ANNALES DE L'I. H. P., SECTION A

J. GINIBRE
M. MOULIN

Hilbert space approach to the quantum mechanical three-body problem

Annales de l'I. H. P., section A, tome 21, nº 2 (1974), p. 97-145
<http://www.numdam.org/item?id=AIHPA_1974__21_2_97_0>

Ann. Inst. Henri Poincaré,
Vol. XXI, n° 2, 1974, p. 97-145

Section A :
Physique théorique.

Hilbert space approach to the quantum mechanical three-body problem

by

J. GINIBRE and M. MOULIN
Laboratoire de Physique Théorique et Hautes Énergies, Orsay, France (*)

ABSTRACT. — We study the quantum mechanical three-body problem in n-dimensional space ($n \geqslant 3$) with pair potentials that decrease at infinity as $|x|^{-(2+\varepsilon)}$. We work in configuration space and use only Hilbert space methods, in particular Kato's theory of smooth operators and Agmon's *a priori* estimates in weighted Hilbert spaces. We recover most of Faddeev's results. We prove in particular that the negative spectrum of H consists, besides the expected absolutely continuous part, of isolated eigenvalues of finite multiplicities which can accumulate at most at zero and at the two-body thresholds from below. The positive singular spectrum is contained in a closed set of measure zero, and the wave operators are asymptotically complete.

INTRODUCTION

The scattering and spectral theory of the quantum mechanical two-body problem has recently been brought into very satisfactory condition in the case of short range potentials, namely potentials $v(x)$ that decrease at infinity as $|x|^{-(1+\varepsilon)}$, $\varepsilon > 0$.

It has been proved long ago that the wave operators

$$\omega_{\pm} = \underset{t \to \mp \infty}{\text{s. lim}} \exp(ith) \exp(-ith_0) \tag{1}$$

(*) Laboratoire associé au Centre National de la Recherche Scientifique. Postal address: Laboratoire de Physique Théorique et Hautes Énergies, Bâtiment 211, Université de Paris-Sud, 91405 Orsay (France).

where h_0 is the free hamiltonian and $h = h_0 + v$, exist in this case (see for instance [1] and references therein contained). Furthermore, the ranges $\mathscr{R}(\omega_\pm)$ of the $\omega_\pm$ are contained in the subspace $\mathfrak{h}_{ac}$ of absolute continuity of h:

$$\mathscr{R}(\omega_\pm) \subseteq \mathfrak{h}_{ac} \tag{2}$$

The problem of proving that equality in fact holds in (2), commonly referred to as that of asymptotic completeness, has been studied by several authors and was finally solved for general short range potentials by Kato ([2], p. 206).

Under general assumptions on v, it is easy to see that the essential spectrum of h is $\sigma_e(h) = [0, \infty)$. The next question is to determine whether this part of the spectrum is absolutely continuous, as physical intuition suggests. The absence of positive eigenvalues has been proved for short range potentials satisfying mild additional regularity conditions [3] [4]. The absence of singular continuous spectrum (together with the absence of positive eigenvalues and with asymptotic completeness) had been proved earlier in the fundamental paper of Ikebe [5], for potentials decreasing at infinity as $|x|^{-(2+\varepsilon)}$. The absence of singular continuous spectrum for $|x|^{-(1+\varepsilon)}$ potentials has been proved recently by Agmon [6], see also [7] and [9].

These results have been extended to some classes of long range potentials under more special assumptions, in particular to repulsive and other potentials by commutator methods [8] [9], and to dilation analytic potentials [10] [11].

The corresponding problems for three-body and more generally N-body systems interacting via two-body forces are still at a much less advanced stage. The existence of the wave operators has been proved by the same methods and for the same potentials as in the two-body case (see for instance [1]). The essential spectrum $\sigma_e(\mathrm{H})$ of the hamiltonian H has been proved to be what physical intuition suggests, namely $\sigma_e(\mathrm{H}) = [\mathrm{E}_0, \infty)$ where E_0 is the lowest two-body threshold [12]. For the other (more difficult) problems, the situation is much less satisfactory. In the three-body case, asymptotic completeness and the absence of singular continuous spectrum have been proved in the fundamental work of Faddeev [13] for potentials that decrease at infinity as $|x|^{-(3+\varepsilon)}$ in three dimensions.

These results have been partially extended to the N-body system by Hepp and coworkers [14] [15] under additional technical assumptions.

The results obtained for repulsive or dilation analytic potentials have been extended to the N-body case [8] [16] [17] [11].

In the present paper, we shall take up the three-body problem in the same spirit as Faddeev, but depart from his methods in the following respects:

(1) Faddeev works in momentum space, and uses Banach spaces of Hölder-continuous functions of the momenta. These spaces are rather

difficult to handle, and have no direct physical meaning. We shall instead work in configuration space, and use exclusively Hilbert spaces. A number of well established techniques is then available, in particular Kato's theory of smooth operators [*18*] and Agmon's method using weighted Hilbert spaces [*6*]. We shall use both.

(2) In the two-body problem, it is convenient to use a symmetrized form of the resolvent equation (see equation (3.1) of this paper). This is even more true in the three-body problem, where this idea has already been introduced by Newton for similar purposes [*19*]. We shall also make use of it.

(3) With the methods described above, it turns out that one can handle potentials that decrease at infinity as $|x|^{-(2+\varepsilon)}$ in n-dimensional space, for $n \geq 3$. With this assumption (more precise formulations of which are contained in section 1.B), we are able to recover most of Faddeev's results, with the exception that we are not able to prove the absence of positive singular continuous spectrum for H.

We prove in particular that the negative spectrum of H consists, besides the expected absolutely continuous part, of isolated eigenvalues of finite multiplicities which can accumulate only at zero and at the two-body thresholds (i. e. at the bound state energies of the two-body subsystems) from below (proposition (7.2)). The positive singular spectrum is contained in a closed set of measure zero (proposition (6.4)), and the wave operators are asymptotically complete (proposition (8.4)).

It appears that for negative energy, the analytical difficulties of the three-body problem are essentially the same as those of the two-body problem. In particular, we can also prove that for potentials that decrease at infinity as $|x|^{-(1+\varepsilon)}$, the negative singular continuous spectrum of H is empty and that the negative point spectrum consists of eigenvalues of finite multiplicities which can accumulate at most at the two-body thresholds and at zero (proposition (7.3)). For such potentials, however, we obtain no information on the positive spectrum.

The paper is organized as follows. Section 1 contains some preliminary definitions and properties, namely kinematics (section 1.A), the conditions on the potentials (section 1.B) and the definition of the Hamiltonian (section 1.C). In section 2, we collect the basic estimates from Agmon's method in the form that is most useful for our purpose, and some extensions which are needed for the three-body case. In section 3, we study the two-body problem, using exclusively the methods that carry over to the three-body problem. This section therefore contains no new result, except perhaps for the fact that for $n \geq 3$ the number of bound states, including those with positive energies, is finite for potentials decreasing at infinity as $|x|^{-(2+\varepsilon)}$. (See [*20*] for estimates on the number of negative energy bound states under similar assumptions on the potential). In section 4, we begin the study of the three-body problem itself by deriving a modified form of the

Faddeev equations and setting up an algebraic formalism to construct the resolvent operator. Special attention is paid to the bound states of the two-body subsystems. In section 5, we derive the basic properties of the kernels of the modified Faddeev equations, namely uniform boundedness, Hölder-continuity and compactness in the closed cut plane, and analyticity in the open cut plane. We then apply the analytic Fredholm theorem to these equations. In section 6 we consider the associated homogeneous equations. We prove that outside of the essential spectrum, their solutions are in one to one correspondence with the bound states of H, while on the essential spectrum, they vanish on the energy shell. We also construct the resolvent operator. In section 7, we study the negative part of the spectrum of H and prove the results mentioned above. In section 8, we express the wave operators, or rather their adjoints, and the spectral projectors of H on absolutely continuous subsets of the spectrum, in terms of the resolvent operator. We then prove asymptotic completeness. Technical estimates are collected in Appendices A and B.

1. PRELIMINARIES

In this section, we collect some definitions and results which will be used throughout the paper. Section 1.A is devoted to kinematics, section 1.B contains the conditions fulfilled by the interactions, and section 1.C is devoted to the definition of the Hamiltonian.

A. Kinematics.

We consider a system of three non relativistic particles in n-dimensional space ($n \geq 3$). Particles will be labelled by latin indices i, j, etc. running from 1 to 3. Pairs of particles will be labelled by greek indices α, β, etc. running over (12), (23), (31). We denote by m_i the mass of particle i, by M the total mass of the system ($M = m_1 + m_2 + m_3$), by m_α the reduced mass of the pair α $\left(m_\alpha^{-1} = m_i^{-1} + m_j^{-1} \text{ if } \alpha = (i, j)\right)$, and by n_α the reduced mass of the pair α and of the third particle $\left(n_\alpha^{-1} = (m_i + m_j)^{-1} + m_k^{-1}\right.$ if $\left.\alpha = (i, j)\right)$. We define μ by:

$$\mu = (m_1 m_2 m_3/M)^{1/2} \tag{1.1}$$

We denote by x_i the position of particle i, by x_α the relative position of the particles in the pair α ($x_\alpha = x_j - x_i$ if $\alpha = (i, j)$) and by y_α the relative position of the third particle with respect to the center of mass of the pair α. We denote by X the set of internal coordinates of the system: $X = (x_\alpha, y_\alpha)$ for any α.

The (x_α, y_α) for different values of α are connected by well-known formulas. Typically:

$$x_{23} = y_3 - m_1(m_1 + m_2)^{-1}x_{12} \tag{1.2}$$

$$y_1 = - m_3(m_2 + m_3)^{-1}y_3 - m_2 M[(m_1 + m_2)(m_2 + m_3)]^{-1}x_{12} \tag{1.3}$$

We shall use the volume element $dX = dx_\alpha dy_\alpha$ (any α) or equivalently $dX = dx_\alpha dx_\beta$ (any $\alpha \neq \beta$) in coordinate space. One easily checks that the Jacobians are equal to one.

After separating out the center-of-mass motion, the classical kinetic energy is given for any α by $\frac{1}{2}(m_\alpha \dot{x}_\alpha^2 + n_\alpha \dot{y}_\alpha^2)$. This suggests the following natural definition of a quadratic form X^2 in coordinate space

$$\mu X^2 = m_\alpha x_\alpha^2 + n_\alpha y_\alpha^2 \tag{1.4}$$

where μ is given by (1.1).

Similarly, we denote by p_i the momentum of particle i, by p_α the relative momentum of the pair α, by q_α the relative momentum of the third particle and the pair α, and by $P = (p_\alpha, q_\alpha)$ the set of internal momenta of the system. The variables p_i, p_α and q_α are conjugate of the variables x_i, x_α and y_α respectively. The (p_α, q_α) for different values of α are connected by formulas analogous to (1.2, 3). The volume element in momentum space is given by $dP = dp_\alpha dq_\alpha$ (any α) $= dp_\alpha dp_\beta$ (any $\alpha \neq \beta$). The kinetic energy in the center-of-mass frame is given for any α by:

$$(2\mu)^{-1}P^2 = (2m_\alpha)^{-1}p_\alpha^2 + (2n_\alpha)^{-1}q_\alpha^2 \tag{1.5}$$

and this serves also as a definition for the quadratic form P^2.

B. Conditions on the interactions.

The three particles are supposed to interact via translation-invariant two-body potentials $v_\alpha(x_\alpha)$. The potentials v_α are supposed to be real measurable functions and to satisfy one of the following two conditions:

$(\mathcal{N})$ $v_\alpha \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for some p and q with $1 \leq q < \frac{n}{2} < p$.

$(\mathcal{A}_{1+\varepsilon})$ v_α can be written as $v_\alpha = (1 + x_\alpha^2)^{-(1+\varepsilon)}w_\alpha$, where $\varepsilon > 0$ and $w_\alpha \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some $p > \frac{n}{2}$ (i. e. w_α can be decomposed as the sum of a function in $L^p(\mathbb{R}^n)$ and a function in $L^\infty(\mathbb{R}^n)$).

Occasionally we shall also use potentials satisfying condition $(\mathcal{A}_{\frac{1}{2}+\varepsilon})$ in obvious notation.

In the special case $n = 3$, we shall also refer to the Rollnik condition: $(\mathcal{R})$ The following quantity is finite:

$$\| v_\alpha \|_{\mathcal{R}}^2 = (4\pi)^{-2} \int | v_\alpha(x)v_\alpha(x') | \, | x - x' |^{-2} dx \, dx' \tag{1.6}$$

The conditions $(\mathcal{N})$ and $(\mathcal{A}_{1+\varepsilon})$ are chosen so as to ensure two properties. First, the local singularities of the potentials are sufficiently weak to be controlled by the kinetic energy in a sense to be made precise in the next section. Second, the potentials decrease at infinity faster than $|x|^{-2}$.

Conditions $(\mathcal{N})$ and $(\mathcal{A}_{1+\varepsilon})$ are not independent. In fact, it is easy to see that $(\mathcal{A}_{1+\varepsilon}) \Rightarrow (\mathcal{N})$ with the same p and with any $q > n/2(1+\varepsilon)$. For $n = 3$, it is known that $(\mathcal{N}) \Rightarrow (\mathcal{R})$ [*21*].

C. The hamiltonian.

Unless otherwise stated, we consider the three particles in the center-of-mass frame, after elimination of the kinetic energy of the center of mass of the system. We define, at least formally

(1) the free hamiltonian:

$$H_0 = (2m_\alpha)^{-1}p_\alpha^2 + (2n_\alpha)^{-1}q_\alpha^2 \tag{1.7}$$

(2) the total hamiltonian:

$$H = H_0 + \sum_\alpha v_\alpha \tag{1.8}$$

(3) the hamiltonian where only the pair α interacts:

$$H_\alpha = H_0 + v_\alpha \tag{1.9}$$

(4) the hamiltonian of the pair α:

$$h_\alpha = (2m_\alpha)^{-1}p_\alpha^2 + v_\alpha \tag{1.10}$$

The free hamiltonian H_0 is self adjoint with an obvious domain $\mathcal{D}(H_0)$. For $n \geq 4$, potentials satisfying $(\mathcal{N})$ are known [*22*] to be small with respect to H_0 in the sense of Kato, namely

(1) $\mathcal{D}(v_\alpha) \supset \mathcal{D}(H_0)$

(2) For any $a > 0$ there exists $b > 0$ such that for any $\psi \in \mathcal{D}(H_0)$:

$$\| v_\alpha\psi \| \leqslant a \| H_0\psi \| + b \| \psi \| \tag{1.11}$$

From this it follows that the H_α and H are self adjoint with the same domain as H_0.

For $n = 3$, the same result holds only if in addition $p \geq 2$. For $\frac{3}{2} < p < 2$, the situation is slightly different, and v_α is not Kato small with respect to H_0 in general. However, v_α is small with respect to H_0 in the sense of quadratic forms, and H and the H_α can be defined as sums of quadratic forms [*21*]. One then has for sufficiently large a:

$$\mathcal{D}((a^2 + H)^{1/2}) = \mathcal{D}((a^2 + H_\alpha)^{1/2}) = \mathcal{D}((1 + H_0)^{1/2}) \subseteq \mathcal{D}(|v_\alpha|^{1/2})$$

Throughout the paper, we shall write equations such as (3.1, 14, 15) where various products of potentials, resolvent operators and sometimes hamiltonians occur. It is easy to check at each stage that these equations make sense in suitable sequences of spaces (cf. [*21*]). We shall not mention this point any further.

2. SOME AUXILIARY SPACES AND ESTIMATES

In this section, we introduce weighted Hilbert spaces and derive some properties of these spaces and in particular of the operator $(\lambda + \Delta)^{-1}$ acting between suitable pairs of such spaces, where Δ is the Laplace operator in $\mathbb{R}^n$. This technique has been used extensively by various authors and in particular by Agmon in the treatment of the two-body Schrödinger Hamiltonian. Most of the results of this section are due to Agmon or are easy extensions of his results ([*6*], see also [7]).

The basic spaces we shall consider are the spaces $L^2_\delta(\mathbb{R}^n)$ defined by:

$$L^2_\delta(\mathbb{R}^n) = \left\{ \varphi : \mathbb{R}^n \to \mathbb{C},\ \| \varphi \|^2_\delta = \int | \varphi(x) |^2 (1 + x^2)^\delta dx < \infty \right\} \quad (2.1)$$

where δ is a real number. $L^2_\delta(\mathbb{R}^n)$ is a Hilbert space with $\| \varphi \|_\delta$ as the norm of φ. This last notation will be used without further comment throughout the paper. The usual scalar product in $L^2(\mathbb{R}^n)$ identifies $L^2_{-\delta}(\mathbb{R}^n)$ with the dual of $L^2_\delta(\mathbb{R}^n)$.

Occasionally we shall use for $L^2_\delta(\mathbb{R}^n)$ equivalent norms obtained by replacing $(1 + x^2)^\delta$ by $(a^2 + x^2)^\delta$ for some $a > 0$ and use the extra freedom in the choice of a to derive certain estimates (see lemma (7.1)).

We shall also need anisotropic L^2_δ spaces defined as follows. Let $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ and $x = (x_1, x_2) \in \mathbb{R}^n$. We shall use

$$L^2_\delta(\mathbb{R}^{n_1}) \otimes L^2(\mathbb{R}^{n_1}) = \left\{ \varphi : \mathbb{R}^n \to \mathbb{C},\ \| \varphi \|^2 = \int | \varphi(x) |^2 (1 + x_1^2)^\delta d^n x < \infty \right\} \quad (2.2)$$

We now derive some properties of these spaces.

A. Restriction to spheres in momentum space.

For positive δ, functions in L^2_δ decrease more rapidly at infinity than functions in L^2, which implies that their Fourier transforms are more regular than functions in L^2. This allows restricting the Fourier transforms to spheres in $\mathbb{R}^n$, for δ sufficiently large, as described below.

We consider first the isotropic case. For any positive k, we define the mapping $\pi(k)$ from functions on $\mathbb{R}^n$ to functions on the unit sphere Ω in $\mathbb{R}^n$ by:

$$(\pi(k)\varphi)(\omega) = k^{(n-1)/2} \hat{\varphi}(k\omega) \quad (2.3)$$

where $\omega \in \Omega$ and $\hat{\varphi}$ is the Fourier transform of φ. $\pi(k)$ is well defined on functions that decrease sufficiently rapidly at infinity (for instance on $L^1(\mathbb{R}^n)$). We denote by $L^2(\Omega)$ the space of square integrable functions on Ω with the invariant measure, hereafter called $d\omega$. If Ω_k is the sphere of radius k in $\mathbb{R}^n$, then the mapping

$$\psi(p) \rightarrow k^{(n-1)/2}\psi(k\omega) \tag{2.4}$$

is an isometry from $L^2(\Omega_k, d^{n-1}p)$ onto $L^2(\Omega, d\omega)$. This is the reason for the introduction of the factor $k^{(n-1)/2}$ in the definition (2.3). The operator $\pi(k)$ satisfies the following properties [6] [7] [23].

PROPOSITION (2.1). — (1) Let $\delta > \frac{1}{2}$. Then $\pi(k)$ is a bounded operator from $L^2_\delta(\mathbb{R}^n)$ to $L^2(\Omega)$ with norm uniformly bounded in k.

(2) Let $\frac{1}{2} < \delta < \frac{n}{2}$. Then there is a constant C (independent of k) such that

$$\| \pi(k) \| \leqslant Ck^{\delta - 1/2} \tag{2.5}$$

(3) $\pi(k)$ is norm Hölder-continuous in k of order $\operatorname{Min}\left(\delta - \frac{1}{2}, 1\right)$.

Proof. — A detailed proof of the proposition can be found in [23]. Here, for completeness, we briefly reproduce the proof of (1) and (2). It is more convenient to consider the adjoint operator $\pi(k)^*$ from $L^2(\Omega)$ to $L^2_{-\delta}(\mathbb{R}^n)$, defined by:

$$(\pi(k)^*\varphi)(x) = k^{(n-1)/2} \int e^{ik\omega . x}\varphi(\omega)d\omega \tag{2.6}$$

By the definition of $L^2_{-\delta}$, we have:

$$\begin{aligned} \| \pi(k)^*\varphi \|^2_{-\delta} &= \int dx(1 + x^2)^{-\delta}k^{n-1} \left| \int d\omega e^{ik\omega . x}\varphi(\omega) \right|^2 \\ &= k^{2\delta - 1} \int dx(k^2 + x^2)^{-\delta} \left| \int d\omega e^{i\omega . x}\varphi(\omega) \right|^2 \end{aligned} \tag{2.7}$$

We decompose the plane wave on the set of eigenprojectors of the angular momentum operator. Let P_l be the projector in $L^2(\Omega)$ on the eigensubspace of the angular momentum with eigenvalue $l(l + n - 2)$ (l positive integer):

$$e^{i\omega . x} = (2\pi)^{n/2} \sum_{l=0}^{\infty} r^{1-n/2} J_{l-1+n/2}(r) P_l \tag{2.8}$$

where $r = |x|$ and J_ν is the Bessel function of order ν.

