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PIERRE FERRERO
OLIVIER DE PAZZIS
DERECK W. ROBINSON

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Section A :
Physique théorique.

Scattering theory with singular potentials. II. The n-body problem and hard cores

by

Pierre FERRERO (*), Olivier de PAZZIS (*) and Derek W. ROBINSON (**)

ABSTRACT. — Results of absolute continuity and asymptotic completeness are proved for n-particles interacting with positive, spherically symmetric, decreasing potentials which are arbitrarily singular at the origin. Hard cores are also allowed. In particular we deduce that the scattering matrix for n hard spheres is unitary. Results for weakly attractive potentials are also given. Our results are valid in all but two dimensional space.

INTRODUCTION

In an earlier paper [*1*], which we refer to as I, one of us developed the theory of scattering to two particles interacting with positive singular potentials. In the present paper we extend many of the results of I to the case of n-particles. We also demonstrate how to handle hard cores. The methods used are similar to those of I and involve extensions of Lavine's work [*2*] [*3*] on positive potentials. We point out that these methods can also be used to discuss weakly attractive potentials and hence extend results of Iorio and O'Carroll [*6*].

(*) Physique Théorique, C. N. R. S., Marseille.
(**) University of Aix-Marseille II, Luminy, Marseille.
Postal address: Centre de Physique Théorique, C. N. R. S., 31, chemin J.-Aiguier, 13274 Marseille Cedex 2 (France).

1. THE HAMILTONIAN

In this section we discuss the definition of a self-adjoint Hamiltonian for n-particles interacting with a positive singular potential and derive an approximation scheme for this Hamiltonian. The discussion parallels part of the discussion of the two body Hamiltonian given in 1. The results for the n-body case are, however, weaker as we only have an analogue of the quadratic form result of 1. This is, nevertheless, sufficient for the subsequent applications to scattering theory, although it leaves open a number of natural questions which we mention at the end of the section.

A. Point particles

We consider n-particles in the configuration space $\mathbb{R}^\nu$ and the corresponding Hilbert space $\mathscr{H}^{(n)}$ of states is given by $L^2(\mathbb{R}^{\nu n})$, *i. e.* $\psi \in \mathscr{H}^{(n)}$ is a function of n variables $(x_1, \ldots, x_n)$, $x_i \in \mathbb{R}^\nu$ such that $\int dx_1 \ldots dx_n \, | \psi |^2 < \infty$. If one wishes to consider particles obeying Bose or Fermi statistics one considers only the subspaces $\mathscr{H}^{(n)}_\pm$ spanned by ψ which are totally symmetric, or antisymmetric, in the coordinates $(x_1, \ldots, x_n)$. This restriction is of no relevance in the following and all our results are valid either on $\mathscr{H}^{(n)}$ or $\mathscr{H}^{(n)}_\pm$. Thus we will ignore symmetry restrictions and work on $\mathscr{H}^{(n)}$.

We associate with each particle a momentum p_i, $i = 1, \ldots, n$ which acts as a differential operator on $\mathscr{H}^{(n)}$:

$$(p_j\psi)(x_1, \ldots, x_n) = - i\nabla_{x_j}\psi(x_1, \ldots, x_n)$$

where $D(p_j) = C_0^\infty(\mathbb{R}^{n\nu})$. The total kinetic energy operator for the system can be introduced as a symmetric operator

$$T_{tot} = \sum_{i=1}^{n} p_i^2$$

The kinetic energy of the center of mass is given by

$$T_{cm} = \frac{1}{n}\left(\sum_{i=1}^{n} p_i\right)^2$$

and the kinetic energy of the relative motion is defined by

$$T_{rel} = T_{tot} - T_{cm}$$

It is well-known that T_{tot} and T_{rel} are positive and essentially self-adjoint and if $\nu > 1$ we will denote their self-adjoint extension by H_{tot} and H_{rel}. If $\nu = 1$

we define H_{tot} and H_{rel} as the self-adjoint operators associated with the closures of the following quadratic forms

$$t_{tot}(\psi) = \sum_{i=1}^{n} \| p_i\psi \|^2$$

$$t_{cm}(\psi) = \frac{1}{n}\left\| \sum_{i=1}^{n} p_i\psi \right\|^2$$

$$t_{rel} = t_{tot} - t_{cm}$$

where $D(t_{tot}) = D(t_{cm}) = D(t_{rel}) = C_0^\infty(\mathbb{R}^n \backslash S_n)$. The set S_n is defined by

$S_n = \{ (x_1, \ldots, x_n) ; (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad x_i = x_j \text{ for some } \quad i \neq j \}$

This choice of H_{tot} and H_{rel} corresponds to particles with point hard cores.

We denote by q_i the operator of multiplication by x_i. The following estimates for H_{tot} and H_{rel} will be of use in the sequel.

LEMMA 1.1. — *The following inequalities are valid, in the sense of quadratic forms*

$$H_{tot} \geqslant H_{rel} \geqslant \sum_{1 \leqslant i < j \leqslant n} \frac{(\nu - 2)^2}{2(n-1)} \frac{1}{| q_i - q_j |^2}$$

Proof. — The following calculations are understood to be between C_0^∞ vectors. We have

$$H_{rel} = \sum_{1 \leqslant i < j \leqslant n} \frac{(p_i + p_j)^2}{2(n-1)} - \frac{1}{n}\left(\sum_{i=1}^{n} p_i\right)^2 + \sum_{1 \leqslant i < j \leqslant n} \frac{(p_i + p_j)^2}{2(n-1)} .$$

