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Verification of Axioms for Euclidean and Relativistic Fields and Haag's Theorem in a Class of $P(\varphi)_2$ -Models (*)

by

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ABSTRACT. — Axioms for Euclidean (Bose) Fields are proposed and shown to suffice for the reconstruction of Relativistic Quantum Fields satisfying the Wightman Axioms. Those axioms are verified for a class of $P(\varphi)_2$ models. They seem to provide a suitable framework for (Bose) Field Theories in Two and three space-time dimensions. It is shown that the $P(\varphi)_2$ -interacting field in the infinite volume limit is not a (generalized) free field or a wick polynomial of a (generalized) free field. If P_1 and P_2 are two interaction polynomials in the region of convergence of the Glimm-Jaffe-Spencer Cluster Expansion then the corresponding infinite volume field theories are different, unless $P_1(\xi) = P_2(\pm \xi + a) + b$.

In this paper we present a simple and short verification of the Wightman axioms [18] [40] for a class of $P(\varphi)_2$ quantum field models with so-called half-Dirichlet boundary conditions [16] and arbitrarily large coupling constant.

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These models have been intensively studied from the point of view of Euclidean Field Theory by Guerra, Rosen and Simon [14] [15] [16] [37], by Nelson [23] [26] and in [6] [7]. Under different conditions but also from the Euclidean point of view they have been studied by Glimm, Jaffe and Spencer in [12] [13] which from the point of view of physics are the most important investigations.

In fact, it seems to be hard to contribute results of some importance which go beyond the masterful investigations of $P(\varphi)_2$ contained in Refs [12] [13] [15] [16] [23] [26] [37]. Nevertheless this paper might have a legitimate motivation: It presents a rather short, simple and reasonably self-contained verification of the Wightman axioms [18] [40] for $P(\varphi)_2$ -models with (half-) Dirichlet boundary conditions which does not involve any sophisticated analysis.

It is tempting to think of these axioms as being ideas in the great Platonic sky of ideas and therefore it seems to be justified to describe a short and easy way to find the shadows of these ideas in the hard world of models (*). The way we want to describe here consists of the following three parts (Sections 1-3):

In Section 1 we formulate three axioms (Axioms A, B, C) for Euclidean fields in terms of a functional J on the Schwartz space $\mathcal{S} = \mathcal{S}_{\text{real}}(\mathbb{R}^d)$ which is called the generating functional of the Schwinger functions [6].

The spirit of Axiom A is the one of Symanzik and Nelson [24] [41]: It guarantees the existence and covariance of Euclidean fields over \mathcal{S} . Axiom B is a rather obvious translation of the Osterwalder-Schrader positivity condition [29] [30] [31] into the language of the generating functional J (See also [17].) Axiom C is motivated by results proven in [6]. It yields the existence of sharp-time fields and a bound on the (time 0 —) fields in terms of the quantum field Hamiltonian obtained from Axioms A and B.

In Section 2 we prove theorems which allow for the reconstruction of relativistic quantum fields from the generating functional J which satisfy the Wightman axioms. We also present a theorem which connects Axioms A, B, and C with Nelson's axioms [24]. Some of the difficult steps in the reconstruction of relativistic quantum fields are of course due to Osterwalder and Schrader [29] [30] [31] and Nelson [24] and there we just outline the simple modifications of their arguments which account for the different starting point of this paper.

In Section 3 we verify the axioms of Section 1 for the $P(\varphi)_2$ models using only some of the simplest, yet most elegant results of [6] [15] [16].

In Section 4 we study the uniqueness of the « Euclidean » and the phy-

(*) [We apologize for the abuse of Platonic philosophy made here.]

sical vacuum. We continue our analysis of the properties of the $P(\varphi)_2$ infinite volume interacting measures started in [7]. We show that these measures are different from Gaussian measures if degree $P > 2$. This proves that the infinite volume interacting fields in the $P(\varphi)_2$ models are *not* free fields (They are not even Wick polynomials of free or generalized free fields). It is also proven that under certain conditions the infinite volume interacting measures associated with polynomials P_1, P_2 , respectively are mutually singular unless $P_1 = P_2$ and physically different unless $P_1(\xi) = P_2(\pm \xi + a) + b$.

The representations of the (time 0-) Weyl relations are investigated and found to be disjoint from Fock representations if degree $P > 2$. Our results prove the non-existence of the interaction picture in the $P(\varphi)_2$ models (Haag's theorem [18] [40]).

SECTION 1

AXIOMS FOR EUCLIDEAN FIELDS

We consider Euclidean fields over a d -dimensional space-time. For the sake of concreteness and because of our applications in Sections 3 and 4 we set $d = 2$ and we only consider the case of one neutral, scalar Bose field. The results of Sections 1 and 2 do not depend on the value of d and generalizations to arbitrary Bose fields (and, *mutatis mutandis*, Fermi fields [28] [29] [30]) can be given. We hope that the axioms proposed in this section are realistic for $d = 2, 3$.

Recall the following definitions:

The real Schwartz space over \mathbb{R}^2 is denoted by $\mathcal{S} (\equiv \mathcal{S}_{\text{real}}(\mathbb{R}^2))$ and \mathcal{S}' denotes its dual. Points in \mathbb{R}^2 are denoted by $\xi = \langle x, t \rangle$. Elements of the Euclidean group $E(2)$ are denoted by β , elements of the « time »-translation subgroup by τ or t and « time »-reflection by \mathfrak{S} . For f in \mathcal{S} we set

$$f_\beta(\xi) = f(\beta^{-1}\xi)$$

We define a closed subspace \mathcal{S}_+ of \mathcal{S} by

$$\mathcal{S}_+ \equiv \{ f | f \in \mathcal{S}, \text{supp } f \subseteq \mathbb{R}_x \times \{ t \geq 0 \} \} \tag{1.1}$$

AXIOM A (*Existence of Euclidean Fields over \mathcal{S}*). — *There exists a functional J on \mathcal{S} such that*

- A1) J is normalized, i. e. $J(0) = 1$.
- A2) J is continuous on \mathcal{S} in the Schwartz space topology.
- A3) J is of positive type (Nelson-Symanzik positive [20] [24] [41]), i. e.

given arbitrary complex numbers c_1, \dots, c_n and test functions f_1, \dots, f_n in \mathcal{S} then

$$\sum_{i,j=1}^n \bar{c}_i c_j J(f_j - f_i) \geq 0 \tag{1.2}$$

A4) J is real, i. e. $J(f) = \overline{J(-f)}$

A5) J is « time-translation » and « time-reflection » invariant, i. e.

$$J(f_t) = J(f) \quad \text{and} \quad J(f_{\mathfrak{g}}) = J(f).$$

A6) J is Euclidean invariant.

Obviously A5) is a special case of A6) and is stated separately for later purposes.

THEOREM 1 (Minlos [20], « reconstruction of Euclidean fields from J »). — There exists a Euclidean invariant probability measure ν on the \mathfrak{S} -algebra generated by the Borel cylinder sets of \mathcal{S}' such that

$$J(f) = \int_{\mathcal{S}'} d\nu(q) e^{iq(f)} \tag{1.3}$$

The measure ν determines a Hilbert space $\mathcal{H}_\nu \equiv L^2(\mathcal{S}', d\nu)$ and the Banach spaces $L^p(\mathcal{S}', d\nu)$, $1 \leq p \leq \infty$.

The operators $\{ e^{i\lambda\Phi(f)} \mid f \in \mathcal{S} \text{ fixed, } \lambda \in \mathbb{R} \}$, where

$$(e^{i\Phi(f)})(q) \equiv e^{iq(f)} \tag{1.4}$$

form a strongly continuous, unitary group on \mathcal{H}_ν with selfadjoint (s. a.) infinitesimal generator $\Phi(f)$, (called the Euclidean field, smeared with test function f in \mathcal{S}).

Remarks. — The function Ω identically 1 on \mathcal{S}' , denoted also by Ω , is a normalized vector in \mathcal{H}_ν which is cyclic for the $*$ algebra generated by the operators $\{ e^{i\Phi(f)} \mid f \in \mathcal{S} \}$. It is called the « Euclidean vacuum ». There exists a strongly continuous, unitary representation T of $E(2)$ on \mathcal{H}_ν such that

$$T_\beta \Omega = \Omega, \quad T_\beta e^{i\Phi(f)} T_\beta^* = e^{i\Phi(f_\beta)} \tag{1.5}$$

for all β in $E(2)$ (The dual mapping of T_β also denoted by T_β is an automorphism of the underlying \mathfrak{S} -algebra, generated by the Borel cylinder sets of \mathcal{S}').

AXIOM B (Osterwalder-Schrader positivity). — The functional J is positive in the sense of Osterwalder and Schrader, i. e. given arbitrary complex numbers c_1, \dots, c_n and test functions f_1, \dots, f_n in \mathcal{S}_+

$$\sum_{i,j=1}^n \bar{c}_i c_j J(f_j - (f_i)_\mathfrak{g}) \geq 0 \tag{1.6}$$

(such an axiom has also been proposed in [17]. It is the natural translation of Osterwalder-Schrader positivity [29] [30] [31] in a probabilistic framework).

AXIOM C (Exponential J-bound and \mathfrak{S}^2 -bound). — C1) The function $J(\lambda_1 f_1 + \lambda_2 f_2)$ (for arbitrary fixed f_1 and f_2 in \mathcal{S}) is once continuously differentiable in λ_1 and λ_2 and

$$\mathfrak{S}^2(f_1, f_2) \equiv - \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} J(\lambda_1 f_1 + \lambda_2 f_2) |_{\lambda_1 = \lambda_2 = 0}$$

is jointly continuous in f_1 and f_2 in the Schwartz space topology. The tempered distribution $\mathfrak{S}^2(h_1 \otimes \delta_{t_1}, h_2 \otimes \delta_{t_2})$, where h_1 and h_2 are in $\mathcal{S}_1 \equiv \mathcal{S}_{\text{real}}(\mathbb{R}^1)$, is a bounded, measurable function of t_1 and t_2 in some (arbitrarily small) open neighbourhood of $t_1 = t_2$.

C2) There exists a real function g in $C_0^\infty(\mathbb{R}^1)$ with the properties

$$0 \leq g(t) \leq 1, g(t) = g(-t), \text{supp } g \subset [-1, 1], g(t) = 1 \text{ if } |t| \leq \frac{1}{2},$$

such that for

$$g_T(t) = \begin{cases} 1, & |t| \leq T \\ g(|t| - T), & \text{otherwise} \end{cases}$$

$$J(\pm ih \otimes g_T) \leq K_h e^{cT}, \text{ for some finite } K_h, \tag{1.8}$$

where h is an arbitrary real function with the property that $||| h ||| \leq 1$ for some norm $||| \cdot |||$ which is continuous on \mathcal{S}_1 , and some finite constant c .

In terms of the measure ν (1.8) states that the Laplace transforms

$$\int_{\mathcal{S}'} d\nu(q) e^{\pm q(h \otimes g_T)}$$

are bounded by $K_h e^{cT}$.

Remarks. — Axiom B yields the construction of a physical Hilbert space \mathcal{H}_w and a positive selfadjoint Hamiltonian H on \mathcal{H}_w with a ground-state Ω (called physical vacuum and notationally not distinguished from the « Euclidean vacuum »). This construction is due to Osterwalder and Schrader [29] [30] [31].

Axiom C1) implies that (time 0—) quantum fields $\varphi(h)$, h in \mathcal{S}_1 , exist and are selfadjoint on some dense domain $D(\varphi(h))$ in \mathcal{H}_w containing the physical vacuum Ω .

Axiom C2) implies that in the sense of densely defined quadratic forms on \mathcal{H}_w

$$\pm \varphi(h) \leq 2 ||| h ||| \cdot (H + c' \cdot I), \tag{1.9}$$

for some finite, positive constant c' .

Inequality (1.9) allows for the construction of Wightman distributions which can be shown to obey Wightman's axioms; (Poincaré invariance follows from the Euclidean invariance of J by arguments due to Nelson [24] [29] [30]).

SECTION 2

**THE RECONSTRUCTION
OF RELATIVISTIC QUANTUM FIELDS
FROM THE GENERATING FUNCTIONAL J**

In this section we outline the reconstruction of relativistic quantum fields from a functional J satisfying Axioms A, B, and C. Following closely Osterwalder and Schrader [29] [30] [31] we first construct the physical Hilbert-space \mathcal{H}_w and the Hamiltonian H . We then construct the (time 0—) quantum field φ and we prove that in the sense of quadratic forms on $\mathcal{H}_w \times \mathcal{H}_w$ the field φ (smeared with a suitable test function) is bounded by the Hamiltonian H . This permits us to reconstruct the functional J and the Schwinger functions from the (time 0—) field φ and the Hamiltonian. The analytic continuation of the Schwinger functions in the time variables to real times can then be done as shown by Nelson [24] (and Osterwalder and Schrader [29] [30] [31]).

Step 1. Construction of the Physical Hilbert Space.

DEFINITIONS. — We let $\mathfrak{M}_{v,\pm}$ denote the von Neumann algebras generated by the operators $\{e^{i\Phi(f)} | f \in \mathcal{S}_+\}$, $\{e^{i\Phi(f)} | f \in \mathcal{S}_+\}$, respectively, on the Hilbert space \mathcal{H}_v . Let $\theta \equiv T_\theta$ denote the unitary « time »-reflection operator on \mathcal{H}_v obtained in Theorem 1. The spaces $\mathcal{H}_{v,\pm}$ are defined to be the closed subspaces $\{F\Omega | F \in \mathfrak{M}_{v,\pm}\}^-$ of \mathcal{H}_v and $E_{v,\pm}$ denote the orthogonal projections onto $\mathcal{H}_{v,\pm}$. We denote the scalar product on $\mathcal{H}_v = L^2(\mathcal{S}', dv)$ by $\langle \dots \rangle$.

We let \mathcal{L}_+ denote the linear space $\{F\Omega | F \in \mathfrak{M}_{v,+}\}$ and equip \mathcal{L}_+ with a new inner product:

$$(G, F)_{\mathcal{L}_+} \equiv \int_{\mathcal{S}'} dv(q) \overline{(\theta G)(q)} F(q) = \langle \theta G, F \rangle, \tag{2.1}$$

for arbitrary F and G in $\mathfrak{M}_{v,+}$.

Let $F = \sum_{i=1}^n c_i e^{i\Phi(f_i)}$, where c_1, \dots, c_n are arbitrary complex numbers and f_1, \dots, f_n are arbitrary test functions in \mathcal{S}_+ . Then

$$\begin{aligned} (F, F)_{\mathcal{L}_+} &= \sum_{i,j=1}^n \bar{c}_i c_j (e^{i\Phi(f_i)}, e^{i\Phi(f_j)})_{\mathcal{L}_+} \\ &= \sum_{i,j=1}^n \bar{c}_i c_j J(f_j - (f_i)_\theta) \geq 0, \text{ by Axiom B.} \end{aligned} \tag{2.2}$$

This proves that $(\cdot, \cdot)_{\mathcal{L}_+}$ is positive semi-definite. Moreover

$$|(G, F)_{\mathcal{L}_+}| \leq \|\theta G\|_{\mathcal{X}_v} \|F\|_{\mathcal{X}_v} = \|G\|_{\mathcal{X}_v} \|F\|_{\mathcal{X}_v}, \tag{2.3}$$

i. e. the topology defined by $\|\cdot\|_{\mathcal{X}_v}$ on $\mathfrak{M}_{v,+}$ is finer than the one defined by $(\cdot, \cdot)_{\mathcal{L}_+}$ on $\{\mathfrak{M}_{v,+}\Omega\}$.

Let \mathcal{N}_+ be the kernel of the inner product $(\cdot, \cdot)_{\mathcal{L}_+}$ in \mathcal{L}_+ . We define \mathcal{H}_w to be the completion of $\mathcal{L}_+/\mathcal{N}_+$ in the inner product $(\cdot, \cdot)_{\mathcal{L}_+}$ and (\cdot, \cdot) to be the induced scalar product on \mathcal{H}_w . Let G be in $\mathfrak{M}_{v,+}$. The equivalence class determined by $G\Omega$ with respect to the kernel \mathcal{N}_+ of the inner product $(\cdot, \cdot)_{\mathcal{L}_+}$ is denoted by $v(G)$, i. e. for each G in $\mathfrak{M}_{v,+}$ $v(G)$ denotes the corresponding vector in \mathcal{H}_w .

Step 2. Construction of the Hamiltonian.

We define a semigroup $\{V_t | t \geq 0\}$ on \mathcal{L}_+ by

$$\begin{aligned} (G, V_t F)_{\mathcal{L}_+} &\equiv (G, (T_t F T_t^*))_{\mathcal{L}_+} \\ &= \langle \theta G, T_t F \rangle \end{aligned} \tag{2.4}$$

which is obviously continuous in t and bounded by $\|G\|_{\mathcal{X}_v} \|F\|_{\mathcal{X}_v}$. It follows from arguments of Osterwalder and Schrader [29] [30] [31] that V_t leaves the kernel \mathcal{N}_+ invariant, V_t is uniformly bounded in t on $\{t \geq 0\}$ and V_t is symmetric with respect to $(\cdot, \cdot)_{\mathcal{L}_+}$ for all $t \geq 0$.

Therefore $\{V_t | t \geq 0\}$ determines uniquely a selfadjoint contraction semigroup $\{\bar{V}_t | t \geq 0\}$ on \mathcal{H}_w . Because of (2.4) and Axiom A this semigroup is weakly continuous in t on $\mathcal{H}_w \times \mathcal{H}_w$.

We set $\Omega := v(I)$ and call it the physical vacuum; (it is not distinguished explicitly from the Euclidean vacuum $\Omega \equiv I$).

LEMMA 2.1. — *The semigroup $\{\bar{V}_t | t \geq 0\}$ leaves Ω invariant. The infinitesimal generator H of $\{\bar{V}_t | t \geq 0\}$ is a positive, selfadjoint operator on \mathcal{H}_w . The semigroup $\{\bar{V}_t | t \geq 0\}$ is positivity preserving in the following sense:*

Let C be the closure of $\{v(F) | F \geq 0, F \in L^2(\mathcal{S}', dv), F \text{ affiliated with } \mathfrak{M}_{v,+}\}$ in the scalar product (\cdot, \cdot) . Then for ψ_1 and ψ_2 in C

$$(\psi_1, \bar{V}_t \psi_2) \geq 0 \tag{2.5}$$

Proof. — The first part of the lemma is obvious. Let us prove the second part. We let F and G be positive functions in $L^2(\mathcal{S}', dv)$ which are affiliated with $\mathfrak{M}_{v,+}$. Then

$$\begin{aligned} (G, V_t F)_{\mathcal{L}_+} &= (G, (T_t F T_t^*))_{\mathcal{L}_+} \\ &= \langle \theta G, T_t F \rangle \\ &= \int_{\mathcal{S}'} dv(q) \overline{(\theta G)(q)} (T_t F T_t^*)(q) \end{aligned}$$

But θG and $T_t F T_t^*$ are obviously positive functions in $L^2(\mathcal{S}', dv)$. Hence $(v(G), \bar{V}_t v(F))$ is non-negative for all $t \geq 0$. Q. E. D.

Step 3. Construction of the (time 0-) Quantum Field.

By Axiom C1) the so-called two point Schwinger function $\mathfrak{S}^2(\xi, \eta)$ exists and is a (translation-invariant) tempered distribution. Following Osterwalder and Schrader [29] [30] [31] one can now show (using steps 1 and 2) that for $\tau \equiv t_2 - t_1 > 0$ the distribution

$$G_{h_1, h_2}(\tau) \equiv \mathfrak{S}^2(h_1 \otimes \delta_{t_1}, h_2 \otimes \delta_{t_2})$$

is real analytic in τ . Moreover for $h_1 = \bar{h}_2 G_{\bar{h}_2, h_2}(\tau)$ is decreasing in τ on $[0, \infty)$.