We substitute (2.8) into (2.7), use the Parseval identity and obtain:

$$\| \pi(k)^*\varphi \|_{-\delta}^2 \leqslant k^{2\delta-1}(2\pi)^n \sum_{l=0}^{\infty} \| P_l\varphi \|^2 \times \int_0^\infty r dr (k^2 + r^2)^{-\delta} | J_{l-1+n/2}(r) |^2 \tag{2.9}$$

$$\leqslant k^{2\delta-1}(2\pi)^n \| \varphi \|^2 \times \operatorname{Sup}_{l \geqslant 0} \int_0^\infty r dr (k^2 + r^2)^{-\delta} | J_{l-1+n/2}(r) |^2 \tag{2.10}$$

By the use of a recurrence relation on the Bessel functions, it is sufficient to have a bound on the integral in (2.10) for $m = l + \frac{n}{2} - 1 \leq 2$. In this case, we use the following bounds on the Bessel functions:

$$\begin{aligned} | J_m(r) | &\leq C r^m \quad \text{for} \quad 0 \leq r \leq 1 \\ | J_m(r) | &\leq C r^{-1/2} \quad \text{for} \quad r \geq 1 \end{aligned} \tag{2.11}$$

for some $C > 0$ independent of m [23].

The contribution of the region $r \leq 1$ to the integral in (2.10) is bounded, for $m = l + \frac{n}{2} - 1 \geq \frac{n}{2} - 1$, by:

$$\int_0^1 r^{n-1}(k^2 + r^2)^{-\delta} dr \leqslant (n - 2\delta)^{-1}(1 + k^2)^{-\delta} \tag{2.12}$$

for $\delta < n/2$.

The contribution of the region $r \geq 1$ is bounded by

$$\int_1^\infty (k^2 + r^2)^{-\delta} dr \leqslant c_\delta (1 + k^2)^{-\delta + 1/2} \tag{2.13}$$

where we have used lemma (2.1.1) below and c_δ is defined by (2.20).

These estimates and a similar estimate of the contribution of the region $r \leq 1$ if $\delta \geq \frac{n}{2}$, when substituted into (2.10), prove parts (1) and (2) of proposition (2.1).

We now give an extension of the previous result to the anisotropic case, which is sufficient for later use in section 8.

PROPOSITION (2.2). — Let $1 \leq \delta < \frac{n_1}{2}$. Then:

(1) The operator $\pi(k)$ defined by (2.3) is bounded from

$$L^2_\delta(\mathbb{R}^{n_1}, dx_1) \otimes L^2(\mathbb{R}^{n_2}, dx_2)$$

to $L^2(\Omega)$ with norm bounded by:

$$\| \pi(k) \| \leqslant C \operatorname{Min} (k^{1/2}, k^{\delta - 1/2}) \tag{2.14}$$

for some constant C independent of k.

(2) $\pi(k)$ is strongly continuous in k.

Proof. — We prove only (1). (2) can be proved by a similar method using proposition (2.1.3).

Let $\varphi \in L^2_\delta(\mathbb{R}^{n_1}, dx_1) \otimes L^2(\mathbb{R}^{n_2}, dx_2)$. Let $n = n_1 + n_2$ and $p = (p_1, p_2)$ where p_1 and p_2 are the Fourier conjugate variables of x_1 and x_2. We decompose p in radial and angular variables $p = (|p|, \omega)$ and similarly $p_i = (|p_i|, \omega_i)$, $i = 1, 2$. Then:

$$\| \pi(k)\varphi \|^2 = k^{n-1} \int d\omega \, | \hat{\varphi}(k\omega) |^2 = \int dp \delta(| p | - k) | \hat{\varphi}(p) |^2 \tag{2.15}$$

Now:

$$\begin{aligned} dp &= dp_1 dp_2 = dp_2 | p_1 |^{n_1 - 1} d | p_1 | d\omega_1 \\ &= dp_2 | p_1 |^{n_1 - 2} | p | d | p | d\omega_1 \end{aligned}$$

Therefore

$$\begin{aligned} \| \pi(k)\varphi \|^2 &= k \int dp_2 k_1^{n_1 - 2} \int d\omega_1 \, | \hat{\varphi}(p_1, p_2) |^2 \\ &= k \int dp_2 k_1^{-1} \| \pi(k_1)\hat{\varphi}(., p_2) \|^2 \end{aligned} \tag{2.16}$$

where $k_1 = (k^2 - p_2^2)^{\frac{1}{2}}$ and the last norm is taken in $L^2(\Omega_1)$ for fixed p_2. By proposition (2.1.1 and 2), there exists a constant C such that:

$$\| \pi(k_1) \| \leqslant C \operatorname{Min} (1, k_1^{\delta - 1/2}) \tag{2.17}$$

Therefore:

$$\| \pi(k)\varphi \|^2 \leqslant C^2 k \int dp_2 \operatorname{Min} (k_1^{-1}, k_1^{2\delta - 2}) \| \varphi(., p_2) \|_\delta^2 \tag{2.18}$$

where the last norm is taken in $L^2_\delta(\mathbb{R}^{n_1}, dx_1)$ for fixed p_2. For $\delta \geq 1$, we have

$$\operatorname{Min} (k_1^{-1}, k_1^{2\delta - 2}) \leqslant \operatorname{Min} (1, k^{2\delta - 2}) \tag{2.19}$$

Substituting (2.19) into (2.18) and integrating over p_2 yields (2.14).

B. Estimates on $(\lambda + \Delta)^{-1}$

In this subsection, we prove that the operator $(\lambda + \Delta)^{-1}$, where Δ is the Laplace operator in $\mathbb{R}^n$, is bounded between suitable pairs of L^2_δ spaces for λ real or complex.

Throughout this section, we shall make use of the following estimates:

LEMMA (2.1). — (1) Let $\delta > 1/2$ and define

$$c_\delta = \int_0^\infty (1 + x^2)^{-\delta} dx \tag{2.20}$$

Then:

$$\int_b^\infty (a^2 + x^2)^{-\delta} dx \leqslant c_\delta (a^2 + b^2)^{-\delta + 1/2} \tag{2.21}$$

(2) Let $0 \leq \delta < 1/2$. Then

$$\begin{aligned} \int_0^b (a^2 + x^2)^{-\delta} dx &\leqslant (1 - 2\delta)^{-1} b (a^2 + b^2)^{-\delta} \\ &< (1 - 2\delta)^{-1} (a^2 + b^2)^{-\delta + 1/2}. \end{aligned} \tag{2.22}$$

Proof. — (1) Let $a/b = \operatorname{tg} \theta$, $0 \leqslant \theta \leqslant \pi/2$. Then:

$$\begin{aligned} (a^2 + b^2)^{\delta - 1/2} \int_b^\infty (a^2 + x^2)^{-\delta} dx &= \int_{\cos \theta}^\infty (\sin^2 \theta + x^2)^{-\delta} dx \\ &= \int_0^\infty (1 + 2x \cos \theta + x^2)^{-\delta} dx \leqslant c_\delta \end{aligned}$$

(2) $$\begin{aligned} \int_0^b (a^2 + x^2)^{-\delta} dx &= b \int_0^1 (a^2 + b^2 x^2)^{-\delta} dx \\ &\leqslant b (a^2 + b^2)^{-\delta} \int_0^1 x^{-2\delta} dx = (1 - 2\delta)^{-1} b (a^2 + b^2)^{-\delta} \end{aligned}$$

The first property of $(\lambda + \Delta)^{-1}$ we shall state is a basic element in Agmon's approach. We reproduce it because we need some information on the constants that appear in it.

PROPOSITION (2.3). — Let $\delta > 1/2$, $\delta' > 1/2$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then the operator $(\lambda + \Delta)^{-1}$ is bounded from $L^2_\delta(\mathbb{R}^n)$ to $L^2_{-\delta'}(\mathbb{R}^n)$. It satisfies the following estimates:

(1) Let $n = 1$, $\varphi \in L^2_\delta(\mathbb{R})$. Then for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$:

$$\left\| \left(\lambda + \frac{d^2}{dx^2} \right)^{-1} \varphi \right\|_{-\delta'}^2 \leqslant |\lambda|^{-1} c_\delta c_{\delta'} \| \varphi \|_\delta^2 \tag{2.23}$$

where c_δ, $c_{\delta'}$ are defined by (2.20).

(2) Let n be arbitrary. There exist constants $c > 0$, $\theta > 0$, independent of λ, such that for all $\varphi \in L^2_\delta(\mathbb{R}^n)$ and all λ with $|\lambda| \geqslant \theta$:

$$\| (\lambda + \Delta)^{-1} \varphi \|_{-\delta'}^2 \leqslant |\lambda|^{-1} c \| \varphi \|_\delta^2 \tag{2.24}$$

Proof. — (1) The operator $\left(\lambda + \frac{d^2}{dx^2} \right)^{-1}$ is represented by the integral kernel

$$(2ik)^{-1} \exp (ik |x - x'|)$$

where $k = \lambda^{1/2}$, Im $k \geqslant 0$. It is therefore a Hilbert Schmidt operator from $L^2_\delta(\mathbb{R})$ to $L^2_{-\delta'}(\mathbb{R})$ with Hilbert Schmidt norm:

$$\begin{aligned} \| (\lambda + d^2/dx^2)^{-1} \|^2_{\mathrm{H.S.}} \\ &= (4 | \lambda |)^{-1} \int_{-\infty}^{\infty} dx\, dx'(1 + x^2)^{-\delta}(1 + x'^2)^{-\delta'} \exp(- \mathrm{Im}\, k | x - x' |) \\ &\leqslant | \lambda |^{-1} c_\delta c_{\delta'} \end{aligned} \tag{2.25}$$

(2) Boundedness for $\lambda \neq 0$ is proved by Agmon. See [7] for a proof. The estimate (2.24) can be proved by keeping track of the λ dependence of the various constants at all stages of the proof.

The next result is closely related to the Rollnik condition ($\mathscr{R}$). Let $n \geq 3$ and represent $\mathbb{R}^n$ as the orthogonal direct sum $\mathbb{R}^3 \oplus \mathbb{R}^{n-3}$. We consider the spaces $L^2_\delta(\mathbb{R}^3) \otimes L^2(\mathbb{R}^{n-3})$ for *the same* orthogonal decomposition and various values of δ.

PROPOSITION (2.4). — Let $\delta > 1/2$, $\delta' > 1/2$, $\delta + \delta' > 2$ and $\lambda \in \mathbb{C}$. Then the operator $(\lambda + \Delta)^{-1}$ is bounded from $L^2_\delta(\mathbb{R}^3) \otimes L^2(\mathbb{R}^{n-3})$ to $L^2_{-\delta'}(\mathbb{R}^3) \otimes L^2(\mathbb{R}^{n-3})$ with norm uniformly bounded with respect to λ.

Proof. — Vectors in $L^2_\delta(\mathbb{R}^3) \otimes L^2(\mathbb{R}^{n-3})$ are represented by functions $\varphi(x) = \varphi(x_1, x_2)$ where $x_1 \in \mathbb{R}^3$, $x_2 \in \mathbb{R}^{n-3}$. We perform a partial Fourier transform on the second variable and obtain functions $\tilde{\varphi}(x_1, p_2)$. In this representation, the operator $(\lambda + \Delta)^{-1}$ is represented as follows:

$$\begin{aligned} ((\lambda + \Delta)^{-1} \tilde{\varphi})(x_1, p_2) \\ &= - (4\pi)^{-1} \int | x_1 - x'_1 |^{-1} \exp(i(\lambda - p_2^2)^{1/2} | x_1 - x'_1 |) \\ &\quad \times \tilde{\varphi}(x'_1, p_2) dx'_1 \end{aligned} \tag{2.26}$$

where Im $(\lambda - p_2^2)^{1/2} \geq 0$. Now for fixed p_2, (2.26) defines a Hilbert-Schmidt operator from $L^2_\delta(\mathbb{R}^3)$ to $L^2_{-\delta'}(\mathbb{R}^3)$, with Hilbert-Schmidt norm:

$$\begin{aligned} c_{\delta,\delta'} &= (4\pi)^{-2} \int dx\, dx'(1 + x^2)^{-\delta}(1 + x'^2)^{-\delta'} | x - x' |^{-2} \\ &= 2^{-1} \int_0^{\infty} r dr\, r' dr'(1 + r^2)^{-\delta}(1 + r'^2)^{-\delta'} \operatorname{Log} \frac{r + r'}{| r - r' |} \end{aligned} \tag{2.27}$$

which is indeed finite for $\delta > 1/2$, $\delta' > 1/2$, $\delta + \delta' > 2$.

In particular, for fixed p_2:

$$\| ((\lambda + \Delta)^{-1} \tilde{\varphi})(., p_2) \|^2_{-\delta'} \leqslant c_{\delta\delta'} \| \tilde{\varphi}(., p_2) \|^2_\delta \tag{2.28}$$

Integrating (2.28) over p_2 yields the result.

We now turn to the following question. Let $\lambda > 0$ and let $\varphi \in L^2_\delta(\mathbb{R}^n)$ for some $\delta > 1/2$. Then the Fourier transform $\hat{\varphi}$ of φ can be restricted to the sphere $p^2 = \lambda$ by proposition (2.1). We assume in addition that this restriction vanishes: $\hat{\varphi}\,|_{p^2=\lambda} = 0$, and try to obtain estimates on the function $(\lambda + \Delta)^{-1}\varphi$ under these assumptions. We consider first the one-dimensional case.

PROPOSITION (2.5). — Let $\lambda > 0$ and $\delta > 1/2$. Let $\varphi \in L^2_\delta(\mathbb{R})$ and $\hat{\varphi}(\pm\lambda^{1/2}) = 0$. Let $\psi = (\lambda + d^2/dx^2)^{-1}\varphi$. Then:

(1) $\psi \in L^2_{\delta-1}(\mathbb{R})$ and for any δ' such that $1/2 < \delta' < \delta \leqslant \delta' + 1/2$, ψ satisfies:

$$\|\psi\|^2_{\delta-1} \leqslant \lambda^{-1} c_{\delta'}[2(\delta - \delta')]^{-1}\|\varphi\|^2_\delta \tag{2.29}$$

where $c_{\delta'}$ is defined by (2.20).

(2) If in addition $\delta > 3/2$, then ψ is bounded in $L^2_{\delta-2}(\mathbb{R})$ uniformly with respect to λ. For any δ' such that $3/2 < \delta' < \delta \leqslant \delta' + 1/2$, ψ satisfies

$$\|\psi\|^2_{\delta-2} \leqslant c_{\delta'-1}[2(\delta - \delta')]^{-1}\|\varphi\|^2_\delta \tag{2.30}$$

where $c_{\delta'-1}$ is defined by (2.20).

Remark. — For $\varphi \in L^2_\delta(\mathbb{R})$, $\delta > 1/2$ and $\lambda > 0$, the function ψ is defined by a limiting process from $(\lambda + i\eta + d^2/dx^2)^{-1}\varphi$ where $\eta \to 0$. The limit exists in some $L^2_{-\delta'}(\mathbb{R})$ with δ' sufficiently large.

Proof of the proposition. — (1) Let $k = \lambda^{1/2}$. ψ is represented by the absolutely convergent integral:

$$\psi(x) = (2ik)^{-1}\int_{-\infty}^{\infty} dx' \exp(ik\,|x - x'|)\varphi(x'). \tag{2.31}$$

Let $x \geq 0$. By assumption:

$$\hat{\varphi}(k) \equiv \int_{-\infty}^{\infty} dx' \exp(-ikx')\varphi(x') = 0$$

Multiplying by $e^{ikx}/2ik$ and subtracting from (2.31), we obtain:

$$\psi(x) = k^{-1}\int_x^{\infty} dx'\varphi(x') \sin[k(x - x')] \tag{2.32}$$

We use the bound $|\sin[k(x - x')]| \leq 1$ and obtain:

$$\begin{aligned} |\psi(x)|^2 &\leqslant \lambda^{-1}\left(\int_x^{\infty} dx'\,|\varphi(x')|\right)^2 \\ &\leqslant \lambda^{-1}\left(\int_x^{\infty} |\varphi(x')|^2(1 + x'^2)^{\delta'}dx'\right)\left(\int_x^{\infty}(1 + x'^2)^{-\delta'}dx'\right) \end{aligned} \tag{2.33}$$

by Schwarz inequality, for any $\delta' > 1/2$,

$$\ldots \leqslant \lambda^{-1}c_{\delta'}(1 + x^2)^{-\delta'+1/2}\int_x^{\infty} |\varphi(x')|^2(1 + x'^2)^{\delta'}dx' \tag{2.34}$$

by lemma (2.1.1).

A similar estimate can be derived for $x < 0$, using $\hat{\varphi}(-k) = 0$. Substituting these estimates into the definition of $\|\psi\|_{\delta-1}$ we obtain:

$$\begin{aligned}\|\psi\|_{\delta-1}^2 &\leqslant \lambda^{-1}c_{\delta'}\int_0^\infty dx(1+x^2)^{\delta-\delta'-1/2}\int_{|x'|>x}|\varphi(x')|^2(1+x'^2)^{\delta'}dx'\\ &= \lambda^{-1}c_{\delta'}\int dx'\,|\varphi(x')|^2(1+x'^2)^{\delta'}\int_0^{|x'|}dx(1+x^2)^{\delta-\delta'-1/2}\\ &\leqslant \lambda^{-1}c_{\delta'}[2(\delta-\delta')]^{-1}\|\varphi\|_\delta^2\end{aligned}$$

for $\delta' < \delta \leqslant \delta' + 1/2$, by lemma (2.1.2).

(2) Let now $\delta > 3/2$. In (2.32), we use the bound

$$|\sin[k(x-x')]| \leqslant k|x-x'|$$

and obtain:

$$\begin{aligned}|\psi(x)|^2 &\leqslant \left(\int_x^\infty dx'\,|\varphi(x')|(x'-x)\right)^2\\ &\leqslant \left(\int_x^\infty dx'\,|\varphi(x')|^2(1+x'^2)^{\delta'}\right)\left(\int_x^\infty dx'(x'-x)^2(1+x'^2)^{-\delta'}\right)\end{aligned}$$

for any $\delta' > 3/2$, by Schwarz inequality,

$$\ldots \leqslant c_{\delta'-1}(1+x^2)^{-\delta'+3/2}\int_x^\infty dx'\,|\varphi(x')|^2(1+x'^2)^{\delta'}$$

by lemma (2.1.1).

The end of the proof is essentially the same as the previous one.

We now extend the previous result to general n.

PROPOSITION (2.6). — Let $\lambda > 0$ and $\delta > 1/2$.

(1) Let $\varphi \in L_\delta^2(\mathbb{R}^n)$ and $\hat{\varphi}(p) = 0$ on the sphere $p^2 = \lambda$. Then

$$\psi \equiv (\lambda + \Delta)^{-1}\varphi \in L_{\delta-1}^2(\mathbb{R}^n)$$

and for some c_1 independent of λ:

$$\|\psi\|_{\delta-1}^2 \leqslant \lambda^{-1}c_1\|\varphi\|_\delta^2 \tag{2.35}$$

(2) Let $\delta > 3/2$. Let $\varphi \in L_\delta^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{n-1})$ and let $\hat{\varphi}(p) = 0$ on the sphere $p^2 = \lambda$. Then $\psi \equiv (\lambda + \Delta)^{-1}\varphi \in L_{\delta-2}^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{n-1})$ and for $3/2 < \delta' < \delta \leqslant \delta' + 1/2$,

$$\|\psi\|^2 \leqslant c_{\delta'-1}[2(\delta-\delta')]^{-1}\|\varphi\|^2 \tag{2.36}$$

where the norms of φ and ψ are taken in the spaces mentioned above.

(3) Let $\delta > 3/2$. Let $\varphi \in L_\delta^2(\mathbb{R}^n)$ and $\hat{\varphi}(p) = 0$ on the sphere $p^2 = \lambda$. Then $\psi \equiv (\lambda + \Delta)^{-1}\varphi \in L_{\delta-2}^2(\mathbb{R}^n)$ and for some c_2 independent of λ:

$$\|\psi\|_{\delta-2}^2 \leqslant c_2\|\varphi\|_\delta^2. \tag{2.37}$$

Proof. — (1) Can be proved from proposition (2.5.1), using a partial Fourier transform and a « cutting and pasting » technique [7].

(2) Follows from proposition (2.5.2) by the use of a partial Fourier transform and of Plancherel theorem.

(3) Follows from (2) with the same constant $C_2 = c_{\delta'-1}[2(\delta - \delta')]^{-1}$ for $\delta \leq 2$. For $\delta > 2$, one uses in addition the inequality

$$(1 + x^2)^{\delta-2} \leqslant \text{Max}\,(1, n^{\delta-3}) \sum_{i=1}^{n} (1 + x_i^2)^{\delta-2}$$

to derive (2.37) with:

$$C_2 = \text{Max}\,(n, n^{\delta-2}) c_{\delta'-1}[2(\delta - \delta')]^{-1}.$$

3. THE TWO-BODY PROBLEM

In this section, we consider the two-body problem.