But it is proved for example in 1, that

$$(p_i - p_j)^2 \geqslant \frac{(\nu - 2)^2}{| q_i - q_j |^2}$$

and hence,

$$\sum_{1 \leqslant i < j \leqslant n} \frac{(p_i - p_j)^2}{2(n-1)} \geqslant \frac{(\nu - 2)^2}{2(n-1)} \sum_{1 \leqslant i < j \leqslant n} \frac{1}{| q_i - q_j |^2} .$$

It remains to prove that the first two terms give a positive contribution. But we have

$$\sum_{1 \leqslant i < j \leqslant n} \frac{(p_i + p_j)^2}{2(n-1)} - \frac{1}{n}\left(\sum_{i=1}^{n} p_i\right)^2$$

$$= \frac{n}{4}\left[\sum_{1 \leqslant i < j \leqslant n} \frac{2(p_i + p_j)^2}{n(n-1)} - \left(\sum_{1 \leqslant i < j \leqslant n} \frac{2(p_i + p_j)}{n(n-1)} \right)^2 \right] \geqslant 0$$

where the inequality follows from the convexity of X^2. The inequality can be extended to the domain of the closed quadratic form $\tilde{h}_{\text{rel}}$ associated with H_{rel} by closure. This completes the proof.

Next we wish to discuss the addition of an interparticle interaction V to H_{rel}. The interaction will be given as amultiplication by a real potential $(x_1, \ldots, x_n) \mapsto v(x_1, \ldots, x_n)$, $(V\psi)(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)\psi(x_1, \ldots, x_n)$ and the domain of V is given by $D(V) = \{ \psi ; \int dx_1 \ldots dx_n v^2 | \psi |^2 < \infty \}$.

At the moment we will not specify further properties of v but in our applications V will always be lower-semi-bounded densely defined, and hence self-adjoint by definition.

We will need to use the quadratic form h_{tot} h_{rel} and V associated with H_{tot}, H_{rel} and V

$$\begin{aligned} h_{\text{tot}}(\psi) &= (\psi, H_{\text{tot}}\psi) \quad , \quad D(h_{\text{tot}}) = D(H_{\text{tot}}) \\ h_{\text{rel}}(\psi) &= (\psi, H_{\text{rel}}\psi) \quad , \quad D(h_{\text{rel}}) = D(H_{\text{rel}}) \\ v(\psi) &= (\psi, V\psi) \quad , \quad D(v) = D(V) \end{aligned}$$

Further if t is any closable quadratic form we denote its closure by $\tilde{t}$.

THEOREM 1.2. — *Let the potential v associated with the interactions be positive, or more generally lower semi-bounded, and such that $v \in L^\infty(K)$ for each compact $K \subset \mathbb{R}^{n\nu} \backslash S_n$ where S_n denotes the singularity set $S_n = \{ (x_1, \ldots, x_n) ; x_\iota \in \mathbb{R}^\nu$, and $x_i = x_j$ for some pair $i \neq j \in (1, 2, \ldots, n) \}$. It follows that if $\nu \neq 2$ the Friedrichs extension and the form sum extension of* $H_{\text{tot}} + v(H_{\text{rel}} + v)$ *are equal,* i. e.

$$\widetilde{h_{\text{tot}} + v} = \tilde{h}_{\text{tot}} + \tilde{v}$$
$$\widetilde{h_{\text{rel}} + v} = \tilde{h}_{\text{rel}} + \tilde{v}$$

Proof. — The proof of the inequalities is very similar to that of theorem 1.2 in 1. Let us prove for instance the second inequality the proof of the first one is actually a little bit simpler.

As $\widetilde{h_{\text{rel}} + v} \subseteq \tilde{h}_{\text{rel}} + \tilde{v}$, to establish equality, it suffices to construct, for every $\varphi \in D(\tilde{h}_{\text{rel}} + \tilde{v})$ a sequence $\varphi_L \in D(H_{\text{rel}}) \cap D(V)$ such that

$$\lim \| \varphi_L - \varphi \| = 0 \qquad \text{and} \qquad \lim (\tilde{h}_{\text{rel}} + \tilde{v})(\varphi_L - \varphi) = 0.$$

Firstly note that if $\varphi \in D(\tilde{h}_{\text{rel}} + \tilde{v})$ has compact support K in $\mathbb{R}^{\nu n} \backslash S_n$, it is easy to construct a sequence $\varphi_L \in D(H_{\text{rel}})$ such that all φ_L have their supports in a compact K' with $K \subset K' \subset \mathbb{R}^{n\nu} \backslash S_n$, and such that

$$\lim_{L \to \infty} \| \varphi_L - \varphi \| = 0$$
$$\lim_{L \to \infty} \tilde{h}_{\text{rel}}(\varphi_L - \varphi) = 0.$$

Then $\lim \tilde{v}(\varphi_L - \varphi) = 0$ follows from the estimates

$$\tilde{v}(\varphi_L - \varphi) \leqslant \sup_{(x_1,\ldots,x_n)\in K'} | v(x_1, \ldots, x_n) | \, \| \varphi_L - \varphi) \|^2$$

Therefore in this case φ is approximated in the desired manner.