By Axiom C1) $G_{\bar{h}_2, h_2}(\tau)$ is bounded in some open neighborhood of $\tau = 0$. Hence

$$G_{\bar{h}_2, h_2}(0) = \lim_{\tau \rightarrow 0^+} G_{\bar{h}_2, h_2}(\tau) = \sup_{\tau > 0} G_{\bar{h}_2, h_2}(\tau) < \infty.$$

By the commutativity of the Euclidean fields (Axiom A and Theorem 1) $\mathfrak{S}^2(f, g) = \mathfrak{S}^2(g, f)$ for all f and g in \mathcal{S} . This property and continuity in τ on $\{\tau \geq 0\}$ yield

$$G_{\bar{h}_2, h_2}(\tau) = G_{h_2, \bar{h}_2}(-\tau).$$

Therefore for a test function h in \mathcal{S}_1 $G_{h, h}(\tau)$ is a bounded, uniformly continuous function of τ on \mathbb{R} .

LEMMA 2.2. — Let $X_{n,s}(t) \equiv \sqrt{\frac{n}{\pi}} e^{-n(t-s)^2}$, $n = 1, 2, \dots$

Then for all h in \mathcal{S}_1 and all real λ

$$s - \lim_{n \rightarrow \infty} e^{i\lambda\Phi(h \otimes X_{n,s})} \equiv e^{i\lambda\Phi(h \otimes \delta_s)} \quad (2.6)$$

exists on \mathcal{X}_v and defines a strongly continuous unitary group in λ . Moreover Ω is in the domain of $\Phi(h \otimes \delta_s)$,

$$\|\Phi(h \otimes \delta_s)\Omega\|_{\mathcal{X}_v} \leq \|h\|_{\mathcal{S}}, \quad (2.7)$$

for some Schwartz space norm $\|\cdot\|_{\mathcal{S}}$,

$$s - \lim_{s' \rightarrow s} e^{i\lambda\Phi(h \otimes \delta_{s'})} = e^{i\lambda\Phi(h \otimes \delta_s)}, \quad s - \lim_{s' \rightarrow s} \Phi(h \otimes \delta_{s'})\Omega = \Phi(h \otimes \delta_s)\Omega \quad (2.8)$$

$$s - \lim_{n \rightarrow \infty} \Phi(h \otimes X_{n,s})\Omega = \Phi(h \otimes \delta_s)\Omega \quad (2.9)$$

Proof. — Obviously $\{X_{n,s}\}_{n=1}^\infty \subset \mathcal{S}_1$ and $X_{n,s} \rightarrow \delta_s$, as $n \rightarrow \infty$, weakly on continuous functions on \mathbb{R} . Since the Euclidean vacuum Ω is cyclic and separating for the operators $\{e^{i\Phi(f)} | f \in \mathcal{S}\}$, it suffices to prove that $I_{n,n'} \equiv \|(e^{i\lambda\Phi(h \otimes X_{n,s})} - e^{i\lambda\Phi(h \otimes X_{n',s})})\Omega\|_{\mathcal{X}_v}^2 \rightarrow 0$, as $n, n' \rightarrow \infty$. Using Duhamel's formula, the commutativity of Euclidean fields and the Schwartz inequality we obtain

$$\begin{aligned} I_{n,n'} &\leq \lambda^2 \|\Phi(h \otimes X_{n,s}) - \Phi(h \otimes X_{n',s})\Omega\|^2 \\ &= \lambda^2 \int dt dt' (X_{n,s}(t) - X_{n',s}(t))(X_{n,s}(t') - X_{n',s}(t')) \\ &\quad \times G_{h,h}(t - t') \end{aligned}$$

which obviously tends to 0, as $n, n' \rightarrow \infty$, since $G_{h,h}(t - t')$ is jointly continuous in t and t' .

Obviously this result also implies that

$$s - \lim_{n \rightarrow \infty} \Phi(h \otimes X_{n,s})\Omega = \Phi(h \otimes \delta_s)\Omega$$

and

$$\|\Phi(h \otimes \delta_s)\Omega\|_{\mathcal{X}_v}^2 = \lim_{n \rightarrow \infty} \|\Phi(h \otimes X_{n,s})\Omega\|_{\mathcal{X}_v}^2 = G_{h,h}(0).$$

By Axiom C1) $\langle \Omega, \Phi(h \otimes X_{n,s})\Phi(h' \otimes X_{n,s})\Omega \rangle$ is a continuous, bilinear functional in h and h' on $\mathcal{S}_1 \times \mathcal{S}_1$, for fixed $n < \infty$. Also it converges to $\langle \Omega, \Phi(h \otimes \delta_s)\Phi(h' \otimes \delta_s)\Omega \rangle$ which therefore is a continuous, bilinear functional in h and h' on $\mathcal{S}_1 \times \mathcal{S}_1$.

Thus there is a Schwartz-space norm $\|\cdot\|_{\mathcal{S}}$ such that

$$\|\Phi(h \otimes \delta_s)\Omega\|^2 \leq \|h\|_{\mathcal{S}}^2, \text{ whence (2.7)}$$

To prove (2.8) it suffices again to show that

$$\begin{aligned} I_{s',s} &\equiv \|(e^{i\lambda\Phi(h \otimes \delta_{s'})} - e^{i\lambda\Phi(h \otimes \delta_s)})\Omega\|_{\mathcal{X}_v}^2 \\ &\leq \lambda^2 \|\Phi(h \otimes \delta_{s'}) - \Phi(h \otimes \delta_s)\Omega\|_{\mathcal{X}_v}^2 \\ &= \lambda^2 [2G_{h,h}(0) - G_{h,h}(s' - s) - G_{h,h}(s - s')] \end{aligned}$$

tends to 0, as $s' \rightarrow s$, which is obvious. This yields (2.8).

Q. E. D.

It is straightforward to show that lemma 2.2 yields:

$$e^{i\lambda\Phi(h \otimes \delta_s)} \in \mathfrak{M}_{v,+}, \text{ all } s \geq 0.$$

For $s = 0, h$ in \mathcal{S}_1 and F in $\mathfrak{M}_{v,+}$ we define

$$e^{i\lambda\varphi(h)}v(F) \equiv v(e^{i\lambda\Phi(h \otimes \delta_0)}F). \tag{2.10}$$

It is easily checked that this equation defines a strongly continuous unitary groupe $\{e^{i\lambda\varphi(h)} \mid \lambda \in \mathbb{R}\}$ on \mathcal{X}_w with a s. a. infinitesimal generator $\varphi(h)$ which is called the (time 0 -) quantum field. Obviously the function $\Phi(h \otimes \delta_s)$ is affiliated with $\mathfrak{M}_{v,+}$ and, because of (2.7), it is in $L^2(\mathcal{S}', dv)$, for all $s \geq 0$. It then follows that

$$\begin{aligned} \varphi(h)\Omega &\equiv \frac{d}{d\lambda} e^{i\lambda\varphi(h)}\Omega \Big|_{\lambda=0} \\ &= \frac{d}{d\lambda} v(e^{i\lambda\Phi(h \otimes \delta_0)}) = v(\Phi(h \otimes \delta_0)) \end{aligned}$$

exists and

$$\|\varphi(h)\Omega\|_{\mathcal{X}_w} = \|\Phi(h \otimes \delta_0)\Omega\|_{\mathcal{X}_v} < \infty, \text{ i. e.}$$

the physical vacuum Ω is in the domain of the s. a. operator $\varphi(h)$. Moreover one can easily show that

$$\bar{V}_\tau \varphi(h)\Omega = \bar{V}_\tau v(\Phi(h \otimes \delta_0)) = v(\Phi(h \otimes \delta_\tau))$$

and hence

$$G_{h_1, h_2}(\tau) = (\varphi(\bar{h}_1)\Omega, \bar{V}_\tau \varphi(h_2)\Omega), \text{ for all } \tau \geq 0.$$

Since the infinitesimal generator H of $\{\bar{V}_\tau \mid \tau \geq 0\}$ is positive, $G_{h_1, h_2}(\tau)$ is analytic in τ for $\operatorname{Re} \tau > 0$ and

$$|G_{h_1, h_2}(\tau)| \leq \sqrt{G_{\bar{h}_1, h_1}(0)G_{\bar{h}_2, h_2}(0)}$$

$$\begin{aligned} \text{Therefore } \mathscr{W}^2(h_1 \otimes \delta_0, h_2 \otimes \delta_t) &\equiv \lim_{s \rightarrow 0^+} G_{h_1, h_2}(s + it) \\ &= (\varphi(\bar{h}_1)\Omega, e^{itH}\varphi(h_2)\Omega) \end{aligned}$$

exists and is a continuous function of t for all h_1 and h_2 in \mathscr{S} . The tempered distribution $\mathscr{W}^2(\xi, \eta)$ is easily shown to be Poincaré-invariant (See e. g. [29] [30] [31] and [24]). It is the two-point Wightman distribution.

At this point we should add some comments on the significance of Axiom C1).

From the well-known spectral representation of the two point Wightman distribution we derive the following spectral representation of the two point Schwinger function:

$$\mathfrak{S}^2(\xi, \eta) = \int_0^\infty d\rho(m^2) S_m^2(\xi - \eta), \quad (2.11)$$

where $S_m^2(\xi - \eta)$ is the kernel of $(-\Delta + m^2)^{-1}$, ρ is a measure supported in $[0, \infty]$ (and in two space-time dimensions $\rho([0, \varepsilon]) \rightarrow 0$, as $\varepsilon \rightarrow 0$, because of the infrared divergencies).

Axiom C1) restricts the growth of the measure ρ as $m^2 \rightarrow \infty$. It holds if e. g. the measure ρ is *finite* (i. e. the theory is canonical in a weak sense, or if $d\rho(x) \leq x^{-\frac{1}{2}-\varepsilon} dx$, as $x \rightarrow \infty$) (*).

Step 4. The φ -Bound and the Construction of the Schwinger Functions.

We have shown in step 3 that for all real functions h with $\|h\|_{\mathscr{S}} < \infty$ $\varphi(h)$ is a s. a. operator on \mathscr{K}_W and that the physical vacuum Ω is in the domain of $\varphi(h) : \|\varphi(h)\Omega\|_{\mathscr{K}_W} = \|\Phi(h \otimes \delta_0)\Omega\|_{\mathscr{K}_V} \leq \|h\|_{\mathscr{S}}$.

Let $||| \cdot |||$ be the norm of Axiom C2) and let h be a real function with $|||h||| < \infty$ and $\|h\|_{\mathscr{S}} < \infty$. The major goal of step 4 is the proof of the following inequality:

There are positive, finite constants c_1 and c_2 such that in the sense of densely defined quadratic forms on $\mathscr{K}_W \times \mathscr{K}_W$

$$\pm \varphi(h) \leq c_1 |||h||| (H + c_2) \quad (2.12)$$

It turns out that, given Axioms A, B, and C1) inequality (2.12) is equivalent to Axiom C2).

We now have to discuss the measurability and selfadjointness of certain functions on \mathscr{S}' of importance for the following arguments with respect to the measure ν .

(*) The author has recently extended all results in Steps 1, 2, 4 to the case where $\int \frac{d\rho(m^2)}{m^2} < \infty$.

Wherever the proof in one of the following statements is omitted we feel it is straightforward and therefore we leave it to the reader. These proofs always consist in applying Duhamel's formula

$$e^A - e^B = \int_0^1 ds e^{sA}(A - B)e^{(1-s)B},$$

the fact that the Euclidean vacuum is cyclic and separating for the algebra generated by $\{ e^{i\Phi(f)} \mid f \in \mathcal{S} \}$ and that it is in the domain of the sharp-time fields $\Phi(h \otimes \delta_s)$, $h \in \mathcal{S}_1$, the Schwartz inequality and the properties of the two point function $\mathfrak{G}^2(\xi, \eta)$ established in step. 3. For the basic techniques of these proofs the reader is referred to [6] (sections 3 and 4).

(a) Let f be a function on \mathbb{R}^2 . We set $f'(x) \equiv f(x, t)$ and we define a norm $|\cdot|_1$ by

$$|f|_1 \equiv \int dt \|f'\|_{\mathcal{S}}$$

Then, obviously,

$$|\mathfrak{G}^2(f, g)| \leq \text{Const } |f|_1 |g|_1 \tag{2.13}$$

and therefore the functional $J(f)$ (of Axiom A) is continuous in f in the norm $|\cdot|_1$ (by Duhamel's formula and the Schwartz inequality). Furthermore, if $\{f_n\}$ is a sequence of functions in \mathcal{S} converging to a real function f in the norm $|\cdot|_1$, then

$$s - \lim_{n \rightarrow \infty} e^{i\lambda\Phi(f_n)} \equiv e^{i\lambda\Phi(f)}$$

exists and defines a unitary group in λ on \mathcal{X}_v . The infinitesimal generator $\Phi(f)$ is s. a. on \mathcal{X}_v and is in $L^2(\mathcal{S}', dv)$, i. e. the Euclidean vacuum is in the domain of $\Phi(f)$, because of (2.13).

EXAMPLE. — Let h be a real function with finite $\|\cdot\|_{\mathcal{S}}$ -norm and let χ_T denote the characteristic function of the interval $[-T, T]$. Then

$$|h \otimes \chi_T|_1 = 2T \|h\|_{\mathcal{S}}$$

is finite. Thus $\Phi(h \otimes \chi_T)$ is a s. a., measurable function in $L^2(\mathcal{S}' dv)$ and hence $e^{\pm\Phi(h \otimes \chi_T)}$ is a s. a. measurable function on \mathcal{S}' . We shall see that,

if in addition $|||h||| \leq \frac{1}{2}, \int_{\mathcal{S}'} dv(q) e^{\pm q(h \otimes \chi_T)} \equiv J(\mp ih \otimes \chi_T)$ is finite, as a consequence of Axiom C2). This will lead to inequality (2.12).

(b) Let $F_n(x) \equiv \begin{cases} x, & x \leq n \\ n, & \text{otherwise} \end{cases}$

From lemma 2.2 we know that for h a real function on \mathbb{R} with $\|h\|_{\mathcal{S}} < \infty$ $\Phi(h \otimes \delta_s)$ is a s. a. function on \mathcal{S}' , for all real s . Therefore $F_n(\Phi(h \otimes \delta_s))$ is a s. a. operator on \mathcal{X}_v bounded from above by n .

By (2.7) $\Phi(h \otimes \delta_s)$ is in $L^2(\mathcal{S}', dv)$ and hence

$$\|(F_n(\Phi(h \otimes \delta_s)) - \Phi(h \otimes \delta_s))\Omega\| \rightarrow 0, \text{ as } n \rightarrow \infty \tag{2.15}$$

Thus

$$e^{i\lambda F_n(\Phi(h \otimes \delta_s))} \rightarrow e^{i\lambda\Phi(h \otimes \delta_s)}, \text{ as } n \rightarrow \infty. \tag{2.16}$$

DEFINITION. — Let C be the class of complex-valued, continuous functions $\{g(t)\}$ on \mathbb{R} with the properties that

$$\text{Im } g \leq 0, \int dt |g(t)| < \infty. \tag{2.17}$$

Let $C_{\mathbb{R}}$ be the class of real valued functions in C .

LEMMA 2.3. — Let g be in C and $\text{supp } g \subseteq [-T, T]$ and let $\tau_{m,N} \equiv 2 \frac{m}{N} T - T$. Then

$$U_N(n, h, g) \equiv \prod_{m=0}^N e^{i \frac{2T}{N} g(\tau_{m,N}) F_n(\Phi(h \otimes \delta_{\tau_{m,N}}))},$$

is a well defined operator on \mathcal{X} , and

$$\begin{aligned} \|U_N(n, h, g)\| &\leq e^{-n \frac{2T}{N} \sum_{m=0}^N \text{Im } g(\tau_{m,N})} \\ s - \lim_{N \rightarrow \infty} U_N(n, h, g) &\equiv e^{i \int d\tau g(\tau) F_n(\Phi(h \otimes \delta_{\tau}))} \end{aligned} \tag{2.18}$$

exists and the norm of this operator is bounded by

$$e^{-n \int d\tau \text{Im } g(\tau)} \tag{2.19}$$

If $\{g_m\}_{m=0}^{\infty}$ is a sequence in C which converges to a function g in the $L^1(\mathbb{R})$ -norm then

$$s - \lim_{m \rightarrow \infty} e^{i \int d\tau g_m(\tau) F_n(\Phi(h \otimes \delta_{\tau}))} \equiv e^{i \int d\tau g(\tau) F_n(\Phi(h \otimes \delta_{\tau}))} \tag{2.20}$$

exists and is bounded.

For g in $C_{\mathbb{R}}$

$$s - \lim_{n \rightarrow \infty} e^{i \int d\tau g(\tau) F_n(\Phi(h \otimes \delta_{\tau}))} = e^{i\Phi(h \otimes g)} \tag{2.21}$$

Finally, for h a real function with $\|h\|_{\mathcal{S}} < \infty$, $\|h\| \leq \frac{1}{2}$ there are finite constants K'_h and c' such that

$$\begin{aligned} 0 \leq \int_{\mathcal{S}'} dv(q) e^{q(h \otimes \chi_T)} &= \lim_{n \rightarrow \infty} \int_{\mathcal{S}'} dv(q) e^{-\int_{-T}^T d\tau F_n(\Phi(h \otimes \delta_{\tau}))} \\ &\leq K'_h e^{c'T} \end{aligned} \tag{2.22}$$

Proof. — It is obvious that $U_N(n, h, g)$ is well defined and

$$\begin{aligned} \|U_N(n, h, g)\| &\leq \left\| \prod_{m=0}^N e^{-n \frac{2T}{N} \text{Im } g(\tau_{m,N})} \right\| \left\| e^{i \frac{2T}{N} \text{Re } g(\tau_{m,N}) F_n(\Phi(h \otimes \delta_{\tau_{m,N}}))} \right\| \\ &= e^{-n \frac{2T}{N} \sum_{m=0}^N \text{Im } g(\tau_{m,N})} \end{aligned}$$

To prove (2.18) it suffices now to show that

$$J_{N,N'} \equiv \|(U_N(n, h, g) - U_{N'}(n, h, g))\|_{\mathcal{X}}^2 \text{ tends to 0, as } N, N' \rightarrow \infty.$$

By Duhamel's formula and the Schwartz inequality

$$J_{N,N'} \leq \text{Const.} \left\| \left[\sum_{m=0}^N \frac{2T}{N} g(\tau_{m,N}) F_n(\Phi(h \otimes \delta_{\tau_{m,N}})) - \sum_{m'=0}^{N'} \frac{2T}{N'} g(\tau_{m',N'}) F_n(\Phi(h \otimes \delta_{\tau_{m',N'}})) \right] \Omega \right\|_{\mathcal{X}_v}^2 \tag{2.23}$$

It is easy to see that

$\langle F_n(\Phi(h \otimes \delta_t))\Omega, F_n(\Phi(h \otimes \delta_{t'}))\Omega \rangle_{\mathcal{X}_v} = (F_n(\varphi(h))\Omega, \bar{V}_{|t-t'|} F_n(\varphi(h))\Omega)_{\mathcal{X}_w}$, (see step 3). Therefore $\langle F_n(\Phi(h \otimes \delta_t))\Omega, F_n(\Phi(h \otimes \delta_{t'}))\Omega \rangle_{\mathcal{X}_v}$ is jointly continuous in t and t' . Using this continuity property and expanding the r. h. s. of (2.23) into a sum of scalar products we see immediately that $J_{N,N'}$ tends to 0, as $N, N' \rightarrow \infty$. This proves (2.18).