The aim is not to give an optimal treatment, but to develop the methods that will carry over to, and derive the results that will be useful for, the three-body case, to be considered in the following sections.

We consider a system of two particles in n dimensional space ($n \geq 3$). We take the relative mass of the particles to be 1 and call v the two-body potential and p the relative momentum of the two particles. The hamiltonian is

$$h = h_0 + v = p^2/2 + v$$

We denote by $g_0(\lambda)$ and $g(\lambda)$ the resolvent operators of h_0 and h:

$$g_0(\lambda) = (\lambda - h_0)^{-1}$$
$$g(\lambda) = (\lambda - h)^{-1}$$

We shall derive general properties of the spectrum $\sigma(h)$ of h by standard time-independent methods. In all this section, the potential is supposed to satisfy condition ($\mathcal{N}$). The resolvent operator can be written formally as:

$$g = g_0 + g_0 \,|\, v \,|^{1/2}(1 - v^{1/2} g_0 \,|\, v \,|^{1/2})^{-1} v^{1/2} g_0 \tag{3.1}$$

where $|\, v \,|^{1/2}$ has the usual meaning and $v^{1/2}$ is defined by $v^{1/2} = v \,|\, v \,|^{-1/2}$. In order to study $g(\lambda)$ it is therefore useful to consider first the operator $a(\lambda) = v^{1/2} g_0(\lambda) \,|\, v \,|^{1/2}$. The main result in this direction is essentially due to Kato [*18*].

PROPOSITION (3.1). — Let v satisfy condition ($\mathcal{N}$). Then, as an operator in $L^2(\mathbb{R}^n)$ the operator $a(\lambda)$ satisfies the following properties.

(1) $a(\lambda)$ is bounded uniformly with respect to $\lambda \in \mathbb{C}$.

(2) $\|\, a(\lambda) \,\|$ tends to zero if $|\, \lambda \,| \to \infty$.

(3) $a(\lambda)$ is norm Hölder-continuous with respect to λ with order $n/2q - 1$ (for $1 < n/2q < 2$) and uniform coefficient (See (3.8)).

(4) $a(\lambda)$ is analytic in λ for $\lambda \notin [0, \infty)$.
(5) $a(\lambda)$ is compact for all $\lambda \in \mathbb{C}$.

Proof. — (1) *Boundedness* (Kato [*18*]). This proof is reproduced for the sake of completeness. Let $\operatorname{Im} \lambda \geqslant 0$. We write:

$$a(\lambda) = -i \int_0^\infty dt e^{it\lambda} v^{1/2} \exp(-ith_0) |v|^{1/2} \tag{3.2}$$

The operator $\exp(-ith_0)$ is unitary in $L^2(\mathbb{R}^n)$. Furthermore, it is represented by the integral kernel:

$$\exp(-ith_0)(x, x') = (2\pi it)^{-n/2} \exp[i(x - x')^2/2t] \tag{3.3}$$

From this it follows that $\exp(-ith_0)$ is bounded from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with norm $(2\pi t)^{-n/2}$. By the Riesz-Thorin theorem ([*24*], p. 525) it is therefore bounded from $L^s(\mathbb{R}^n)$ to $L^{s'}(\mathbb{R}^n)$ for $1 \leqslant s \leqslant 2$, $s^{-1} + s'^{-1} = 1$, with norm $(2\pi t)^{n/2 - n/s}$. We now estimate $\| v^{1/2} \exp(-ith_0) |v|^{1/2} \|$ for $v \in L^p(\mathbb{R}^n)$. Let $\varphi \in L^2(\mathbb{R}^n)$. By Hölder inequality, $|v|^{1/2}\varphi \in L^s(\mathbb{R}^n)$ with $s^{-1} = 2^{-1} + (2p)^{-1}$ and

$$\| \, |v|^{1/2}\varphi \|_{L^s} \leqslant \| v \|_{L^p}^{1/2} \| \varphi \|$$

Therefore $\exp(-ith_0)|v|^{1/2}\varphi \in L^{s'}$ with $s'^{-1} = 2^{-1} - (2p)^{-1}$ and

$$\| \exp(-ith_0)|v|^{1/2}\varphi \|_{L^{s'}} \leqslant (2\pi t)^{-n/2p} \| v \|_{L^p}^{1/2} \| \varphi \|$$

By another application of Hölder inequality, we obtain:

$$\| v^{1/2} \exp(-ith_0) |v|^{1/2} \| \leqslant (2\pi t)^{-n/2p} \| v \|_{L^p} \tag{3.4}$$

Therefore, if $v \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, we obtain:

$$\| a(\lambda) \| \leqslant \int_0^\infty dt \exp(-t \operatorname{Im} \lambda) \times \operatorname{Min}((2\pi t)^{-n/2p} \| v \|_{L^p}, id(p \to q)) \tag{3.5}$$

The estimate containing p (resp q) ensures the convergence of the integral for $t \to 0$ (resp $t \to \infty$). The bound thereby obtained is uniform in λ for $\operatorname{Im} \lambda \geqslant 0$. A similar proof holds for $\operatorname{Im} \lambda \leqslant 0$.

(2) *Behaviour at infinity.* We consider first the simple case where $|\operatorname{Im} \lambda| \to \infty$ or where $\operatorname{Re} \lambda \to -\infty$. It follows from the estimate (3.5) that $\| a(\lambda) \|$ tends to zero when $\operatorname{Im} \lambda \to \infty$. The same result holds for $\operatorname{Im} \lambda \to -\infty$. On the other hand, for $\operatorname{Re} \lambda \leqslant 0$, one can use the representation

$$a(\lambda) = -\int_0^\infty dt e^{\lambda t} v^{1/2} \exp(-th_0) |v|^{1/2} \tag{3.6}$$

It follows from (3.6) by similar estimates that $\| a(\lambda) \| \to 0$ when $\operatorname{Re} \lambda \to -\infty$.

We next consider the more difficult case where $\operatorname{Re} \lambda \to \infty$, but $|\operatorname{Im} \lambda|$ does not. Since $\| a(\lambda) \|$ is bounded in terms of $\| v \|_{L^p}$ and $\| v \|_{L^q}$ uniformly in λ, it is sufficient to prove that $\| a(\lambda) \| \to 0$ for a set of potentials which is dense in $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. We choose the subset of potentials of the form $v(x) = (1 + x^2)^{-\delta} w(x)$ for some $\delta > 1/2$, where $w \in L^\infty(\mathbb{R}^n)$. It is therefore sufficient to prove that $\| (1 + x^2)^{-\delta/2} g_0(\lambda)(1 + x^2)^{-\delta/2} \|$ tends to zero when $|\lambda|$ tends to infinity. This follows from proposition (2.3.2).

(3) *Hölder continuity*. Let $1 < n/2q < 2$, $\operatorname{Im} \lambda \geqslant 0$, $\operatorname{Im} \lambda' \geqslant 0$. Then:

$$| e^{it\lambda} - e^{it\lambda'} | \leqslant \operatorname{Min} (2, t | \lambda - \lambda' |) \tag{3.7}$$

Therefore, we obtain from (3.4, 7):

$$\begin{aligned} \| a(\lambda) - a(\lambda') \| &\leqslant \| v \|_{L^q} \int_0^\infty dt (2\pi t)^{-n/2q} \operatorname{Min} (2, t | \lambda - \lambda' |) \\ &= 4 \| v \|_{L^q} (4\pi)^{-n/2q} (2 - n/2q)^{-1} (n/2q - 1)^{-1} | \lambda - \lambda' |^{n/2q - 1} \end{aligned} \tag{3.8}$$

by an elementary computation.

(4) *Analyticity* in norm follows from weak analyticity, which follows in turn from the fact that the integrand in the representation (3.2) of the matrix elements of $a(\lambda)$ is analytic in λ, and from the previous estimates.

(5) *Compactness*. It is sufficient to prove compactness for λ real negative with $|\lambda|$ large. Compactness in the open cut plane $\mathbb{C} \backslash [0, \infty)$ follows from analyticity [*25*, App. 3], and compactness on the cut follows from uniform Hölder continuity in λ.

Let $\lambda < 0$. We use the representation (3.6) for $a(\lambda)$ and split the integral as $\int_0^\infty = \int_0^a + \int_a^\infty$. From estimates similar to but simpler than those used in the proof of boundedness, it follows easily that the first integral tends to zero in norm when a tends to zero, and that the second integral is norm convergent. It is therefore sufficient to prove that the integrand is compact, namely that $v^{1/2} \exp(- th_0) | v |^{1/2}$ is compact for all $t > 0$. Since this operator is norm continuous as a function of v for

$$v \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n),$$

it is sufficient to prove compactness if v satisfies the additional condition $v \in L^2(\mathbb{R}^n)$. Now for $v \in L^2(\mathbb{R}^n)$, we prove below that $v^{1/2} \exp(-th_0) | v |^{1/2}$ is a Hilbert-Schmidt operator, and therefore compact. In fact, from the representation of $\exp(- th_0)$ by the integral kernel:

$$\exp(- th_0)(x, x') = (2\pi t)^{-n/2} \exp [- (x - x')^2/2t] \tag{3.9}$$

it follows that:

$$\begin{aligned} \| v^{1/2} \exp(- th_0) | v |^{1/2} \|_{\text{H.S.}}^2 &= (2\pi t)^{-n} \int dx\, dx' | v(x) | \; | v(x') | \exp [- (x - x')^2/t] \\ &= (4\pi t)^{-n/2} \langle | v |, \exp(- th_0/2) | v | \rangle \\ &\leqslant (4\pi t)^{-n/2} \| v \|_{L^2}^2 \end{aligned} \tag{3.10}$$

since $\exp(- th_0/2)$ is a bounded operator in $L^2(\mathbb{R}^n)$.

Remark (3.1). — For $n = 3$, all the statements in proposition (3.1), except for Hölder continuity, hold true for $v \in \mathscr{R}$, the Rollnik class. In addition, $a(\lambda)$ is a Hilbert-Schmidt operator in this case (see [*21*] and references therein quoted), and is continuous in λ in Hilbert-Schmidt norm.

Remark (3.2). — We recall that an operator a is said to be h_0-smooth if the following quantity is finite [*18*]:

$$\| a \|_{h_0}^2 \equiv \operatorname*{Sup}_{\lambda \in \mathbb{C}} \pi^{-1} | \operatorname{Im} \lambda | \; \| a g_0(\lambda) \|^2 < \infty \tag{3.11}$$

A sufficient condition for a to be h_0-smooth is that $a g_0(\lambda) a^*$ be bounded uniformly in λ. In fact, $\| a g_0(\lambda) \| = \| g_0(\bar{\lambda}) a^* \|$ and:

$$| \operatorname{Im} \lambda | \; \| g_0(\lambda) a^* \varphi \|^2 = | \operatorname{Im} \langle \varphi, a g_0(\lambda) a^* \varphi \rangle | \leqslant \| a g_0(\lambda) a^* \| \; \| \varphi \|^2 \tag{3.12}$$

It follows from this remark and from proposition (3.1.1) that $v^{1/2}$ and $| v |^{1/2}$ are h_0-smooth. Actually this was the main motivation for proving this proposition in [*18*].

Proposition (3.1) enables us to apply the analytic Fredholm theorem to invert the operator $(1 - a(\lambda))$ [*26*, p. 201]. Let ξ be the set of $\lambda \in \mathbb{C}$ for which the homogeneous equation $\varphi = a(\lambda)\varphi$ has a solution in $L^2(\mathbb{R}^n)$. Let $\xi_+ = \xi \cap [0, \infty)$ and $\xi_- = \xi \backslash \xi_+$ (Notice that ξ_+ is the union of two subsets, obtained by considering $[0, \infty)$ as the upper and lower lips of the cut). Then:

PROPOSITION (3.2). — (1) ξ is bounded and closed. ξ_- is discrete.

(2) $(1 - a(\lambda))^{-1}$ exists as a bounded operator in $L^2(\mathbb{R}^n)$ for all $\lambda \notin \xi$, is meromorphic in the open cut plane $\mathbb{C} \backslash [0, \infty)$ with poles at the points of ξ_-, and is uniformly bounded and uniformly Hölder continuous in λ on the closed subsets of the closed cut plane not intersecting ξ.

From the representation (3.1) and proposition (3.2), one can derive a number of properties of $\sigma(h)$ and $g(\lambda)$. The next result will be stated without proof (see for instance [*21*] for equivalent proofs under different assumptions).

PROPOSITION (3.3). — (1) ξ_- is real and finite.

(2) ξ_- coincides with the negative part of $\sigma(h)$. The latter is discrete and is in fact the discrete spectrum $\sigma_d(h)$.

Remark (3.3). — The fact that ξ_- is finite follows from the argument of Schwinger (see for instance [*21*], p. 86). For $n = 3$, it is sufficient that $v \in \mathscr{R}$, and one obtains in addition an upper bound on the number of negative energy bound states.

Some information is also available on and near the cut.

PROPOSITION (3.4). — (1) ξ_+ is a bounded closed set of Lebesgue measure zero.

(2) ξ_+ contains the positive part of the point spectrum of h:

$$\sigma_p(h) \cap [0, \infty) \subset \xi_+$$

(3) The operator $v^{1/2}g(\lambda) | v |^{1/2}$ as an operator in $L^2(\mathbb{R}^n)$ is compact, uniformly bounded, and uniformly Hölder continuous in λ on the closed subsets of the closed cut plane not intersecting ξ.

(4) The same properties hold for $g(\lambda)$ as an operator from $L^2_\delta(\mathbb{R}^n)$ to $L^2_{-\delta}(\mathbb{R}^n)$ for any $\delta > 1$.

(5) The part of the spectrum of h in $[0, \infty)\backslash\xi_+$ is absolutely continuous. Moreover, the spectral projector of h on $[0, \infty)\backslash\xi_+$ is the absolutely continuous projector of h.

Proof. — (1) ξ_+ is bounded and closed because ξ is. That ξ_+ has Lebesgue measure zero is a result of Kuroda [*27*]. (See [*21*], p. 127 for a proof).

(2) Let $\lambda \geqslant 0$, $h\psi = \lambda\psi$. Let $\eta \neq 0$. By assumption:

$$(\lambda + i\eta - h_0)\psi = v\psi + i\eta\psi \tag{3.13}$$

We apply $g_0(\lambda + i\eta)$ to (3.13) and obtain:

$$\psi = g_0(\lambda + i\eta)v\psi + i\eta g_0(\lambda + i\eta)\psi\,. \tag{3.14}$$

Define $\varphi = v^{1/2}\psi$. Then $\varphi \in L^2(\mathbb{R}^n)$. In fact, if $n \geqslant 4$ then

$$\psi \in \mathscr{D}(h) = \mathscr{D}(h_0) \subset \mathscr{D}(v) \subset \mathscr{D}(v^{1/2})\,.$$

If $n = 3$, then $\psi \in \mathscr{D}(h) \subset \mathscr{D}((a^2 + h)^{1/2}) = \mathscr{D}((1 + h_0)^{1/2}) \subset \mathscr{D}(v^{1/2})$. φ satisfies:

$$\varphi = v^{1/2}g_0(\lambda + i\eta) | v |^{1/2}\varphi + i\eta v^{1/2}g_0(\lambda + i\eta)\psi \tag{3.15}$$

Let now $\eta \to 0$. The first term in the RHS is continuous in η and tends to $a(\lambda)\varphi$ by proposition (3.1.3). It suffices to prove that the second term tends to zero. Now $v^{1/2}$ is h_0-smooth by remark (3.2). Therefore:

$$\| \eta v^{1/2}g_0(\lambda + i\eta) \|^2 \leqslant \pi | \eta | \;\; \| v^{1/2} \|^2_{h_0} \tag{3.16}$$

This completes the proof.

(3) Follows immediately from (3.1) and proposition (3.2.2).

(4) Follows from (3) by noticing that $(1 + x^2)^{-\delta}$ satisfies condition $(\mathscr{A}_\delta)$, and therefore condition $(\mathscr{N})$ if $\delta > 1$.

(5) The first statement follows from (2) and (4) by using the representation:

$$\| e_{[a,b]}\varphi \|^2 = \lim_{\eta\downarrow 0} (2\pi i)^{-1} \int_a^b d\lambda \langle \varphi, (g(\lambda - i\eta) - g(\lambda + i\eta))\varphi \rangle \tag{3.17}$$

where $e_{[a,b]}$ is the spectral projector of h on $[a, b]$, for $[a, b] \subset (0, \infty)\backslash\xi_+$ and $\varphi \in L^2_\delta(\mathbb{R}^n)$ for some $\delta > 1$.

The second statement follows from the first and from (1).

Remark (3.4). — It follows from Proposition (3.4.3) that $| v |^{1/2}$ is

h-smooth on any interval $[a, b] \subset (0, \infty)\backslash\xi_+$. From this fact, from remark (3.2) and from proposition (3.4.5), it follows that the wave operators exist and are asymptotically complete [28].

In the end of this section, we shall obtain additional properties of the singular set ξ_+ and of the spectrum $\sigma(h)$ under slightly stronger assumptions on the potential: from now on, we assume v to satisfy condition $(\mathscr{A}_{1+\varepsilon})$. Then:

PROPOSITION (3.5). — Let v satisfy $(\mathscr{A}_{1+\varepsilon})$. Then ξ is finite and consists of eigenvalues of h with finite multiplicities, plus possibly $\lambda = 0$. In particular h as a finite number of bound states. The singular continuous spectrum $\sigma_{cs}(h)$ of h is empty.

Remark (3.5). — It has been proved by Agmon ([6], see also [7]) that if v satisfies condition $(\mathscr{A}_{1/2+\varepsilon})$, the singular set ξ_+ is countable with $\lambda = 0$ as the only possible accumulation point. Therefore the continuous singular spectrum is empty. The main tools of Agmon's method are the estimates of propositions (2.3) and (2.6.1) for the free resolvent operator. These estimates are not uniform in λ and cannot be applied near $\lambda = 0$. This is why ξ_+ can accumulate at zero in this case. The proof of proposition (3.5) closely follows Agmon's, and we shall give only a brief sketch of it. It differs from Agmon's by the use of the estimates of propositions (2.4) and (2.6.3) which are uniform in λ, and can therefore be applied near $\lambda = 0$.

Sketch of the proof of proposition (3.5). — One first shows that each solution of the homogeneous equation $\varphi = a(\lambda)\varphi$ gives rise to an eigenstate $\psi = g_0(\lambda)\,|\,v\,|^{1/2}\varphi$ with eigenvalue λ. By repeated use of proposition (2.4), one first shows that $\psi \in L^2_{-(1+\varepsilon)/2}(\mathbb{R}^n)$ with

$$\|\psi\|_{-(1+\varepsilon)/2} \leqslant D_1 \|\varphi\| \tag{3.18}$$

where the constant D_1 is independent of λ.

Then, from

$$0 = \operatorname{Im}\left\langle \varphi, \frac{v}{|v|}\varphi \right\rangle = \lim_{\eta \downarrow 0} \eta \,\|\, g_0(\lambda + i\eta)\,|\,v\,|^{1/2}\varphi \,\|^2 \tag{3.19}$$

and from proposition (2.1.1), one deduces that $\widehat{|v|^{1/2}\varphi}$ vanishes on the sphere $p^2 = \lambda$. Then, using proposition (2.6.3) repeatedly, one shows that $\psi \in L^2_\varepsilon(\mathbb{R}^n)$ with:

$$\|\psi\|_\varepsilon^2 \leqslant C_2 \|\,|\,v\,|^{1/2}\varphi\,\|_{2+\varepsilon}^2 \leqslant D_2 \|\varphi\|^2 \tag{3.20}$$

where D_2 is independent of λ.

From this it follows that ψ is an eigenvector of h, and that the set of eigenvectors of h is compact, and therefore finite dimensional. This completes the proof.

4. MODIFIED FADDEEV EQUATIONS

In this section, we begin the study of the three-body problem, and more precisely we introduce the appropriate generalization to the three-body case of the operator $a(\lambda)$ used in Section 3. This operator will be obtained from a suitably modified version of the Faddeev equations, as $a(\lambda)$ was obtained by symmetrizing the Lippmann-Schwinger equation. This method has already been used by Newton [*19*] for similar purposes.

The physical Hilbert space of the system is $\mathcal{H} = L^2(\mathbb{R}^{2n})$. Here $\mathbb{R}^{2n}$ is the space of internal coordinates $X = (x_\alpha, y_\alpha)$ of the system. We have already defined (Section (1.C)) the free hamiltonian H_0, the total hamiltonian H, the hamiltonians H_α of the three-body system with only the pair α interacting, and the hamiltonians h_α of the two-body subsystems. The corresponding resolvent operators are denoted by $G_0(\lambda)$, $G(\lambda)$, $G_\alpha(\lambda)$ and $g_\alpha(\lambda)$ respectively. For instance:

$$G(\lambda) = (\lambda - H)^{-1} \tag{4.1}$$

and similarly for the others. In all this section, the two-body potentials are assumed to fulfill condition $(\mathcal{N})$. In addition, we make the following assumptions on the two-body subsystems:

$(\mathcal{S}_+)$ For each α, the set $\xi_{+\alpha}$ of non negative values of λ for which the operator $a_\alpha(\lambda)$ has the eigenvalue 1 is empty.