Let f_L be the following sequence of functions, already introduced in 1:

$$f_L \in C_0^\infty(\mathbb{R}^\nu) \quad ; \quad 0 \leqslant f_L \leqslant 1 \quad ; \quad f_L(x) = 1 \quad \text{if} \quad \frac{1}{L} < | x | < L \, ;$$

$$f_L(x) = 0 \quad \text{if} \quad | x | > 2L \quad \text{or} \quad | x | < \frac{1}{2L} \, ;$$

$$| \nabla f_L | < \frac{q}{| x |} \quad \text{if} \quad | x | < \frac{1}{L} \quad ; \quad | \nabla f_L | < 1 \quad \text{if} \quad | x | > L.$$

Now if $\varphi \in D(\tilde{h}_{rel} + \tilde{v})$ then $\varphi_L = F_L\varphi = \prod_{1\leqslant L<j\leqslant n} f_L(x_i - x_j)\varphi$ is also in $D(\tilde{h}_{rel} + \tilde{v})$ and it has compact support in $\mathbb{R}^{\nu n}\backslash S_n$. Thus the φ_L can be approximated in the desired manner from the discussion of the above paragraph. It suffices then to prove that

$$\lim_{L\to\infty} \| \varphi_L - \varphi \| = 0 \qquad \lim_{L\to\infty} \tilde{h}_{rel}(\varphi_L - \varphi) = 0 \qquad \lim_{L\to\infty} \tilde{v}(\varphi_L - \varphi) = 0$$

The first and third conditions follow from the dominated convergence theorem. To prove the second condition we note that $\varphi \in D(\tilde{h}_{rel})$ implies that there exists a linear change of coordinates $x_i' = \Sigma \alpha_{ij} x_j$ such that $\nabla_{x_i'}\varphi$ exists in the sense of distribution, $\nabla_{x_i'}\varphi \in L^2(\mathbb{R}^{\nu n})$ and $\tilde{h}_{rel}(\varphi) = \sum_{i=1}^{n} \| \nabla_{x_i'}\varphi \|^2$.

We used the fact that is a positive bilinear form in the p_i. Therefore

$$\tilde{h}_{rel}(\varphi_L - \varphi) = \sum_{i=1}^{n} \| (\nabla_{x_i'}F_L)\varphi - (1 - F_L)\nabla_{x_i'}\varphi \|^2$$

$$\leqslant \sum_{i=1}^{n} 2 \| (\nabla_{x_i'}F_L)\varphi \|^2 + 2 \| (1 - F_L)\nabla_{x_i'}\varphi \|^2$$

Now $\lim \| (1 - F_L)\nabla_{x_i'}\varphi \| = 0$ from the dominated convergence theorem. Furthermore

$$\sum_{i=1}^{n} \| (\nabla_{x_i'}F_L)\varphi \|^2 \leqslant \text{const.} \sum_{1\leqslant i<j\leqslant n} \int dx_1 \ldots dx_n \, | \nabla_{x_i} f_L(x_i - x_j) |^2 \, | \varphi |^2$$

$$\leqslant \text{const.} \sum_{1\leqslant i<j\leqslant n} \left\{ \int_{|x_i - x_j| \leqslant 1/L} dx_1 \ldots dx_n \, | \nabla_{x_i} f_L(x_i - x_j) |^2 \, | \varphi |^2 \right.$$

$$\left. + \int_{|x_i - x_j| \geqslant L} dx_1 \ldots dx_n \, | \nabla_{x_i} f_L(x_i - x_j) |^2 \, | \varphi |^2 \right\}$$

As

$$\sum_{1 \leqslant i < j \leqslant n} \int_{|x_i - x_j| \geqslant L} dx_1 \ldots dx_n \, | \nabla_{x_i} f_L(x_i - x_j) |^2 \, | \varphi |^2$$

$$\leqslant \sum_{1 \leqslant i < j \leqslant n} \int dx_1 \ldots dx_n \, | \varphi |^2$$

This term converges to zero as $L \to \infty$. On the other hand

$$\sum_{1 \leqslant i < j \leqslant n} \int_{|x_i - x_j| \leqslant \frac{1}{L}} dx_1 \ldots dx_n \, | \nabla_{x_i} f_L(x_i - x_j) |^2 \, | \varphi |^2$$

$$\leqslant a^2 \sum_{1 \leqslant i < j \leqslant n} \int_{|x_i - x_j| \leqslant 1/L} dx_1 \ldots dx_n \frac{| \varphi |^2}{| x_i - x_j |^2}$$

and it follows from lemma 1.1 that $\varphi \in D(| x_i - x_j |^{-1})$ so that this term converges to zero. This completes the proof.

We recall now the following theorem of approximation, which is proved in 1.

THEOREM 1.3. — *Let* V *be a densely defined interaction operator associated with a positive potential v. Assume* $H_{tot} + V$ (*resp.* $H_{rel} + V$) *is densely defined and that its Friedrichs extension and form sum extension are equal.*

Let $\{ V_L \}_{L \geqslant 0}$ *be a sequence of interactions associated with bounded potentials* v_L *such that*

1. $0 = v_0(x_1, \ldots, x_n) \leqslant v_1(x_1, \ldots, x_n) \leqslant \ldots \leqslant v_L(x_1, \ldots, x_n) \leqslant \ldots \leqslant v(x_1, \ldots, x_n)$
2. $\lim_{L \to \infty} v_L(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$

It follows that $H_{tot} + V_L$ (*resp.* $H_{rel} + V_L$) *is self-adjoint on* $D(H_{tot})$ (*resp.* $D(H_{rel})$) *and* $H_{tot} + V_L$ ($H_{rel} + V_L$) *converges to the Friedrichs extension of* $H_{tot} + V$ ($H_{rel} + V$) *in the strong resolvent sense.*

B. Hard core particles

In this subsection we discuss the definition of the Hamiltonian for particles with hard cores. The discussion is similar to the foregoing but differences enter essentially because the hard cores give rise to a choice of « free » Hamiltonians each corresponding to a different choice of boundary condition on the cores.

The Hilbert space $\mathscr{H}_a^{(n)}$ for n particles with spherical hard cores of diameter « a » is a closed subspace of $\mathscr{H}^{(n)}$ spanned by $\psi \in \mathscr{H}^{(n)}$ which vanish on the singularity set $S_a^{(n)}$, where $S_a^{(n)} = \{ (x_1, \ldots, x_n); x_i \in \mathbb{R}^\nu, | x_i - x_j | < a$ for some pair $i \neq j \in (1, 2, \ldots, n) \}$ *i. e.* $\psi(x_1, \ldots, x_n) = 0$ whenever the distance between any pair of the coordinates x_i which represent the positions of the particle centres, is smaller than a.