Since g is continuous $e^{-n \frac{T}{N} \sum_{m=0}^N \text{Im} g(\tau_{m,N})} \rightarrow e^{-n \int_{-T}^T dt \text{Im} g(t)}$, whence (2.19).
Now

$$\begin{aligned} & \| (e^{i \int d\tau g_m(\tau) F_n(\Phi(h \otimes \delta_\tau))} - e^{i \int d\tau g_{m'}(\tau) F_n(\Phi(h \otimes \delta_\tau))}) \Omega \|_{\mathcal{X}_v}^2 \\ & \leq C_n \int d\tau d\tau' |g_m(\tau) - g_{m'}(\tau)| |g_m(\tau') - g_{m'}(\tau')| \\ & \quad \times \langle F_n(\Phi(h \otimes \delta_\tau))\Omega, F_n(\Phi(h \otimes \delta_{\tau'}))\Omega \rangle_{\mathcal{X}_v} \\ & \leq C_n \|g_m - g_{m'}\|_1^2 \|h\|_{\mathcal{F}}^2 \end{aligned}$$

which tends to 0 as $m, m' \rightarrow \infty$.

Here $C_n \equiv e^{2n \sup_m \|g_m\|_1}$.

Similar reasoning yields (2.21) (Hint:

$$\| (F_n(\Phi(h \otimes \delta_\tau)) - \Phi(h \otimes \delta_\tau))\Omega \|^2 \rightarrow 0, \text{ as } n \rightarrow \infty,$$

uniformly in τ).

We are left with proving (2.22).

Let F be a ν -measurable, integrable function on \mathcal{S}' . We set

$$\langle F \rangle_\nu \equiv \int_{\mathcal{S}'} d\nu(q) F(q)$$

and define $\Phi_n(h, f) \equiv \int d\tau F_n(\Phi(h \otimes \delta_\tau)) f(\tau)$ where f is in C . Let h be a real function on \mathbb{R} with $\|h\|_{\mathcal{F}} < \infty$ and $\|h\| \leq \frac{1}{2}$ and let g and g_T be the functions defined in Axiom C2).

DEFINITION :

$$g_1(t) = \begin{cases} g(-t - T), & t \leq -T \\ 0, & t \geq -T \end{cases}, \quad g_2 = \chi_T,$$

$$g_3(t) = \begin{cases} g(t - T), & t \geq T \\ 0, & t \leq T \end{cases}, \quad g_4(t) = \begin{cases} g(t), & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

Then for all $j = 1, \dots, 4$ $e^{\Phi_n(\pm h, g_j)}$ is a well defined bounded operator on \mathcal{X}_v . We claim that in the $L^2(\mathcal{S}', dv)$ -norm

$$\lim_{n \rightarrow \infty} e^{\Phi_n(\pm h, g_1 + g_3)} = e^{\Phi(\pm h \otimes (g_1 + g_3))} \tag{2.24}$$

It is obvious that the sequence $\{\Phi_n(\pm h, g_1 + g_3)\}_{n=0}^\infty$ is increasing and $\|\Phi(\pm h \otimes (g_1 + g_3)) - \Phi_n(\pm h, g_1 + g_3)\|_{\Omega}^2_{\mathcal{X}_v} \leq \|g_1 + g_3\|_1^2 \|\varphi(h) - F_n(\varphi(h))\|_{\Omega}^2_{\mathcal{X}_w}$

which tends to 0, as $n \rightarrow \infty$. Hence

$$\lim_{n \rightarrow \infty} \Phi_n(\pm h, g_1 + g_3) = \sup_n \Phi_n(\pm h, g_1 + g_3) = \Phi(\pm h \otimes (g_1 + g_3))$$

in $L^2(\mathcal{S}', dv)$ and thus v -almost everywhere on \mathcal{S}' .

Hence $\lim_{n \rightarrow \infty} e^{\Phi_n(\pm h, g_1 + g_3)} = \sup_n e^{\Phi_n(\pm h, g_1 + g_3)} = e^{\Phi(\pm h \otimes (g_1 + g_3))}$, v -almost everywhere on \mathcal{S}' . It suffices therefore to prove that $\{e^{\Phi_n(\pm h, g_1 + g_3)}\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(\mathcal{S}', dv)$.

Now

$$\| (e^{\Phi_n(\pm h, g_1 + g_3)} - e^{\Phi_m(\pm h, g_1 + g_3)}) \Omega \|_{\mathcal{X}_v}^2 \leq \langle e^{2\Phi_n(\pm h, g_1 + g_3)} \rangle_v + \langle e^{2\Phi_m(\pm h, g_1 + g_3)} \rangle_v - 2 \langle e^{\Phi_n(\pm h, g_1 + g_3)} e^{\Phi_m(\pm h, g_1 + g_3)} \rangle_v$$

By the same arguments as given above

$$\lim_{m, n \rightarrow \infty} e^{\Phi_n(\pm h, g_1 + g_3)} e^{\Phi_m(\pm h, g_1 + g_3)} = \sup_{m, n} (\dots) = e^{2\Phi(\pm h \otimes (g_1 + g_3))},$$

v -almost everywhere on \mathcal{S}' .

Therefore, by the monotone convergence theorem, it suffices to establish uniform bounds on

$$\langle e^{2\Phi_n(\pm h, g_1 + g_3)} \rangle_v \quad \text{and} \quad \langle e^{\Phi_n(\pm h, g_1 + g_3)} e^{\Phi_m(\pm h, g_1 + g_3)} \rangle_v$$

Since $\langle e^{\Phi_n(\dots)} e^{\Phi_m(\dots)} \rangle_v \leq \langle e^{2\Phi_n(\dots)} \rangle_v^{\frac{1}{2}} \langle e^{2\Phi_m(\dots)} \rangle_v^{\frac{1}{2}}$ bounds on the first integral above yield bounds on the second one.

Now

$$\begin{aligned} 0 &\leq \langle e^{2\Phi_n(\pm h, g_1 + g_3)} \rangle_v \\ &= (v(e^{2\Phi_n(\pm h, g_4)}), \bar{V}_{2T} v(e^{2\Phi_n(\pm h, g_4)}))_{\mathcal{X}_w} \\ &\leq \|v(e^{2\Phi_n(\pm h, g_4)})\|_{\mathcal{X}_w}^2 \\ &= \langle e^{2\Phi_n(\pm h, g_4 + (g_4) \otimes)} \rangle_v = \langle e^{2\Phi_n(\pm h, g)} \rangle_v \\ &\leq \langle e^{2\Phi(\pm h \otimes g)} \rangle_v \equiv J(\mp i2h \otimes g) \end{aligned}$$

which is finite by Axiom C2). This completes the proof of (2.24) and shows that $e^{\pm \Phi(h \otimes (g_1 + g_3))}$ is in $L^2(\mathcal{S}', dv)$.

The proof of (2.22) is now easy:

$$\begin{aligned} \langle e^{\Phi_n(\pm h, \chi_T)} \rangle_v &= \langle e^{\Phi_n(\pm h, \chi_T) \pm \Phi(h \otimes (g_1 + g_3))} e^{\mp \Phi(h \otimes (g_1 + g_3))} \rangle_v \\ &\leq \langle e^{2\Phi_n(\pm h, \chi_T)} e^{\pm 2\Phi(h \otimes (g_1 + g_3))} \rangle_v^{\frac{1}{2}} \langle e^{\mp 2\Phi(h \otimes (g_1 + g_3))} \rangle_v^{\frac{1}{2}} \\ &\leq J(\pm i2h \otimes g)^{\frac{1}{2}} \langle e^{2[\Phi_n(\pm h, \chi_T) \pm \Phi(h \otimes (g_1 + g_3))]} \rangle_v^{\frac{1}{2}} \end{aligned}$$

As before we show that

$$0 \leq \lim_n e^{2[\Phi_n(\pm h, \chi_T) \pm \Phi(h \otimes (g_1 + g_3))]} = \sup_n e^{2[\Phi_n(\pm h, \chi_T) \pm \Phi(h \otimes (g_1 + g_3))]} = e^{\pm 2\Phi(h \otimes g_T)}$$

and

$$0 \leq \lim_{n \rightarrow \infty} e^{2\Phi_n(\pm h, \chi_T)} = \sup_n (\dots) = e^{\pm 2\Phi(h \otimes \chi_T)}$$

v -almost everywhere. Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle e^{2[\Phi_n(\pm h, \chi_T) \pm \Phi(h \otimes (g_1 + g_3))]} \rangle_v^{\frac{1}{2}} &= \langle e^{\pm 2\Phi(h \otimes g_T)} \rangle_v^{\frac{1}{2}} \\ &\equiv J(\mp i2h \otimes g_T)^{\frac{1}{2}} \leq K_{2h} e^{\frac{\epsilon}{2} T} \end{aligned}$$

and hence $\langle e^{\Phi_n(\pm h, \chi_T)} \rangle_v \leq \text{Const } e^{\frac{\epsilon}{2} T}$ and by the monotone convergence theorem $\langle e^{\pm \Phi(h \otimes \chi_T)} \rangle_v \leq \text{Const } e^{\frac{\epsilon}{2} T}$. (2.25)

Q. E. D.

Construction of the perturbed Hamiltonian and φ -Bound.

We define a linear subspace

$$\mathcal{D} = \{ \theta \in \mathcal{X}_W \mid \theta = v(F) \text{ for some } F \in \mathfrak{M}_{v,+} \} \text{ for } \mathcal{X}_W. \quad (2.26)$$

From (2.3) and the construction of \mathcal{X}_W we know that \mathcal{D} is dense in \mathcal{X}_W .

One can now construct semigroups $\{ P_t^\pm \mid t \geq 0 \}$ on \mathcal{D} .

Let

$$\begin{aligned} P_t^{\pm, N} \theta &\equiv (\bar{V}_{t/N} e^{t/NF_n(\varphi(\pm h))})^N \theta \\ &= (\bar{V}_{t/N} e^{t/NF_n(\varphi(\pm h))})^{N-1} v(T_{t/N}(e^{t/NF_n(\Phi(\pm h \otimes \delta_0))} F) T_{t/N}^*) \\ &= (\bar{V}_{t/N} e^{t/NF_n(\varphi(\pm h))})^{N-1} v(e^{t/NF_n(\Phi(\pm h \otimes \delta_{t/N}))} F_{t/N}), \end{aligned}$$

where $F_{t/N} \equiv T_{t/N} F T_{t/N}^*$,

$$= \dots = v(e^{t/N \sum_{m=1}^N F_m(\Phi(\pm h \otimes \delta_{m,t/N}))} F_t)$$

Obviously

$$\begin{aligned} \| P_t^{\pm, N} \theta \| &\leq \| \bar{V}_{t/N} \|^N \cdot \| e^{t/NF_n(\varphi(\pm h))} \| \cdot \| \theta \| \\ &\leq e^{m t} \| \theta \| \end{aligned} \quad (2.27)$$

From lemma 2.3, (2.18) we know that

$$s - \lim_{N \rightarrow \infty} e^{t/N \sum_{m=1}^N F_m(\Phi(\pm h \otimes \delta_{m,t/N}))} F = e^{\Phi_n(\pm h, \chi_t^+)} F,$$

where χ_t^+ is the characteristic function of $[0, t]$, in the norm of \mathcal{X}_v . Hence, by (2.3)

$$P_t^\pm \theta \equiv s - \lim_{N \rightarrow \infty} P_t^{\pm, N} \theta = v(e^{\Phi_n(\pm h, \chi_t^+)} F_t) \text{ exists,} \quad (2.28)$$

where the limit is with respect to the strong topology on \mathcal{X}_W . It now follows

from (2.27), Theorem 1, the construction of \mathcal{X}_w and \bar{V}_t that $\{P_t^\pm \mid t \geq 0\}$ are weakly continuous, s. a., exponentially bounded semigroups on \mathcal{X}_w . Hence they have s. a., infinitesimal generators A_\pm which are bounded below by $-n$.

Since $e^{\Phi_n(\pm h, \lambda t^\pm)}$ is in $\mathfrak{M}_{v,+}$ and since F_t is in $\mathfrak{M}_{v,+}$ if F is, for all $t \geq 0$, we conclude that the semigroups $P_t^\pm \mid t \geq 0$ leave \mathcal{D} invariant. Thus $\mathcal{D}_\pm \equiv \{P_{t_0}^\pm \theta \mid \theta \in \mathcal{D}, t_0 > 0\}$ are cores for A_+, A_- , respectively. If ψ is in \mathcal{D}_\pm there exists a $G \in \mathfrak{M}_{v,+}$ such that $\psi = v(G)$. Therefore

$$\begin{aligned} \frac{1}{t}(P_t^\pm - I)\psi &= \frac{1}{t}v(e^{\Phi_n(\pm h, \lambda t^\pm)}G_t - G) \\ &= v\left(\frac{1}{t}(e^{\Phi_n(\pm h, \lambda t^\pm)} - I)G_t\right) + v\left(\frac{1}{t}(G_t - G)\right) \end{aligned} \tag{2.29}$$

The l. h. s. of (2.28) tends to $-A_\pm$, as $t \rightarrow 0$.

Since G_t is in $\mathfrak{M}_{v,+}$, i. e. $\|G_t\|_{L^\infty(\mathcal{S}^1, dv)} < \infty$, and therefore

$$\begin{aligned} \|v(F_n(\Phi(\pm h \otimes \delta_0))G)\|_{\mathcal{X}_w} &\leq \|F_n(\Phi(\pm h \otimes \delta_0))G\Omega\|_{\mathcal{X}_v} \\ &\leq \|F_n(\Phi(\pm h \otimes \delta_0))\Omega\|_{\mathcal{X}_v} \|G\|_{L^\infty(\mathcal{S}^1, dv)} \\ &\leq \|\varphi(\pm h)\Omega\| \|G\|_{L^\infty(\mathcal{S}^1, dv)} < \infty \end{aligned}$$

the first term on the r. h. s. of (2.29) has a limit which is given by

$$v(F_n(\Phi(\pm h \otimes \delta_0))G) = F_n(\varphi(\pm h))\psi.$$

Thus the second term on the r. h. s. must have a limit.

This limit is equal to

$$\begin{aligned} \lim_{t \rightarrow 0} v\left(\frac{1}{t}(G_t - G)\right) &= \frac{1}{t}(\bar{V}_t - I)\psi \\ &= -H\psi \end{aligned}$$

Therefore on the core \mathcal{D}_\pm

$$-A_\pm = -(H - F_n(\varphi(\pm h))) \tag{2.30}$$

THEOREM 2.4. — Suppose that h is a real function on \mathbb{R} with $\|h\|_{\mathcal{S}}$ and $\|h\|$ finite. Then in the sense of densely defined, quadratic forms on $\mathcal{X}_w \times \mathcal{X}_w$

$$\pm \varphi(h) \leq 2 \|h\| \cdot \left(H + \frac{c}{4}\right)$$

Proof. — We show that for real h with $\|h\|_{\mathcal{S}} < \infty$ and $\|h\| \leq \frac{1}{2}$

$$A_\pm \equiv [H - F_n(\varphi(\pm h))]^- \geq -\frac{c}{4}, \tag{2.31}$$

uniformly in $n < \infty$.

Thus on the quadratic form domain $Q(H)$ of H

$$F_n(\varphi(\pm h)) \leq H + \frac{c}{4}, \text{ uniformly in } n < \infty. \tag{2.32}$$

Now $Q(H) \cap \mathcal{D}$, where \mathcal{D} is defined in (2.26), is dense in $Q(H)$ in the norm $\|\psi\|_1 = \|H^\pm \psi\|_{\mathcal{X}_w} + \|\psi\|_{\mathcal{X}_w}$ (Recall that \mathcal{D} contains a core for H, H^\pm .)

Since $\|([F_n(\varphi(\pm h)) - \varphi(\pm h)]\Omega)\|_{\mathcal{X}_w} \rightarrow 0$, as $n \rightarrow \infty$,

$$s - \lim_{n \rightarrow \infty} F_n(\varphi(\pm h)) = \pm \varphi(h) \text{ on } \mathcal{D}.$$

From this and (2.32) we now conclude that

$$\pm \varphi(h) \leq H + \frac{c}{4} \text{ on } Q(H).$$

Since $\varphi(h)$ is linear in h , this proves the theorem.

We still must prove (2.31). From (2.27) we know that

$$A_\pm = [H - F_n(\varphi(\pm h))]^-$$

is bounded below by $-n$. Thus

$$\delta E_n(\pm h) \equiv \inf \text{spec } [H - F_n(\varphi(\pm h))]^- > -\infty.$$

From the spectral theorem and the properties of \mathcal{D} it follows that, given $\varepsilon > 0$, there exists a unit vector ψ_ε in \mathcal{D} such that

$$\langle \psi_\varepsilon, e^{-2T(H - F_n(\varphi(\pm h)))^-} \psi_\varepsilon \rangle \geq e^{-2T[\delta E_n(\pm h) + \varepsilon]},$$

for all positive T .

Since ψ_ε is in \mathcal{D} , there is a G_ε in $\mathfrak{M}_{v,+}$ such that

$$\psi_\varepsilon = v(G_\varepsilon).$$

Thus

$$\begin{aligned} & \langle e^{-T[H - F_n(\varphi(\pm h))]^-} \psi_\varepsilon, e^{-T[H - F_n(\varphi(\pm h))]^-} \psi_\varepsilon \rangle \\ &= \langle v(e^{\Phi_n(\pm h, \chi^\dagger)}(G_\varepsilon)_T), v(e^{\Phi_n(\pm h, \chi^\dagger)}(G_\varepsilon)_T) \rangle \\ &= \langle \theta(e^{\Phi_n(\pm h, \chi^\dagger)}(\bar{G}_\varepsilon)_T) e^{\Phi_n(\pm h, \chi^\dagger)}(G_\varepsilon)_T \rangle_v \\ &= \langle \theta(\bar{G}_\varepsilon)_T(G_\varepsilon)_T e^{\Phi_n(\pm h, \chi^\dagger)} \rangle_v \\ &\leq \|\theta(\bar{G}_\varepsilon)_T(G_\varepsilon)_T\|_{L^\infty(\mathcal{S}', d\nu)} \langle e^{\Phi_n(\pm h, \chi^\dagger)} \rangle_v \\ &\leq \|G_\varepsilon\|_{L^\infty(\mathcal{S}', d\nu)}^2 J(\mp ih \otimes \chi_T) \\ &\leq \text{Const.} \|G_\varepsilon\|_{L^\infty(\mathcal{S}', d\nu)}^2 e^{\frac{\varepsilon}{2}T}, \text{ by (2.25)} \end{aligned}$$

Thus

$$e^{-2T[\delta E_n(\pm h) + \varepsilon]} \leq \text{Const} \|G_\varepsilon\|_{L^\infty(\mathcal{S}', d\nu)}^2 e^{\frac{\varepsilon}{2}T}$$

Taking logarithms and passing to the limit $T = \infty$ we obtain

$$-\delta E_n(\pm h) - \varepsilon \leq \frac{c}{4}$$

Since $\varepsilon > 0$ is arbitrarily small this proves (2.31). Thus the theorem is proven. Q. E. D.

COROLLARY 2.5. — For all $t \geq 0$ and for $||| h ||| \leq \frac{1}{2}$

$$s - \lim_{n \rightarrow \infty} e^{-t(H - F_n(\varphi(\pm h)))^-} = e^{-t(H \mp \varphi(h))^-} \text{ exists} \tag{2.33}$$

and determines an exponentially bounded, weakly continuous semigroup on \mathcal{X}_W . For $G \in \mathfrak{M}_{v,+}$

$$e^{-t(H \mp \varphi(h))^-} v(G) = v(e^{\pm \Phi(h \otimes \chi_t^*)} G) \tag{2.34}$$

Proof. — Equation (2.33) follows directly from Theorem 2.4 and its proof. Equation (2.34) follows from (2.28), Theorem 2.4 and its proof. Q. E. D.