$(\mathcal{S}_-)$ The negative discrete spectrum $\xi_{-\alpha}$ of h_α consists of one single eigenvalue $-\chi_\alpha^2 < 0$ with multiplicity one.

We have seen in Section 3 that if v_α satisfies instead of $(\mathcal{N})$ the slightly stronger condition $(\mathcal{A}_{1+\varepsilon})$ then ξ_α is finite and coincides with the point spectrum of h_α, plus possibly $\lambda = 0$. In this case $(\mathcal{S}_+)$ reduces to the assumption that $a_\alpha(0)$ does not have the eigenvalue 1 and that h_α has no positive eigenvalues. The last property can be shown to follow from mild additional regularity assumptions on the potentials [*3*] [*4*]. However, since the argument and these assumptions are of a fairly different nature from those in this paper, we prefer to state $(\mathcal{S}_+)$ instead of them.

From proposition (3.3), we already know that $\xi_{-\alpha}$ is finite. The only effect of $(\mathcal{S}_-)$ is to simplify the notations and spare irrelevant indices. It can be dropped with only trivial changes in the subsequent results and proofs. Actually some of the results will be stated without assuming $(\mathcal{S}_-)$, in particular in Section 7.

We now proceed to derive the appropriate generalization of equation (3.1). We start from the Faddeev equations. Let $V = \sum_\alpha v_\alpha$ and define the operator T by:

$$T = V + VGV \tag{4.2}$$

Then:

$$G = G_0 + G_0 T G_0 . \tag{4.3}$$

It follows from (4.2) that T can be expressed by:

$$T = \sum_{\alpha,\beta} M_{\alpha\beta} \tag{4.4}$$

where:

$$M_{\alpha\beta} = v_\alpha \delta_{\alpha\beta} + v_\alpha G v_\beta \tag{4.5}$$

so that:

$$G = G_0 + G_0 \sum_{\alpha,\beta} M_{\alpha\beta} G_0 . \tag{4.6}$$

Intuitively, $M_{\alpha\beta}$ is the contribution to T of all the terms in the perturbation expansion or in the multiple collision expansion [*29*] for which the first pair of particles to interact on the right is β, and the last pair of particles to interact on the left is α.

It can be shown that the $M_{\alpha\beta}$ satisfy the following equations (Faddeev equations) [*13*]:

$$M_{\alpha\beta} = T_\alpha \delta_{\alpha\beta} + T_\alpha G_0 \sum_{\gamma \neq \alpha} M_{\gamma\beta} \tag{4.7}$$

where the T_α are defined by:

$$T_\alpha = v_\alpha + v_\alpha G_\alpha v_\alpha \tag{4.8}$$

These equations hold under mild restrictions on the potentials for all λ in the intersection of the resolvent sets of the various hamiltonians.

We now modify these equations as follows. Define:

$$T'_\alpha = v_\alpha^{1/2} + v_\alpha^{1/2} G_\alpha v_\alpha \tag{4.9}$$

so that:

$$T_\alpha = | v_\alpha |^{1/2} T'_\alpha \tag{4.10}$$

(We recall that $v_\alpha^{1/2} \equiv v_\alpha | v_\alpha |^{-1/2}$).

We introduce new operators $L_{\alpha\beta}$, which intuitively are obtained from $M_{\alpha\beta}$ by removing $| v_\alpha |^{1/2}$ on the left and T'_β on the right, so that:

$$M_{\alpha\beta} = | v_\alpha |^{1/2} L_{\alpha\beta} T'_\beta \tag{4.11}$$

Formally, $L_{\alpha\beta}$ is defined by:

$$L_{\alpha\beta} = \delta_{\alpha\beta}(1 - v_\alpha^{1/2} G_0 | v_\beta |^{1/2}) + v_\alpha^{1/2}(G - G v_\beta G_0) | v_\beta |^{1/2} \tag{4.12}$$

and (4.11) is proved by substituting (4.12) into its RHS, comparing with (4.5) and using the Lippman-Schwinger equation repeatedly. It follows from (4.6) and (4.11) that:

$$G = G_0 + G_0 \sum_{\alpha,\beta} | v_\alpha |^{1/2} L_{\alpha\beta} v_\beta^{1/2} G_\beta \tag{4.13}$$

This is the generalization of (3.1) we were looking for. The $L_{\alpha\beta}$ can be proved to satisfy the following set of equations:

$$L_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{\gamma\neq\alpha} v_\alpha^{1/2} G_\alpha \mid v_\gamma \mid^{1/2} L_{\gamma\beta} \tag{4.14}$$

This is an equation for the set of the $L_{\alpha\beta}$, the kernel of which is the appropriate generalization of the operator $a(\lambda)$. Note that the Faddeev equations (4.7) are obtained immediately by substituting (4.14) into (4.11).

It will be clear from the estimates in Section 5 that all equations (4.9-14) make sense for potentials satisfying ($\mathcal{N}$), for any λ in the intersection of the resolvent sets of the various hamiltonians. In particular, the $L_{\alpha\beta}$ are bounded operators in $\mathcal{H}$ with these assumptions. It is natural to consider (4.14) as an operator equation in the direct sum of three copies of $\mathcal{H}$, namely $\bigoplus_\alpha \mathcal{H}_\alpha$. The family of the $L_{\alpha\beta}$ defines in a natural way an operator in this space. Let $\Phi_\alpha^{(0)}$ be the components of a vector in this space. Then (4.14) can be regarded as an equation for the vector with components:

$$\Phi_\alpha = \sum_\beta L_{\alpha\beta} \Phi_\beta^{(0)}$$

namely:

$$\Phi_\alpha = \Phi_\alpha^{(0)} + v_\alpha^{1/2} G_\alpha \sum_{\beta\neq\alpha} \mid v_\beta \mid^{1/2} \Phi_\beta \tag{4.15}$$

The space $\bigoplus_\alpha \mathcal{H}_\alpha$ would be adequate to study this equation if there were no two-body bound states. In the presence of two-body bound states, however, we need a more elaborate construction.

We first introduce some auxiliary functions. Let the two-body subsystems satisfy conditions ($\mathcal{S}_+$) and ($\mathcal{S}_-$) and let ψ_α be the eigenfunction of h_α with eigenvalue $-\chi_\alpha^2$.

The only property of ψ_α that we need is that $\psi_\alpha \in L_\delta^2(\mathbb{R}^n)$ for all δ, where $L_\delta^2(\mathbb{R}^n)$ is defined by (2.1) [*30*]. Let $\theta \geqslant 0$. We define functions $\rho_\alpha(y_\alpha)$ by:

$$\rho_\alpha^2(y_\alpha) = \int dx_\alpha \mid \psi_\alpha(x_\alpha) \mid^2 (1 + x_\alpha^2)^\theta \sum_{\beta\neq\alpha} \mid v_\beta(x_\beta) \mid \tag{4.16}$$

where the integration is performed for fixed y_α.

These functions satisfy the following properties:

LEMMA (4.1). — (1) Let $v_\beta \in L^p(\mathbb{R}^n)$ for all β and some $p \geqslant 1$. Then $\rho_\alpha^2 \in L^p(\mathbb{R}^n)$, and

$$\| \rho_\alpha^2 \|_{L^p} \leqslant \| \psi_\alpha \|_\theta^2 \sum_{\beta\neq\alpha} \| v_\beta \|_{L^p} \tag{4.17}$$

(2) Let $v_\beta(x) = (1 + x^2)^{-(1+\varepsilon)}w_\beta(x)$ with $w_\beta \in L^p(\mathbb{R}^n)$ for all β and some $p \geqslant 1$. Then $\rho_\alpha^2(y) = (1 + y^2)^{-(1+\varepsilon)}\sigma_\alpha^2(y)$ where $\sigma_\alpha^2 \in L^p(\mathbb{R}^n)$ and:

$$\| \sigma_\alpha^2 \|_{L^p} \leqslant 2^{1+\varepsilon} \| \psi_\alpha \|_{\theta+1+\varepsilon}^2 \sum_{\beta\neq\alpha} \| w_\beta \|_{L^p} \tag{4.18}$$

(3) Let $n = 3$ and $v_\beta \in \mathcal{R}$ for all β. Then $\rho_\alpha^2 \in \mathcal{R}$ and:

$$\| \rho_\alpha^2 \|_{\mathcal{R}} \leqslant \| \psi_\alpha \|_\theta^2 \sum_{\beta\neq\alpha} \| v_\beta \|_{\mathcal{R}} \tag{4.19}$$

Proof. — (1) Let $p^{-1} + p'^{-1} = 1$, $\varphi \geqslant 0$, $\varphi \in L^{p'}(\mathbb{R}^n)$. Then:

$$\begin{aligned}\int \rho_\alpha^2(y_\alpha)\varphi(y_\alpha)dy_\alpha &= \int dX \, | \psi_\alpha(x_\alpha) |^2(1 + x_\alpha^2)^\theta \sum_{\beta\neq\alpha} | v_\beta(x_\beta) | \, \varphi(y_\alpha) \\ &\leqslant \int dx_\alpha \, | \psi_\alpha(x_\alpha) |^2(1 + x_\alpha^2)^\theta \, \| \varphi \|_{L^{p'}} \sum_{\beta\neq\alpha} \| v_\beta \|_{L^p} \\ &= \| \psi_\alpha \|_\theta^2 \, \| \varphi \|_{L^{p'}} \sum_{\beta\neq\alpha} \| v_\beta \|_{L^p} \end{aligned} \tag{4.20}$$

by Hölder inequality, and from the fact that $dX = dx_\alpha dy_\alpha = dx_\alpha dx_\beta$, which follows from (1.2, 3).

(2) Is proved in the same way as (1), using in addition the bound:

$$(1 + y_\alpha^2) \leqslant 2(1 + x_\alpha^2)(1 + x_\beta^2) \tag{4.21}$$

(3) Is proved by substituting the definition (4.16) into (1.6), changing the order of integrations, and using Schwarz inequality.

We now come back to the system (4.15). We assume the two-body subsystems to satisfy conditions $(\mathcal{S}_+)$ and $(\mathcal{S}_-)$. Let P_α be the projection operator in $\mathcal{H}$ on the subspace that consists of the $\psi_\alpha(x_\alpha) \otimes \varphi(y_\alpha)$ where φ ranges over $L^2(\mathbb{R}^n)$. We separate out from G_α the singularity produced by the two-body bound state by writing:

$$G_\alpha = G'_\alpha + P_\alpha g_{0\alpha} \tag{4.22}$$

where we define:

$$g_{0\alpha} \equiv g_{0\alpha}(\lambda) = (\lambda + \chi_\alpha^2 - q_\alpha^2/2n_\alpha)^{-1} \tag{4.23}$$

In order to make the separation explicit in (4.15), we introduce new functions $\Phi_{\alpha i}$ $(i = 0, 1)$ satisfying a new system of equations. This system will be equivalent to (4.15) when λ lies in the intersection of the resolvent sets of the various hamiltonians.

Suppose that (Φ_α) is a solution of (4.15) and that $\Phi_\alpha^{(0)}$ is decomposed as follows:

$$\Phi_\alpha^{(0)} = \Phi_{\alpha 0}^{(0)} + v_\alpha^{1/2} P_\alpha g_{0\alpha} \rho_\alpha \Phi_{\alpha 1}^{(0)} \tag{4.24}$$

where by ρ_α we mean the multiplication operator by the function defined in (4.16), and where $\Phi_{\alpha 1}^{(0)} = P_\alpha \Phi_{\alpha 1}^{(0)} \in \mathscr{R}(P_\alpha)$. Define:

$$\left\{ \begin{aligned} \Phi_{\alpha 0} &= \Phi_{\alpha 0}^{(0)} + v_\alpha^{1/2} G'_\alpha \sum_{\beta \neq \alpha} |v_\beta|^{1/2} \Phi_\beta \\ \Phi_{\alpha 1} &= \Phi_{\alpha 1}^{(0)} + \rho_\alpha^{-1} P_\alpha \sum_{\beta \neq \alpha} |v_\beta|^{1/2} \Phi_\beta \end{aligned} \right. \tag{4.25}$$

Then Φ_α can be recovered from the $\Phi_{\alpha i}$ $(i = 0, 1)$ by a formula similar to (4.24):

$$\Phi_\alpha = \Phi_{\alpha 0} + v_\alpha^{1/2} P_\alpha g_{0\alpha} \rho_\alpha \Phi_{\alpha 1} \tag{4.26}$$

Furthermore the $\Phi_{\alpha i}$ satisfy the following set of equations:

$$\begin{aligned} \Phi_{\alpha 0} &= \Phi_{\alpha 0}^{(0)} + v_\alpha^{1/2} G'_\alpha \sum_{\beta \neq \alpha} |v_\beta|^{1/2} \Phi_{\beta 0} + v_\alpha^{1/2} G'_\alpha \sum_{\beta \neq \alpha} v_\beta P_\beta g_{0\beta} \rho_\beta \Phi_{\beta 1} \\ \Phi_{\alpha 1} &= \Phi_{\alpha 1}^{(0)} + \rho_\alpha^{-1} P_\alpha \sum_{\beta \neq \alpha} |v_\beta|^{1/2} \Phi_{\beta 0} + \rho_\alpha^{-1} P_\alpha \sum_{\beta \neq \alpha} v_\beta P_\beta g_{0\beta} \rho_\beta \Phi_{\beta 1} \end{aligned} \tag{4.27}$$

Conversely, if the $(\Phi_{\alpha i})$ form a solution of the system (4.27) and if Φ_α is defined by (4.26), then the Φ_α satisfy the original system (4.15).

Instead of studying the equation (4.15) in the space $\bigoplus_\alpha \mathscr{H}_\alpha$, we shall rather consider the system (4.27) in the space:

$$\overline{\mathscr{H}} = \bigoplus_\alpha \bigoplus_{i=0,1} \mathscr{H}_{\alpha i}$$

A vector in this space will be denoted $\Phi = \bigoplus_{\alpha, i} \Phi_{\alpha i}$. It should be kept in mind in the considerations of the following sections that we consider only vectors in the subspace defined by $\Phi_{\alpha 1} = P_\alpha \Phi_{\alpha 1} \in \mathscr{R}(P_\alpha)$ for all α. The space $\overline{\mathscr{H}}$ is unnecessarily « large », and each summand $\mathscr{H}_{\alpha 1}$ could be replaced by $L^2(\mathbb{R}^n, dy_\alpha)$. We have kept $\overline{\mathscr{H}}$ for notational convenience. The system (4.27) will be regarded as an equation for Φ in $\overline{\mathscr{H}}$:

$$\Phi = \Phi^{(0)} + A(\lambda)\Phi \tag{4.28}$$

The operator $A(\lambda)$ defined by (4.27, 28) will play a similar role in the three-body case as the operator $a(\lambda)$ in the two-body case. This operator will be studied in detail in the next section.

We conclude this section by indicating briefly a possible extension of the previous construction to the case where one drops assumption $(\mathscr{S}_-)$. Let $-\chi_{\alpha i}^2$, $1 \leqslant i \leqslant b_\alpha$, be the negative eigenvalues of h_α, and $P_{\alpha i}$ the projection operator in $\mathscr{H}$ on the subspace spanned by the $\psi(x_\alpha) \otimes \varphi(y_\alpha)$ where ψ ranges over the corresponding eigensubspace of h_α and φ over $L^2(\mathbb{R}^n)$, let $P_\alpha = \sum_i P_{\alpha i}$, and let

$$g_{0\alpha i}(\lambda) = (\lambda + \chi_{\alpha i}^2 - q_\alpha^2/2n_\alpha)^{-1} \tag{4.29}$$

Define $\rho_{\alpha i}$ by:

$$\rho_{\alpha i}^2(y_\alpha) = \mathrm{Tr}_\alpha \left[(1 + x_\alpha^2)^\theta P_{\alpha i} \sum_{\beta \neq \alpha} | v_\beta | \right] \tag{4.30}$$

where Tr_α denotes the partial trace in the space $L^2(\mathbb{R}^n, dx_\alpha)$. We use the same space $\overline{\mathscr{H}}$ as before. Let $\Phi = (\Phi_{\alpha i}) \in \overline{\mathscr{H}}$ (here again, i takes the values 0 and 1). Actually, we consider only the subspace of those Φ for which $\Phi_{\alpha 1} \in \mathscr{R}(P_\alpha)$ for all α. The system of equations that generalizes (4.27) is then obtained immediately from:

$$\Phi_\alpha = \Phi_{\alpha 0} + v_\alpha^{1/2} \sum_i P_{\alpha i} g_{0\alpha i} \rho_{\alpha i} \Phi_{\alpha 1} \tag{4.31}$$

$$\begin{cases} \Phi_{\alpha 0} = \Phi_{\alpha 0}^{(0)} + v_\alpha^{1/2} G'_\alpha \sum_{\beta \neq \alpha} | v_\beta |^{1/2} \Phi_\beta \\ \Phi_{\alpha 1} = \Phi_{\alpha 1}^{(0)} + \sum_i \rho_{\alpha i}^{-1} P_{\alpha i} \sum_{\beta \neq \alpha} | v_\beta |^{1/2} \Phi_\beta \end{cases} \tag{4.32}$$

where:

$$G'_\alpha = (1 - P_\alpha) G_\alpha \tag{4.33}$$

5. PROPERTIES OF THE OPERATOR A(λ)

In this section, we prove that the operator $A(\lambda)$ defined by (4.27, 28) satisfies properties similar to those derived for $a(\lambda)$ in proposition (3.1). In all this section and in the next one, we assume again that all potentials satisfy condition $(\mathscr{N})$ and that all two-body subsystems satisfy conditions $(\mathscr{S}_+)$ and $(\mathscr{S}_-)$. We consider successively the various elements $A_{\alpha i, \beta j}(\lambda)$ as operators in $\mathscr{H}$, for $i, j = 0, 1$.

We consider first the operator:

$$A_{\alpha 0, \beta 0}(\lambda) = v_\alpha^{1/2} G'_\alpha(\lambda) | v_\beta |^{1/2} \tag{5.1}$$

PROPOSITION (5.1). — The operator $A_{\alpha 0, \beta 0}$ satisfies the following properties:

(1) $A_{\alpha 0, \beta 0}$ is bounded, uniformly with respect to $\lambda \in \mathbb{C}$.

(2) $\| A_{\alpha 0, \beta 0} \|$ tends to zero if $| \mathrm{Im}\, \lambda | \to \infty$ or if $\mathrm{Re}\, \lambda \to -\infty$.

(3) $A_{\alpha 0,\beta 0}$ is norm Hölder-continuous with respect to λ with order $n/2q - 1$ and uniform coefficient.

(4) $A_{\alpha 0,\beta 0}$ is analytic in λ for $\lambda \notin [0, \infty)$.

(5) $A_{\alpha 0,\beta 0}$ is compact for all $\lambda \in \mathbb{C}$.

Proof. — Consider first the equation

$$G_\alpha = G_0 + G_\alpha v_\alpha G_0 \tag{5.2}$$

Multiplying both sides from the left by $1 - P_\alpha$, we obtain:

$$G'_\alpha = (1 - P_\alpha)G_0 + G'_\alpha v_\alpha G_0 \tag{5.3}$$

Substituting (5.3) into (5.1), we obtain:

$$A_{\alpha 0,\beta 0} = (1 + v_\alpha^{1/2} G'_\alpha \mid v_\alpha \mid^{1/2}) v_\alpha^{1/2} G_0 \mid v_\beta \mid^{1/2} - v_\alpha^{1/2} P_\alpha G_0 \mid v_\beta \mid^{1/2} \tag{5.4}$$

Now:

$$G'_\alpha(\lambda) = g'_\alpha(\lambda - q_\alpha^2/2n_\alpha) \tag{5.5}$$

It follows from proposition (3.4), from the definition of G'_α, from (5.5) and from assumptions $(\mathscr{S}_+)$ and $(\mathscr{S}_-)$ that the operator $v_\alpha^{1/2} G'_\alpha \mid v_\alpha \mid^{1/2}$ is analytic in the complex plane cut along $[0, \infty)$, uniformly bounded and uniformly Hölder-continuous. Therefore the first term in the RHS of (5.4) will obey all the statements of proposition (5.1) provided the second factor $v_\alpha^{1/2} G_0 \mid v_\beta \mid^{1/2}$ does.

The second term in the RHS of (5.4) can be written as

$$[v_\alpha^{1/2} P_\alpha (1 + x_\alpha^2)^{(1+\varepsilon)/2}][(1 + x_\alpha^2)^{-(1+\varepsilon)/2} G_0 \mid v_\beta \mid^{1/2}] \tag{5.6}$$

The first factor is a fixed (i. e. λ independent) bounded operator, and the whole term will satisfy proposition (5.1) provided the second factor does. The latter is obtained from $v_\alpha^{1/2} G_0 \mid v_\beta \mid^{1/2}$ by replacing v_α by $(1 + x_\alpha^2)^{-(1+\varepsilon)}$ and can therefore be studied by the same method.