Next consider the differential operators T_{tot}, T_{rel} introduced in the previous subsection, restricted to $C_0^\infty(\mathbb{R}^{\nu n}\backslash S_a^{(n)})$. These operators are no longer essentially self-adjoint and hence it is not possible to reproduce H_{tot} and H_{rel} in the same unique manner as before. Instead we will define H_{tot}^a and H_{rel}^a as the Friedrichs extensions of T_{tot} and T_{rel} (This corresponds to the definition previously adopted for point particles when $\nu = 1$). It is well-known that this definition implies that H_{tot}^a and H_{rel}^a are differential operators such that

$$\psi \in D(H_{tot}^a), \qquad \text{or} \qquad \psi \in D(H_{rel}^a)$$

only if $\psi(x_1, \ldots, x_n) = 0$ for $(x_1, \ldots, x_n) \in \partial S_a^{(n)}$ where $\partial S_a^{(n)}$ denotes the surface of $S_a^{(n)}$.

These new definitions lead to a result similar to lemma 1.1.

LEMMA 1.4. — *If $\nu \neq 2$ the following inequalities are valid.*

$$H_{tot}^a \geqslant H_{rel}^a \geqslant \sum_{1 \leqslant i < j \leqslant n} \frac{1}{2(n-1)} \frac{1}{(|q_i - q_j| - a)^2}$$

in the sense of quadratic forms.

Proof. — The proof is identical to the proof of lemma 1.1 except we use $(p_i - p_j)^2 \geqslant (|q_i - q_j| - a)^2$ which is valid if $\nu \neq 2$.

Let us indicate how this inequality is deduced. It suffices to consider $p^2 = -\nabla^2$ acting on $L^2(\mathbb{R}^\nu)$. If $\varphi \in C_0^\infty(\mathbb{R}^\nu)$ and we decompose with respect to spherical harmonics with coefficients $|x| \to \varphi_L(|x|)$ and then introduce $\psi_L(|x|) = |x|^{(\nu-1)/2}\varphi_L(|x|)$ we find that

$$\|\nabla\varphi\|^2 = \sum_{L \geqslant 0} \int_0^\infty dr \overline{\psi}_L(r)\left[-\frac{d^2}{dr^2} + \frac{1}{r^2}\left(\frac{(\nu-2)^2}{4} - \frac{1}{4} + L(L+1)\right)\right]\psi_L(r)$$

$$\geqslant - \sum_{L \geqslant 0} \int_0^\infty dr \overline{\psi}_L(r) \frac{d^2}{dr^2} \psi_L(r) \qquad \text{for} \qquad \nu \neq 2.$$

But if $\varphi(x) = 0$ for $|x| \leqslant a$ we then find

$$\|\nabla\varphi\|^2 \geqslant \sum_{L \geqslant 0} \int_a^\infty dr \frac{|\psi_L(r)|^2}{4(r-a)^2} = \left(\varphi, \frac{1}{4(|q| - a)^2}\varphi\right)$$

We have used the well-known inequality $-d^2/dr^2 \geqslant 1/4r^2$ and a translation of coordinates.

We can now formulate the analogue of theorem 1.3.

THEOREM 1.5. — *Let the potential v associated with the interaction be positive, or more generally lower semi-bounded, and such that $v \in L^\infty(K)$ for each compact $K \subset \mathbb{R}^{\nu n} \backslash S_a^{(n)}$. It follows that, if $\nu \neq 2$ the Friederichs extension and the form sum extension of $H_{tot}^a + V(H_{rel}^a + V)$ are equal,* i. e.

$$\widetilde{h_{tot}^a + v} = \tilde{h}_{tot}^a + \tilde{v}$$

$$\widetilde{h_{rel}^a + v} = \tilde{h}_{rel}^a + \tilde{v}$$

Proof. — The proof is identical to the proof of theorem 1.3 if one replaces the sequence of functions f_L by the following

$$f_L(x) \in C_0^\infty(\mathbb{R}^\nu) \quad ; \quad 0 \leqslant f_L \leqslant 1 \quad ; \quad f_L(x) = 1 \quad \text{if} \quad a + \frac{1}{L} \leqslant |x| \leqslant L \; ;$$

$$f_L(x) = 0 \quad \text{if} \quad |x| > 2L \quad \text{or} \quad |x| < a + \frac{1}{2L} \quad ; \quad |\nabla f_L| < \frac{c}{|x| - a}$$

$$\text{if} \quad |x| < a + 1/L \quad ; \quad |\nabla f_L| < 1 \quad \text{if} \quad |x| > L.$$

The main result of this subsection is the following:

THEOREM 1.6. — *Let V be a densely defined interaction operator on $\mathcal{H}_a^{(n)}$ associated with a positive potential v. Assume $H_{tot}^a + V(H_{rel}^a + V)$ is densely defined and that its Friederichs extension and form sum extension are equal. Let $\{V_L\}_{L \geqslant 0}$ be a sequence of interactions associated with bounded potentials v_L such that*

1. $0 = v_0(x_1, \ldots, x_n) \leqslant v_1(x_1, \ldots, x_n) \leqslant \ldots \leqslant v_L(x_1, \ldots, x_n) \leqslant \ldots \leqslant v(x_1, \ldots, x_n)$
2. $\lim_{L \to \infty} v_L(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$

Finally let $\{U_m\}_{m \geqslant 0}$ be the set of interactions on $\mathcal{H}^{(n)}$ defined by the potentials

$$\begin{aligned} u_m(x_1, \ldots, x_m) &= m \qquad (x_1, \ldots, x_n) \in S_a^{(n)} \\ &= 0 \qquad (x_1, \ldots, x_n) \notin S_a^{(n)}. \end{aligned}$$

