon\mathscr{L}_{\mathrm{Y}}\tilde{g}_{ab}$$

cannot affect such an S. Therefore, conformal metrics $\tilde{g}_{ab}$ and $\tilde{g}_{ab}+\varepsilon\mathscr{L}_{\mathrm{Y}}\tilde{g}_{ab}$ are identified as referring to the same point of $\tilde{\mathscr{S}}$. Likewise, if two infinitesimally differing Riemannian metrics g_{ab} and g'_{ab} differ by

$$\delta g_{ab} = \varepsilon\mathscr{L}_{\mathrm{X}}g_{ab} = \varepsilon(\mathrm{KX})_{ab}$$

for some X^a, then g_{ab} and g'_{ab} define the same point of ordinary superspace $\mathscr{S}$. Therefore, in constructing « tangent vectors » h_{ab} to $\mathscr{S}$ one need only consider transverse tensors $h^t_{ab}(\delta g_{ab} = \varepsilon h^t_{ab})$. Since $g^{ab}\delta\tilde{g}_{ab} = 0$, we may therefore assume without loss of generality that tangent vectors to $\tilde{\mathscr{S}}$ have the form $\tilde{h}^{\mathrm{TT}}_{ab}$ where $\delta\tilde{g}_{ab} = \varepsilon\tilde{h}^{\mathrm{TT}}_{ab}$, $\tilde{h}_{ab} = g^{-1/3}h_{ab}$.

This gives a further simple interpretation of the decomposition theorem provided in Section 5. In this light, the theorem says that the true change of conformal three-geometry $(\tilde{h}^{\mathrm{TT}}_{ab})$ is generated only by the true change of the full Riemannian three-geometry (h^t_{ab}) and not from its longitudinal part $h^l_{ab} = \mathscr{L}_{\mathrm{X}}g_{ab}$ because $\tilde{h}^{\mathrm{TT}}_{ab} = (g^{-1/3}h^t_{ab})^{\mathrm{TT}}$ and $(h^l_{ab})^{\mathrm{TT}} = 0$. Furthermore, from (4.4) we see that the presence in h^t_{ab} of a non-constant trace may be regarded from the viewpoint of conformal geometry as generating a a further pure « gauge-like » effect not influencing the underlying conformal geometry itself, i. e., a term of the form $g^{-1/3}(\mathrm{LM})_{ab}$.

We can now consider briefly the structure of $\tilde{\mathscr{S}}$ and demonstrate by a simple argument that this structure is not « stable » in the sense that the neighborhoods of conformal metrics with symmetries [13] (conformal Killing vectors) have a smaller dimensionality than « generic » neighborhoods (no symmetries). Thus, symmetric conformal geometries lie on « boundaries » of $\tilde{\mathscr{S}}$.

Consider two conformal geometries, represented by $\tilde{a}_{ab}$ and $\tilde{b}_{ab}$ differing slightly from a given conformal geometry represented by $\tilde{g}_{ab}$. Thus,

$$\tilde{a}_{ab} = \tilde{g}_{ab} + \varepsilon\tilde{h}_{ab} \tag{6.2}$$

$$\tilde{b}_{ab} = \tilde{g}_{ab} + \varepsilon\tilde{l}_{ab} \tag{6.3}$$

where both $\tilde{h}_{ab}$ and $\tilde{l}_{ab}$ are transverse-traceless.

To see the effect of the existence of a conformal symmetry, assume that $\tilde{l}_{ab}$ is chosen in the form

$$\tilde{l}_{ab} = (\tilde{h}_{ab} + \mathscr{L}_{\mathrm{X}}\tilde{h}_{ab})^{\mathrm{TT}} \tag{6.4}$$

where X^a is assumed to be a *conformal Killing vector* of $\tilde{g}_{ab}$. Of course, we have that $\tilde{h}_{ab} = \tilde{h}^{\mathrm{TT}}_{ab}$ by hypothesis. But although $\mathscr{L}_{\mathrm{X}}\tilde{h}_{ab}$ is tracefree,

$$g^{ab}\mathscr{L}_{\mathrm{X}}\tilde{h}_{ab} = 0, \tag{6.5}$$

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[13] Here I am considering only continuous symmetries.

it is not transverse in general because

$$\nabla^b \mathscr{L}_X \tilde{h}_{ab} = - \tilde{h}_{ab} \nabla^b \nabla_c X^c. \tag{6.6}$$