It is therefore sufficient to prove proposition (5.1) for the operator $v_\alpha^{1/2} G_0 \mid v_\beta \mid^{1/2}$. We therefore restrict our attention to this operator.

(1) *Boundedness* was proved in [*31*]. The proof is reproduced for completeness. Let $\operatorname{Im} \lambda \geqslant 0$, $\alpha = (12)$, $\beta = (23)$. We use the representation:

$$v_{12}^{1/2} G_0(\lambda) \mid v_{23} \mid^{1/2} = -i \int_0^\infty dt e^{i\lambda t} v_{12}^{1/2} \exp(-itH_0) \mid v_{23} \mid^{1/2} \tag{5.7}$$

so that:

$$\| v_{12}^{1/2} G_0(\lambda) \mid v_{23} \mid^{1/2} \| \leqslant \int_0^\infty dt e^{-t \operatorname{Im} \lambda} \| v_{12}^{1/2} \exp(-itH_0) \mid v_{23} \mid^{1/2} \| \tag{5.8}$$

provided the integral converges.

For the sake of this argument, it is convenient to consider the three-body system in the space $L^2(\mathbb{R}^{3n})$ without separating the center of mass motion. Then:

$$\exp(-itH_0) = [\exp(-itp_3^2/2m_3)]\,[id_2]\,[id_1] \tag{5.9}$$

The first and last factors are unitary and commute with v_{12} and v_{23} respectively, and therefore drop out of (5.8).

Assume now that v_{12} and $v_{23} \in L^p(\mathbb{R}^n)$. Let $\varphi \in L^2(\mathbb{R}^{3n})$ and take partial norms in $L^2(\mathbb{R}^n, dx_2)$ for fixed x_1 and x_3. From (3.4) we obtain:

$$\| v_{12}^{1/2} \exp(-itp_2^2/2m_2) | v_{23} |^{1/2} \varphi \|_{L^2(x_2)}^2 \leqslant (2\pi t/m_2)^{-n/p} \| v_{12} \|_{L^p} \| v_{23} \|_{L^p} \| \varphi \|_{L^2(x_2)}^2 \quad (5.10)$$

Integration over x_1 and x_3 yields:

$$\| v_{12}^{1/2} \exp(-itH_0) | v_{23} |^{1/2} \|^2 \leqslant (2\pi t/m_2)^{-n/p} \| v_{12} \|_{L^p} \| v_{23} \|_{L^p} \quad (5.11)$$

The end of the proof is identical with that of proposition (3.1.1).

(2), (3), (4). The proof is identical with that of the corresponding statements in the two-body case.

(5) *Compactness.* It is essential here that the center of mass motion be separated out. The proof is almost identical with that in the two-body case, with the following modification: we want to prove that

$$v_{12}^{1/2} \exp(-tH_0) | v_{23} |^{1/2}$$

is a Hilbert-Schmidt operator for a set of potentials which is dense in $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. We now take $v_\alpha \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. We represent $\exp(-tH_0)$ by the following integral kernel:

$$\exp(-tH_0)(X, X') = (2\pi t/\mu)^{-n} \times \exp\left[-(2Mt)^{-1} \sum_\alpha m_i m_j (x_\alpha - x'_\alpha)^2 \right] \quad (5.12)$$

where $\alpha = (ij)$ and μ is defined by (1.1). Therefore:

$$\| v_{12}^{1/2} \exp(-tH_0) | v_{23} |^{1/2} \|_{H.S.}^2 = (2\pi t/\mu)^{-2n} \int dX dX' | v_{12}(x_{12}) | \, | v_{23}(x'_{23}) | \exp\left[-(Mt)^{-1} \sum_\alpha m_i m_j (x_\alpha - x'_\alpha)^2 \right] \quad (5.13)$$

Now $dX = dx_{12} dy_3 = dx_{12} dx_{23}$ and similarly $dX' = dx'_{12} dx'_{23}$. We obtain an upper bound by dropping the term with $\alpha = (13)$ in the exponent, namely

$$\ldots \leqslant \| v_{12} \|_{L^1} \| v_{23} \|_{L^1} (4\pi t)^{-n} (m_1 m_3)^{n/2}. \quad (5.14)$$

This completes the proof.

We now turn to the element

$$A_{\alpha 1, \beta 0} = \rho_\alpha^{-1} P_\alpha | v_\beta |^{1/2} \quad (5.15)$$

In view of the applications in section 7, we give a slightly stronger result than needed in this and the next sections.

PROPOSITION (5.2). — $A_{\alpha 1, \beta 0}$ is bounded from $L^2_{-\theta}(\mathbb{R}^n, dx_\alpha) \otimes L^2_\delta(\mathbb{R}^n, dy_\alpha)$ to $L^2_\nu(\mathbb{R}^n, dx_\alpha) \otimes L^2_\delta(\mathbb{R}^n, dy_\alpha)$ for any real δ, ν, with norm bounded by:

$$\| A_{\alpha 1, \beta 0} \| \leqslant \| \psi_\alpha \|_\nu^2 \quad (5.16)$$

Proof. — We recall that θ is the index that occurs in the definition of ρ_α (4.16).

Since $A_{\alpha 1,\beta 0}$ commutes with multiplication by functions of y_α, it is sufficient to consider the case $\delta = 0$. Let $\varphi \in L^2_{-\theta}(\mathbb{R}^n, dx_\alpha) \otimes L^2(\mathbb{R}^n, dy_\alpha)$. Then:

$$\| A_{\alpha 1,\beta 0}\varphi \|^2 = \| \psi_\alpha \|_v^2 \int dy_\alpha \rho_\alpha^{-2}(y_\alpha) \left| \int dx_\alpha \psi_\alpha(x_\alpha) \, | v_\beta(x_\beta) |^{1/2} \varphi(X) \right|^2$$
$$\leqslant \| \psi_\alpha \|_v^2 \int dX (1 + x_\alpha^2)^{-\theta} \, | \varphi(X) |^2$$
$$\times \rho_\alpha^{-2}(y_\alpha) \int dx'_\alpha \, | \psi_\alpha(x'_\alpha) |^2 (1 + x'^2_\alpha)^\theta \, | v_\beta(x'_\beta) |$$

by Schwarz inequality applied to the integration over x_α, and where the last integration is performed for fixed $y_\alpha = y'_\alpha$,

$$\ldots \leqslant \| \psi_\alpha \|_v^2 \int dX (1 + x_\alpha^2)^{-\theta} \, | \varphi(X) |^2$$

by the definition of ρ_α. This completes the proof.

We now turn to the element

$$A_{\alpha 0,\beta 1}(\lambda) = v_\alpha^{1/2} G'_\alpha v_\beta P_\beta g_{0\beta} \rho_\beta \tag{5.17}$$

PROPOSITION (5.3). — $A_{\alpha 0,\beta 1}(\lambda)$ satisfies the same properties as stated in proposition (5.1) for $A_{\alpha 0,\beta 0}$ with the only exception that analyticity in λ holds only in the complex plane cut along $[-\chi_\beta^2, \infty)$.

Proof. — We use again (5.3):

$$A_{\alpha 0,\beta 1} = [(1 + v_\alpha^{1/2} G'_\alpha \, | v_\alpha |^{1/2}) v_\alpha^{1/2} - v_\alpha^{1/2} P_\alpha] G_0 v_\beta P_\beta g_{0\beta} \rho_\beta \tag{5.18}$$

Now $g_{0\beta}$ commutes with v_β and P_β. Furthermore:

$$G_0(\lambda) g_{0\beta}(\lambda) = (\chi_\beta^2 + p_\beta^2/2m_\beta)^{-1} [G_0(\lambda) - g_{0\beta}(\lambda)] \tag{5.19}$$

Therefore:

$$G_0(\lambda) g_{0\beta}(\lambda) v_\beta P_\beta = [g_{0\beta}(\lambda) - G_0(\lambda)] P_\beta \,. \tag{5.20}$$

Finally:

$$A_{\alpha 0,\beta 1} = (1 + v_\alpha^{1/2} G'_\alpha \, | v_\alpha |^{1/2}) [v_\alpha^{1/2} g_{0\beta} P_\beta \rho_\beta - v_\alpha^{1/2} G_0 P_\beta \rho_\beta]$$
$$- v_\alpha^{1/2} P_\alpha g_{0\beta} P_\beta \rho_\beta + v_\alpha^{1/2} P_\alpha G_0 P_\beta \rho_\beta \tag{5.21}$$

The factor $(1 + v_\alpha^{1/2} G'_\alpha \, | v_\alpha |^{1/2})$ has already been studied in the proof of proposition (5.1). The first term in the square bracket can be written as:

$$(v_\alpha^{1/2} \rho_\beta^{-1} P_\beta)(\rho_\beta g_{0\beta} \rho_\beta P_\beta) \tag{5.22}$$

Up to an absolute value of v_α, the first factor is $A^*_{\beta 0,\alpha 1}$ and is therefore a λ-independent bounded operator with norm less than one by proposition (5.2).

The second factor is the tensor product of a fixed compact operator P_β

in $L^2(\mathbb{R}^n, dx_\beta)$ with the operator $\rho_\beta g_{0\beta}\rho_\beta$ in $L^2(\mathbb{R}^n, dy_\beta)$. The latter has all the required properties by proposition (3.1) and lemma (4.1).

The second term in the square bracket in (5.21) is $v_\alpha^{1/2}G_0\rho_\beta P_\beta$ and satisfies all the required properties, by an application of proposition (5.1) to the factor $v_\alpha^{1/2}G_0\rho_\beta$, and lemma (4.1).

The next term in (5.21) can be written for some $\delta > 1$ as:

$$(v_\alpha^{1/2}P_\alpha(1 + y_\beta^2)^{\delta/2}P_\beta)((1 + y_\beta^2)^{-\delta/2}g_{0\beta}\rho_\beta P_\beta) \tag{5.23}$$

The first factor is a λ independent compact (in fact Hilbert-Schmidt) operator in $\mathcal{H}$, as can be easily seen by an elementary computation using (4.21), while the second factor is analogous to the second one in (5.22). Finally, the last term in (5.21) can be written as:

$$(v_\alpha^{1/2}P_\alpha(1 + x_\alpha^2)^{\delta/2})((1 + x_\alpha^2)^{-\delta/2}G_0\rho_\beta)P_\beta \tag{5.24}$$

and the second factor is controlled by another application of proposition (5.1). This completes the proof.

We finally consider the operator

$$A_{\alpha 1,\beta 1}(\lambda) = \rho_\alpha^{-1}P_\alpha v_\beta P_\beta g_{0\beta}\rho_\beta \tag{5.25}$$

PROPOSITION (5.4). — Let $\theta > \delta + 1$. Then, as an operator from $\mathcal{H}$ to $L_\nu^2(\mathbb{R}^n, dx_\alpha) \otimes L_\delta^2(\mathbb{R}^n, dy_\alpha)$, $A_{\alpha 1,\beta 1}$ is uniformly bounded, compact, uniformly Hölder continuous and analytic in the complex plane cut along $[-\chi_\beta^2, \infty)$. In addition $\| A_{\alpha 1,\beta 1} \|$ tends to zero when $|\lambda| \to \infty$.

If $n = 3$, and if v_α and v_β satisfy condition $(\mathcal{R})$, $A_{\alpha 1,\beta 1}$ is a Hilbert-Schmidt operator.

Proof. — It suffices to consider the following operator from $\mathcal{H}$ to $\mathcal{H}$:

$$(1 + y_\alpha^2)^{\delta/2}P_\alpha\rho_\alpha^{-1}v_\beta P_\beta g_{0\beta}\rho_\beta = [(1 + y_\alpha^2)^{\delta/2}P_\alpha\rho_\alpha^{-1}v_\beta P_\beta(1 + y_\beta^2)^{(\theta-\delta)/2}] \times [(1 + y_\beta^2)^{(\delta-\theta)/2}g_{0\beta}\rho_\beta P_\beta] \tag{5.26}$$

The second factor has all the required properties by proposition (3.1). It suffices to show that the first one B is bounded in $\mathcal{H}$.

Let $\varphi = P_\beta\varphi \in \mathcal{H}_{\beta 1}$. Then:

$$\| B\varphi \|^2 = \int dy_\alpha \rho_\alpha^{-2}(y_\alpha)(1 + y_\alpha^2)^\delta \left| \int dx_\alpha \psi_\alpha(x_\alpha)v_\beta(x_\beta)(1 + y_\beta^2)^{(\theta-\delta)/2}\varphi(X) \right|^2$$

$$\leqslant 2^\theta \int dy_\alpha \rho_\alpha^{-2}(y_\alpha) \left| \int dx_\alpha \, |\psi_\alpha(x_\alpha)| \, (1 + x_\alpha^2)^{\theta/2} \, |v_\beta(x_\beta)| \, (1 + x_\beta^2)^{\theta/2} \, |\varphi(X)| \right|^2$$

by repeated use of (4.21),

$$\leqslant 2^\theta \int dX \, |\varphi(X)|^2(1 + x_\beta^2)^\theta \, |v_\beta(x_\beta)| \times \rho_\alpha^{-2}(y_\alpha) \int dx'_\alpha \, |\psi_\alpha(x'_\alpha)|^2(1 + x'^2_\alpha)^\theta \, |v_\beta(x'_\beta)|$$

by Schwarz inequality applied to the x_α integration for fixed y_α,

$$\ldots \leqslant 2^\theta \| v_\beta^{1/2}\psi_\beta \|_\theta^2 \, \| \varphi \|^2$$

since the second line is less than one by the definition of ρ_α.

It follows from propositions (5.1) to (5.4) that the operator $A(\lambda)$ is bounded in $\overline{\mathcal{H}}$ uniformly with respect to λ, uniformly Hölder continuous in λ, and analytic in the complex plane cut along $\sigma_e = [E_0, \infty)$ where $E_0 = \underset{\alpha}{\text{Min}} (-\chi_\alpha^2)$. Furthermore $\| A^2(\lambda) \|$ tends to zero when $|\text{Im}\,\lambda| \to \infty$ or when $\text{Re}\,\lambda \to -\infty$ and $A^2(\lambda)$ is compact for all λ in the closed cut plane.

We can therefore apply the analytic Fredholm theorem ([*26*], p. 201) to invert the operator $[1 - A(\lambda)]$. Let ξ be the set of values of λ for which the homogeneous equation $\Phi = A(\lambda)\Phi$ has a solution in $\overline{\mathcal{H}}$. Let

$$\xi_d = \xi \cap (\mathbb{C}\backslash[E_0, \infty))$$

and $\xi_e = \xi \cap [E_0, \infty)$. Then (see [*34*] for the fact that the Fredholm alternative holds when only some power of the relevant operator is compact):

PROPOSITION (5.5). — (1) ξ is closed. $\text{Re}\,\lambda$ is bounded from below and $|\text{Im}\,\lambda|$ is bounded for $\lambda \in \xi$. ξ_d is discrete.

(2) The operator $[1 - A(\lambda)]^{-1}$ exists as a bounded operator in $\overline{\mathcal{H}}$ for all $\lambda \notin \xi$, is meromorphic in the open cut plane $\mathbb{C}\backslash[E_0, \infty)$ with poles at the points of ξ_d, and is uniformly bounded and uniformly Hölder continuous in λ on the compact subsets of the closed cut plane not intersecting ξ.

Further information on ξ, and consequently on $[1 - A(\lambda)]^{-1}$, on the spectrum $\sigma(H)$ and on the resolvent operator $G(\lambda)$, will be obtained in the following sections after studying the homogeneous equation $\Phi = A(\lambda)\Phi$.

6. THE HOMOGENEOUS EQUATION $\Phi = A(\lambda)\Phi$

In this section, we study the homogeneous equation $\Phi = A(\lambda)\Phi$ and derive additional properties of the singular set ξ, relating it to the spectrum of H. We also construct the resolvent operator $G(\lambda)$.

Let $\sigma_p(H)$ be the point spectrum of H. Then:

PROPOSITION (6.1). — $\sigma_p(H) \subset \xi$. More precisely, let $H\psi = \lambda\psi$, $\lambda \in \mathbb{R}$, $\psi \in \mathcal{D}(H)$. Define $\Phi \in \overline{\mathcal{H}}$ by:

$$\Phi_{\alpha 0} = v_\alpha^{1/2}(1 - P_\alpha)\psi \tag{6.1}$$

$$\Phi_{\alpha 1} = \rho_\alpha^{-1} P_\alpha \sum_{\beta \neq \alpha} v_\beta \psi \tag{6.2}$$

Then Φ satisfies the homogeneous equation $\Phi = A(\lambda)\Phi$ (If $\lambda \in \sigma_e$, this means that $\Phi = A(\lambda \pm i0)\Phi$).

Proof. — We consider first the simple case where $\lambda \notin \sigma_e$. It follows from $H\psi = \lambda\psi$ that:

$$\psi = G_\alpha(\lambda) \sum_{\beta \neq \alpha} v_\beta \psi \tag{6.3}$$

Therefore:

$$(1 - P_\alpha)\psi = G'_\alpha(\lambda) \sum_{\beta \neq \alpha} v_\beta \psi \tag{6.4}$$

and

$$P_\alpha \psi = P_\alpha g_{0\alpha}(\lambda) \sum_{\beta \neq \alpha} v_\beta \psi \tag{6.5}$$

$$P_\alpha \psi = g_{0\alpha}(\lambda) \rho_\alpha \Phi_{\alpha 1} \tag{6.6}$$

by (6.2). It follows from (6.1) and (6.4) that:

$$\Phi_{\alpha 0} = v_\alpha^{1/2} G'_\alpha(\lambda) \sum_{\beta \neq \alpha} | v_\beta |^{1/2} (\Phi_{\beta 0} + v_\beta^{1/2} P_\beta \psi) \tag{6.7}$$

From (6.1) and (6.2), we obtain:

$$\Phi_{\alpha 1} = \rho_\alpha^{-1} P_\alpha \sum_{\beta \neq \alpha} | v_\beta |^{1/2} (\Phi_{\beta 0} + v_\beta^{1/2} P_\beta \psi) \tag{6.8}$$

Substituting (6.6) into the RHS of (6.7) and (6.8) yields immediately $\Phi = A(\lambda)\Phi$.

We now turn to the case where $\lambda \in \sigma_e$. Let $\eta \neq 0$. It follows from $H\psi = \lambda\psi$ that:

$$\psi = G_\alpha(\lambda + i\eta) \sum_{\beta \neq \alpha} v_\beta \psi + i\eta G_\alpha(\lambda + i\eta)\psi . \tag{6.9}$$

We have introduced η is order to use G_α inside its regularity domain. In all subsequent equations of the proof, it will be understood that *all* resolvent operators are taken at the value $\lambda + i\eta$ and the argument will be omitted. From (6.9), we obtain as before:

$$(1 - P_\alpha)\psi = G'_\alpha \sum_{\beta \neq \alpha} v_\beta \psi + i\eta G'_\alpha \psi \tag{6.10}$$

$$P_\alpha \psi = g_{0\alpha} P_\alpha \sum_{\beta \neq \alpha} v_\beta \psi + i\eta g_{0\alpha} P_\alpha \psi \tag{6.11}$$

$$P_\alpha \psi = g_{0\alpha} \rho_\alpha \Phi_{\alpha 1} + i\eta g_{0\alpha} P_\alpha \psi \tag{6.12}$$

It follows from (6.1) and (6.10) that:

$$\Phi_{\alpha 0} = v_\alpha^{1/2} G'_\alpha \sum_{\beta \neq \alpha} | v_\beta |^{1/2} (\Phi_{\beta 0} + v_\beta^{1/2} P_\beta \psi) + i\eta v_\alpha^{1/2} G'_\alpha \psi \tag{6.13}$$

We now substitute (6.12) into (6.13) and (6.8) and obtain:

$$\Phi_{\alpha 0} = (A(\lambda + i\eta)\Phi)_{\alpha 0} + i\eta v_\alpha^{1/2} G'_\alpha \psi + i\eta v_\alpha^{1/2} G'_\alpha \sum_{\beta \neq \alpha} v_\beta P_\beta g_{0\beta} \psi \tag{6.14}$$

$$\Phi_{\alpha 1} = (A(\lambda + i\eta)\Phi)_{\alpha 1} + i\eta \rho_\alpha^{-1} P_\alpha \sum_{\beta \neq \alpha} v_\beta P_\beta g_{0\beta} \psi \tag{6.15}$$

We now let $\eta \to 0$. Since $A(\lambda)$ is Hölder continuous in λ, it suffices to prove that the three correction terms in (6.14) and (6.15) tend to zero when $\eta \to 0$. This is done in appendix A.

We now turn to the converse problem. This is a simple task if $\lambda \notin \sigma_e$.

PROPOSITION (6.2). — Let $\lambda \in \xi$, $\lambda \notin \sigma_e$, and let $\Phi = A(\lambda)\Phi$. Then λ is an eigenvalue of H.