It follows that $H_{tot} + U_m + V_m(H_{rel} + U_m + V_m)$ is self-adjoint on $D(H_{tot})(D(H_{rel}))$ and converges to the Friederichs extension $H_{tot}^F(H_{rel}^F)$ of $H_{tot}^a + V(H_{rel}^a + V)$ in the strong resolvent sense. Explicitly

$$\lim_{m \to \infty} \left\| \left((H_{tot} + U_m + V_m + E)^{-1} - (H_{tot}^F + E)^{-1} \right) \psi \right\| = 0 \quad , \quad \psi \in \mathcal{H}_a^{(n)}$$

$$\lim_{m \to \infty} \left\| (H_{tot} + U_m + V_m + E)^{-1} \psi \right\| = 0 \quad , \quad \psi \notin \mathcal{H}_a^{(n)\perp}$$

for all $E > 0$ and similarly for H_{rel}.

Proof. — The sequence of bounded operators $(H_{tot} + U_m + V_m + E)^{-1}$ is monotonically decreasing and hence converges strongly. But if

$$\psi \in \mathscr{H}_a^{(n)\perp} \quad , \quad (\psi, (H_{tot} + U_m + V_m + E)^{-1}\psi) \leqslant (m + E)^{-1} \| \psi \|^2$$

which converges to zero as $m \to \infty$.

Next we shall consider the convergence of $(H_{tot} + U_m + E)^{-1}$ on $\mathscr{H}_a^{(n)}$. Firstly we note that the restriction of $\lim (H_{tot} + U_m + E)^{-1}$ to $\mathscr{H}_a^{(n)}$ is an operator R(E) on $\mathscr{H}_a^{(n)}$ and secondly we wish to identify this limit as $(H_{tot}^a + E)^{-1}$. In fact the foregoing estimate and the Schwartz inequality imply immediately that $\lim (H_{tot} + U_m + E)^{-1}$ is a positive bounded self-adjoint operator R(E) on $\mathscr{H}_a^{(n)}$.

The next step consists in showing that R(E) is the resolvent of some operator. Let us prove that R(E) is an injective mapping. Suppose that $R(E)\varphi = 0$ for some $\varphi \in \mathscr{H}_a^{(n)}$. One has for some $\psi \in D(H_{tot})$

$$0 = ((H_{tot} + E)\psi, R(E)\varphi) = \lim_{m\to\infty} ((H_{tot} + E)\psi, (H_{tot} + U_m + E)^{-1}\varphi)$$

Using the resolvent equation this yields:

$$(\psi, \varphi) = \lim_{m\to\infty} (U_m\psi, (H_{tot} + U_m + E)^{-1}\varphi)$$

Taking $\psi \in C_0^\infty(\mathbb{R}^{\nu n}\backslash S_a^{(n)})$ one sees that $U_m\psi$ is identically zero and hence $(\psi, \varphi) = 0$, this implies $\varphi = 0$. Next note that the range of R(E) is dense in $\mathscr{H}_a^{(n)}$ because R(E) is an injective self-adjoint operator. Therefore we can consider the operator $R(E)^{-1}$ with domain,

$$D(R(E)^{-1}) = \{ \varphi \in \mathscr{H}_a^{(n)} \ ; \ \varphi = \lim_{m\to\infty} (H_{tot} + U_m + E)^{-1}\psi \text{ for some } \psi \in H_a^{(n)} \}.$$

It remains to show that $R(E)^{-1} - E = H_{tot}^a$. Let $\varphi \in C_0^\infty(\mathbb{R}^{\nu n}\backslash S_a^{(n)})$ then φ can also be considered as a vector $\varphi \in \mathscr{H}^{(n)}$ with the property that φ vanishes on the singularity set $S_a^{(n)}$; with this identification one has $\varphi \in D(H_{tot})$ and $H_{tot}\varphi = H_{tot}^a\varphi$. Next for each $\psi \in \mathscr{H}^{(n)}$ one can use the resolvent equation to deduce that

$$(\psi, (H_{tot}+U_m+E)^{-1}(H_{tot}+E)\varphi) = (\psi, (1+(H_{tot}+U_m+E)^{-1}U_m)\varphi) = (\psi, \varphi)$$

Therefore taking the limit we find that

$$(\psi, R(E)(H_{tot}^a + E)\varphi) = (\psi, \varphi) \quad , \quad \psi \in \mathscr{H}^{(n)}, \varphi \in C_0^\infty(\mathbb{R}^{\nu n}\backslash S_a^{(\nu)})$$

This demonstrates that $h_{tot}^a(R(E)\psi, \varphi) + E(R(E)\psi, \varphi) = (\psi, \varphi)$. But as $C_0^\infty(\mathbb{R}^{\nu n}\backslash S_a^{(n)})$ is a core for h_{tot}^a we conclude that

$$R(E)\psi \in D(H_{tot}^a) \qquad \text{and} \qquad (H_{tot}^a + E)R(E)\psi = \psi.$$

Next we remark that the above argument can be respected to conclude that $(H_{tot} + U_m + V_n + E)^{-1}$ converges to $(H^a_{tot} + V_n + E)^{-1}$ as $m \to \infty$. Theorem 1.3 can then be applied to deduce that $(H^a_{tot} + V_n + E)^{-1}$ converges to $(H^F_{tot} + E)^{-1}$ as $n \to \infty$. However by monotonicity the diagonal limit $m = n \to \infty$ is identical to this double limit.