Equations (6.5) and (6.6) follow from the fact that, by assumption, $\mathscr{L}_X g_{ab} = 0$, which is equivalent to $\mathscr{L}_X g_{ab} = \frac{2}{3} g_{ab} \nabla_x X^c$. Alternatively, (6.6) results from the fact that, not $\mathscr{L}_X h_{ab}^{TT}$, but rather $\mathscr{L}_X h_{ab}^{TT} + \frac{1}{3} h_{ab}^{TT} \nabla_c X^c$, is TT if X^a is a conformal Killing vector [4]. Using (3.1), we can put

$$\mathscr{L}_X \tilde{h}_{ab} = (\mathscr{L}_X \tilde{h}_{ab})^{TT} + g^{-1/3}(LS)_{ab} \tag{6.7}$$

for a vector S^a satisfying

$$\nabla^b[g^{-1/3}(LS)_{ab}] = \nabla^b[- \tilde{h}_{ab} \nabla_c X^c] \tag{6.8}$$

from (6.6) and $\nabla^b \tilde{h}_{ab} = 0$. Therefore, because $\tilde{h}_{ab} = g^{-1/3} h_{ab}$, (6.8) can be written

$$\nabla^b(LS)_{ab} = \nabla^b(- h_{ab} \nabla_c X^c) \tag{6.9}$$

Hence, (6.9) can be solved uniquely for $(LS)_{ab}$, inasmuch as the right-hand side is the divergence of a tracefree symmetric tensor and is therefore orthogonal to X^a, the conformal symmetry vector.

From (6.2) we have

$$\mathscr{L}_X \tilde{a}_{ab} = \varepsilon \mathscr{L}_X \tilde{h}_{ab} \tag{6.10}$$

since $\mathscr{L}_X \tilde{g}_{ab} = 0$ by hypothesis. This is the key point, as we shall now demonstrate. For we have from (6.3),

$$\tilde{b}_{ab} = \tilde{g}_{ab} + \varepsilon \tilde{h}_{ab} + \varepsilon(\mathscr{L}_X \tilde{h}_{ab})^{TT} \tag{6.11}$$

or, using (6.2) and (6.7),

$$\tilde{b}_{ab} = \tilde{a}_{ab} + \varepsilon \mathscr{L}_X \tilde{h}_{ab} - \varepsilon g^{-1/3}(LS)_{ab}. \tag{6.12}$$

Now (6.10) and $g^{-1/3}(LS)_{ab} = \mathscr{L}_S \tilde{g}_{ab}$ show that

$$\tilde{b}_{ab} = \tilde{a}_{ab} + \mathscr{L}_X \tilde{a}_{ab} - \varepsilon \mathscr{L}_S \tilde{g}_{ab} \tag{6.13}$$

Let us examine the final term. We have

$$\varepsilon \mathscr{L}_S \tilde{g}_{ab} = \varepsilon \mathscr{L}_S(\tilde{a}_{ab} - \varepsilon \tilde{h}_{ab})$$

Hence, neglecting the term of order ε^2, we obtain

$$\tilde{b}_{ab} = \tilde{a}_{ab} + \mathscr{L}_X \tilde{a}_{ab} - \varepsilon \mathscr{L}_S \tilde{a}_{ab} = \tilde{a}_{ab} + \mathscr{L}_w \tilde{a}_{ab}, \tag{6.14}$$

where $W^a = X^a - \varepsilon S^a$. Observe that $\mathscr{L}_w \tilde{a}_{ab}$ is a term of the first order in ε as expected. This argument shows that, although $\varepsilon(\mathscr{L}_X \tilde{h}_{ab})^{TT}$ is transverse-traceless with respect to g_{ab}, it may be written in purely longitudinal form *with respect to the metric* a_{ab}, if we neglect terms of order ε^2. Thus, to the necessary order a_{ab} and b_{ab} differ in scale and in coordinatization, but not in the intrinsic conformally invariant geometries which they respectively define. This result can only be obtained when the original metric g_{ab} admits a conformal Killing vector X^a. If there is no symmetry, different transverse-traceless tensors, such as $\tilde{h}_{ab}$ and $\tilde{l}_{ab}$, always generate transformations to inequivalent conformal geometries. Thus, one obtains the conclusion that the dimensionality of the tangent space to $\tilde{\mathscr{S}}$ at a conformal geometry with symmetry is *less* than that of a conformal geometry without symmetry. Of course, one must use caution in drawing conclusions about the dimensionality of such infinite-dimensional spaces. However, the present method of reasoning may be used in describing tangent vectors of ordinary superspace with an analogous conclusion: Riemannian geometries on closed manifolds admitting Killing vectors have tangent spaces in of fewer dimensions than geometries without symmetries. A precise version of the result for superspace is given in the « stratification theorem » [*5*]. We see that conformal superspace is also « stratified » by the presence of conformal geometries with symmetries. More importantly, the presence or absence of conformal symmetries turns out to be of crucial importance in determining the structure of gravitational phase space [*14*].

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