Proof. — Define Φ_α by (4.26). (Notice that $g_{0\alpha}$ is bounded and analytic near λ). It then follows by an elementary computation, as mentioned in Section 4, that the Φ_α satisfy the equations:

$$\Phi_\alpha = v_\alpha^{1/2} G_\alpha \sum_{\beta \neq \alpha} | v_\beta |^{1/2} \Phi_\beta \tag{6.16}$$

Define:

$$\psi = G_0(\lambda) \sum_\alpha | v_\alpha |^{1/2} \Phi_\alpha \tag{6.17}$$

From:

$$G_\alpha = G_0 + G_0 v_\alpha G_\alpha \tag{6.18}$$

it follows that:

$$\Phi_\alpha = v_\alpha^{1/2} G_0 \sum_{\beta \neq \alpha} | v_\beta |^{1/2} \Phi_\beta + v_\alpha^{1/2} G_0 | v_\alpha |^{1/2} v_\alpha^{1/2} G_\alpha \sum_{\beta \neq \alpha} | v_\beta |^{1/2} \Phi_\beta$$
$$= v_\alpha^{1/2} G_0 \sum_\beta | v_\beta |^{1/2} \Phi_\beta = v_\alpha^{1/2} \psi \tag{6.19}$$

From (6.17) and (6.19) we obtain $\psi = G_0 V \psi$ and the result follows by applying $\lambda - H_0$ to both sides of this relation.

From propositions (6.1) and (6.2) and the results in section 5, we can obtain a number of properties of ξ, $\sigma(H)$ and $G(\lambda)$. We consider first the case where $\lambda \notin \sigma_e$. Then (cf. proposition (3.3)):

PROPOSITION (6.3). — (1) $\xi_d = \sigma_p(H) \cap (\mathbb{C}\backslash\sigma_e)$. ξ_d is real, bounded from below, and discrete, with E_0 as its only possible accumulation point.

(2) ξ_d is the discrete spectrum $\sigma_d(H)$ of H, and σ_e is the essential spectrum $\sigma_e(H)$.

Proof. — (1) Follows from propositions (5.5), (6.1) and (6.2).

(2) Is in essence the theorem of Hunziker [*12*] [*21*] [*32*]. From (4.13), proposition (5.5) and some algebraic manipulations, it follows that $G(\lambda)$ is meromorphic in $\mathbb{C}\backslash\sigma_e$ with poles at the points of ξ_d and compact residues. From this and proposition (6.2), it follows that $\sigma_e(H) \subset \sigma_e$ and $\sigma_d(H) \supset \xi_d$.

It remains to prove that $\sigma_e(H) \supset \sigma_e$. This can be done by other methods [*12*].

We next construct $G(\lambda)$ in general, including the case where λ lies on the cut σ_e, and collect the information that is readily available in this situation (cf. proposition 3.4):

PROPOSITION (6.4). — (1) ξ_e is a closed set of Lebesgue measure zero.

(2) ξ_e contains the part of the point spectrum of H contained in σ_e:

$$\sigma_p(H) \cap \sigma_e \subset \xi_e$$

(3) Let $\delta > 1$. As an operator from $L^2_\delta(\mathbb{R}^{2n})$ to $L^2_{-\delta}(\mathbb{R}^{2n})$, $G(\lambda)$ is compact, uniformly bounded and uniformly Hölder continuous, on the compact subsets of the closed cut plane (along σ_e) not intersecting ξ.

(4) The part of the spectrum of H in $\sigma_e\backslash\xi_e$ is absolutely continuous. Moreover, the spectral projector of H on $\overline{\sigma_e\backslash\xi_e}$ is the absolutely continuous projector of H.

Proof. — (1) The proof is the same as that of proposition (3.4.1).

(2) is a repetition of part of proposition (6.1).

(3) *Construction of* $G(\lambda)$. Let $\delta > 1$. In this argument, it is appropriate to modify the definition of the ρ_α. We shall replace (4.16) by:

$$\rho_\alpha^2(y_\alpha) = \int dx_\alpha \, | \psi_\alpha(x_\alpha) |^2 (1 + x_\alpha^2)^\theta \sum_{\beta \neq \alpha} | v_\beta(x_\beta) | + (1 + y_\alpha^2)^{-\delta} \quad (6.20)$$

It is easy to check that this does not affect any of the results in sections 5 and 6.

We now define a mapping $J(\lambda)$ from $L^2_\delta(\mathbb{R}^{2n})$ to $\mathscr{H}$. Let $\varphi \in L^2_\delta(\mathbb{R}^{2n})$. We define $\Phi^{(0)} = J(\lambda)\varphi$ by:

$$\begin{cases} \Phi^{(0)}_{\alpha 0} = v_\alpha^{1/2} G'_\alpha \varphi \\ \Phi^{(0)}_{\alpha 1} = \rho_\alpha^{-1} P_\alpha \varphi \end{cases} \quad (6.21)$$

The operator $J(\lambda)$ satisfies the following properties:

LEMMA (6.1). — $J(\lambda)$ is a bounded operator from $L^2_\delta(\mathbb{R}^{2n})$ to $\mathscr{H}$ for all λ. $J(\lambda)$ is analytic in λ in the complex plane cut along $[0, \infty)$. $J(\lambda)$ is uniformly bounded and uniformly Hölder continuous in λ in the closed cut plane.

Proof. — For the components $\Phi_{\alpha 0}^{(0)}$, the result follows through the use of (5.3) and proposition (5.1) from the properties of the two-body systems obtained in section 3, and from assumptions $(\mathscr{S}_+)$ and $(\mathscr{S}_-)$. For the components $\Phi_{\alpha 1}^{(0)}$, the result follows from the fact that the function $\rho_\alpha^{-1}(y_\alpha)(1 + X^2)^{-\delta/2}$ is uniformly bounded because of (6.20).

We now construct $G(\lambda)$ as follows. Let $\lambda \notin \xi$.

Let $\varphi \in L_\delta^2(\mathbb{R}^{2n})$. We define $\Phi^{(0)} = J(\lambda)\varphi$ and use it as the inhomogeneous term in (4.28) to construct $\Phi = [1 - A(\lambda)]^{-1}\Phi^{(0)}$. We finally substitute this Φ into the RHS of the equation

$$G\varphi = G_0\Big(\varphi + \sum_\alpha (|v_\alpha|^{1/2}\Phi_{\alpha 0} - \rho_\alpha\Phi_{\alpha 1})\Big) + \sum_\alpha g_{0\alpha}\rho_\alpha\Phi_{\alpha 1} \tag{6.22}$$

The latter satisfies all the properties stated in (3), by lemma (6.1), proposition (5.5.2) and proposition (5.1).

It remains to identify it with $G\varphi$. By analytic continuation, it is sufficient to do so for Re (λ) large and negative. In this case, we define $\Phi_\alpha^{(0)}$ and Φ_α by (4.24) and (4.26): the $\Phi_\alpha^{(0)}$ and Φ_α lie in $\mathscr{H}$.

From (4.24) and the definition (6.21) of $J(\lambda)$, it follows that

$$\Phi_\alpha^{(0)} = v_\alpha^{1/2}G_\alpha\varphi \tag{6.23}$$

Furthermore $\Phi_\alpha^{(0)}$ and Φ_α satisfy (4.15), as mentioned in section 4. Therefore, from (4.13):

$$G\varphi = G_0\varphi + G_0\sum_\alpha |v_\alpha|^{1/2}\Phi_\alpha \tag{6.24}$$

We finally substitute (4.26) into (6.24), use (5.20), and obtain (6.22).

(4) The proof is the same as that of proposition (3.4.5).

For $\lambda \in \sigma_e$, we can proceed a little further towards a converse of proposition (6.1). The following result means intuitively that solutions of the homogeneous equation vanish on the energy shell.

PROPOSITION (6.5). — Let $\lambda \in \xi_e$ and let $\Phi = A(\lambda + i0)\Phi$. Then:

$$\lim_{\eta\downarrow 0} \eta \,\|\, G_0(\lambda + i\eta)\sum_\alpha (|v_\alpha|^{1/2}\Phi_{\alpha 0} - \rho_\alpha\Phi_{\alpha 1})\,\|^2 = 0 \tag{6.25}$$

$$\lim_{\eta\downarrow 0} \eta \,\|\, g_{0\alpha}(\lambda + i\eta)\rho_\alpha\Phi_{\alpha 1}\,\|^2 = 0 \tag{6.26}$$

Proof. — Let $\eta > 0$ and define:

$$\Phi'(\eta) = \Phi - A(\lambda + i\eta)\Phi \tag{6.27}$$

By the Hölder continuity of $A(.)$, $\Phi'(\eta)$ tends to zero in norm in $\overline{\mathscr{H}}$ when $\eta \downarrow 0$.

Equation (6.27) is a special case of (4.28) with $\Phi'(\eta)$ as the inhomogeneous term. Define, in analogy with (4.26):

$$\Phi_\alpha(\eta) = \Phi_{\alpha 0} + v_\alpha^{1/2} P_\alpha g_{0\alpha}(\lambda + i\eta)\rho_\alpha \Phi_{\alpha 1} \tag{6.28}$$

By the same algebra as in section 4, $\Phi_\alpha(\eta)$ satisfy the equations

$$\Phi_\alpha(\eta) = v_\alpha^{1/2} G_\alpha(\lambda + i\eta) \sum_{\beta \neq \alpha} | v_\beta |^{1/2} \Phi_\beta(\eta) + \Phi'_\alpha(\eta) \tag{6.29}$$

where $\Phi'_\alpha(\eta)$ is defined in terms of $\Phi'(\eta)$ by a formula similar to (6.28).

Multiplying both sides of (6.29) by $(1 - v_\alpha^{1/2} G_0(\lambda + i\eta) | v_\alpha |^{1/2})$ we obtain by the same computation as in the derivation of (6.24):

$$\Phi_\alpha(\eta) = v_\alpha^{1/2} G_0(\lambda + i\eta) \sum_\beta | v_\beta |^{1/2} \Phi_\beta(\eta) + (1 - v_\alpha^{1/2} G_0(\lambda + i\eta) | v_\alpha |^{1/2}) \Phi'_\alpha(\eta) \tag{6.30}$$

We multiply both sides of (6.30) by $v_\alpha | v_\alpha |^{-1}$, take the scalar product with $\Phi_\alpha(\eta)$, sum over α and take the imaginary part:

$$0 = \operatorname{Im} \sum_\alpha \langle \Phi_\alpha(\eta), v_\alpha | v_\alpha |^{-1} \Phi_\alpha(\eta) \rangle = \eta \, || G_0(\lambda + i\eta) \sum_\alpha | v_\alpha |^{1/2} \Phi_\alpha(\eta) ||^2 + \sum_\alpha \operatorname{Im} \langle \Phi_\alpha(\eta), v_\alpha | v_\alpha |^{-1} (1 - v_\alpha^{1/2} G_0(\lambda + i\eta) | v_\alpha |^{1/2}) \Phi'_\alpha(\eta) \rangle \tag{6.31}$$

From (6.28) and (5.20), we obtain:

$$G_0(\lambda + i\eta) \sum_\alpha | v_\alpha |^{1/2} \Phi_\alpha(\eta) = G_0(\lambda + i\eta) \sum_\alpha (| v_\alpha |^{1/2} \Phi_{\alpha 0} - \rho_\alpha \Phi_{\alpha 1}) + \sum_\alpha g_{0\alpha}(\lambda + i\eta) \rho_\alpha \Phi_{\alpha 1} \tag{6.32}$$

We substitute (6.32) into (6.31) and expand the $|| \cdot ||^2$. We obtain a sum of diagonal terms which are precisely the expressions in the LHS of (6.25) and (6.26), and cross terms. Therefore, in order to prove proposition (6.4), it is sufficient to show that (*i*) the cross terms, (*ii*) the last term in the RHS of (6.31) tend to zero when $\eta \downarrow 0$. This is done in appendix B.

7. ABSENCE OF NEGATIVE SINGULAR CONTINUOUS SPECTRUM

In this section, we shall obtain additional information on ξ_e and on $\sigma(H)$ and in particular on the negative part of these sets.

We first assume that the v_α satisfy condition $(\mathscr{A}_{1+\varepsilon})$ and that the two-body subsystems satisfy conditions $(\mathcal{S}_+)$ and $(\mathcal{S}_-)$. Since condition $(\mathcal{A}_{1+\varepsilon})$ is stronger than condition $(\mathcal{N})$, all the results in sections 4, 5 and 6 are available. We then obtain the following refinement of proposition (6.5):

PROPOSITION (7.1). — Assume conditions $(\mathcal{A}_{1+\varepsilon})$, $(\mathcal{S}_+)$ and $(\mathcal{S}_-)$. Let $\lambda \in \xi_e$ and $\Phi = A(\lambda + i0)\Phi$. Then:

$$\int d\mathrm{P}\delta(\lambda - \mathrm{P}^2/2\mu)\left|\sum_\alpha (\widehat{|v_\alpha|^{1/2}\Phi_{\alpha 0}} - \widehat{\rho_\alpha\Phi_{\alpha 1}})\right|^2 = 0 \tag{7.1}$$

$$\int d\mathrm{P}\delta(\lambda + \chi_\alpha^2 - q_\alpha^2/2n_\alpha)\,|\widehat{\rho_\alpha\Phi_{\alpha 1}}|^2 = 0 \tag{7.2}$$

where μ and P^2 are defined by (1.1) and (1.5).

Proof. — Let

$$\varphi = \sum_\alpha (|v_\alpha|^{1/2}\Phi_{\alpha 0} - \rho_\alpha\Phi_{\alpha 1}) \tag{7.3}$$

We consider first the case where $v_\alpha = w_\alpha(1 + x^2)^{-(1+\varepsilon)}$ and $w_\alpha \in \mathrm{L}^\infty(\mathbb{R}^n)$ for all α. In this case, φ is a sum of six terms, each of which belongs to some space $\mathrm{L}^2_{1+\varepsilon}(\mathbb{R}^n) \otimes \mathrm{L}^2(\mathbb{R}^n)$. Therefore, by proposition (2.2.1), $\hat{\varphi}$ can be restricted to spheres in momentum space, and the LHS of (6.25) can be written as:

$$\lim_{\eta \downarrow 0} \eta \int_0^\infty dk[(\lambda - k^2/2\mu)^2 + \eta^2]^{-1}\,||\,\pi(k)\varphi\,||^2 \tag{7.4}$$

where $\pi(k)$ is the operator of restriction of $\hat{\varphi}$ to the sphere $\mathrm{P}^2 = k^2$ as defined by (2.3). The last factor in the integrand is a continuous function of k. One can therefore take the limit $\eta \downarrow 0$, thereby obtaining (7.1).

In the general case where $w_\alpha \in \mathrm{L}^p(\mathbb{R}^n) + \mathrm{L}^\infty(\mathbb{R}^n)$ one can write for instance:

$$\begin{aligned} \eta\,||\,\mathrm{G}_0(\lambda + i\eta)\varphi\,||^2 &= \mathrm{Im}\,\langle \varphi, \mathrm{G}_0(\lambda + i\eta)\varphi \rangle \\ &= -\,\mathrm{Im}\,(1 + \lambda + i\eta)\langle \varphi, \mathrm{G}_0(\lambda + i\eta)\mathrm{G}_0(-1)\varphi \rangle \\ &= (1 + \lambda)\eta\,||\,\mathrm{G}_0(\lambda + i\eta)[-\mathrm{G}_0(-1)]^{1/2}\varphi\,||^2 + 0(\eta) \end{aligned} \tag{7.5}$$

Now one sees easily that $[-\mathrm{G}_0(-1)]^{1/2}\,|\,w_\alpha\,|^{1/2}$ is bounded in all the relevant spaces $\mathrm{L}^2_{1+\varepsilon}(\mathbb{R}^n) \otimes \mathrm{L}^2(\mathbb{R}^n)$. One then obtains (7.1) in the equivalent form:

$$\int d\mathrm{P}\delta(\lambda - \mathrm{P}^2/2\mu)\,|\,(1 + \lambda)^{1/2}(1 + \mathrm{P}^2/2\mu)^{-1/2}\hat{\varphi}(\mathrm{P})\,|^2 = 0\,.$$

The proof of (7.2) is similar to the previous one.

We are now able to apply Agmon's method for negative energies. In this case, assumptions $(\mathcal{S}_+)$ and $(\mathcal{S}_-)$ are not needed. We obtain:

PROPOSITION (7.2). — Let the v_α satisfy condition $(\mathcal{A}_{1+\varepsilon})$. Then the negative part of ξ consists of the negative point spectrum of H, plus possibly the two-body thresholds. The negative point spectrum of H consists of eigenvalues of finite multiplicities which can accumulate at most at zero and at the two-body thresholds from below. The negative singular continuous spectrum is empty.

Proof. — We assume for simplicity that $w_\alpha \in L^\infty(\mathbb{R}^n)$ for all α and that $(\mathcal{S}_-)$ holds. The proof is similar to that of proposition (3.5). Let $\lambda \in \xi_e$, $\lambda < 0$ and let $\Phi = A(\lambda)\Phi$, $\Phi \in \overline{\mathcal{H}}$. By the same argument as in proposition (3.5), it is sufficient to show that ψ defined by (4.26) and (6.17) belongs to $L^2_\varepsilon(\mathbb{R}^{2n})$ with norm uniformly bounded in λ for λ in a semi-closed interval $(a, b] \subset (-\infty, 0)$ not containing any two-body threshold. The first step is to show that Φ has suitable decrease properties at infinity. In order to do this, we need the following lemma:

LEMMA (7.1). — For $\lambda < 0$, the operators $G_0(\lambda)$ and $G'_\alpha(\lambda)$ are bounded operators in $L^2_\delta(\mathbb{R}^n, dx_\alpha) \otimes L^2_{\delta'}(\mathbb{R}^n, dy_\alpha)$ and in $L^2_\delta(\mathbb{R}^{2n}, dX)$ with norm uniform in λ on the compact subsets of $(-\infty, 0)$.

Lemma (7.1) will be proved below. We first use it to prove proposition (7.2). Consider the homogeneous equation $\Phi = A(\lambda)\Phi$ (cf. (4.27)) where the elements of $A(\lambda)$ are given by (5.1), (5.15), (5.21) and (5.25). By assumption, $\Phi_{\alpha i} \in L^2(\mathbb{R}^{2n}) = \mathcal{H}$ for all α and i. From (5.1) and lemma (7.1), it follows that $A_{\alpha 0,\beta 0}\Phi_{\beta 0} \in L^2_{1+\varepsilon}(\mathbb{R}^{2n})$. Consider next $A_{\alpha 0,\beta 1}\Phi_{\beta 1}$. The crucial term is the contribution of (5.22). From lemma (4.1) and proposition (2.4), it follows that

$$\rho_\beta g_{0\beta}\rho_\beta P_\beta \Phi_{\beta 1} \in L^2_\nu(\mathbb{R}^n, dx_\beta) \otimes L^2_\varepsilon(\mathbb{R}^n, dy_\beta)$$

for some $\nu \geqslant \varepsilon$. From this fact, from proposition (5.2) and lemma (7.1), it follows that the contribution of (5.22) to $A_{\alpha 0,\beta 1}\Phi_{\beta 1}$ also belongs to $L^2_\varepsilon(\mathbb{R}^{2n})$. Therefore, $\Phi_{\alpha 0} \in L^2_\varepsilon(\mathbb{R}^{2n})$. We inject this result into the equation for $\Phi_{\alpha 1}$. By proposition (5.2), the term $A_{\alpha 1,\beta 0}\Phi_{\beta 0}$ belongs to $L^2_\varepsilon(\mathbb{R}^{2n})$, while the term $A_{\alpha 1,\beta 1}\Phi_{\beta 1}$ belongs to $L^2_1(\mathbb{R}^{2n})$ for $\theta > 2$ by proposition (5.4). Finally, all the $\Phi_{\alpha i}$ belong to $L^2_\varepsilon(\mathbb{R}^{2n})$. We now iterate this procedure, the crucial term being treated by the use of proposition (2.4), until we obtain $\Phi_{\alpha i} \in L^2_\delta(\mathbb{R}^{2n})$ for some $\delta > 1/2$. The estimates on $\| \Phi_{\alpha i} \|_\delta$ obtained at this stage are uniform in λ on any compact subset of $(-\infty, 0)$, by lemma (7.1) and the fact that the estimate in proposition (2.4) is uniform in λ.

We then iterate again. The crucial term $\rho_\beta g_{0\beta}\rho_\beta P_\beta$ is estimated by the use of propositions (2.6.3) and (7.1) if $\lambda > -\chi^2_\beta$. If $\lambda < -\chi^2_\beta$ we simply use the fact that $g_{0\beta}$ is bounded in $L^2_\delta(\mathbb{R}^n)$ for any δ, uniformly in λ for $\lambda \leqslant b < -\chi^2_\beta$. We can pick up an ε at each iteration, and obtain finally that $\Phi_{\alpha i} \in L^2_1(\mathbb{R}^{2n})$ for all α and i. It then follows from (4.26) by a last application of propositions (2.6.3) and (7.1) that $\Phi_\alpha \in L^2_\varepsilon(\mathbb{R}^{2n})$, and therefore from (6.17) and lemma (7.1) that $\psi \in L^2_\varepsilon(\mathbb{R}^{2n})$. The number of iterations is finite, the basic estimate from proposition (2.6.3) is uniform in λ. Therefore the required uniformity in λ follows from that in lemma (7.1). This completes the proof of proposition (7.2) in this case. The proof extends easily to the general case where $w_\alpha \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ (cf. the proof of proposition (7.1)) and where $(\mathcal{S}_-)$ is relaxed.