The result for H^a_{rel} can now be deduced by noting that the Friederichs extension and the form sum extension of H^a_{rel} and H^a_{cm} coincide. As in 1 the importance of this approximation theorem is the fact that if $H_n \to H$ in the strong resolvent sense then $\exp\{iH_n t\}$ converges strongly to $\exp\{iHt\}$ uniformly for t in any interval of $\mathbb{R}$.

The above results concerning equality of form extensions are weaker than the results obtained for two particles in 1. For two particles $H_{rel} + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{2\nu}\backslash S^2)$ if the interaction is sufficiently repulsive. The natural conjecture for n-particle is that this self-adjointness pertains whenever

$$v(x_1, \ldots, x_n) \geqslant \sum_{1 \leqslant i < j \leqslant n} \frac{C_\nu}{|x_i - x_j|^2}$$

and $v \in L^2_{loc}(\mathbb{R}^{n\nu}\backslash S^n)$, where $C_\nu = \max(0, 1 - (\nu - 2)^2/4)$. If $\nu \geqslant 4$ then the result can be proved as in 1, but the physically interesting cases $\nu \leqslant 3$ are more difficult.

Finally we remark that it is not clear whether the assumption $v \in L^\infty_{loc}(\mathbb{R}^{n\nu}\backslash S^n)$ is necessary in theorem 1.2 or whether this could be replaced by $v \in L^2(\mathbb{R}^{n\nu}\backslash S^n)$.

2. POSITIVE DECREASING INTERACTIONS

In this section we consider only these interactions V given by translationally invariant two body potentials v_{ij} of the form

$$(V\psi)(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} v_{ij}(x_i - x_j)\psi(x_1, \ldots, x_n)$$

Using the methods of Lavine [2] [3], the techniques of 1 and the results of the previous section one can now derive the following.

THEOREM 2.1. — *Let* V *be an interaction given by two-body potentials* v_{ij}, *with following properties.*

1. $v_{ij} \geqslant 0$,
2. v_{ij} *is spherically symmetric,*
3. v_{ij} *is decreasing*, i. e.

$$v_{ij}(\lambda\,|\,x_i - x_j\,|) \leqslant v_{ij}(|\,x_i - x_j\,|) \quad , \quad \lambda \geqslant 1$$

Assume either that $v_{ij} \in L^{\infty}_{loc}(\mathbb{R}^{\nu}\backslash\{0\})$ *or* $v_{ij} \in L^{\infty}_{loc}(\mathbb{R}^{\nu}\backslash S_a)$ *where* $S_a = \{x; |x| < a\}$. *In the first case define* H *as the Friederichs extension of* $H_{rel} + V$ *in the second case define* H^a *as the Friederichs extension of* $H^a_{rel} + V$. *It follows that* H *and* H^a *have absolutely continuous spectrum if* $\nu \neq 2$.

THEOREM 2.2. — *Adopt the same assumptions and definitions as in theorem 2.1. Further let* v_{ij} *satisfy*

$$\limsup_{|x_i - x_j| \to \infty} |x_i - x_j|^3 v_{ij}(x_i - x_j) = 0$$

It follows that the Møller matrices $\Omega_{\pm}(H, H_{rel})$, $\Omega_{\pm}(H_{rel}, H)$ *exist as unitary mappings on* $\mathcal{H}^{(n)}$ *and the Møller matrices* $\Omega_{\pm}(H^a, H_{rel})$, $\Omega_{\pm}(H_{rel}, H^a)$ *exist as unitary mappings between* $\mathcal{H}^{(n)}$ *and* $\mathcal{H}^{(n)}_a$ *if* $\nu \neq 2$.

Proof. — The proof of these two theorems is very similar to the proof given in 1 for two particles. We will sketch this proof for the case of point particles, the proof for hard core particles is identical except for notations.

First let $v_{ij}^{(L)}$ be a sequence of bounded potentials which approximate v_{ij} in the sense of theorem 1.3. We choose the $v_{ij}^{(L)}$ to be spherically symmetric and decreasing. Let $H^{(L)}$ denote the Hamiltonian defined with $v_{ij}^{(L)}$ replacing v_{ij}. Then $H^{(L)}$ converges to H in the strong resolvent sense. In particular if B is a bounded operator which is $H^{(L)}$ smooth uniformly in L, *i. e.*

$$\int_{-\infty}^{\infty} dt \left\| B e^{-iH^{(L)}t} \psi \right\|^2 < C \| \psi \|^2$$

with C independent of L and ψ, then B is also H smooth.

Next we use Lavine's commutator estimates [*2*], adapted slightly in the manner of 1, to construct operators B which are uniformly $H^{(L)}$ smooth. These estimates are obtained with the aid of a progress operator

$$A_g = \sum_{1 \leq i < j \leq n} A_{gij}$$

$$A_{g_{ij}} = g_{ij}(q_{ij}) \frac{q_{ij}}{|q_{ij}|} \cdot p_{ij} + p_{ij} \cdot \frac{q_{ij}}{|q_{ij}|} g_{ij}(q_{ij})$$

Here we have introduced relative coordinates

$$q_{ij} = q_i - q_j \quad , \quad p_{ij} = (p_i - p_j)/2$$

and
$$g_{ij}(x) = \int_0^x d\rho\, h_{ij}^2(\rho)$$

with
$$h_{ij}(\rho) = 1/(1 + \rho^2)^{\frac{1}{4}+\varepsilon}$$

By calculating the commutators $i[H^{(L)}, A_g]$ and estimating a lower bound in the manner of 1 and [2], one concludes that the following operators are uniformly $H^{(L)}$-smooth and hence H-smooth

$$h_{ij}^3(H+1)^{-\frac{1}{2}} \quad , \quad h_{ij}p_{ij}(H+1)^{-\frac{1}{2}} \quad , \quad p_{ij}h_{ij}(H+1)^{-\frac{1}{2}}$$

The conclusion of theorem 2.1 then follows from Kato's results [4] which state that if B is H-smooth the range of B^* is contained in the subspace of absolute continuity of H. But the range of $h_{ij}^3(H+1)^{-\frac{1}{2}}$ is dense.