Bounds on the J-functional and Construction of the Schwinger Functions.

Let J be the functional defined in Axiom A. We want to show that $J(\lambda f)$, f in \mathcal{S} , is the boundary value of a function $J(\xi f)$ which is analytic in ξ on the domain $\Sigma_f \equiv \{ \xi \in \mathbb{C} \mid |\operatorname{Im} \xi| < 1/|f|_{\mathcal{S}} \}$, where $|\cdot|_{\mathcal{S}}$ is some norm on real functions over \mathbb{R}^2 which is continuous on \mathcal{S} .

It then follows from these analyticity properties that the m^{th} moments

$$\begin{aligned} \mathfrak{S}^m(f_1, \dots, f_m) &\equiv \left\langle \prod_{j=1}^m \Phi(f_j) \right\rangle_v = (-i)^m \frac{\partial^m}{\partial \lambda_1 \dots \partial \lambda_m} \\ &\times J \left(\sum_{j=1}^m \lambda_j f_j \right) \Big|_{\lambda_1 = \dots = \lambda_m = 0} \end{aligned} \tag{2.35}$$

of the measure v exist for all positive, finite integers m and arbitrary test-functions f_1, \dots, f_m in $\mathcal{S}(\mathbb{R}^2)$ and that they are continuous, multilinear functionals on $\mathcal{S}(\mathbb{R}^2)^{\times m}$. By the nuclear theorem $\mathfrak{S}^m(x_1, t_1, \dots, x_m, t_m)$ is a tempered distribution. It is called the m -point Schwinger function (or Euclidean Green's function).

We may therefore call J the *generating functional of the Schwinger functions*, [6]. Let f be a function over \mathbb{R}^2 and $f^t(x) \equiv f(x, t)$. We define

$$|f|_{\mathcal{S}} \equiv \int dt ||| f^t ||| + \sup_t ||| f^t ||| \tag{2.36}$$

Let f be in $\mathcal{S}(\mathbb{R}^2)$ with $\operatorname{supp} f \subset \mathbb{R} \times [-T, T]$ for some $T < \infty$ and such that $|\operatorname{Im} f|_{\mathcal{S}} < \frac{1}{2}$. Let $\tau_{n,N} = -T + \frac{2n}{N}T$.

It follows now from Lemma 2.3, Theorem 2.4, and Corollary 2.5 that

$$\lim_{N \rightarrow \infty} \left(\Omega, \prod_{n=0}^N e^{-\frac{2T}{N} [H + i\varphi(f^{*n,N})]^-} \Omega \right) = \langle e^{i\Phi(f)} \rangle_v \equiv J(f) \text{ exists,} \quad (2.37)$$

and by Theorem 2.4 $H - \varphi(\text{Im } f^{*n,N}) \geq c \|\text{Im } f^{*n,N}\|$, for

$$\|\text{Im } f^{*n,N}\| < |\text{Im } f|_{\mathcal{S}} \leq \frac{1}{2}$$

$$\begin{aligned} \left(\Omega, \prod_{n=0}^N e^{-\frac{2T}{N} [H + i\varphi(f^{*n,N})]^-} \Omega \right) &\leq \prod_{n=0}^N \left\| e^{-\frac{2T}{N} [H - \varphi(\text{Im } f^{*n,N})]^-} \right\| \\ &\leq e^{\frac{2T}{N} \sum_{n=0}^N c \|\text{Im } f^{*n,N}\|} \end{aligned}$$

Thus $|J(f)| \leq J(i \text{Im } f) \leq e^{c|\text{Im } f|_{\mathcal{S}}}$, for $|\text{Im } f|_{\mathcal{S}} \leq \frac{1}{2}$. (2.38)

In particular, if f is in \mathcal{S} and $|\text{Im } \xi| |f|_{\mathcal{S}} < \frac{1}{2}$

$$|J(\xi f)| \leq J(i(\text{Im } \xi) f) < e^{c|\text{Im } \xi| |f|_{\mathcal{S}}} \quad (2.39)$$

It is shown in [6], Lemma 3.2, that (2.39) implies continuity of $J(f)$ in f with respect to the norm $|\cdot|_{\mathcal{S}}$. Moreover for functions

f_1, \dots, f_m with $|f_j|_{\mathcal{S}} < \infty, j = 1, \dots, m$

$$\mathfrak{S}^m(f_1, \dots, f_m) = \left\langle \prod_{j=1}^m \Phi(f_j) \right\rangle_v = \frac{(-i)^m \partial^m}{\partial \lambda_1 \dots \partial \lambda_m} J \left(\sum_{j=1}^m \lambda_j f_j \right) \Big|_{\lambda_1 = \dots = \lambda_m = 0}$$

exists, is linear in f_1, \dots, f_m and

$$|\mathfrak{S}^m(f_1, \dots, f_m)| \leq c^m m! \prod_{j=1}^m |f_j|_{\mathcal{S}}, \text{ for some finite } c \quad (2.40)$$

and all m , by the Cauchy formula for analytic functions. See [6], Theorem 3.8 (c). This proves that $\mathfrak{S}^m(x_1, t_1, \dots, x_m, x_t)$ is tempered and determines the order of this distribution in dependence of m .

Let $\underline{t}(f) \equiv \inf \{ t | \exists x \text{ such that } \langle x, t \rangle \in \text{supp } f \}$, where f is some function on \mathbb{R}^2 , and let

$$\bar{t}(f) \equiv \sup \{ t | \exists x \text{ such that } \langle x, t \rangle \in \text{supp } f \}.$$

Let v be the completion of $\mathcal{S}(\mathbb{R}^2)$ in the norm $|\cdot|_{\mathcal{S}}$ and let v_d be the class of functions on \mathbb{R}^2 which are bounded and piecewise continuous and such

that $-\infty < \underline{t}(f) < \bar{t}(f) < \infty$, there exists a time ordered sequence of disjoint open intervals $\Delta_1, \dots, \Delta_{n(f)}$ on \mathbb{R} such that

$$\left(\bigcup_{l=1}^{n(f)} \Delta_l \right)^- = [\underline{t}(f), \bar{t}(f)] \quad \text{and} \quad f(x, t) = f^{\Delta_l}(x),$$

for $t \in \Delta_l$ and some function f^{Δ_l} with $||| f^{\Delta_l} ||| < \infty$, all $l = 1, \dots, n(f)$. Clearly each f in \mathcal{V} is the limit of a sequence $\{f_n\}_{n=0}^\infty \subset \mathcal{V}_d$ in the norm $|\cdot|_{\mathcal{F}}$ (2.41)

Let now f_1, \dots, f_m be in \mathcal{V}_d and

$$\bar{t}(f_1) < \underline{t}(f_2) < \bar{t}(f_2) < \underline{t}(f_3) < \dots < \bar{t}(f_{m-1}) < \underline{t}(f_m)$$

Using equation (2.37), corollary 2.5 and the φ -bound proven in Theorem 2.4 we conclude that

$$\begin{aligned} \mathfrak{S}^m(f_1, \dots, f_m) &= (-i)^m \frac{\partial^m}{\partial \lambda_1, \dots, \partial \lambda_m} \lim_{N \rightarrow \infty} \\ &\quad \left(\Omega, \prod_{n=0}^N e^{-\frac{2T}{N} [H + i \sum_{j=1}^m \lambda_j \varphi(f_j^{(n)})]} \Omega \right) \Big|_{\lambda_1 = \dots = \lambda_m = 0} \\ &= \int \prod_{i=1}^m dt_i \left(\Omega, \prod_{j=1}^m (\varphi(f_j^{t_j}) \bar{V}_{t_{j+1}-t_j}) \Omega \right), \end{aligned} \tag{2.42}$$

where $T > \max \{ |\underline{t}(f_1)|, |\bar{t}(f_m)| \}$, by standard arguments. Let h_1, \dots, h_m be real functions on \mathbb{R} with $||h_i||_{\mathcal{F}} < \infty$ and $||| h_i ||| \leq \infty$, $i = 1, \dots, m$.

Then it follows from (2.42) and (2.41) that in the sense of distributions in t_1, \dots, t_m

$$\mathfrak{S}^m(h_1 \otimes \delta_{t_1}, \dots, h_m \otimes \delta_{t_m}) = \left(\Omega, \prod_{j=1}^m (\varphi(h_j) \bar{V}_{t_{j+1}-t_j}) \Omega \right). \tag{2.43}$$

provided $t_1 < t_2 < \dots < t_m$. We set $\tau_i = t_{i+1} - t_i$, $i = 1, \dots, m$. The r. h. s. of (2.43) is the restriction of a function $F_{h_1, \dots, h_m}(\tau_1, \dots, \tau_{m-1})$ analytic in $\tau_1, \dots, \tau_{m-1}$ on $\mathfrak{I}_{m-1} \equiv \{ \langle \tau_1, \dots, \tau_{m-1} \rangle \mid \text{Re } \tau_i > 0 \}$ to the set $\{ \tau_1 = t_2 - t_1 > 0, \dots, \tau_{m-1} = t_m - t_{m-1} > 0 \}$. By Theorem 2.4

$$|F_{h_1, \dots, h_m}(\tau_1, \dots, \tau_{m-1})| \leq 2^m \prod_{j=1}^m ||| h_j ||| \frac{e^{\frac{\varepsilon}{4} |\tau_j|}}{|\text{Re } \tau_j|} \tag{2.44}$$

We can now proceed along the lines explored by Nelson in the basic paper [24] and get the following:

THEOREM 2.6. — *The Schwinger functions $\mathfrak{S}^m(x_1, t_1, \dots, x_m, t_m)$ obtained from a generating functional J which satisfies Axioms A, B, and C are the Wightman functions (denoted by $\mathcal{W}^m(x_1, it_1, \dots, x_m, it_m)$) at the Eucli-*

dean points $\{ \langle x_1, it_1, \dots, x_m, it_m \rangle \mid \langle x_i, t_i \rangle \neq \langle x_j, t_j \rangle \text{ for } i \neq j, x_j \text{ and } t_j \text{ real for all } j \}$ of a unique relativistic Quantum Field Theory satisfying all Wightman axioms (with the possible exception of the uniqueness of the vacuum).

The Hilbert space, the (time 0 -) quantum field and the energy momentum operator (H, P) obtained from the Wightman distributions

$$\{ \mathcal{W}^m(x_1, t_1, \dots, x_m, t_m) \}_{m=0}^\infty$$

by Wightman's reconstruction theorem [18] [40] are the same as the Hilbert space \mathcal{H}_w , the (time 0 -) field ϕ , the infinitesimal generator H of the s. a. contraction semigroup ∇_t (and the infinitesimal generator P of the unitary space-translation group $\{ T_x \mid x \in \mathbb{R} \}$ defined in a natural way on \mathcal{H}_w) obtained in steps 1, 2, and 3 of Section 2.

Remarks Concerning the Second Part of Theorem 2.6. — It is obvious that from the bounds (2.38), (2.39) on $|J(\xi f)|$ it follows that the « Euclidean vacuum » Ω is an analytic vector for the fields $\{ \Phi(f) \mid |f|_\varphi < \infty \}$. Thus the vectors $e^{i\Phi(f)}\Omega \in \mathcal{H}_v$ and $v(e^{i\Phi(f)}) \in \mathcal{H}_w$ (with $\text{supp } f \subseteq \mathbb{R} \times \{ t \geq 0 \}$) can be obtained by power series expansion of $e^{i\Phi(f)}$, provided $|f|_\varphi$ is sufficiently small. But this, the analyticity properties of the Schwinger functions in the time variables and the Reeh-Schlieder theorem [18] [40] for some complex neighbourhood of the Euclidean points imply that the space \mathcal{H}_w , the (time 0 -) field ϕ and the energy momentum operator (H, P) obtained from the functional J by the constructions in steps 1, 2, and 3 of Section 2 are the same as the ones one gets from the Schwinger functions

$$\{ \mathcal{S}^m(\xi_1, \dots, \xi_m) \}_{m=0}^\infty$$

by Osterwalder-Schrader reconstruction [29] [30], or Nelson's reconstruction [24]. By results of [29] [30] this proves the second part of Theorem 2.6. All other details for the proof of Theorem 2.6 follow directly from our results in Section 2 and refs [24] [29] [30].

Further Consequences of Axioms A, B, and C and Connections to Nelson's Axioms.

1) Suppose that Axioms A1)-A5), B and C hold. Then all the results of Section 2 (with the exception of the Euclidean invariance of the Schwinger functions and the Poincaré-invariance of the Wightman distributions $\{ \mathcal{W}^m \}_{m=0}^\infty$ obtained in Theorem 2.6) remain true. Thus Axioms A1)-A5), B and C always imply the existence of quantum fields reconstructed from the Wightman distributions [18] [40] (and acting as densely defined operators on the space \mathcal{H}_w of step 1).

If in addition to Axioms A1)-A5) B and C, the Schwinger functions are

Euclidean invariant then the functional J is *Euclidean invariant*. For, if $|\xi| < \frac{1}{2}|f|_{\mathcal{S}^1}$, where $|f|_{\mathcal{S}}$ is finite, then by (2.39) and (2.40)

$$J(\xi f) = \sum_{m=0}^{\infty} \frac{(i\xi)^m}{m!} \mathfrak{S}^m(f, \dots, f), \tag{2.45}$$

which is Euclidean invariant. Thus Axioms A, B and C hold.

2) Suppose that Axioms A1)-A5), B and C1) hold and that in the sense of densely defined quadratic forms on $\mathcal{H}_W \times \mathcal{H}_W$

$$\pm \varphi(h) \leq \| \| h \| \| (H + c) \tag{2.46}$$

Then for arbitrary, real functions g, g_T such as specified in Axiom C2) or for $g_T = \chi_T$ and for arbitrary real function h on \mathbb{R}^1 with $\| \| h \| \| \leq 1$

$$J(\pm ih \otimes g_T) \leq e^{c\|g_T\|_1} \leq e^{2c} e^{2cT}, \tag{2.47}$$

whence Axiom C2).

Proof. — Let $T' \equiv T + 1$. Application of (2.37) yields

$$\begin{aligned} 0 \leq J(\pm ih \otimes g_T) &\equiv \langle e^{\mp \Phi(h \otimes g_T)} \rangle_{\nu} \\ &= \lim_{N \rightarrow \infty} \left(\Omega, \prod_{n=0}^N e^{-2T'/N[H \pm g(\tau_{n,N})\varphi(h)]} - \Omega \right) \\ &\leq \lim_{N \rightarrow \infty} \prod_{n=0}^N \| e^{-\frac{2T'}{N}[H + g(\tau_{n,N})\varphi(h)]} \| \\ &= \lim_{N \rightarrow \infty} e^{c \frac{2T'}{N} \left[\sum_{n=0}^N g(\tau_{n,N}) \right]}, \\ &= e^{c\|g_T\|_1} \end{aligned}$$

because of (2.46) and the inequality $0 \leq g(\tau_{n,N}) \leq 1$.

Q. E. D.

3) Suppose that Axioms A1)-A5), B and C hold and let f be a real function on \mathbb{R}^2 with $|f|_{\mathcal{S}} < \infty$. Then there exists a $\delta(f) > 0$ such that $\text{Re } J(\xi f) > 0$, provided $|\xi| < \delta(f)$. This is true, since $J(0) = 1$ and $J(\xi f)$ is continuous in ξ in some neighbourhood of $\text{Im } \xi = 0$.

Therefore $\log J(\xi f)$ is analytic in ξ on $\{z \in \mathbb{C} \mid |z| < \delta(f)\}$. It is the generating functional of the truncated Schwinger functions

$\mathfrak{S}^m(x_1, t_1, \dots, x_m, t_m)^T, [I]$. Using the analyticity properties of $\log J(\xi f)$ in ξ and Cauchy's formula [6] we obtain the estimates

$$|\mathfrak{S}^m(f_1, \dots, f_m)^T| \leq K^m m! \prod_{j=1}^m |f_j|_{\mathcal{S}}, \text{ for some finite } K. \tag{2.48}$$

(For the techniques, see [6]). A beautiful analysis of the generating functionals of truncated Schwinger functions and vertex functions is given in [1].

4) Suppose that Axioms A1)-A5, B and C hold and let f be in \mathcal{S} . Then the quantum fields

$$\varphi_w(f) \equiv \int dt e^{iHt} \varphi(f') e^{-iHt}, \tag{2.49}$$

where the integral is defined in the weak sense on $Q(H)$ (by Theorem 2.4), are essentially s. a. on $C^\infty(H)$. See [10] and [22] for a proof.

Time ordered and retarded products of the quantum fields φ_w (smeared with test functions in $\mathcal{S}(\mathbb{R}^2)$) exist as densely defined operators on \mathcal{X}_w . Their vacuum expectation values exist and are tempered distributions. See [22] for a proof.

5) Let \mathcal{X}_w be the Hilbert space, H the Hamiltonian — s. a. and positive — and Ω the physical vacuum in \mathcal{X}_w , i. e. the groundstate of H , associated with some quantum field φ_w over \mathcal{S} .

Suppose that the (time 0-) fields $\varphi(h) = \varphi_w(h \otimes \delta_0)$ exist and are s. a. for all test functions h in \mathcal{S}_1 and that

$$(\Omega, \varphi(h)^2 \Omega)_{\mathcal{X}_w} \text{ is a continuous functional on } \mathcal{S}_1.$$

Assume moreover that the vacuum Ω is cyclic for the von Neumann algebra $\mathfrak{M}(0)$ generated by the operators $\{ e^{i\varphi(h)} \mid h \in \mathcal{S}_1 \}$ on \mathcal{X}_w . Finally, assume that for arbitrary, positive operators F and G in $\mathfrak{M}(0)$

$$(F\Omega, e^{-iH}G\Omega)_{\mathcal{X}_w} \geq 0 \tag{2.50}$$

Under these assumptions the following holds

THEOREM 2.7. — (a) Let T be some finite real number and

$$\tau_{n,N} = -T + \frac{2n}{N}T.$$

Let f be in \mathcal{S} . Then

$$J(f) = \lim_{T \rightarrow \infty} \left[\lim_{N \rightarrow \infty} \left(\Omega, \prod_{n=0}^N (e^{-\frac{2T}{N}H} e^{i\frac{2T}{N}\varphi(f^{(\tau_{n,N}))})} \Omega) \right)_{\mathcal{X}_w} \right]$$

exists and obeys Axioms A1)-A5), B and C1).

(b) If moreover $\pm \varphi(h) \leq \| \| h \| \| (H + c)$ on $Q(H)$ (the quadratic form domain of H), for some norm $\| \| \cdot \| \|$ continuous on \mathcal{S}_1 and some finite c , then Axiom C2) holds.

(c) The (Euclidean) Field Theory obtained from the functional J of (a) by reconstruction (Theorem 1) obeys a special form of Nelson's axioms [24] (discussed in [38]), where the field is only assumed to transform covariantly under « time »-translations and the Markov property can only be shown to hold for strips and half-planes parallel to the x -axis (See also [6], theorem 3.1.)

(d) If J is a functional satisfying Axioms A, B and C and if the physical vacuum Ω in the Hilbert space \mathcal{X}_w is cyclic for the von Neumann algebra $\mathfrak{M}(0)$ generated by the operators $\{ e^{i\varphi(h)} = e^{i\Phi(h \otimes \delta_0)} \mid h \text{ in } \mathcal{S}_1 \}$ then all hypotheses

of Theorem 2.7, (a)-(c), in particular (2.50), hold and the Euclidean Field Theory obtained from J obeys a special form of Nelson's axioms [24] [38] [6], where the Markov property is only known to hold for strips and half-planes.