Proof of lemma (7.1). — We give the proof for G'_α, the case of G_0 is simpler. We first assume $h_\alpha \geqslant 0$ so that $G'_\alpha = G_\alpha$. In order to prove that G_α is bounded in the space

$$L_f^2(\mathbb{R}^{2n}) = \left\{ \varphi : \|\varphi\|_f^2 = \int dX f(X)^2 |\varphi(X)|^2 < \infty \right\} \tag{7.6}$$

where f is some smooth strictly positive function, it is sufficient to show that for some constant $C \geqslant 0$ and all $\varphi \in \mathcal{H}$:

$$\| fG_\alpha\varphi \| \leqslant C \| f\varphi \| \tag{7.7}$$

One easily sees that a sufficient condition for (7.7) to hold is that for some constant C and all ψ in the form domain of $f(H_\alpha - \lambda)f^{-1}$:

$$\|\psi\|^2 \leqslant C |\langle \psi, f(H_\alpha - \lambda)f^{-1}\psi \rangle| \tag{7.8}$$

Now

$$f(H_\alpha - \lambda)f^{-1} = H_\alpha - \lambda + \frac{1}{2}[[f, H_\alpha], f^{-1}] + \frac{1}{2}[[f, H_\alpha], f^{-1}]_+ \tag{7.9}$$

Taking the matrix element $\langle \psi, .\psi \rangle$, we obtain for $\lambda < 0$:

$$|\langle \psi, f(H_\alpha - \lambda)f^{-1}\psi \rangle| \geqslant \mathrm{Re}\,\langle \psi, f(H_\alpha - \lambda)f^{-1}\psi \rangle$$
$$= \left\langle \psi, \left(H_\alpha - \lambda + \frac{1}{2}[[f, H_\alpha], f^{-1}]\right)\psi \right\rangle \geqslant (|\lambda| - \tau)\|\psi\|^2 \tag{7.10}$$

where:

$$\tau = \operatorname{Sup} f^{-2} |\nabla f|^2 \tag{7.11}$$

Typically, f has the form:

$$f(X) = (a^2 + x_\alpha^2)^{\delta/2}(a^2 + y_\alpha^2)^{\delta'/2} \tag{7.12}$$

and one checks easily that τ is finite and tends to zero as $a \to \infty$. This proves (7.8) and therefore lemma (7.1) in this case.

We next relax the assumption $h_\alpha \geqslant 0$. It is sufficient to prove that

$$\| fG'_\alpha\varphi \| \leqslant C \| f\varphi \| \tag{7.13}$$

for all φ in $\mathcal{H}$, or equivalently to prove (7.7) for φ such that $P_\alpha\varphi = 0$. Assume this to hold and let $\psi = fG'_\alpha\varphi$. We want to prove (7.8). By the same argument as before, we obtain:

$$|\langle \psi, f(H_\alpha - \lambda)f^{-1}\psi \rangle| \geqslant \langle \psi, (H_\alpha - \lambda - \tau)\psi \rangle$$
$$= \langle \psi, (1 - P_\alpha)H_\alpha\psi \rangle + (|\lambda| - \tau)\|\psi\|^2 + \langle P_\alpha\psi, H_\alpha P_\alpha\psi \rangle \tag{7.14}$$

Now the first term in the RHS is positive, while in the last one we use the fact that $H_\alpha \geqslant -\chi_\alpha^2$. Therefore:

$$\ldots \geqslant (|\lambda| - \tau)\|\psi\|^2 - \chi_\alpha^2 \|P_\alpha\psi\|^2 \tag{7.15}$$

Now:

$$P_\alpha\psi = [P_\alpha, f]G'_\alpha\varphi = [P_\alpha, f]f^{-1}\psi \tag{7.16}$$

It is easy to check that with f defined by (7.12), $\|[P_\alpha, f]f^{-1}\|$ tends to zero when $a \to \infty$. From this and (7.15), we obtain (7.8). This completes the proof.

It has been proved by Yafaev [33] under similar assumptions on the potentials (essentially $|x|^{-(2+\varepsilon)}$ decrease at infinity) that the negative point spectrum of H is finite. We do not obtain this result here, since we are not able to extend Schwinger's argument to exclude the possible accumulation of eigenvalues at the two-body thresholds from below.

It appears clearly in the proof of proposition (7.2) that for negative energies, all the difficulties come from the two-body subsystems. It follows from this fact that a similar result holds under Agmon's original assumption $(\mathscr{A}_{1/2+\varepsilon})$. Again $(\mathscr{S}_+)$ and $(\mathscr{S}_-)$ are not needed.

PROPOSITION (7.3). — Let the potentials v_α satisfy $(\mathscr{A}_{1/2+\varepsilon})$. Then the negative point spectrum of H consists of eigenvalues with finite multiplicities, which can accumulate only at $\lambda = 0$ and at the two-body thresholds. The negative singular continuous spectrum of H is empty.

Sketch of proof. — We use again equations similar to (4.27, 28), with the following difference. Under assumption $(\mathscr{A}_{1/2+\varepsilon})$ it is unreasonable to assume that the two-body subsystems have a finite number of bound states. One therefore splits G_α as follows. Let $\eta > 0$ and let $e_\alpha(-\eta)$ be the spectral projector of h_α on the half line $(-\infty, -\eta)$. One defines

$$g'_\alpha = g_\alpha[1 - e_\alpha(-\eta)] \qquad \text{and} \qquad G'_\alpha(\lambda) = g'_\alpha(\lambda - q_\alpha^2/2n_\alpha)$$

and separates out the contribution of the two-body bound states with energies less than $-\eta$ as in (4.32). There is a finite number of them. For λ in the complex plane cut along $[-\eta, \infty)$ it follows easily from Agmon's estimates as expressed in particular by proposition (2.3) and from the fact that the G_0 and G'_α are taken inside their analyticity domain that the new equation can be studied by the same method as (4.28). The main difference is that our methods do not give any information for λ near the positive real axis. One can study the negative singular set ξ_e associated with the homogeneous equation as in propositions (6.4), (7.1) and (7.2). One can apply the iteration method described in the proofs of propositions (3.5) and (7.2), using now the λ dependent estimates of proposition (2.6.1). The estimates are uniform in λ in each compact subset of $(-\infty, -\eta)$ not containing any two-body threshold. With these differences and qualifications, the proof is a repetition of that of proposition (7.2).

8. WAVE OPERATORS AND ASYMPTOTIC COMPLETENESS

In this section, we shall derive the expression of the wave operators in terms of the resolvent operator, and prove asymptotic completeness. The Kato-Lavine method using H-smoothness of the interaction is not directly applicable in the present situation. In fact, we have a multichannel problem, and the absolutely continuous subspace of H is expected to be the direct sum of the subspaces of the various channels. In order to separate these subspaces, we would need to construct the corresponding projectors first, and therefore the wave operators themselves. We shall therefore follow the more traditional route opened by Ikebe [5] and followed by Faddeev [13]. In contrast with Faddeev's approach however, we shall make use of the fact that the existence of the wave operators can be proved directly. This avoids the detour of deriving basic properties of these operators which follow immediately from their time-dependent definition, and not from their expression in terms of the resolvent operators.

We first recall the well-known existence results, under the assumptions of this paper.

PROPOSITION (8.1). — Let the potentials satisfy condition ($\mathcal{N}$), where we assume in addition that $p \geqslant 2$. Then the following strong limits exist:

$$\Omega_{0\pm} = \underset{t\to\mp\infty}{\text{s.}\lim} \exp(it\mathrm{H}) \exp(-it\mathrm{H}_0) \tag{8.1}$$

$$\Omega_{\alpha\pm} = \underset{t\to\mp\infty}{\text{s.}\lim} \exp(it\mathrm{H}) \exp(-it\mathrm{H}_\alpha)\mathrm{P}_\alpha \tag{8.2}$$

They are isometric from $\mathscr{H}$ (resp $\mathscr{R}(\mathrm{P}_\alpha) \subset \mathscr{H}$) into $\mathscr{H}$. They intertwine H and H_0 (resp H and H_α). The ranges of the Ω_+ are orthogonal. The same holds for the Ω_-.

Remark (8.1). — For the proof with more general interactions, see [1]. The assumption $p \geqslant 2$ is an additional restriction only for $n = 3$. In this case, it implies that $\mathscr{D}(\mathrm{V}) \supset \mathscr{D}(\mathrm{H}) = \mathscr{D}(\mathrm{H}_0)$ (see section 1C). If $n = 3$ and $3/2 < p < 2$, the direct existence proof can be extended easily to Ω_0 by a perturbation argument, which however does not seem to work as simply for the Ω_α.

The next step is to obtain an expression for the Ω^* in terms of the resolvent operator. We now assume conditions ($\mathscr{A}_{1+\varepsilon}$), ($\mathscr{S}_+$) and ($\mathscr{S}_-$) to hold, so that the results of sections (4), (5) and (6) are available.

PROPOSITION (8.2). — Let conditions ($\mathscr{A}_{1+\varepsilon}$), ($\mathscr{S}_+$) and ($\mathscr{S}_-$) hold. Let $\varphi \in \mathrm{L}^2_\delta(\mathbb{R}^{2n})$ for some $\delta > 1$. Define $\Phi(\lambda) \in \overline{\mathscr{H}}$ by:

$$\Phi(\lambda) = [1 - \mathrm{A}(\lambda)]^{-1}\mathrm{J}(\lambda)\varphi \tag{8.3}$$

as in the proof of proposition (6.4.3).

Define:

$$\varphi_0(\lambda) = \varphi + \sum_\alpha [\,|\,v_\alpha\,|^{1/2}\Phi_{\alpha 0}(\lambda) - \rho_\alpha \Phi_{\alpha 1}(\lambda)] \tag{8.4}$$

Then the $\Omega^*\varphi$ are given in momentum-space variables by:

$$\widehat{\Omega^*_{0\pm}\varphi}(\mathrm{P}) = \hat{\varphi}_0(\mathrm{P}, \lambda = \mathrm{P}^2/2\mu \pm i0) \tag{8.5}$$

$$\widehat{\Omega^*_{\alpha\pm}\varphi}(\mathrm{P}) = \widehat{\rho_\alpha\Phi_{\alpha 1}}(\mathrm{P}, \lambda = q_\alpha^2/2n_\alpha - \chi_\alpha^2 \pm i0) \tag{8.6}$$

The $\widehat{\Omega^*_{0\pm}\varphi}$ (resp. $\widehat{\Omega^*_{\alpha\pm}\varphi}$) have restrictions to spheres $\mathrm{P}^2 = k^2$ (resp. $q_\alpha^2 = k^2$) which belong to the L^2-spaces of the corresponding spheres, and are continuous functions of k for $k^2/2\mu \notin \xi_e$ (resp. $k^2/2n_\alpha - \chi_\alpha^2 \notin \xi_e$).

Remark (8.2). — We recall that $\Phi_{\alpha 1} = \mathrm{P}_\alpha \Phi_{\alpha 1}$ so that the p_α dependence of $\widehat{\Omega^*_\alpha\varphi}$ is contained in a trivial factor $\hat{\psi}_\alpha$. The relevant statements at the end of proposition (8.2) refer to the q_α dependence only.

Proof of proposition (8.2). — We concentrate on the Ω_- and drop the subscript $-$. The Ω_+ can be studied similarly.

We first consider Ω_0. It follows from (8.1) that Ω_0 can be represented as the abelian limit:

$$\Omega_0 = \lim_{\eta \downarrow 0} \eta \int_0^\infty dt e^{-\eta t} \exp(it\mathrm{H}) \exp(-it\mathrm{H}_0) \tag{8.7}$$

Let now $\psi \in \mathscr{H}$ be such that $\hat{\psi}$ is a bounded continuous function of P, and that the set $\mathrm{Supp}_2(\hat{\psi}) = \{\mathrm{P}^2/2\mu : \mathrm{P} \in \text{support of } \hat{\psi}\}$ is compact and does not intersect ξ_e.

Let γ be a closed counter-clockwise contour in the complex plane around this set, with $|\,\mathrm{Im}\, z\,| \leqslant \eta/2$ for all z in γ. Then:

$$\psi = (2\pi i)^{-1} \int_\gamma dz \mathrm{G}_0(z)\psi\,. \tag{8.8}$$

Therefore:

$$\Omega_0\psi = \lim_{\eta \downarrow 0} \eta \int_0^\infty dt e^{-\eta t}(2\pi i)^{-1} \int_\gamma dz \exp[it(\mathrm{H} - z)]\mathrm{G}_0(z)\psi \tag{8.9}$$

$$= \lim_{\eta \downarrow 0} - i\eta(2\pi i)^{-1} \int_\gamma dz \mathrm{G}(z - i\eta)\mathrm{G}_0(z)\psi \tag{8.10}$$

where the integral is norm convergent.

We now take the scalar product of $\Omega_0\psi$ with $\varphi \in \mathrm{L}^2_\delta(\mathbb{R}^{2n})$:

$$\langle \varphi, \Omega_0\psi \rangle = \lim_{\eta \downarrow 0} - i\eta(2\pi i)^{-1} \int_\gamma dz \langle \mathrm{G}(\bar{z} + i\eta)\varphi, \mathrm{G}_0(z)\psi \rangle \tag{8.11}$$

We define $\Phi(\lambda)$ by (8.3). Then, by (6.22) and (8.4):

$$G(\bar{z} + i\eta)\varphi = G_0(\bar{z} + i\eta)\varphi_0(\bar{z} + i\eta) + \sum_{\alpha} g_{0\alpha}(\bar{z} + i\eta)\rho_\alpha\Phi_{\alpha 1}(\bar{z} + i\eta) \quad (8.12)$$

We substitute (8.12) into (8.11). We shall prove below that the contribution of the last sum from (8.12) vanishes in the limit $\eta \downarrow 0$. Assuming this for the moment, we obtain:

$$\langle \varphi, \Omega_0\psi \rangle = \lim_{\eta\downarrow 0} - i\eta(2\pi i)^{-1} \int_\gamma dz \langle \varphi_0(\bar{z} + i\eta), G_0(z - i\eta)G_0(z)\psi \rangle \quad (8.13)$$

$$= \lim_{\eta\downarrow 0} (2\pi i)^{-1} \int_\gamma dz \langle \varphi_0(\bar{z} + i\eta), G_0(z)\psi \rangle \quad (8.14)$$

by the use of the first resolvent identity, and the fact that one of the terms drops out by analyticity. Now φ_0 can be restricted to spheres in momentum space by proposition (2.2), (and some trivial addition to take into account the local singularities of the potentials). The same holds for ψ by assumption. Therefore:

$$\langle \varphi, \Omega_0\psi \rangle = \lim_{\eta\downarrow 0} (2\pi i)^{-1} \int_\gamma dz \int_0^\infty dk(z - k^2/2\mu)^{-1} \times \langle \pi(k)\varphi_0(\bar{z} + i\eta), \pi(k)\psi \rangle \quad (8.15)$$

where the last factor is uniformly bounded with compact support in k. We interchange the order of integrations and use the fact that $\varphi_0(\bar{z} + i\eta)$ is analytic for $z \in \gamma$:

$$\langle \varphi, \Omega_0\psi \rangle = \lim_{\eta\downarrow 0} \int_0^\infty dk \langle \pi(k)\varphi_0(k^2/2\mu + i\eta), \pi(k)\psi \rangle \quad (8.16)$$

Now $\varphi_0(k^2/2\mu + i\eta)$ is Hölder-continuous in η uniformly in k for $k^2/2\mu \in \operatorname{supp}_2 \hat{\psi}$, so that the limit $\eta \downarrow 0$ can be taken inside the integral:

$$\langle \varphi, \Omega_0\psi \rangle = \int_0^\infty dk \langle \pi(k)\varphi_0(k^2/2\mu + i0), \pi(k)\psi \rangle \quad (8.17)$$

From (8.17) and from the fact that the set of ψ under consideration is dense in $\mathscr{H}$, we obtain (8.5).

The last statements of proposition (8.2) follow essentially from proposition (2.2).

It remains to be shown that the contribution of the last sum in (8.12) to $\langle \varphi, \Omega_0\psi \rangle$ vanishes in the limit $\eta \downarrow 0$. A typical terms is:

$$\lim_{\eta\downarrow 0} (2\pi i)^{-1} \int_\gamma d\bar{z} \langle \rho_\alpha\Phi_{\alpha 1}(z + i\eta), - i\eta g_{0\alpha}(z - i\eta)G_0(z)\psi \rangle \quad (8.18)$$

Now:

$$g_{0\alpha}(z - i\eta)G_0(z) = (p_\alpha^2/2m_\alpha + \chi_\alpha^2 - i\eta)^{-1}[G_0(z) - g_{0\alpha}(z - i\eta)] \quad (8.19)$$

The term $g_{0\alpha}$ does not contribute to the integral because of the analyticity of $\Phi_{\alpha 1}(\bar{z} + i\eta)$ and we are left with:

$$\lim_{\eta \downarrow 0} - i\eta(2\pi i)^{-1} \int_\gamma dz \times \langle (p_\alpha^2/2m_\alpha + \chi_\alpha^2 + i\eta)^{-1}\rho_\alpha\Phi_{\alpha 1}(\bar{z} + i\eta), G_0(z)\psi \rangle \quad (8.20)$$

Using the same method as for (8.14), one sees that the integral has a well-defined limit when $\eta \downarrow 0$ and therefore the whole term tends to zero with η.

We now turn to the proof of (8.6). Since it is very similar to that of (8.5), we only sketch the basic steps. Let $\psi = \psi_\alpha \otimes \psi' \in \mathscr{R}(P_\alpha)$ where $\hat{\psi}'$ is a bounded continuous function of q_α such that the set

$$\mathrm{Supp}_2\, \hat{\psi}' = \{ q_\alpha^2/2n_\alpha - \chi_\alpha^2 : q_\alpha \in \mathrm{Supp}\, \hat{\psi}' \}$$

is compact and does not intersect ξ_e, and let γ be a contour around this set as before. Then:

$$\psi = (2\pi i)^{-1} \int_\gamma dz g_{0\alpha}(z) P_\alpha \psi \quad (8.21)$$

We substitute (8.21) into (8.22):

$$\Omega_\alpha \psi = \lim_{\eta \downarrow 0} \eta \int_0^\infty dt e^{-\eta t} \exp(itH) \exp(-itH_\alpha) P_\alpha \psi \quad (8.22)$$

and obtain:

$$\Omega_\alpha \psi = \lim_{\eta \downarrow 0} - i\eta(2\pi i)^{-1} \int_\gamma dz G(z - i\eta) g_{0\alpha}(z) P_\alpha \psi \quad (8.23)$$

We take the scalar product with φ and substitute (8.12) into the result. By the same argument as before, the contributions from all terms in (8.12) vanish in the limit $\eta \downarrow 0$, except for that of the term with $g_{0\alpha}(\bar{z} + i\eta)$. Using again the first resolvent identity and analyticity, we obtain:

$$\langle \varphi, \Omega_\alpha \psi \rangle = \lim_{\eta \downarrow 0} (2\pi i)^{-1} \int_\gamma dz \langle \rho_\alpha \Phi_{\alpha 1}(\bar{z} + i\eta), g_{0\alpha}(z) P_\alpha \psi \rangle \quad (8.24)$$

After performing the trivial partial scalar product over the variable p_α, one is left with the same problem as before, where however q_α replaces P. The end of the proof is identical with the previous one.

Proposition (8.2) gives an explicit construction of the various $\Omega^*\varphi$ for a dense set of vectors φ. For a general $\varphi \in \mathscr{H}$, the $\Omega^*\varphi$ are obtained by a limiting process similar to the extension of the Fourier transform from $L^1 \cap L^2$ to L^2. For a general $\varphi \notin L_\delta^2(\mathbb{R}^{2n})$, equations (8.3-6) are not expected to make sense in general, and the regularity properties of $\widehat{\Omega^*\varphi}$ stated in the proposition are not expected to hold.

Proposition (8.2) would be the starting point for an eigenfunction expansion for H. Heuristically, $\widehat{\Omega_{0\pm}^*\varphi}(P)$ (resp. $\widehat{\Omega_{\alpha\pm}^*\varphi}(., q_\alpha)$) is the scalar product of φ with the distorted plane wave that is an eigenfunction of H with eigenvalue $P^2/2\mu$ (resp. $q_\alpha^2/2n_\alpha - \chi_\alpha^2$) and corresponds at $t \to \mp\infty$ with a plane wave $\exp(iP.X)$ (resp. $\psi_\alpha \exp(iq_\alpha y_\alpha)$). We shall not pursue the matter further here.