To complete the proof of theorem 2.2 one argues in the manner of 1. First one proves that the Møller matrices

$$\Omega_\pm(H, H_{rel}) = s.\lim_{t\to\pm\infty} e^{iHt}e^{-iH_{rel}t}$$

exist. This is a known result, see for example [5], and it is established in two parts. First let χ_{ij} denote a positive C^∞ function such that

$$\chi_{ij}(x_{ij}) = 1 \quad , \quad |x_{ij}| < R$$
$$\chi_{ij}(x_{ij}) = 0 \quad , \quad |x_{ij}| > R+1$$

and introduce

$$F = 1 - \prod_{1\leqslant i<j\leqslant n}(1-\chi_{ij})$$

Thus $F = 1$ in a neighbourhood of the singularity set $S^{(n)}$ and $F = 0$ far from this set. One shows that

$$s.\lim_{t\to\pm\infty} e^{iHt}Fe^{-iH_{rel}t} = 0$$

and

$$s.\lim_{t\to\pm\infty} e^{iHt}(1-F)e^{-iH_{rel}t} = \Omega_\pm(H, H_{rel})$$

Next one establishes by use of the smoothness estimates that the following limits exist

$$s.\lim_{t\to\pm\infty} e^{iH_{rel}t}Fe^{-iHt} = 0$$

and

$$s.\lim_{t\to\pm\infty} (H_{rel}+1)^{-\frac{1}{2}}e^{iH_{rel}t}(1-F)e^{-iHt} = \widehat{\Omega}_\pm(H_{rel}, H).$$

For example the first limit is given as follows

$$\begin{aligned}\|e^{iH_{rel}t}Fe^{-iHt}(H+1)^{-1}\psi\|^2 &\leqslant ((H+1)^{-1}\psi, e^{iHt}Fe^{-iHt}(H+1)^{-1}\psi)\\ &\leqslant \sum_{1\leqslant i<j\leqslant n}((H+1)^{-1}\psi, e^{iHt}\chi_{ij}e^{-iHt}(H+1)^{-1}\psi)\\ &\leqslant C\sum_{1\leqslant i<j\leqslant n}((H+1)^{-1}\psi, e^{iHt}h_{ij}^6e^{-iHt}(H+1)^{-1}\psi)\\ &= C\sum_{1\leqslant i<j\leqslant n}\|h_{ij}^3e^{-iHt}(H+1)^{-1}\psi\|^2\end{aligned}$$

where C is a suitably chosen constant.

But from the second expression we see that the first derivative, with respect to t is uniformly bounded and from the last expression and smoothness the function is integrable. Thus it tends to zero at infinity.

The existence of $\tilde{\Omega}_\pm(H_{rel}, H)$ is proved in exactly the same manner as the two body case, see 1, lemma 3.3. Finally using the arguments of 1, lemma 3.4 and theorem 3.5 one concludes from the existence of $\Omega_\pm(H, H_{rel})$ and $\tilde{\Omega}_\pm(H_{rel}, H)$ that the Møller matrices $\Omega_\pm(H_{rel}, H)$ exist. We omit the details.

In [3] Lavine has shown that if V is relatively bounded with respect to H_{rel} and $v_{ij}(x) = 0(|x|^{-1-\varepsilon})$ the conclusions of theorem 2.2 are still valid. In 1 this result was generalized to singular potentials for two-particles. We have been unable to find a proof in the n-particle case which is valid for singular potentials.

3. WEAK ATTRACTION

In the previous section we have established asymptotic completeness in the n-particle problem for repulsive interactions. Apparently the only other result of this type which has been established from first principles is the result of Iorio and O'Carroll [6]. These authors show that if $\nu \geqslant 3$ and $v_{ij} \in L^{\nu/2+\varepsilon}(\mathbb{R}^\nu) \cap L^{\nu/2-\varepsilon}(\mathbb{R}^\nu)$ for some $\varepsilon > 0$ and $H = H_0 + \lambda V$ defined as a form sum then the Møller matrices $\Omega_\pm(H, H_0)$, $\Omega_\pm(H_0, H)$ exist and are unitary for all $|\lambda| < \lambda_0/n(n-1)$ where λ_0 is a constant independent of n. In particular the spectrum of $H_0 + \lambda V$ is then absolutely continuous. Thus the interactions considered by these authors can be partly (or wholly) attractive but their results are only valid when the attraction is too weak to form bound states.

We will now show that the methods that we have used can also be applied to obtain a weak coupling result of this nature. The basic estimate is an improvement of the commutator estimates used for decreasing potentials. We give this result for bounded potentials, lower semi-bounded potentials can be handled by approximation.

LEMMA 3.1. — *Let* V *be an interaction associated with bounded two body potentials* v_{ij} *which are spherically symmetric and once differentiable. Let* A_g *denote the progress operator defined in section 2, and set* $H = H_{rel} + V$. *One has for* $\varphi \in C_0^\infty$ *and* $\nu \neq 2$

$$i(\varphi, [H, A_g]\varphi) \geqslant \sum_{1 \leqslant i < j \leqslant n} (\varphi, C_{ij}\varphi)$$

where

$$C_{ij} = 4h_{ij}p_{ij}^2 h_{ij} + (1 + 4\varepsilon)\big(|q_{ij}|^2 - 1\big)/\big(1 + |q_{ij}|^2\big)^{\frac{5}{2}+2\varepsilon} + 2_{ij}\|g\|_\infty(n-1)\min\big(0, -v'_{ij}\big)$$