Proof. — Part (a) of Theorem 2.7 is proven in [6], Section 3, up to the verification of Axiom B (See also [37].) The proof of Axiom B is as follows:

Let f_1, \dots, f_n be arbitrary test functions in \mathcal{S} with the property that $\text{supp } f_j \subseteq \mathbb{R} \times \{t \geq 0\}$, $j = 1, \dots, n$,

and let c_1, \dots, c_n be arbitrary complex numbers. We define a sequence $\{\theta_{T,N}(c, f)\}$ of vectors in \mathcal{X}_W by

$$\theta_{T,N}(c, f) \equiv \sum_{l=1}^n c_l \prod_{m=\lceil \frac{N}{2} \rceil}^N e^{-\frac{2T}{N}H} e^{i\frac{2T}{N}\phi(f_l^{(m,n)})} \Omega,$$

where $\lceil a \rceil$ is the smallest integer bigger than a . By a straightforward calculation we get

$$0 \leq (\theta_{T,N}(c, f), e^{\frac{2T}{N}H} \theta_{T,N}(c, f))_{\mathcal{X}_W} = \sum_{l, l'=1}^n \bar{c}_l c_{l'} \left(\Omega, \prod_{m=0}^N e^{-\frac{2T}{N}H} e^{i\frac{2T}{N}\phi(f_l^{(m,n)} - (f_{l',9})^{(m,n)})} \Omega \right)_{\mathcal{X}_W}$$

(for N odd) which by the first part of (a) tends to $\sum_{l, l'=1}^n \bar{c}_l c_{l'} J(f_l - f_{l',9})$,

as we let tend first $N \rightarrow \infty$ and then $T \rightarrow \infty$. Thus Axiom B holds and the proof of (a) is complete.

Part (b) now follows from (2.47). Part (c) is proven in [38]. The basic reason for part (c) to be true is the fact that under the conditions of Theorem 2.7 $\{e^{-tH} | t \geq 0\}$ is the transition function of a conservative Markov process on the spectrum of the algebra $\mathfrak{M}(0)$ and is s. a. on \mathcal{X}_W . Such a Markov process allows for the construction of a *path-space measure* which has the *Markov property* and is essentially given by the Fourier transform of $J(\cdot)$ (See also [6], Section 3; [5] for the terminology.)

The first part of (d) follows easily from steps 1, 2 (Lemma 2.1), and 3 of this section. Since under the assumption of (d) the hypotheses of Theorem 2.7 (a)-(c) are true, part (c) applies and yields Nelson's axioms with the Markov property for strips and half-planes parallel to the x -axis. Euclidean invariance then yields the more general Markov property stated in (d). Q. E. D.

**Remark concerning the Axioms
of Osterwalder and Schrader [29] [30] [31].**

Axioms A, B, and C imply the Osterwalder-Schrader Axioms in the forms given in refs. [29] [31] up to the uniqueness of the vacuum (This

follows from estimate (2.40).) The Osterwalder-Schrader Axioms are therefore clearly deeper and more esthetical. These Axioms in the form of ref. [30] (Axioms (EO')-(E3)) are verified for the $P(\varphi)_2$ -models studied in Section 3 in [6] and for the class of $P(\varphi)_2$ -models constructed in [12] [13] in ref. [30].

SECTION 3

VERIFICATION OF AXIOMS A, B, AND C FOR A CLASS OF $P(\varphi)_2$ -MODELS

In this section we verify our axioms for the well-known $P(\varphi)$ -models with half-Dirichlet boundary conditions [16] in a two dimensional space-time [6] [7] [16] [26] [27]. Our results and the main techniques are based on estimates derived in [6] and on the beautiful correlation inequalities of Guerra, Rosen, and Simon [16].

We start with a *probabilistic definition* of these models [23] [16]. Let m be some positive, real number and let $S_m(\xi - n)$ be the kernel of the s. a. operator $(-\Delta + m^2)^{-1}$ on the space $L^2(\mathbb{R}^2)$. Let f and g be in the real Sobolev space \mathcal{X}_{-1} , i. e. $\|(-\Delta + 1)^{-\frac{1}{2}}f\|_2$ and $\|(-\Delta + 1)^{-\frac{1}{2}}g\|_2$ are finite. We define

$$\langle f, g \rangle_m \equiv ((-\Delta + m^2)^{-\frac{1}{2}}f, (-\Delta + m^2)^{-\frac{1}{2}}g)_{L^2(\mathbb{R}^2)} \tag{3.1}$$

We let J_m^0 be the functional given by

$$J_m^0(f) \equiv \exp [-\langle f, f \rangle_m], f \text{ in } \mathcal{S} \text{ (or in } \mathcal{X}_{-1}\text{)}. \tag{3.2}$$

It is well-known that J_m^0 satisfies Axioms A, B, and C [16] [17] [23] [25] [37], etc.

We let $dv_m^0(q)$ be the measure and Φ the Euclidean field over \mathcal{S} obtained from the functional J_m^0 by reconstruction; Theorem 1. Let m_0 be some fixed real number, $0 \leq m_0 \leq m$. We define Wick-monomials of the fields $\Phi(f)$, f in \mathcal{X}_{-1} (with respect to the measure $dv_{m_0}^0(q)$ on \mathcal{S}') as follows:

$$:\Phi(f)^n: \equiv \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j n!}{2^j j! (n-2j)!} \langle f, f \rangle_{m_0}^j \Phi(f)^{n-2j} \tag{3.3}$$

It is known that one can let tend f to δ_ξ and still the l. h. s. of (3.3) makes sense as a densely defined sesquilinear form on $L^2(\mathcal{S}', dv_m^0) \times L^2(\mathcal{S}', dv_m^0)$ (for all finite $m \geq m_0$). The form

$$:\Phi^n:(h) \equiv \int d^2\xi : \Phi(\xi)^n : h(\xi), \tag{3.4}$$

where the integral on the r. h. s. of (3.4) is defined in the weak sense, is in $L^p(\mathcal{S}', dv_m^0)$, $1 \leq p \leq \infty$, for all functions h in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Moreover, if n is even and h positive,

$$e^{-\Phi^n:(h)} \text{ is in } L^p(\mathcal{S}', dv_m^0), 1 \leq p \leq \infty \tag{3.5}$$

(See [8] [16] [21]).

Let Λ be a compact set in \mathbb{R}^2 with continuous, piecewise smooth boundaries which coincides with the closure $\overline{\Lambda_{\text{int}}}$ of its interior Λ_{int} and let χ_Λ be the characteristic function of Λ . We define the measure

$$dv_{m_0, m}^\Lambda \equiv \frac{1}{z_{m_0, m}^\Lambda} e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda)} dv_m^0, \quad (3.6)$$

where
$$z_{m_0, m}^\Lambda \equiv \int_{\mathcal{S}'} e^{\frac{1}{2}(m^2 - m_0^2) : q^2 : (\chi_\Lambda)} dv_m^0(q)$$

which is known to be finite as long as $m_0 < m$. We let $J_{m_0, m}^\Lambda$ be the Fourier transform of $dv_{m_0, m}^\Lambda$.

Let
$$\Lambda_\theta \equiv \{ \langle x, t \rangle \mid \langle x, -t \rangle \in \Lambda \}$$
 and
$$\Lambda_\pm \equiv \{ \langle x, t \rangle \in \Lambda \mid t \geq 0 \}$$
 (3.7)

LEMMA 3.1. — Suppose that $\Lambda = \Lambda_\theta$. Then for all finite $m > m_0$ $J_{m_0, m}^\Lambda$ satisfies Axioms A1)-A4), B and C and

$$J_{m_0, m}^\Lambda(f) = J_{m_0, m}^\Lambda(f_\theta) \quad (3.8)$$

Proof. — It is obvious that $J_{m_0, m}^\Lambda$ satisfies Axioms A1)-A4) and C. Clearly

$$\begin{aligned} J_{m_0, m}^\Lambda(f) &= \frac{1}{z_{m_0, m}^\Lambda} \langle e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda)} e^{i\Phi(f)} \rangle_{v_m^0} \\ &= \frac{1}{z_{m_0, m}^\Lambda} \langle e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda)} e^{i\Phi(f_\theta)} \rangle_{v_m^0} \\ &= \frac{1}{z_{m_0, m}^\Lambda} \langle e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda)} e^{i\Phi(f_\theta)} \rangle_{v_m^0} \\ &= J_{m_0, m}^\Lambda(f_\theta), \text{ whence (3.8).} \end{aligned}$$

We are left with verifying Axiom B. Let f_1, \dots, f_n be arbitrary test functions in \mathcal{S} with $\text{supp } f_l \subseteq \mathbb{R} \times \{t \geq 0\}$, $l = 1, \dots, n$, and let c_1, \dots, c_n be arbitrary complex numbers. Then

$$\begin{aligned} \sum_{l, l'=1}^n \bar{c}_l c_{l'} J_{m_0, m}^\Lambda(f_{l'} - f_{l, \theta}) &= \sum_{l, l'=1}^n \bar{c}_l c_{l'} \langle e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda)} e^{i\Phi(f_{l'} - f_{l, \theta})} \rangle_{v_m^0} \frac{1}{z_{m_0, m}^\Lambda} \\ &= \frac{1}{z_{m_0, m}^\Lambda} \left\langle \left[\sum_{l=1}^n \bar{c}_l e^{-i\Phi(f_{l, \theta})} \right] e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda, \theta)} \right. \\ &\quad \left. \times e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_\Lambda)} \left[\sum_{l'=1}^n c_{l'} e^{i\Phi(f_{l'})} \right] \right\rangle_{v_m^0} \quad (3.9) \end{aligned}$$

Since $\Lambda = \Lambda_+ \cup \Lambda_- = \Lambda_+ \cup \Lambda_{+, \mathfrak{g}}$ and hence

$$\theta \left(e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_{\Lambda})} \left[\sum_{l=1}^n c_l e^{i\Phi(f_l)} \right] \right) = e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_{\Lambda, \mathfrak{g}})} \left[\sum_{l=1}^n \bar{c}_l e^{-i\Phi(f_l, \mathfrak{g})} \right]$$

and since $e^{\frac{1}{2}(m^2 - m_0^2) : \Phi^2 : (\chi_{\Lambda})}$ is affiliated with $\mathfrak{M}_{v_m, +}$ the r. h. s. of (3.9) is positive. Q. E. D.

Remark. — Let $S_{m_0, m}^\Lambda(\xi, \eta)$ be the kernel of the operator

$$(-\Delta + m^2 - (m^2 - m_0^2)\chi_\Lambda)^{-1} \text{ on } L^2(\mathbb{R}^2).$$

By straightforward arguments [37] we see that

$$J_{m_0, m}^\Lambda(f) = \exp [-(f, S_{m_0, m}^\Lambda * f)_{L^2(\mathbb{R}^2)}], \text{ } f \text{ in } \mathcal{S} \tag{3.10}$$

DEFINITION. — Let $S_{m_0, D}^\Lambda(\xi, \eta)$ be the kernel of the operator $(-\Delta_{\Lambda, D} + m_0^2)^{-1}$ on the space $L^2(\Lambda)$, where $\Delta_{\Lambda, D}$ is the Laplacian with Dirichlet boundary conditions at $\partial\Lambda$.

Let f be in $C_0^\infty(\Lambda)$ (i. e. f is C^∞ , $\text{supp } f \subseteq \Lambda$ and $f = 0$ on $\partial\Lambda$). We define

$$J_{m_0, D}^\Lambda(f) \equiv \exp [-(f, S_{m_0, D}^\Lambda * f)_{L^2(\Lambda)}] \tag{3.11}$$

LEMMA 3.2. — For f in $C_0^\infty(\Lambda)$

$$J_{m_0, D}^\Lambda(f) = \lim_{m \rightarrow \infty} J_{m_0, m}^\Lambda(f)$$

Proof. — By (3.10) and (3.11) it suffices to check that for f in $C_0^\infty(\Lambda)$

$$(f, S_{m_0, m}^\Lambda * f) \rightarrow (f, S_{m_0, D}^\Lambda * f), \text{ as } m \rightarrow \infty.$$

This is well-known from two dimensional quantum mechanics (Use an eigenfunction expansion of $S_{m_0, m}^\Lambda$; the continuous part of spec

$$(-\Delta + m^2 - (m^2 - m_0^2)\chi_\Lambda)$$

is negligible as $m \rightarrow \infty$).

Q. E. D.

COROLLARY 3.3. — If $\Lambda = \Lambda_{\mathfrak{g}}$ then $J_{m_0, D}^\Lambda$ satisfies Axioms A1)-A4) B and C1) with the test function space \mathcal{S} replaced by the nuclear space $C_0^\infty(\Lambda)$ and

$$J_{m_0, D}^\Lambda(f) = J_{m_0, D}^\Lambda(f_{\mathfrak{g}}).$$

Remarks. — By Theorem 1, $J_{m_0, D}^\Lambda$ is the Fourier transform of a measure $dv_{m_0, D}^\Lambda$ on the dual $C_0^\infty(\Lambda)'$ of $C_0^\infty(\Lambda)$. Again Φ denotes the field on $C_0^\infty(\Lambda)'$ associated with $J_{m_0, D}^\Lambda$. From (3.11) it is obvious that for f in $C_0^\infty(\Lambda)$ $J_{m_0, D}^\Lambda(\xi f)$ is entire analytic in ξ and that

$$|J_{m_0, D}^\Lambda(\xi f)| \leq J_{m_0, D}^\Lambda(i(\text{Im } \xi)f)$$

By the first and second Griffiths inequalities [16]

$$J_{m_0, D}^\Lambda(i(\operatorname{Im} \xi)f) \leq J_{m_0, D}^\Lambda(-i|\operatorname{Im} \xi| |f|) \leq J_{m_0}^0(-i|\operatorname{Im} \xi| |f|) = \exp[|\operatorname{Im} \xi|^2 \langle f, f \rangle_{m_0}] \tag{3.12}$$

and hence (see [6], Lemma 3.2) $J_{m_0, D}^\Lambda(f)$ is continuous in f in the norm

$$|f|_{m_0} \equiv \sqrt{\langle f, f \rangle_{m_0}},$$

uniformly in Λ . Therefore $J_{m_0, D}^\Lambda(\cdot)$ has the following trivial extension to the space \mathcal{S} : For f in \mathcal{S} we set

$$J_{m_0, D}^\Lambda(f) \equiv J_{m_0, D}^\Lambda(f \cdot \chi_\Lambda). \tag{3.13}$$

From (3.12) and (3.13) we conclude that $J_{m_0, D}^\Lambda$ satisfies Axioms A1-A4, B and C, and $J_{m_0, D}^\Lambda(f) = J_{m_0, D}^\Lambda(f_\beta)$, provided $\Lambda = \Lambda_\beta$.

DEFINITION. — Let P be a real polynomial

$$P(\xi) = \sum_{l=0}^n a_l \xi^{2l} + \mu \xi, \tag{3.14}$$

where $a_n > 0$ and (without loss of generality [16]) $\mu \geq 0$. We set

$$V_\Lambda = V_\Lambda(P) \equiv \int d^2\xi : P(\Phi) : (\xi) \chi_\Lambda(\xi) \tag{3.15}$$

and

$$V_{\Lambda_\pm} \equiv \int d^2\xi : P(\Phi) : (\xi) \chi_{\Lambda_\pm}(\xi)$$

and Wick ordering is defined as in (3.3).

The following is well-known [8] [16] [21] :

$$\left. \begin{aligned} &V_\Lambda = V_{\Lambda_+} + V_{\Lambda_-}, V_\Lambda \text{ and } V_{\Lambda_\pm} \text{ are in } L^p(\mathcal{S}', dv_{m_0, D}^\Lambda) \\ \text{and } &e^{-V_\Lambda}, e^{-V_{\Lambda_\pm}} \text{ are in } L^p(\mathcal{S}', dv_{m_0, D}^\Lambda), 1 \leq p \leq \infty \end{aligned} \right\} \tag{3.16}$$

DEFINITION.

$$Z_{m_0, D}^{\Lambda, P} \equiv \int_{\mathcal{S}'} dv_{m_0, D}^\Lambda e^{-V_\Lambda(P)}$$

$$dv_{m_0, D}^{\Lambda, P} \equiv \frac{1}{Z_{m_0, D}^{\Lambda, P}} e^{-V_\Lambda(P)} dv_{m_0, D}^\Lambda \tag{3.17}$$

We let $J_{m_0, D}^{\Lambda, P}$ denote the Fourier transforms of the measure $dv_{m_0, D}^{\Lambda, P}$, and we set

$$J_{m_0, D}^{\Lambda, P}(f) \equiv J_{m_0, D}^{\Lambda, P}(f \cdot \chi_\Lambda), \text{ all } f \text{ in } \mathcal{S} \tag{3.18}$$

which is well defined because of (3.12) and (3.16).

Obviously

$$\begin{aligned} Z_{m_0, D}^{\Lambda, P} dv_{m_0, D}^{\Lambda, P} &= e^{-V_{\Lambda_-}} e^{-V_{\Lambda_+}} dv_{m_0, D}^\Lambda \\ &= e^{-V_{\Lambda_+}} e^{-V_{\Lambda_-}} dv_{m_0, D}^\Lambda \\ &= \theta(e^{-V_{\Lambda_+}}) e^{-V_{\Lambda_+}} dv_{m_0, D}^\Lambda, \end{aligned} \tag{3.19}$$

provided $\Lambda = \Lambda_\beta$. Moreover $e^{-V_{\Lambda_+}}$ is affiliated with $\mathfrak{M}_{v_{m_0, D, +}^\Lambda}$. Hence

LEMMA 3.4. — Let $\Lambda = \Lambda_g$ be compact (with continuous piecewise smooth boundaries and $\Lambda = \overline{\Lambda_{int}}$).

Then $J_{m_0,D}^{\Lambda,P}$ satisfies Axioms A1)-A4), B, and C and

$$J_{m_0,D}^{\Lambda,P}(f) = J_{m_0,D}^{\Lambda,P}(f_g)$$

Proof. — With the exception of Axiom C this is a direct consequence of corollary 3.3 (3.12) (3.13) (3.16) and (3.19). It is obvious that $J_{m_0,D}^{\Lambda,P}(\xi f)$ is entire analytic in ξ and

$$\begin{aligned} |J_{m_0,D}^{\Lambda,P}(\xi f)| &\leq \frac{1}{Z_{m_0,D}^{\Lambda,P}} \langle e^{-V_\Lambda} e^{-Im \xi \Phi(f)} \rangle_{V_{m_0,D}^\Lambda} \\ &\leq \frac{1}{Z_{m_0,D}^{\Lambda,P}} \langle e^{-2V_\Lambda} \rangle_{V_{m_0,D}^\Lambda}^{\frac{1}{2}} \langle e^{-2 Im \xi \Phi(f)} \rangle_{V_{m_0,D}^\Lambda}^{\frac{1}{2}} \end{aligned} \tag{3.20}$$

which together with (3.12) and (3.16) yields Axiom C2).

$$\langle \Phi(h \otimes \delta_t) \Phi(h \otimes \delta_{t'}) \rangle_{V_{m_0,D}^{\Lambda,P}} \leq \frac{1}{Z_{m_0,D}^{\Lambda,P}} \langle (\Phi(h \otimes \delta_t) \Phi(h \otimes \delta_{t'}))^2 \rangle_{V_{m_0,D}^\Lambda}^{\frac{1}{2}} \times \langle e^{-2V_\Lambda} \rangle^{\frac{1}{2}}$$

and this together with (3.12) and (3.16) yields C1).