We now turn to the question of asymptotic completeness. Let E_{ac} be the spectral projector of H on the subspace $\mathcal{H}_{ac}$ of absolute continuity. From the intertwining properties of the Ω, we know that

$$\mathcal{H}_{ac} \equiv \mathcal{R}(E_{ac}) \supseteq \mathcal{R}(\Omega_{0\pm}) \oplus \bigoplus_\alpha \mathcal{R}(\Omega_{\alpha\pm}) \qquad (8.25)$$

We shall prove asymptotic completeness in the sense that equality holds in (8.25).

The first step is to construct the spectral projectors of H on intervals not intersecting ξ_e.

PROPOSITION (8.3). — Let $[a, b] \subset \sigma_e$, $[a, b] \cap \xi_e = \emptyset$. Let $\varphi \in L_\delta^2(\mathbb{R}^{2n})$ for some $\delta > 1$, and define $\Phi(\lambda)$ and $\varphi_0(\lambda)$ by (8.3) and (8.4). Then:

$$\| E_{[a,b]}\varphi \|^2 = \int_{a \leqslant P^2/2\mu \leqslant b} dP\, |\, \hat{\varphi}_0(P, P^2/2\mu \pm i0)\,|^2$$
$$+ \sum_\alpha \int_{a \leqslant q_\alpha^2/2n_\alpha - \chi_\alpha^2 \leqslant b} dP\, |\, \widehat{\rho_\alpha \Phi_{\alpha 1}}(P, q_\alpha^2/2n_\alpha - \chi_\alpha^2 \pm i0)\,|^2 \qquad (8.26)$$

Proof. — Since $a, b \notin \sigma_p(H)$ by proposition (6.4.2), we have for all φ in $\mathcal{H}$ (cf. (3.17)):

$$\| E_{[a,b]}\varphi \|^2 = \lim_{\eta \downarrow 0} (\eta/\pi) \int_a^b d\lambda\, \| G(\lambda + i\eta)\varphi \|^2 \qquad (8.27)$$

We consider first the case where $v_\alpha = w_\alpha(1 + x^2)^{-(1+\varepsilon)}$ and $w_\alpha \in L^\infty(\mathbb{R}^n)$ for all α. Let $\varphi \in L_\delta^2(\mathbb{R}^{2n})$. We substitute (8.12) into (8.27). We obtain diagonal terms and cross terms. We consider first the diagonal term with $G_0(\lambda + i\eta)$. By the same argument as in the proof of proposition (7.1), it can be rewritten as:

$$\lim_{\eta \downarrow 0} (\eta/\pi) \int_a^b d\lambda \int_0^\infty dk[(\lambda - k^2/2\mu)^2 + \eta^2]^{-1} \| \pi(k)\varphi_0(\lambda + i\eta) \|^2 \qquad (8.28)$$

By Fubini's theorem, we can interchange the integrations over λ and k. Since $[a, b] \cap \xi_e = \emptyset$, the vector $\pi(k)\varphi_0(\lambda + i\eta)$ is norm continuous in λ and η for all $\lambda \in [a, b]$ and $\eta \geqslant 0$ by lemma (6.1), proposition (5.5.2) and proposition (2.2.1). Furthermore, it satisfies the estimate:

$$\| \pi(k)\varphi_0(\lambda + i\eta) \|^2 \leqslant Ck \| \varphi \|_\delta^2 \qquad (8.29)$$

for some constant C independent of λ and η for $\lambda \in [a, b]$.

By an elementary computation, one checks that the integral over λ:

$$\int_a^b d\lambda(\eta/\pi)[(\lambda - k^2/2\mu)^2 + \eta^2]^{-1} \| \pi(k)\varphi_0(\lambda + i\eta) \|^2$$

is bounded uniformly in η by an integrable function of k ([5], p. 24). By Lebesgue's dominated convergence theorem, one can therefore take the limit $\eta \downarrow 0$ inside the k integral. Since $\| \pi(k)\varphi_0(\lambda + i\eta) \|^2$ is (Hölder) continuous in λ and η for fixed k, one can then perform the limit $\eta \downarrow 0$ in the λ integral [5], thereby obtaining for (8.28) the expression:

$$\int_{a \leqslant k^2/2\mu \leqslant b} dk \, \| \pi(k)\varphi_0(k^2/2\mu) \|^2 \tag{8.30}$$

This is precisely the first term in the RHS of (8.26).

By the same argument, the three diagonal terms with $g_{0\alpha}$ coming from (8.12) can be shown to yield the last sum in (8.26).

We finally consider the cross terms. The contributions to $\eta \| G(\lambda + i\eta)\varphi \|^2$ of the terms $(0, \alpha)$ and (α, β) are respectively:

$$2\eta \operatorname{Re} \langle \varphi_0(\lambda + i\eta), G_0(\lambda - i\eta) g_{0\alpha}(\lambda + i\eta)\rho_\alpha \Phi_{\alpha 1}(\lambda + i\eta) \rangle \tag{8.31}$$

and

$$2\eta \operatorname{Re} \langle \rho_\alpha \Phi_{\alpha 1}(\lambda + i\eta), g_{0\alpha}(\lambda - i\eta) g_{0\beta}(\lambda + i\eta)\rho_\beta \Phi_{\beta 1}(\lambda + i\eta) \rangle \tag{8.32}$$

These quantities are shown to tend to zero with η in Appendix B. This completes the proof for $w_\alpha \in L^\infty(\mathbb{R}^n)$. The general case is dealt with by the same trick as in proposition (7.1).

We can now prove a generalized Parseval identity, from which asymptotic completeness follows immediately.

PROPOSITION (8.4). — For all φ in $\mathscr{H}$,

$$\| E_{ac}\varphi \|^2 = \| \Omega^*_{0\pm}\varphi \|^2 + \sum_\alpha \| \Omega^*_{\alpha\pm}\varphi \|^2 \tag{8.33}$$

The wave operators are asymptotically complete in the sense that equality holds in (8.25).

Proof. — Since ξ_e is closed, $\sigma_e \backslash \xi_e$ is open and is therefore a denumerable union of closed intervals not intersecting ξ_e and with disjoint interiors. Taking the sum of the contributions of these intervals as given by proposition (8.3), and using proposition (8.2) and the fact that ξ_e has Lebesgue measure zero, we obtain (8.33) for all $\varphi \in L^2_\delta(\mathbb{R}^{2n})$, and therefore for all $\varphi \in \mathscr{H}$ by continuity.

The second statement follows from the first and proposition (8.1). This completes the proof.

ACKNOWLEDGMENTS

One of us (J. G.) is grateful to B. Simon for several illuminating discussions (or rather private lectures) at early stages of this work, as well as for the generous communication of unpublished material.

Note added in proof:

After this work was completed, we received a preprint *Asymptotic completeness in two-and three particle quantum mechanical systems* by Lawrence E. THOMAS, where similar results are derived by similar methods. The equations used by Thomas are slightly different from ours, the basic estimates rely partly on the Sobolev inequality, and asymptotic completeness is proved by the use of spectral integrals.

APPENDIX A

CONTROL OF THE CORRECTION TERMS IN (6.14) AND (6.15)

By the use of (5.3), the first correction term in (6.14) can be written as:

$$i\eta v_\alpha^{1/2}G'_\alpha(\lambda + i\eta)\psi = (1 + v_\alpha^{1/2}G'_\alpha \,|\, v_\alpha \,|^{1/2})i\eta v_\alpha^{1/2}G_0(\lambda + i\eta)\psi - v_\alpha^{1/2}P_\alpha(1 + x_\alpha^2)^\delta i\eta(1 + x_\alpha^2)^{-\delta}G_0(\lambda + i\eta)\psi \quad (A.1)$$

$1 + v_\alpha^{1/2}G'_\alpha \,|\, v_\alpha \,|^{1/2}$ is bounded uniformly in λ and η, as can be seen in the proof of proposition (5.1), and so is $v_\alpha^{1/2}P_\alpha(1 + x_\alpha^2)^\delta$ for any real δ. Now for $\delta > 1/2$, $(1 + x_\alpha^2)^{-\delta}$ and $|\, v_\alpha \,|^{1/2}$ are H_0-smooth by proposition (5.1.1) and remark (3.2).

Therefore:

$$\|\, i\eta v_\alpha^{1/2}G_0(\lambda + i\eta)\,\| \leqslant (\eta\pi)^{1/2}\,\|\, v_\alpha^{1/2}\,\|_{H_0}$$

and

$$\|\, i\eta(1 + x_\alpha^2)^{-\delta}G_0(\lambda + i\eta)\,\| \leqslant (\eta\pi)^{1/2}\,\|\,(1 + x_\alpha^2)^{-\delta}\,\|_{H_0}$$

This proves that the first correction term tends to zero as $\eta \to 0$.

In the second correction term $i\eta v_\alpha^{1/2}G'_\alpha \sum_{\beta \neq \alpha} v_\beta P_\beta g_{0\beta}\psi$, one can replace G'_α by G_0, by another use of (5.3). Now:

$$\eta v_\alpha^{1/2}G_0 v_\beta P_\beta g_{0\beta} = \eta v_\alpha^{1/2}g_{0\beta}P_\beta - \eta v_\alpha^{1/2}G_0P_\beta \quad (A.2)$$

It suffices to prove that the RHS of (A.2) tends to zero in norm as $\eta \to 0$. Now:

$$\eta v_\alpha^{1/2}g_{0\beta}P_\beta = [v_\alpha^{1/2}\rho_\beta^{-1}P_\beta][\eta P_\beta \rho_\beta g_{0\beta}(\lambda + i\eta)]$$

the first factor is $A_{\beta 1,\alpha 0}^*$ and is therefore bounded, by proposition (5.2), and the second factor is bounded in norm by $(\eta\pi)^{1/2}\,\|\,\rho_\beta\,\|_{h_0}$ because ρ_β is h_0-smooth by lemma (4.1) and remark (3.2).

On the other hand:

$$\|\,\eta v_\alpha^{1/2}G_0P_\beta\,\| \leqslant (\eta\pi)^{1/2}\,\|\, v_\alpha^{1/2}\,\|_{H_0}$$

Therefore the second correction term tends to zero in norm as $\eta \to 0$.

The correction term in (6.15) can be written as:

$$i\eta\rho_\alpha^{-1}P_\alpha v_\beta P_\beta g_{0\beta}\psi = (P_\alpha\rho_\alpha^{-1}v_\beta P_\beta\rho_\beta^{-1})(i\eta\rho_\beta g_{0\beta}(\lambda + i\eta)\psi) \quad (A.3)$$

$P_\alpha\rho_\alpha^{-1}v_\beta P_\beta\rho_\beta^{-1}$ is bounded uniformly in λ and η, and the last factor is bounded as above, by another use of h_0-smoothness of ρ_β. This proves that the third correction term tends to zero as $\eta \to 0$.

APPENDIX B

i) CONTROL OF THE CROSS TERMS IN (6.31).

They are of two different types:

$$\eta \langle G_0(\lambda + i\eta) \sum_{\alpha} (|v_\alpha|^{1/2}\Phi_{\alpha 0} - \rho_\alpha \Phi_{\alpha 1}), g_{0\beta}(\lambda + i\eta) P_\beta \rho_\beta \Phi_{\beta 1} \rangle \quad (B.1)$$

and

$$\eta \langle P_\alpha g_{0\alpha}(\lambda + i\eta)\rho_\alpha \Phi_{\alpha 1}, P_\beta g_{0\beta}(\lambda + i\eta)\rho_\beta \Phi_{\beta 1} \rangle, \quad (\alpha \neq \beta). \quad (B.2)$$

For instance consider the term of (B.1):

$$\eta \langle G_0(\lambda + i\eta) |v_\alpha|^{1/2}\Phi_{\alpha 0}, g_{0\beta}(\lambda + i\eta) P_\beta \rho_\beta \Phi_{\beta 1} \rangle \quad (B.3)$$

The other terms are quite similar.

$P_\beta G_0 |v_\alpha|^{1/2}$ is a uniformly bounded operator in λ; this can be seen by writing it in the form $[P_\beta(1 + x_\beta^2)^\delta] [(1 + x_\beta^2)^{-\delta} G_0 |v_\alpha|^{1/2}]$ where the first factor is bounded for any δ, and the second factor is uniformly bounded in λ for $\delta > 1/2$ by a proof similar to that of proposition (5.1.1).

Therefore the absolute value of (B.3) is bounded by

$$(\eta\pi)^{1/2} \| P_\beta G_0 |v_\alpha|^{1/2} \| \quad \| \rho_\beta \|_{h_0} \| \Phi_{\alpha 0} \| \quad \| \Phi_{\beta 1} \|$$

by the use of h_0-smoothness of ρ_β (see appendix A). This tends therefore to zero as $\eta \to 0$.

On the other hand $P_\alpha g_{0\alpha}(\lambda + i\eta)\rho_\alpha \Phi_{\alpha 1} \in L^2_\theta(x_\alpha) \otimes L^2_{-\delta}(y_\alpha)$ for any $\delta > 1/2$ uniformly in λ and η. Using twice (4.21) one deduces that the scalar product in (B.2) is bounded uniformly in λ and η for $\theta \geqslant 2\delta$. Therefore the cross term (B.2) tends to zero as $\eta \to 0$.

ii) CONTROL OF THE CORRECTION TERM IN (6.31).

It can be written as a sum over α of:

$$\langle \Phi_\alpha(-\eta), v_\alpha |v_\alpha|^{-1}(1 - v_\alpha^{1/2} G_0(\lambda + i\eta)|v_\alpha|^{1/2})\Phi'_\alpha(\eta) \rangle + \langle \Phi_\alpha(\eta) - \Phi_\alpha(-\eta), \textit{idem} \rangle \quad (B.4)$$

Now:

$$(1 - v_\alpha^{1/2} G_0(\lambda + i\eta)|v_\alpha|^{1/2}) v_\alpha^{1/2} P_\alpha g_{0\alpha}(\lambda + i\eta)\rho_\alpha = v_\alpha^{1/2} G_0(\lambda + i\eta)\rho_\alpha P_\alpha \quad (B.5)$$

and

$$\rho_\alpha G_0(\lambda - i\eta) v_\alpha^{1/2} |v_\alpha|^{1/2} g_{0\alpha}(\lambda - i\eta) P_\alpha = \rho_\alpha [g_{0\alpha}(\lambda - i\eta) - G_0(\lambda - i\eta)] P_\alpha \quad (B.6)$$

by using (5.20).

Therefore, substituting (B.5) and (B.6) in the first term of (B.4) as many times as necessary yields:

$$\begin{aligned} &\langle \Phi_{\alpha 0}, v_\alpha |v_\alpha|^{-1}(1 - v_\alpha^{1/2} G_0 |v_\alpha|^{1/2})\Phi'_{\alpha 0}(\eta) \rangle + \langle |v_\alpha|^{1/2} G_0 \rho_\alpha P_\alpha \Phi_{\alpha 1}, \Phi'_{\alpha 0}(\eta) \rangle \\ &\quad + \langle \rho_\alpha g_{0\alpha} \rho_\alpha P_\alpha \Phi_{\alpha 1}, \Phi'_{\alpha 1}(\eta) \rangle - \langle \rho_\alpha G_0 \rho_\alpha P_\alpha \Phi_{\alpha 1}, \Phi'_{\alpha 1}(\eta) \rangle \\ &\quad + \langle v_\alpha |v_\alpha|^{-1} \rho_\alpha G_0 v_\alpha^{1/2} P_\alpha \Phi_{\alpha 0}, \Phi'_{\alpha 1}(\eta) \rangle \end{aligned} \quad (B.7)$$

Each term in (B.7) tends to zero when $\eta \to 0$ as being the scalar product in $\mathscr{H}$ of a vector bounded in norm with respect to η (by estimates similar to those of the proofs of propositions (3.1.1) and (5.1.1)) with a vector that tends to zero in norm in $\mathscr{H}$. On the other hand, the second term in (B.4) can be written as:

$$\langle 2i\eta |v_\alpha|^{1/2} P_\alpha g_{0\alpha}(\lambda - i\eta) g_{0\alpha}(\lambda + i\eta)\rho_\alpha \Phi_{\alpha 1}, (1 - v_\alpha^{1/2} G_0(\lambda + i\eta)|v_\alpha|^{1/2})\Phi'_\alpha(\eta) \rangle \quad (B.8)$$

Then, using (6.28), (B.5) and (B.6), we can rewrite the scalar product in (B.8) as:

$$\langle 2i\eta g_{0\alpha}\rho_\alpha P_\alpha \Phi_{\alpha 1}, G_0 \,|\, v_\alpha \,|^{1/2} \Phi'_{\alpha 0}(\eta) + (g_{0\alpha} - G_0)\rho_\alpha P_\alpha \Phi'_{\alpha 1}(\eta) \rangle \quad (B.9)$$

Therefore, the absolute value of (B.9) is bounded by:

$$2\pi \| \rho_\alpha \|_{h_0} \| \Phi_{\alpha 1} \| [\| v_\alpha^{1/2} \|_{H_0} \| \Phi'_{\alpha 0}(\eta) \| + (\| \rho_\alpha \|_{h_0} + \| \rho_\alpha \|_{H_0}) \| \Phi'_{\alpha 1}(\eta) \|] \quad (B.10)$$

Each term in (B.10) tends to zero as $\eta \rightarrow 0$ because $\Phi'_{\alpha 0}(\eta)$ and $\Phi'_{\alpha 1}(\eta)$ tend to zero in $\mathscr{H}$. This completes the proof.

REFERENCES

[1] S. T. Kuroda, *Nuov. Cim.*, t. XII, 1959, p. 431.
[2] T. Kato, *Proc. Intern. Conf. on Functional Analysis and related topics*, Tokyo Univ. Press, 1970.
[3] S. Agmon, *Jour. Anal. Math.*, t. **23**, 1970, p. 1.
[4] B. Simon, *Comm. Pure Appl. Math.*, t. **22**, 1969, p. 531.
[5] T. Ikebe, *Arch. Ratl. Mech. Anal.*, t. **5**, 1960, p. 1.
[6] S. Agmon, International Congress of Mathematicians, Nice, 1970.
[7] M. Reed and B. Simon, *Methods of modern mathematical Physics*, Academic Press, New York, Vol. III, *in preparation.*
[8] R. Lavine, *Commun. Math. Phys.*, t. **20**, 1971, p. 301.
[9] R. Lavine, *J. Func. Anal.*, t. **12**, 1973, p. 30.
[10] J. Aguilar and J. M. Combes, *Commun. Math. Phys.*, t. **22**, 1971, p. 269.
[11] B. Simon, *Commun. Math. Phys.*, t. **27**, 1972, p. 1.
[12] W. Hunziker, *Helv. Phys. Acta*, t. **39**, 1966, p. 451.
[13] L. D. Faddeev, *Trudy Steklov Math. Inst.*, t. **69**, 1963.
[14] K. Hepp, *Helv. Phys. Acta*, t. **42**, 1969, p. 425.
[15] A. Schtalheim, *Helv. Phys. Acta*, t. **44**, 1971, p. 642.
[16] R. Lavine, *J. Math. Phys.*, t. **14**, 1973, p. 376.
[17] E. Balslev and J. M. Combes, *Commun. Math. Phys.*, t. **22**, 1971, p. 280.
[18] T. Kato, *Math. Ann.*, t. **162**, 1966, p. 258.
[19] R. G. Newton, *J. Math. Phys.*, t. **12**, 1971, p. 1552.
[20] A. Martin, *Helv. Phys. Acta*, t. **45**, 1972, p. 140.
[21] B. Simon, *Quantum Mechanics for Hamiltonians defined as quadratic forms*, Princeton Univ. Press, Princeton, 1971.
[22] E. Nelson, *J. Math. Phys.*, t. **5**, 1964, p. 332.
[23] U. Greifenegger, K. Jörgens, J. Weidmann and M. Winkler, *Streutheorie für Schrödinger operatoren*, preprint, 1972.
[24] N. Dunford and J. Schwartz, *Linear operators*, Interscience, New York, 1958, Vol. I.
[25] B. Simon, *Helv. Phys. Acta*, t. **43**, 1970, p. 607.
[26] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York, 1973, Vol. I.
[27] S. T. Kuroda, *Jour. Anal. Math.*, t. **20**, 1967, p. 57.
[28] R. Lavine, *Indiana Univ. Math. Jour.*, t. **21**, 1972, p. 643.
[29] K. M. Watson, *Phys. Rev.*, t. **89**, 1953, p. 575.
[30] A. J. O'Connor, *Commun. Math. Phys.*, t. **32**, 1973, p. 319.
[31] R. J. Iorio and M. O'Carroll, *Commun. Math. Phys.*, t. **27**, 1972, p. 137.
[32] J. M. Combes, *Commun. Math. Phys.*, t. **12**, 1969, p. 283.
[33] D. R. Yafaev, *Dokl. Akad. Nauk SSSR*, t. **206**, 1972, p. 68 ; *Transl. Sov. Phys. Dokl.*, t. **17**, 1973, p. 849.
[34] L. V. Kantorovitch and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, London, 1964.

(Manuscrit reçu le 22 avril 1974)