Proof. — One has

$$i[\mathrm{H}, \mathrm{A}_g] = i[\mathrm{H}_{\mathrm{rel}}, \mathrm{A}_g] + i[\mathrm{V}, \mathrm{A}_g].$$

One shows by the calculations of 1 that the first term is bounded below by the first two terms of C_{ij}. Let us consider

$$\begin{aligned} \mathrm{C}'_{ij} &= i[\mathrm{V}_{ij}, \mathrm{A}_g] \\ &= i[\mathrm{V}_{ij}, \mathrm{A}_{g_{ij}}] + i \sum_{k \neq i, k \neq j} [\mathrm{V}_{ij}, \mathrm{A}_{g_{ik}} + \mathrm{A}_{g_{jk}}]. \end{aligned}$$

But one has

$$i[\mathrm{V}_{ij}, \mathrm{A}_{g_{ij}}] = -2v'_{ij}g_{ij}$$

and

$$\begin{aligned} i[\mathrm{V}_{ij}, \mathrm{A}_{g_{ik}} + \mathrm{A}_{g_{jk}}] &= -v'_{ij}\frac{q_{ij}}{|q_{ij}|} \cdot \left(g_{ik}\frac{q_{ik}}{|q_{ik}|} + g_{jk}\frac{q_{kj}}{|q_{nj}|}\right) \\ &\geqslant \min(0, -v'_{ij}) 2 \| g \|_\infty . \end{aligned}$$

Therefore

$$i[\mathrm{V}_{ij}, \mathrm{A}_g] \geqslant 2(n-1) \| g \|_\infty \min(0, -v'_{ij})$$

and the proof is complete.

Using this estimation one can deduce the absence of bound states and asymptotic completeness for potentials which are possibly attractive. For example one has the following result.

THEOREM 3.2. — *Let* V *be an interaction given by two body potentials* v_{ij} *with the following properties.*

1. v_{ij} *is spherically symmetric and lower semi-bounded.*
2. v_{ij} *is once continuously differentiable on* $\mathbb{R}^\nu \backslash \{ 0 \}$.
3. $$\limsup_{|x_i - x_j| \to \infty} |x_i - x_j|^3 \, | v_{ij}(x_i - x_j) | = 0$$
4. *There is a* $\delta > 0$ *such that*

$$\min(0, -v'_{ij}) \leqslant \frac{(2-\delta)(\nu-2)^2 h_{ij}^2}{|q_{ij}|^2} \frac{1}{2 \| g \|_\infty (n-1)}$$

Let H *denote the Friederichs extension of* $\mathrm{H}_{\mathrm{rel}} + \mathrm{V}$. *It follows, for* $\nu \neq 2$, *that the spectrum of* H *is absolutely continuous and the Møller matrices* $\Omega_\pm(\mathrm{H}, \mathrm{H}_{\mathrm{rel}})$, $\Omega_\pm(\mathrm{H}_{\mathrm{rel}}, \mathrm{H})$ *exist as unitary mappings on* $\mathcal{H}^{(n)}$.

Proof. — The proof is again similar to previous proofs and so we will only sketch the outlines.

First note that v_{ij} can only have a positive singularity at the origin and that v_{ij} can be decomposed as a sum of a positive decreasing potential and a bounded potential which satisfies the assumptions of the theorem. The positive decreasing potential can be approximated from below by bounded decreasing potentials and the approximation procedures of theorem 1.2 will apply.

Next using the bound on v'_{ij}, the bound

$$p_{ij}^2 \geqslant (v - 2)^2/|q_{ij}|^2$$

and the results of Lemma 3.1 we find that H (or its approximates) satisfy

$$i[\mathrm{H}, \mathrm{A}_g] \geqslant c_1 h_{ij} p_{ij}^2 h_{ij} + c_2 h_{ij}^6$$

where c_1, $c_2 > 0$, for all pairs i, j. Next we have that $\mathrm{V} \geqslant -\mathrm{B}$ for some $\mathrm{B} \in \mathbb{R}$ and hence

$$\mathrm{H}_{\mathrm{rel}} + 1 \leqslant \mathrm{H} + \mathrm{B} + 1$$

It then follows as before that

$$h_{ij}^3(\mathrm{H} + \mathrm{B} + 1)^{-\frac{1}{2}} \quad , \quad p_{ij} h_{ij}(\mathrm{H} + \mathrm{B} + 1)^{-\frac{1}{2}}, \text{ etc.},$$

are H-smooth. This establishes that the spectrum of H is absolutely continuous. The proof of asymptotic completeness is essentially unchanged.

Remark 1. — The conditions on v_{ij} in the above theorem allow v_{ij} to have an arbitrary positive singularity at the origin. If one is only interested in v_{ij} which are Kato perturbations of $\mathrm{H}_{\mathrm{rel}}$ the conditions on v_{ij} can be weakened. It is no longer necessary that v_{ij} is lower semi-bounded and one can allow $v_{ij}(x_{ij}) = 0(|x_{ij}|^{-2-\varepsilon})$ at infinity. The proof of asymptotic completeness then follows from the work of Lavine [3] with the modified estimates given above.

Remark 2. — One allows under the conditions of remark 1 potentials v_{ij} which are $0(|x_{ij}|^{-2-\varepsilon})$ at infinity. This is in a sense the optimal behaviour at infinity because negative potentials with a slower decrease inevitably lead to the formation of bound states, cf. [6]. Results of the above type are interpretable as weak coupling statements in the sense of Iorio and O'Carroll [6]. Note however that in the latter authors result the coupling constant λ is restricted as a function of the number of particles n; by a condition of the form $|\lambda| < \lambda_0/n(n-1)$. Our result gives only a restriction of the type $|\lambda| < \lambda_0/(n-1)$. From stability considerations this latter behaviour should be optimal.

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