Q. E. D.

We now must show how one can pass to the limit $\Lambda = \mathbb{R}^2$ and verify Axioms A, B, and C for the limit functional

$$J_{m_0,D}^P \equiv J_{m_0,D}^{\mathbb{R}^2,P}$$

The basic ingredients for this are the correlation inequalities of Guerra, Rosen and Simon [16] (and Nelson [26]) and uniform bounds on the functional $J_{m_0,D}^{\Lambda,P}(\xi f)$ established in [6]. In order to get these uniform bounds we have to compare the functional $J_{m_0,D}^{\Lambda,P}$ with a functional $J_{m_0}^{\Lambda',P}$ which is defined to be the Fourier transform of the measure

$$d\nu_{m_0}^{\Lambda',P} \equiv \frac{1}{Z_{m_0}^{\Lambda',P}} e^{-V_{\Lambda'}(P)} d\nu_{m_0}^0,$$

where $Z_{m_0}^{\Lambda',P} \equiv \int_{\mathcal{S}'} d\nu_{m_0}^0 e^{-V_{\Lambda'}(P)}$ and this is finite for compact Λ' .

We define a norm $||| \cdot |||$ on functions on \mathbb{R}^1 by

$$||| h ||| \equiv \left(\int dx (1 + x^2) |h(x)|^2 \right)^{\frac{1}{2}} \tag{3.21}$$

and a norm $|\cdot|_{\mathcal{S}}$ on functions on \mathbb{R}^2 by

$$|f|_{\mathcal{S}} \equiv \int dt ||| f^t ||| + \sup_t ||| f^t |||. \tag{3.22}$$

We now choose $\Lambda' = \Lambda_1^T \equiv [-l/2, l/2] \times [-T/2, T/2]$. It is shown in [6], subsection 3.4, by use of the transfer matrix method [16] [37], or, in other

words, by use of the spatially cutoff $P(\varphi)_2$ quantum field Hamiltonian H_l [9] that

$$J_l(\xi f) \equiv \lim_{T \rightarrow \infty} J_{m_0}^{\Lambda, P}(\xi f)$$

exists for all complex ξ and real f with finite norm $|f|_{\mathcal{F}}$ and, for arbitrary but fixed f , is entire analytic in ξ . The functional J_l satisfies Axioms A1)-A5), B, and C.

Actually, it is shown in [6], sections 2 and 3, that the Hamiltonian H_l (which is densely defined, positive and s. a. on the usual Fock space \mathcal{F} [9]) obeys the hypotheses of Theorem 2.7 with

$$\mathcal{H}_w = \mathcal{F}, H = H_l, \Omega = \Omega_l \equiv \text{unique ground state of } H_l,$$

and hence yields the existence of the functional J_l which satisfies Theorem 2.7 (a) (b) and (c).

The following uniform bounds are proven in [6], subsections 3.4, 3.5 :

$$|J_l(\xi f)| \leq J_l(i(\text{Im } \xi) f) \leq Ae^{B|\text{Im } \xi|^2 |f|_{\mathcal{F}}^2}, \tag{3.23}$$

for some finite A and B independent of l . Let g and g_T be real test functions on \mathbb{R} such as specified in Axiom C2). Let h be a real function on \mathbb{R} with $|||h||| \leq 1$. Then there exist finite constants K and c independent of l such that

$$J_l(\pm ih \otimes g_T) \leq Ke^{cT}. \tag{3.24}$$

If ν_l denotes the measure on \mathcal{S}' whose Fourier transform is the functional J_l then

$$|\mathfrak{S}_l^2(h_1 \otimes \delta_{t_1}, h_2 \otimes \delta_{t_2})| \equiv |\langle \Phi(h_1 \otimes \delta_{t_1}) \Phi(h_2 \otimes \delta_{t_2}) \rangle_{\nu_l}| \leq C |||h_1||| \cdot |||h_2|||, \tag{3.25}$$

for some finite C independent of l (See [6] formulas (3.42) and (4.16).)

DEFINITION. — Let $I_\Lambda \equiv \sup \{ |x| \mid x \text{ in } \mathbb{R}, \exists t \text{ such that } \langle x, t \rangle \in \Lambda \}$.

LEMMA 3.5. — Suppose that the polynomial P is such as specified in (3.14) (such that the Griffiths inequalities of Guerra, Rosen and Simon [16] apply). Then

$$|J_{m_0, D}^{\Lambda, P}(\xi f)| \leq J_{m_0, D}^{\Lambda, P}(i(\text{Im } \xi) f) \leq J_{m_0, D}^{\Lambda, P}(-i|\text{Im } \xi| |f|) \tag{3.26}$$

$$J_{m_0, D}^{\Lambda, P}(-i|\text{Im } \xi| |f|) \leq J_{I_\Lambda}(-i|\text{Im } \xi| |f|) \leq Ae^{B|\text{Im } \xi|^2 |f|_{\mathcal{F}}^2}, \tag{3.27}$$

$$J_{m_0, D}^{\Lambda, P}(\pm ih \otimes g_T) \leq Ke^{cT}, \tag{3.28}$$

where h and g_T are such as in (3.24). Let f be a non-negative function with $|f|_{\mathcal{F}} < \infty$ and let λ be a positive real number. If $\Lambda \subset \Lambda'$ (compact) then

$$0 \leq J_{m_0, D}^{\Lambda, P}(-i\lambda f) \leq J_{m_0, D}^{\Lambda', P}(-i\lambda f) \tag{3.29}$$

Finally

$$\begin{aligned} & |\langle \Phi(h_1 \otimes \delta_{t_1}) \Phi(h_2 \otimes \delta_{t_2}) \rangle_{\nu_{m_0, D}^{\Lambda, P}}| \\ & \leq \langle \Phi(|h_1| \otimes \delta_{t_1}) \Phi(|h_2| \otimes \delta_{t_2}) \rangle_{\nu_{m_0, D}^{\Lambda, P}} \\ & \leq \langle \Phi(|h_1| \otimes \delta_{t_1}) \Phi(|h_2| \otimes \delta_{t_2}) \rangle_{\nu_l} \leq C |||h_1||| \cdot |||h_2||| \end{aligned} \tag{3.30}$$

Proof. — We know that for arbitrary, fixed f with $|f|_{\mathcal{F}}$ finite, the functions $J_{m_0, D}^{\Lambda, P}(\xi f)$ and $J_{l_\Lambda}(\xi f)$ are entire analytic in ξ . Therefore they have a power series expansion at $\xi = 0$ which converges absolutely for arbitrary ξ :

$$J_{m_0, D}^{\Lambda, P}(\xi f) = \sum_{m=0}^{\infty} \frac{(i\xi)^m}{m!} \mathfrak{S}_{m_0, D}^{m, \Lambda, P}(f, \dots, f)$$

$$J_{l_\Lambda}(\xi f) = \sum_{m=0}^{\infty} \frac{(i\xi)^m}{m!} \mathfrak{S}_{l_\Lambda}^m(f, \dots, f),$$

where

$$\mathfrak{S}_{m_0, D}^{m, \Lambda, P}(f_1, \dots, f_m) \equiv (-i)^m \frac{\partial^m}{\partial \xi_1, \dots, \partial \xi_m} J_{m_0, D}^{\Lambda, P} \left(\sum_{i=1}^m \xi_i f_i \right) \Big|_{\xi_1 = \dots = \xi_m = 0}$$

and $\mathfrak{S}_{l_\Lambda}^m(f_1, \dots, f_m)$ are the m -point Schwinger functions.

Let f_1, \dots, f_m be functions on \mathbb{R}^2 with $|f_j|_{\mathcal{F}} < \infty, j = 1, \dots, m$. It is shown in [6] that the estimate (3.23) implies

$$|\mathfrak{S}_{l_\Lambda}^m(f_1, \dots, f_m)| \leq D^m \sqrt{m!} \prod_{j=1}^m |f_j|_{\mathcal{F}}, \tag{3.31}$$

for some finite constant D and uniformly in $l_\Lambda \in [0, \infty)$. By the first Griffiths inequality [16]

$$|\mathfrak{S}_{m_0, D}^{m, \Lambda, P}(f_1, \dots, f_m)| \leq \mathfrak{S}_{m_0, D}^{m, \Lambda, P}(|f_1|, \dots, |f_m|), \tag{3.32}$$

whence (3.26), and by the second Griffiths inequality [16]

$$\mathfrak{S}_{m_0, D}^{m, \Lambda, P}(|f_1|, \dots, |f_m|) \leq \mathfrak{S}_{l_\Lambda}^m(|f_1|, \dots, |f_m|) \tag{3.33}$$

and for $\Lambda \subset \Lambda'$

$$\mathfrak{S}_{m_0, D}^{m, \Lambda, P}(|f_1|, \dots, |f_m|) \leq \mathfrak{S}_{m_0, D}^{m, \Lambda', P}(|f_1|, \dots, |f_m|), \tag{3.34}$$

which together with (3.23) yields (3.27) (3.29), respectively. Inequality (3.28) follows from (3.27) (for $f = h \otimes g_T$ and $\xi = \pm i$) and from (3.24). Finally (3.30) is a consequence of (3.33) for $m = 2$ and the estimate (3.25).

Remark. — The applicability of the Griffiths inequalities (3.32)-(3.34) is due to our special choice (3.14) of the interaction polynomial P and is not possible for general polynomials [16].
Q. E. D.

COROLLARY 3.6. — *For fixed complex $\xi \{ J_{m_0, D}^{\Lambda, P}(\xi f) \}_{\Lambda \subset \mathbb{R}^2}$ is a family of functionals which is uniformly bounded in Λ for arbitrary functions f with $|f|_{\mathcal{F}}$ finite and is equicontinuous in f in the norm $|\cdot|_{\mathcal{F}}$. For fixed, real f with $|f|_{\mathcal{F}} < \infty \{ J_{m_0, D}^{\Lambda, P}(\xi f) \}_{\Lambda \subset \mathbb{R}^2}$ is a family of entire analytic functions of ξ which is uniformly bounded in Λ in absolute value by $Ae^{B|\operatorname{Im}\xi|^2|f|_{\mathcal{F}}^2}$.*

Proof. — The first part of the corollary is proven in [6] (section 3, lemma 3.2 and theorem 3.8). The second part of the corollary follows from lemma 3.4 (3.20) and (3.23).
Q. E. D.

THEOREM 3.7 (Main result). — 1) Let $\{\Lambda_j\}_{j=1}^\infty$ be a monotone increasing sequence of compact regions (with continuous, piecewise smooth boundaries and the properties that $\Lambda_j = \overline{\Lambda_{j,\text{int}}}$, $\Lambda_j \subset \Lambda_{j+1}$) converging to \mathbb{R}^2 . Then for arbitrary, real function f with $|f|_{\mathcal{S}}$ finite and an arbitrary, complex number ξ ,

$$J_{m_0, D}^P(\xi f) \equiv \lim_{j \rightarrow \infty} J_{m_0, D}^{\Lambda_j, P}(\xi f) \text{ exists}$$

and is independent of the sequence $\{\Lambda_j \mid \text{properties of } \Lambda_j \text{ as specified}\}_{j=1}^\infty$ chosen.

2) The limit functional $J_{m_0, D}^P$ satisfies Axioms A, B, and C of Section 1.

Proof. — Let f_1 and f_2 be arbitrary, non-negative functions with $|f_1|_{\mathcal{S}}$ and $|f_2|_{\mathcal{S}}$ finite and let λ_1, λ_2 be non-negative, real numbers. Then by (3.29) and (3.27) of lemma 3.5

$$\begin{aligned} \lim_{j \rightarrow \infty} J_{m_0, D}^{\Lambda_j, P}(-i(\lambda_1 f_1 + \lambda_2 f_2)) &\equiv J_{m_0, D}^P(-i(\lambda_1 f_1 + \lambda_2 f_2)) \\ &= \sup J_{m_0, D}^{\Lambda_j, P}(-i(\lambda_1 f_1 + \lambda_2 f_2)) \end{aligned} \quad (3.35)$$

exists. Also (3.29) and (3.35) imply that $J_{m_0, D}^P(-i(\lambda_1 f_1 + \lambda_2 f_2))$ is independent of the sequence $\{\Lambda_j\}_{j=1}^\infty$ (and hence *Euclidean invariant*), by standard arguments [16].

Obviously $J_{m_0, D}^{\Lambda_j, P}(\xi_1 f_1 + \xi_2 f_2)$ is jointly entire analytic in ξ_1 and ξ_2 (for all $j < \infty$) and by (3.27)

$$|J_{m_0, D}^{\Lambda_j, P}(\xi_1 f_1 + \xi_2 f_2)| \leq A e^{B(|\text{Im} \xi_1| |f_1| + |\text{Im} \xi_2| |f_2|)}_{\mathcal{S}}, \text{ all } j < \infty.$$

Hence, by (3.35) and twice applying Vitali's theorem, we conclude that

$$\lim_{j \rightarrow \infty} J_{m_0, D}^{\Lambda_j, P}(\xi_1 f_1 + \xi_2 f_2) \equiv J_{m_0, D}^P(\xi_1 f_1 + \xi_2 f_2)$$

exists for arbitrary complex ξ_1 and ξ_2 , is entire analytic in ξ_1 and ξ_2 and is independent of the sequence $\{\Lambda_j\}_{j=1}^\infty$. Now, if f is a real function with $|f|_{\mathcal{S}} < \infty$ and if

$$f_+ \equiv \frac{1}{2} \{f + |f|\}, f_- \equiv \frac{1}{2} \{-f + |f|\}, \text{ then } |f_{\pm}|_{\mathcal{S}} \leq |f|_{\mathcal{S}} < \infty$$

and hence

$$J_{m_0, D}^P(\xi f) = \lim_{j \rightarrow \infty} J_{m_0, D}^{\Lambda_j, P}(\xi f_+ - \xi f_-)$$

exists and is independent of the sequence $\{\Lambda_j\}_{j=1}^\infty$. This proves Part 1).

Proof of 2). — It follows from the independence of the limit functional $J_{m_0, D}^P(f)$ of the sequence $\{\Lambda_j\}_{j=1}^\infty$ that this functional is Euclidean invariant [16] [26]. It then follows from corollary 3.6 that $J_{m_0, D}^P(f)$ is continuous in f in the norm $|\cdot|_{\mathcal{S}}$ and therefore in the topology of \mathcal{S} . Hence, by Part 1) and lemma 3.4 $J_{m_0, D}^P$ satisfies Axiom A. Choosing the sequence $\{\Lambda_j\}_{j=1}^\infty$ such that $\Lambda_{j, \mathcal{S}} = \Lambda_j$, for all j , and applying lemma 3.4 proves Axiom B. Finally Axiom C follows from lemma 3.5 (3.28) and (3.30). Q. E. D.

Remark. — Since $J_{m_0, D}^P(\xi f)$ is entire analytic in ξ for $|f|_{\mathcal{S}} < \infty$ and

$|J_{m_0,D}^P(\xi f)| \leq A e^{B|\text{Im}\xi|^2 |f|^2}$, the bounds (3.31) extend to the Schwinger functions $\mathfrak{S}_{m_0,D}^{m,P}(f_1, \dots, f_m)$ obtained from the generating functional $J_{m_0,D}^P$ and yield an improvement of the bounds (2.40) [6].

SECTION 4

THE UNIQUENESS

OF THE « EUCLIDEAN » AND THE PHYSICAL VACUUM ; THE $P(\varphi)_2$ INTERACTING FIELDS ARE NOT FREE FIELDS

In this section we derive some more detailed properties of the infinite-volume generating functionals $J_{m_0,D}^P$ constructed in Theorem 3.7. By Theorem 1 these functionals are the Fourier transforms of probability measures $\nu_{m_0,D}^P$ on the \mathfrak{S} -algebra generated by the Borel cylinder sets of \mathcal{S}' .

These measures are called the infinite-volume $P(\varphi)$ interacting measures.

The first question we want to answer is under what conditions the « Euclidean vacuum » defined in Theorem 1 and the physical vacuum defined in Step 2 of Section 2 are unique in a sense defined below.

THEOREM 4.1. — *Let $P_4(\xi) = a\xi^4 + b\xi^2 + \mu\xi$, where $a > 0$ (and for convenience $\mu > 0$ [16]). Let $\nu_{m_0,D}^{P_4}$ be the corresponding infinite-volume interacting measure. Then*

1) *The « Euclidean vacuum » (i. e. the function I on \mathcal{S}') is the only vector in $\mathcal{K}_v \equiv L^2(\mathcal{S}', d\nu_{m_0,D}^{P_4})$ which is invariant under the (« time-translation » subgroup of the) Euclidean group $E(2)$.*

2) *The physical vacuum Ω in \mathcal{K}_W (defined in Step 2, Section 2) is the only Poincaré-invariant vector in \mathcal{K}_W .*

Proof. — We use the following proposition which is essentially due to Araki [1]:

PROPOSITION D. — *Let J be a functional satisfying Axioms A, B and C. Then (a) the « Euclidean vacuum » Ω is unique in the sense of Theorem 4.1, 1) if and only if for arbitrary test functions f and g in \mathcal{S}*

$$|J(f + g_s) - J(f)J(g)| \rightarrow 0, \text{ as } |s| \rightarrow \infty, \tag{4.1}$$

where $g_s(x, t) \equiv g(x, t - s)$

(b) *If (4.1) holds then the physical vacuum is the only Poincaré-invariant vector in the Wightman Hilbert space \mathcal{K}_W constructed from the functional J in Section 2.*

Proof of Proposition D. — Part (a) is proven by Araki in [1]. For the proof of part (b) notice that from condition (4.1) it follows that

$$\begin{aligned} &|\mathfrak{S}^{k+l}(f_1, \dots, f_k, (f_{k+1})_s, \dots, (f_{k+l})_s) - \mathfrak{S}^k(f_1, \dots, f_k) \\ &\mathfrak{S}^l(f_{k+1}, \dots, f_{k+l})| \end{aligned}$$

tends to 0, as $|s| \rightarrow \infty$, where $\mathfrak{S}^k(\xi_1, \dots, \xi_k)$ is the k -point Schwinger function obtained from the generating functional J as in Section 2. But these Cluster properties tell us that the eigen value 0 of the Hamiltonian H obtained in Step 2, Section 2, is simple. See [29] [30] [31] [36]. This completes the proof of Proposition D.

We now come back to the proof of Theorem 4.1. We must prove the Cluster properties (4.1).

From the bound (3.31) on the m -point Schwinger functions we know that for all finite s $J_{m_0, D}^{P_4}(f + g_s)$ is given by

$$J_{m_0, D}^{P_4}(f + g_s) = \sum_{i, k=0}^{\infty} \frac{i^{l+k}}{l! k!} \mathfrak{S}_{m_0, D}^{k+l, P_4}(\underbrace{f, \dots, f}_k, \underbrace{g_s, \dots, g_s}_l) \quad (4.2)$$

and the r. h. s. of (4.2) converges absolutely.

We show that the r. h. s. of (4.2) converges absolutely, *uniformly in s*. For

$$\begin{aligned} & \sum_{i, k=0}^{\infty} \frac{1}{l! k!} |\mathfrak{S}_{m_0, D}^{k+l, P_4}(\underbrace{f, \dots, f}_k, \underbrace{g_s, \dots, g_s}_l)| \\ & \leq \sum_{i, k=0}^{\infty} \frac{1}{l! k!} \mathfrak{S}_{m_0, D}^{k+l, P_4}(\underbrace{|f|, \dots, |f|}_k, \underbrace{|g_s|, \dots, |g_s|}_l), \text{ by (3.32)} \\ & \leq \sum_{i, k=0}^{\infty} \frac{1}{l! k!} [\mathfrak{S}_{m_0, D}^{2k, P_4}(\underbrace{|f|, \dots, |f|}_{2k})]^{\frac{1}{2}} [\mathfrak{S}_{m_0, D}^{2l, P_4}(\underbrace{|g|, \dots, |g|}_{2l})]^{\frac{1}{2}} \\ & \leq \sum_{i, k=0}^{\infty} \frac{1}{l! k!} D^{k+l} \sqrt{\sqrt{(2k)!} \sqrt{(2l)!}} |f|_{\mathcal{S}}^k |g|_{\mathcal{S}}^l, \text{ by (3.31)} \\ & \leq \sum_{i, k=0}^{\infty} \frac{(2D)^{k+l}}{\sqrt{l! k!}} |f|_{\mathcal{S}}^k |g|_{\mathcal{S}}^l, \end{aligned}$$

which converges for all finite $|f|_{\mathcal{S}}$ and $|g|_{\mathcal{S}}$. It therefore suffices to show that for arbitrary f and g in \mathcal{S}

$$|\mathfrak{S}_{m_0, D}^{k+l, P_4}(\underbrace{f, \dots, f}_k, \underbrace{g_s, \dots, g_s}_l) - \mathfrak{S}_{m_0, D}^{k, P_4}(\underbrace{f, \dots, f}_k) \mathfrak{S}_{m_0, D}^{l, P_4}(\underbrace{g, \dots, g}_l)| \quad (4.3)$$

tends to 0, as $s \rightarrow \infty$, for arbitrary $k < \infty$ and $l < \infty$.

But the convergence (4.3) has been shown by Simon in [36] [37] by using the beautiful Lee-Yang theorem proven in [35] [27] (which is only known to be true for $P = P_4$!). This and Proposition D complete the proof of Theorem 4.1.

Q. E. D.

Remarks. — (i) Theorem 4.1, 2) has also been proven by Glimm, Jaffe and Spencer [12] [13] under different hypotheses on the polynomial P. They establish a stronger form of the Cluster properties (4.3) (yielding the existence of a positive mass gap). But this and the bounds (3.27) (proven in [6] under the hypotheses of [12] [13]) imply Theorem 4.1, 1).

(ii) COROLLARY 4.2. — *Let J be a functional on \mathcal{S} obeying Axiom A1)-A5) and hypothesis (4.1) of Proposition D. Then the measure ν obtained from J by Theorem 1 is ergodic under the action of the group $\{T_t | t \in \mathcal{H}\}$ defined in (1.5) (set $\beta = t$), where \mathcal{H} denotes the « time-translation » subgroup E(2).*

Proof. — Since J satisfies hypothesis (4.1) of Proposition D and Axiom A1)-A5), the function I identically 1 on \mathcal{S}' , i. e. the « Euclidean » vacuum is the only vector in $L^2(\mathcal{S}', d\nu)$ which is invariant under $\{T_t | t \in \mathcal{H}\}$. Hence, if F is a ν -measurable, positive function with $\int_{\mathcal{S}'} F d\nu = 1$ and the measure $F d\nu$ is invariant under $\{T_t | t \in \mathcal{H}\}$, then $F = I$. Q. E. D.

THEOREM 4.3. — *Let ν and μ be two probability measures defined on the \mathfrak{S} -algebra generated by the Borel cylinder sets of \mathcal{S}' (for short PBC measures) which are invariant and ergodic under the action of the group $\{T_t | t \in \mathcal{H}\}$ of automorphisms of the underlying \mathfrak{S} -algebra.*

Then either $d\nu = d\mu$ or ν and μ are mutually singular PBC measures. If only μ is assumed to be ergodic under $\{T_t | t \in \mathcal{H}\}$ then there exists a number $\lambda \in [0, 1]$ and a measure μ_s which is invariant under $\{T_t | t \in \mathcal{H}\}$ such that μ and μ_s are mutually singular and $d\nu = \lambda d\mu_s + (1 - \lambda)d\mu$.

Remarks. — This theorem is well-known from the theory of group representations (*). For the convenience of the reader we present a short proof:

Obviously there exists a measure μ_s such that μ_s and μ are mutually singular PBC measures, a μ -measurable, positive function F on \mathcal{S}' with

$$\int_{\mathcal{S}'} d\mu(q)F(q) = 1 \text{ and a real number } \lambda \in [0, 1] \text{ such that}$$

$$d\nu = \lambda d\mu_s + (1 - \lambda)F d\mu$$

Let χ_μ and χ_{μ_s} be characteristic functions on \mathcal{S}' such that

$$\int_{\mathcal{S}'} d\mu_s(q)\chi_{\mu_s}(q) = 1, \int_{\mathcal{S}'} d\mu_s(q)\chi_\mu(q) = 0, \int_{\mathcal{S}'} d\mu(q)\chi_{\mu_s}(q) = 0.$$

(*) See e. g. : G. W. MACKEY, *Induced Representations and Quantum Mechanics*, W. A. Benjamin, New York, 1968 (Chapter 5, Section 5.2).

Then for all $t \in \mathcal{H}$

$$\begin{aligned} \int_{\mathcal{S}'} dv(q) \chi_\mu(q) (T_t \chi_{\mu_s})(q) &= (1 - \lambda) \int_{\mathcal{S}'} d\mu(q) F(q) (T_t \chi_{\mu_s})(q) \\ &= (1 - \lambda) \int_{\mathcal{S}'} d\mu(q) (T_{-t} F)(q) \chi_{\mu_s}(q) = 0, \end{aligned}$$

since μ is invariant under $\{T_t | t \in \mathcal{H}\}$.

We now let $|t| \rightarrow \infty$ and use the ergodicity of ν to conclude that

$$0 = \int_{\mathcal{S}'} dv(q) \chi_\mu(q) (T_t \chi_{\mu_s})(q) \\ \left[\int_{\mathcal{S}'} dv(q) \chi_\mu(q) \right] \cdot \left[\int_{\mathcal{S}'} dv(q) \chi_{\mu_s}(q) \right] = (1 - \lambda)\lambda, \text{ as } |t| \rightarrow \infty.$$

Thus $\lambda = 0$ or $\lambda = 1$. If $\lambda = 0$ then clearly $F = 1$, since both ν and μ are invariant under $\{T_t | t \in \mathcal{H}\}$.

This proves the first part of the theorem.

If ν is not ergodic but still invariant under $\{T_t | t \in \mathcal{H}\}$ then

$$d\nu = \lambda d\mu_s + (1 - \lambda) F d\mu, \quad \lambda \in (0, 1),$$

and it is standard to show that the measures $d\mu_s$ and $F d\mu$ must be invariant under $\{T_t | t \in \mathcal{H}\}$. Since μ is ergodic, $F = 1$. This completes the proof of Theorem 4.3.

Theorem 4.3 has non-commutative generalizations:

EXAMPLE. — Let \mathfrak{A} be a C^* -algebra and $\{\tau_t | t \in \mathcal{H}\}$ a representation of \mathcal{H} as C^* automorphisms of \mathfrak{A} . Let ω_1 and ω_2 be two states on \mathfrak{A} which are invariant and ergodic under $\{\tau_t | t \in \mathcal{H}\}$, i. e.

$$\omega_j(A \tau_t(B)) \rightarrow \omega_j(A) \omega_j(B), \text{ as } |t| \rightarrow \infty, \text{ for } j = 1, 2.$$

Then either $\omega_1 = \omega_2$, or the G. N. S. representations π_{ω_1} and π_{ω_2} of \mathfrak{A} are *disjoint*. If the support of the measure $\int e^{-ipt} \omega_j(A * \tau_t(A)) dt$ is contained in the interval $[a, \infty)$ for some fixed $a > -\infty$ and all $A \in \mathfrak{A}$, $j = 1, 2$, then the representations π_{ω_j} , $j = 1, 2$, of \mathfrak{A} are *irreducible*. Under conditions stated in [33] (e. g. asymptotic abelianess) each \mathcal{H} -invariant state ω on \mathfrak{A} can be uniquely decomposed into pure \mathcal{H} -invariant states. The following applications of Theorem 4.3 seem to be new.

DEFINITIONS. — Let m_0 be a fixed, positive, bare mass. We define $C(m_0)$ to be the class of real, lower bounded polynomials P with $P(0) = 0$ for which the Schwinger functions $\{\mathfrak{S}_{m_0, D}^{k, P}(\xi_1, \dots, \xi_k)\}_{k=0}^\infty$ can be constructed by a convergent Cluster expansion in the sense of refs. [12] [13] (and have exponential cluster properties).

— Let $:$ denote Wick ordering with respect to the Gaussian mea-

sure $v_{m_0}^0$ and let P be a polynomial, g a test function in \mathcal{S} . We define a function Ψ_g^P on \mathcal{S}' by

$$\Psi_g^P(q) \equiv e^{-\frac{1}{2}(g, (-\Delta + m_0^2)g)} e^{-q((-\Delta + m_0^2)g)} e^{-U_g^P(q)}, \tag{4.4}$$

where
$$U_g^P(q) \equiv \int d^2\xi [: P(q + g) : (\xi) - : P(q) : (\xi)] \tag{4.5}$$

We define $C_1(m_0)$ to be the class of real, lower bounded polynomials P with $P(0) = 0$ for which the interacting measure $v_{m_0, D}^P$ exists and has the property that the Radon-Nikodym derivatives

$$\frac{dv_{m_0, D}^P(q + g)}{dv_{m_0, D}^P(q)}$$

exist, are positive $v_{m_0, D}^P$ -integrable functions on \mathcal{S}' and

$$\frac{dv_{m_0, D}^P(q + g)}{dv_{m_0, D}^P(q)} = \Psi_g^P(q), \text{ for all } g \text{ in } \mathcal{S}. \tag{4.6}$$

It is shown in [7] (Theorem 4.5) that the classes $C_1(m_0)$ and $C_1(m_0) \cap C(m_0)$ are non-empty. More precisely, if P is an arbitrary real, lower bounded polynomial with $P(0) = 0$ then there exist a positive λ such that $\lambda \cdot P \in C(m_0)$ [12] [13] and a positive λ_1 such that $\lambda_1 \cdot P \in C_1(m_0)$ (*). If $v_{m_0, D}^P$ is in $C_1(m_0) \cap C(m_0)$ then this measure is ergodic under the action of $\{ T_t \mid t \in \mathcal{R} \}$ (Theorem 4.1).

Let $\{ \Pi(g), g \in \mathcal{S} \}$ denote the s. a. Euclidean momenta constructed in [7] which are canonically conjugate to the Euclidean fields. If F is a $v_{m_0, D}^P$ -measurable function then

$$(e^{i\Pi(g)}F)(q) = F(q - g)\sqrt{\Psi_g^P(q)} \tag{4.7}$$

CONJECTURE. — If $v_{m_0, D}^P$ is in $C_1(m_0) \cap C(m_0)$ then it is ergodic under the action of the group $\{ e^{i\Pi(g)} \mid g \in \mathcal{S} \}$. This conjecture would have important consequences concerning the Markov property of the measure $v_{m_0, D}^P$ for half planes; Theorem 2.7 and [7].

— Let ν be some PBC measure on \mathcal{S}' and let $:\ :_\nu$ denote Wick ordering with respect to the measure ν [16] [37]. Let Q be an arbitrary, real polynomial. Suppose

$$\int_{\mathcal{S}'} d\nu(q) e^{i : Q(q) :_\nu(f)}$$

is continuous in f on \mathcal{S} . Then this functional satisfies the hypotheses of Theorem 1 and hence there exists a measure ν_Q such that

$$\int_{\mathcal{S}'} d\nu_Q(q) e^{iq(f)} \equiv \int_{\mathcal{S}'} d\nu(q) e^{i : Q(q) :_\nu(f)}, \tag{4.8}$$

for all f in \mathcal{S} .

(*) Recent results of the author imply that $\lambda_1 = 1$, i. e. if P is a real, lower bounded polynomial, $P \in C_1(m_0)$.

COROLLARY 4.4. — 1) Let P be a polynomial with degree $P > 3$ such that the infinite volume interacting PBC measure $\nu_{m_0, D}^P$ exists (Section 3; [6], subsection 4.1). Let ν_C^0 be an arbitrary Gaussian PBC measure on \mathcal{S}' with covariance operator C which is invariant and ergodic under the action of the group $\{T_t | t \in \mathcal{H}\}$. Let Q be some polynomial.

Then for all Q the measures $d\nu_{C, Q}^0$ are invariant and ergodic under $\{T_t | t \in \mathcal{H}\}$.

The measures $\nu_{m_0, D}^P$ and $\nu_{C, Q}^0$ are mutually singular (i. e. $\nu_{m_0, D}^P$ does not define a (generalized) free, Euclidean field or a Wick polynomial of such a field).

2) Let $\nu_{m_0, D}^{P_1}$ and $\nu_{m_0, D}^{P_2}$ be in $C_1(m_0)$ and assume that the measure $\nu_{m_0, D}^{P_1}$ is ergodic under $\{T_t | t \in \mathcal{H}\}$ (e. g. $\nu_{m_0, D}^{P_1} \in C(m_0) \cap C_1(m_0)$). Let $Q(\xi) = a\xi + b$, where a and b are arbitrary real numbers and let $\nu_{m_0, D, Q}^{P_1}$ be defined as in (4.8).

Then the measures $\nu_{m_0, D, Q}^{P_1}$ and $\nu_{m_0, D}^{P_2}$ are mutually singular, unless $P_1(\xi) = P_2(\pm \xi \mp b) + d, a = \pm 1$. In particular $\nu_{m_0, D}^{P_1}$ and $\nu_{m_0, D}^{P_2}$ are mutually singular, unless $P_1 = P_2$.

Proof. — Proof of 1): It is rather obvious that the measures $\nu_{C, Q}^0$ are invariant and ergodic under $\{T_t | t \in \mathcal{H}\}$. By Theorem 4.3

$$d\nu_{m_0, D}^P = \lambda d\nu_s + (1 - \lambda)d\nu_{C, Q}^0, \lambda \in [0, 1],$$

where ν_s and $\nu_{C, Q}^0$ are mutually singular, \mathcal{H} -invariant PBC measures.

We now show that there exists a positive, measurable function G on \mathcal{S}' such that:

G is $\nu_{m_0, D}^P$ -integrable, yet G is not $\nu_{C, Q}^0$ -integrable.

Since $\lambda \int d\nu_s(q)G(q) \geq 0$, this implies that $\lambda = 1$ and hence $\nu_{m_0, D}^P$ and ν_C^0 are mutually singular. We distinguish two cases:

Case 1. — $Q(\xi) = a\xi + b$.

There exist constants c_1 and c_2 such that

$$:\Phi^2(x):_{\nu_{C, Q}^0} = :\Phi^2(x): + c_1\Phi(x) + c_2$$

The measures $\nu_{C, Q}^0$ and $\nu_{m_0, D}^P$ are mutually singular if c_1 and c_2 are not finite. We may therefore assume that they are finite.

We now choose $G = e^{:\Phi^2:_{\nu_{C, Q}^0}(f)}$, where f is some function in \mathcal{S} . It follows from result proven in [6] (subsections 2.2, 3.4, 3.5, and 4.1) that

$$\int_{\mathcal{S}'} d\nu_{m_0, D}^P(q)G(q) = \int_{\mathcal{S}'} d\nu_{m_0, D}^P(q)e^{[q^2:(f) + c_1q(f) + c_2 \int f d^2\xi]}$$

is finite for all f in \mathcal{S} . However, it is a well-known fact with an easy proof that there exist functions f in \mathcal{S} such that the function $e^{q^2:_{\nu_{C, Q}^0}(f)}$ is not $\nu_{C, Q}^0$ -integrable.

Case 2. — Degree $Q > 1$. Here we choose $G = e^{\Phi(f)}$, $f \in \mathcal{S}$. We know from Section 3 and from ref. [6] (subsection 4.1) that

$$\int_{\mathcal{S}'} dv_{m_0, D}^P(q) e^{q(f)} \text{ is finite for all } f \text{ in } \mathcal{S}.$$

On the other hand one knows that there exist functions f in \mathcal{S} such that $e^{Q(q) : v_C^0(f)}$ is not v_C^0 -integrable. Therefore $e^{q(f)}$ is not $v_{C, Q}^0$ -integrable. This completes the proof of 1).

Proof of 2): If $v_{m_0, D}^{P_1}$ is ergodic under $\{T_t | t \in \mathcal{H}\}$ then obviously $v_{m_0, D, Q}^{P_1}$ is ergodic under $\{T_t | t \in \mathcal{H}\}$, as well. By Theorem 4.3 there exists a measure v_s and a number $\lambda \in [0, 1]$ such that

$$dv_{m_0, D}^{P_2} = \lambda dv_s + (1 - \lambda) dv_{m_0, D, Q}^{P_1},$$

where v_s and $v_{m_0, D, Q}^{P_1}$ are mutually singular PBC measures. Therefore there exists a characteristic function χ on \mathcal{S}' such that

$$\int_{\mathcal{S}'} dv_s(q) \chi(q) = 0, \quad \int_{\mathcal{S}'} dv_{m_0, D, Q}^{P_1}(q) \chi(q) = 1$$

and
$$\chi dv_{m_0, D}^{P_2} = (1 - \lambda) dv_{m_0, D, Q}^{P_1} \tag{4.9}$$

We now assume that $\lambda \neq 1$.

It is easy to see that the measure $v_{m_0, D, Q}^{P_1}$ is quasi-invariant under the group $\{e^{i\Pi(g)} | g \in \mathcal{S}\}$ since $v_{m_0, D}^{P_1}$ is so (by hypothesis) and $Q(\xi) = a\xi + b$. Therefore the Radon-Nikodym derivative

$$\Psi_{g, Q}^{P_1}(q) \equiv \frac{dv_{m_0, D, Q}^{P_1}(q + g)}{dv_{m_0, D, Q}^{P_1}(q)}$$

exists and

$$\Psi_{g, Q}^{P_1}(q) = \Psi_{\frac{1}{a}g}^{P_1}\left(\frac{1}{a}q - \frac{b}{a}\right). \text{ We set } \alpha \equiv \frac{1}{a}, \beta \equiv -\frac{b}{a}$$

Since $\Psi_{g, Q}^{P_1}$ exists we conclude that

$$\chi(q + g) = \chi(q), \text{ a. e.}, \tag{4.10}$$

where a. e. means « $v_{m_0, D, Q}^{P_1}$ -almost everywhere ». From (4.9) and (4.10) we conclude that

$$\Psi_g^{P_2}(q) = \Psi_{g, Q}^{P_1}(q) = \Psi_{\alpha g}^{P_1}(\alpha q + \beta), \text{ a. e.} \tag{4.11}$$

Let P' denote the first derivative of the polynomial P and let

$$: P'(\gamma\Phi + \delta) : (g) \equiv \int d^2\xi : P'(\gamma\Phi + \delta) : (\xi)g(\xi).$$

We set

$$\begin{aligned} F_g(\Phi) &\equiv (\alpha\Phi + \beta)(-\Delta + m_0^2)\alpha g \\ &+ : P'_1(\alpha\Phi + \beta) : (\alpha g) - : P'_2(\Phi) : (g) - \Phi(-\Delta + m_0^2)g \\ &= (\alpha^2 - 1)\Phi(-\Delta + m_0^2)g + \alpha\beta \cdot (-\Delta + m_0^2)g \\ &+ : P'_1(\alpha\Phi + \beta) : (\alpha g) - : P'_2(\Phi) : (g) \end{aligned}$$

From (4.11) we get

$$\frac{d}{d\mu} (\Psi_{\mu g}^{P_2}(q) - \Psi_{\mu \alpha g}^{P_1}(\alpha q + \beta)) |_{\mu=0} = F_g(q) = 0, \text{ a. e.}$$

and hence (using the results of [7], section 4)

$$[\Pi(g_1), [\Pi(g_2), \dots [\Pi(g_k), F_g(\Phi)] \dots]](q) = 0, \text{ a. e.} \tag{4.12}$$

for arbitrary k and arbitrary test functions g, g_1, \dots, g_k .

(The Euclidean momenta Π are defined in (4.7); the class $C_1(m_0)$ is studied in [7].)

But (4.12) implies:

$$(\alpha^2 - 1)(h, (-\Delta + m_0^2)g) = 0, \forall h, g \text{ in } \mathcal{S}, \text{ i. e. } \alpha = \pm 1,$$

and $\alpha P'_1(\alpha\xi + \beta) - P'_2(\xi) = 0$ and therefore $P_1(\pm \xi + \beta) = P_2(\xi) + P_1(\beta)$, since $P_1(0) = P_2(0) = 0$.

Thus $\lambda = 1$ (see (4.9)), unless $P_1(\pm \xi + \beta) = P_2(\xi) + P_1(\beta)$, $a = \pm 1$, b arbitrary. The equation $P_1 = P_2$, $P_1 \notin C(m_0)$, does not necessarily imply $\lambda = 0$, because there might exist several disjoint phases. This completes the proof of Corollary 4.4. Q. E. D.

Remarks. — It is easy to see that Corollary 4.4, 1) implies that the physical $P(\varphi)_2$ field φ_w obtained from Theorem 2.6 by Wightman's reconstruction theorem is not a (generalized) free field or a Wick polynomial of a (generalized) free field. It is presumably not in the tempered Borchers class of a free field with some arbitrary positive mass.

— The techniques of Theorem 4.3 and Corollary 4.4 can also be applied to the $P(\varphi)_2$ interacting measures $\nu_{m_0, D}^{\Lambda, l, P}$ with finite space cutoff l but no cutoff in the time-direction.

— One gets more explicit information on the support of the infinite volume interacting measures by using continuity properties of the functional $J_{m_0, D}^P(f)$ in f . See [3] [32]. If P is in $C(m_0)$ then $J_{m_0, D}^P$ is continuous on the Sobolev space

$$\mathcal{X}_{-1} = \{ g \in \mathcal{S}' \mid \| (-\Delta + 1)^{-\frac{1}{2}} g \|_2 < \infty \},$$

for the shifted field

$$\Phi'(\xi) \equiv \Phi(\xi) - \int_{\mathcal{S}'} dv_{m_0, D}^P(q) q(\xi)$$

Proof. — The two-point Schwinger function $\mathfrak{E}_{m_0, D}^{2, P}$ has the representation

$$\mathfrak{E}_{m_0, D}^{2, P}(\xi, \eta) = \int_{m^*}^{\infty} d\rho(m^2) [-\Delta + m^2]^{-1}(\xi - \eta)$$

for some measure ρ on $[0, \infty)$. Since the $P(\varphi)_2$ model defines a *canonical* quantum field theory for $P \in C(m_0)$ we have $\int_{m^*}^{\infty} d\rho(m^2) = 1$. By the mass-

gap theorem of refs. [12] [13] $m^* > 0$. Hence $\mathfrak{S}_{m_0, D}^{2, P}(f, f)$ is continuous on \mathcal{H}_{-1} . Q. E. D.

For more general continuity properties, see [11]. For consequences concerning the support of $v_{m_0, D}^P$, see [3] [32].

— Let P_1 and P_2 be polynomials in $C_1(m_0) \cap C(m_0)$. Then the measures $v_{m_0, D, Q}^{P_1}$ (defined by (4.8)) and $v_{m_0, D}^{P_2}$ are *mutually singular*, unless $Q(\xi) = \pm \xi + b$ and $P_1(\pm \xi + b) + d = P_2(\xi)$.

Proof. — The case where $Q(\xi) = a\xi + b$ has already been analyzed in Corollary 4.4, 2). We therefore assume that $\text{degree } Q > 1$. Clearly the measures $v_{m_0, D, Q}^{P_1}$ and $v_{m_0, D}^{P_2}$ are invariant and ergodic under $\{T_t \mid t \in \mathcal{H}\}$, since they are in the class $C(m_0)$. Hence they are mutually singular, unless they are equal. From the above remark we know that the two point function $\mathfrak{S}_{m_0, D}^{2, P_2}(f, f)$ is continuous in f on \mathcal{H}_{-1} . However, from estimates proven in [7], we know that

$$\int_{\mathcal{S}'} dv_{m_0, D, Q}^{P_1}(q)(q(f))^2$$

is *not* defined on all distributions f in \mathcal{H}_{-1} . Therefore the measures $v_{m_0, D}^{P_2}$ and $v_{m_0, D, Q}^{P_1}$ are *not* equal. Q. E. D.

We now want to discuss the consequences of Theorem 4.3 and Corollary 4.4 concerning the (time 0-) and the relativistic quantum $P(\varphi)_2$ fields. Although our results here are not very deep they represent a possibly instructive example of how one can pass from a purely Euclidean statement to a statement about the relativistic quantum fields. The example we want to present here is:

Corollary 4.4 $\Leftrightarrow \begin{cases} \text{Non-existence of the interaction picture,} \\ \text{(i. e. Haag's theorem [18] [40]).} \end{cases}$

We start our discussion with an analysis of the infinite volume (time 0-) interacting measure. Let J be a functional on \mathcal{S} obeying Axioms A, B, and C, \mathcal{H}_W the relativistic Hilbert space and Ω the physical vacuum obtained in Theorem 2.6. We know from Section, step 3, that for h a real-valued function on \mathbb{R}^1 with $\|h\|_{\mathcal{S}} < \infty$ the (time 0-) fields $\varphi(h) = \Phi(h \otimes \delta_0)$ exist and are selfadjoint on the spaces $L^2(\mathcal{S}', dv)$ and \mathcal{H}_W . Moreover $\|\varphi(h)\Omega\| \leq \|h\|_{\mathcal{S}}$; see (2.7).

It therefore follows that the functional

$$j(h) \equiv J(h \otimes \delta_0) \tag{4.13}$$

has the properties

A1') $j(0) = 1$, A2') j is continuous on \mathcal{S}_1 (more precisely $j(h)$ is continuous in h in the norm $\|\cdot\|_{\mathcal{S}}$), A3') j is of positive type, A4') j is real, and A5') j is invariant under the group \mathcal{P} of space-translations and under space-reflections. By Theorem 1, j is the Fourier transform of some \mathcal{P} -invariant, space-reflection invariant PBC measure μ on \mathcal{S}'_1 . The following result follows directly from [1].

PROPOSITION D'. — If Ω is the only Poincaré-invariant state in \mathcal{X}_w then

$$|j(h + h'_x) - j(h)j(h')| \rightarrow 0, \text{ as } |x| \rightarrow \infty,$$

where $h_x(y) \equiv h(y - x)$, and the measure μ is \mathcal{P} -ergodic. Theorem 4.3 tells us that two \mathcal{P} -invariant, \mathcal{P} -ergodic measures μ_1 and μ_2 on \mathcal{S}'_1 are either equal or mutually singular.

APPLICATION. — Let

$$P \in C(m_0) \text{ or } P(\xi) = a\xi^4 + b\xi^2 + \mu\xi, \quad a > 0 \quad \mu \neq 0 \quad (4.14)$$

For this class of polynomials we know that the hypothesis of Proposition D' is true and its conclusions are therefore true for the functional $j_{m_0}^P(h) \equiv J_{m_0, D}^P(h \otimes \delta_0)$. The measure associated with $j_{m_0}^P$ is denoted by $\mu_{m_0}^P$ and is \mathcal{P} -invariant and \mathcal{P} -ergodic.

COROLLARY 4.5. — 1) Let $P \in C(m_0)$. There exists a positive number λ_0 such that the PBC measures $\mu_{m_0}^{\lambda, P}$ and μ_C^0 are mutually singular for $0 \leq \lambda < \lambda_0$, where μ_C^0 is an arbitrary Gaussian PBC measure on \mathcal{S}'_1 with covariance C which is \mathcal{P} -invariant and \mathcal{P} -ergodic.

2) Let P and P_1 be in $C(m_0)$ and let $\lambda_1 = \sup \{ \lambda \mid \lambda P_1 \in C(m_0) \}$. Then the set $\{ \lambda \mid 0 \leq \lambda \leq \lambda_1, \mu_{m_0}^{\lambda, P_1} \equiv \mu_{m_0}^P \}$ is countable.

Proof. — 1) It is proven in Section 4 of ref. [6] that all the moments of the measure $\mu_{m_0}^P$ (i. e. the (time 0-) Wightman distributions \equiv (time 0-) Schwinger functions)

$$\rho_{m_0}^{k, P}(h_1, \dots, h_k) \equiv \int_{\mathcal{S}'} d\mu_{m_0}^P(\chi) \prod_{i=1}^k \chi(h_i) \quad h_j \in \mathcal{S}_1, \quad j = 1, \dots, k$$

exist and are tempered.

Let now $P \in C(m_0)$. Then by results of Dimock [4] and the uniform estimates on $\rho_{m_0}^{k, P}(h_1, \dots, h_k)$ in section 4 of [6] the perturbation series for the moments $\rho_{m_0}^{k, \lambda, P}(h_1, \dots, h_k)$ in λ is asymptotic at $\lambda = 0$.

This result enables us to show that the truncated four-point function $\rho_{m_0}^{4, \lambda, P}(x_1, \dots, x_4)^T$ (defined in the usual, inductive way [18]) is different from 0 if the coupling constant λ is small enough. However all the truncated moments of the measure μ_C^0 are known to vanish. Application of Theorem 4.3 (with \mathcal{S}' replaced by \mathcal{S}'_1 and \mathcal{H} replaced by \mathcal{P}) completes the proof of 1).

2) Let $P_1 \in C(m_0)$. It follows from the Cluster expansion in the form of ref. [13] that the (time 0-) Wightman distributions $\rho_{m_0}^{k, \lambda, P_1}(h_1, \dots, h_k)$ are analytic in λ in some complex neighbourhood of $(0, \lambda_1)$, where λ_1 is some positive number depending on P_1 .

By Dimock's results [4] the functions $\rho_{m_0}^{k, \lambda, P_1}(h_1, \dots, h_k)$ are not constant in λ . Therefore the equations

$$\rho_{m_0}^{k, \lambda, P_1}(x_1, \dots, x_k) = \rho_{m_0}^{k, P}(x_1, \dots, x_k), \quad \text{for all } k = 1, 2, \dots$$

have at most countably many solutions in the interval $[0, \lambda_1]$. This and Theorem 4.3 complete the proof of Corollary 4.5. Q. E. D.

There exist various generalizations of Corollary 4.5.

EXAMPLES. — Corollary 4.5, 2) still holds if one replaces $\mu_{m_0}^{\lambda P_1}$ by $\mu_{m_0, Q}^{\lambda P_1}$ (see (4.8)) for some real polynomial Q .

— Let P_1 and P_2 be in $C(m_0) \cap C_1(m_0)$. Suppose the measures $\nu_{m_0, D}^{P_1}$ and $\nu_{m_0, D}^{P_2}$ have the Markov property ([24] and Theorem 2.7) for half planes. Then $\mu_{m_0}^{P_1} = \mu_{m_0}^{P_2}$ if and only if $\nu_{m_0, D}^{P_1} = \nu_{m_0, D}^{P_2}$, i. e. $P_1 = P_2$, by Corollary 4.4, 2).

Finally let us discuss the representation of the canonical (time 0-) commutation relations on the physical Hilbert space \mathcal{X}_W . If $P \in C(m_0)$ it is known [10] [12] that there exist selfadjoint momenta $\pi(g)$, $g \in \mathcal{S}_1$, which are canonically conjugate to the (time 0-) fields $\varphi(h)$, $h \in \mathcal{S}_1$, i. e. the Weyl relations

$$e^{i\varphi(h)}e^{i\pi(g)} = e^{i\pi(g)}e^{i\varphi(h)}e^{-i(h,g)}$$

hold on the Hilbert space \mathcal{X}_W and $\pi(h) = i[H, \varphi(h)]$ (which is essentially selfadjoint on any core for H). Let \mathfrak{A} be the C^* -algebra generated by the operators

$$\{ e^{i\varphi(h)}e^{i\pi(g)} \mid h, g \text{ in } \mathcal{S}_1 \}.$$

The representation π_W of the Weyl algebra \mathfrak{A} on \mathcal{X}_W is irreducible; see [7] [10] [12].

Using results proven in [2] [19] we can show that Corollary 4.5, 1) implies that the representation π_W of \mathfrak{A} is disjoint from the representation $\pi_{\mathcal{F}}$ of \mathfrak{A} on the usual Fock space \mathcal{F} , provided $P \in C(m_0)$ and $0 < \lambda < \lambda_0$. Similar consequences follow from Corollary 4.5, 2).

We can prove a similar, yet more interesting result if we use the non-commutative version of Theorem 4.3 and Corollary 4.4, 1): If $P \in C(m_0)$ the physical vacuum $\Omega \equiv \Omega_{m_0}^P$ in \mathcal{X}_W is known to be unique [12] [13], and the state $\omega_{m_0}^P$ defined by

$$\omega_{m_0}^P(e^{i\varphi(h)}e^{i\pi(g)}) \equiv (\Omega_{m_0}^P, e^{i\varphi(h)}e^{i\pi(g)}\Omega_{m_0}^P),$$

where $\pi(h) = i[H, \varphi(h)]$ and H is the $P(\varphi)_2$ Hamiltonian, is then \mathcal{P} -invariant and \mathcal{P} -ergodic. We set

$$\omega_{m, Q}(e^{i\varphi(h)}e^{i\pi(g)}) \equiv (\Omega_0, e^{iQ(\varphi_m(h))}e^{iQ(\pi_m(g)) - b}\Omega_0),$$

where Ω_0 is the Fock vacuum, φ_m is the free (time 0-) field of mass $m > 0$ and π_m is its canonically conjugate momentum, $Q(\xi) = \pm \xi + b$.

THEOREM 4.6. — Let $P \in C(m_0)$. Then the states $\omega_{m_0}^P$ and $\omega_{m, Q}$ determine disjoint, irreducible representations of the Weyl algebra \mathfrak{A} , unless $b = 0$, $P = 0$, $m_0 = m$.

Proof. — Suppose the representations determined by the states $\omega_{m_0}^P$ and $\omega_{m, Q}$ are not disjoint. Then they are unitarily equivalent, since they are irreducible. Since the states $\omega_{m_0}^P$ and $\omega_{m, Q}$ are invariant and ergodic

under $\{T_x | x \in \mathcal{P}\}$, $\omega_{m_0}^P = \omega_{m,Q}$, by the non-commutative version of Theorem 4.3. The state $\omega_{m,Q}$ uniquely determines the expectation values

$$\left(\Omega_0, \prod_{j=1}^M (\pm \varphi_m(h_j) + b) \prod_{i=1}^N (\pm \pi_m(g_i)) \Omega_0 \right)$$

By Haag's theorem [40] these expectation values *uniquely* determine the Wightman distributions of a free field with mass m .

Therefore the states $\omega_{m_0}^P$ and $\omega_{m,Q}$ determine identical Wightman distributions and hence identical Schwinger functions. But in the case at hand the Schwinger functions determine the infinite volume (interacting) measure uniquely, and we conclude that $v_{m_0}^P = v_{m,Q}^0$. But this contradicts Corollary 4.4, 1) unless $b = 0$, $P = 0$, $m = m_0$. Q. E. D.

Theorem 4.6 tells us that

$$\omega_{m_0}^P = \omega_{m,Q} \text{ if and only if } v_{m_0}^P = v_{m,Q}^0,$$

or, equivalently, $b = 0$, $P = 0$, $m = m_0$.

This means that the interaction picture does *not* exist if degree $P > 2$.

It is shown in [7], section 6, that the Araki-functional [1]

$$\omega_{m_0,Q}^{\lambda,P}(e^{i\varphi(h)}e^{i\pi(g)}), \text{ } h \text{ and } g \text{ in } \mathcal{S}_1$$

has a perturbation series in λ which is asymptotic at $\lambda = 0$. The result is based on ref. [4].

Unfortunately such a result is not yet proven for the S-matrix, and there is no abstract theorem which tells us that Corollary 4.4, 1) and Theorem 4.6 imply $S \neq I$. We hope that the results of this section stimulate an analysis of the properties of the S-matrix, whose existence is established in [12].

Concluding Remarks.

We believe that this or the other result presented in this paper must be known by different authors. Axioms A and B have also been proposed in [17] and are of course inspired by refs. [24] [29] [30] [31]. Section 2 is partially based on results proven in [24] [29] [30]. Yet, Axiom C and a detailed analysis of its consequences, in particular a *rigorous construction* of (time 0-) quantum fields from the Euclidean fields, as well as the reconstruction of relativistic quantum fields satisfying Wightman's axioms from the generating functional of the Schwinger functions alone seem to be new. This functional is the natural object to consider from a probabilistic point of view. The additional structure with respect to the Osterwalder-Schrader axioms given through Axiom A turns out to be useful to distinguish a given theory satisfying Axioms A, B, and C from the one of the (generalized) free field. This is illustrated by our results in Section 4.

Section 3 contains apparently the first systematic verification of axioms

for Euclidean fields and of the Wightman axioms for the $P(\varphi)_2$ models with (half) Dirichlet boundary conditions.

Section 4 tells us that if two interaction polynomials P_1 and P_2 (with convergent Cluster expansion) do not satisfy $P_2(\xi) = P_1(\pm \xi + a) + b$ they determine different models in the sense that the Wightman distributions are different. The Wightman distributions remain different if one replaces one of the quantum fields by a Wick polynomial. The corresponding Euclidean measures are mutually singular. The interacting $P(\varphi)_2$ quantum field is never a (generalized) free field or a Wick polynomial of a (generalized) free field.

R. Schrader has informed the author of independent results similar to but slightly weaker than the ones proven in Corollary 4.4, 1) [34]. His techniques of proof are different from ours. B. Simon and J. Rosen have also obtained independently results which seem to agree essentially with the ones summarized in Corollary 4.4, 1). Their techniques are similar to ours. However, Corollaries 4.4, 2), 4.5 and Theorem 4.6 appear to be new.

We conclude with a problem: If $S_{m_0}^P$ denotes the S-matrix of a $P(\varphi)_2$ model. Show:

- 1) $S_{m_0}^P \neq I$ if degree $P > 2$.
- 2) The perturbation series of $S_{m_0}^{\lambda, P}$ in λ is asymptotic at $\lambda = 0$.
- 3) $S_{m_0}^{P_1} \neq S_{m_0}^{P_2}$ unless $P_1(\xi) = P_2(\pm \xi + a) + b$.

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Note added in typescript.

Since the date when this paper was completed the following interesting generalization of Corollary 4.4 has been found:

THEOREM. — *Let P_1 and P_2 be normalized, real, lower bounded polynomials such as in (3.14). Then the infinite volume interacting measures $dv_{m_0}^{P_1}$ and $dv_{m_0}^{P_2}$ are mutually singular unless $P_1 = P_2$.*

The relativistic quantum fields φ_{P_1} and φ_{P_2} obtained from the measures $dv_{m_0}^{P_1}$, $dv_{m_0}^{P_2}$, respectively, by reconstruction according to Theorem 2.6 belong to different Borchers classes (see Ref. 4) unless $P_1(x) = P_2(\pm x + a) + b$ (In particular, if $P \neq 0$, $\deg P > 3$ then φ_P is not in the Borchers class of the free field.)

The proof of this Theorem—which will appear in a forthcoming paper—

is based on the following *result*: Each component of an infinite volume interacting $P(\varphi)_2$ measure $dv_{m_0}^P$ ergodic under the « time »-translation group \mathcal{H} satisfies Axioms A, B, C, the hypotheses of Proposition D and belongs to the class $C_1(m_0)$ (see Section 4, definitions following Theorem 4.3).

Furthermore we make use of the fact that φ_P is a *canonical field*.

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