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Formulation of analytical mechanics in general relativity

by

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ABSTRACT. — An intrinsic formulation of Analytical Mechanics is given. After studying notions of connection and Riemannian structure, a deep analysis of the concept of acceleration naturally leads to Lagrangian and Hamiltonian formulations of Dynamics. Comparison is made with Classical Dynamics.

RÉSUMÉ. — Nous donnons une formulation intrinsèque de la Mécanique Analytique. Après avoir étudié la notion de connection et de structure riemannienne, une analyse approfondie du concept d'accélération conduit naturellement à la formulation lagrangienne et hamiltonienne de la dynamique. Nous faisons la comparaison avec la dynamique classique.

INTRODUCTION

My purpose is to give a rigorous setting for Analytical Mechanics in General Relativity, using the language of modern Differential Geometry.

I believe that a more systematic effort in this direction can lead, not only to a mathematical formalization of Physics, but also to a deeper understanding of it. Sometimes, generalizations of physical theories arise simply by trivial comparison with classical ones. Actually, Geometry can play an interesting role in binding some crucial points of Physics.

One of the most important features of modern Mathematics is the idea of structure. It is in this spirit that I wrote this paper.

The setting of the law of motion is performed in several steps, each one

involving only a specific structure. This fact leads naturally to a comparison of General Relativistic with Classical Mechanics. The differences are, in fact, less than those which physical expositions commonly show and are centered at the different nature of time in the two theories. Discrepancy reduces essentially to the phase-space and to some restrictions on forces. The law of motion is here postulated without any assumption of conservation laws or other, but only assuming the classical law, properly written in the new context.

In Classical Analytical Mechanics, space-time is $M \times \mathbb{R}$, where M is a Riemannian manifold. The original affine structure, which is destroyed by constraints, is important in describing forces (action at distance hypothesis and III principle), but has not a special role in the law of motion and its consequences (see [10]). Furthermore, the product $M \times \mathbb{R}$ is strictly conditioned to the choice of a frame of reference and has not, in its own right, an interesting Riemannian structure.

In General Relativistic Analytical Mechanics, space-time is a manifold M , with a hyperbolic normal structure. Topologically it needs not to be trivial: this fact has a cosmological nature and it is not due to constraints. Time appears, at least locally, if we introduce a frame of reference by means of a vector field on M (see [11]), which splits M into a product. But this splitting is necessary only for operative physical interpretation of the theory. For this reason this paper may look different from analogous treatments in physical texts. In fact, in our discussion, operative problems are not involved and hence objects as the time and the space-part of the law of motion, mouvement mass ($m\sqrt{1-v^2}$), mass-energy equivalence, ... do not appear explicitly. Only proper rest mass and proper standard time are involved. This time is characterized, imposing, by means of the metric, that motion have positive velocity, or, more exactly, that it be normalized. The first condition leads to consider as phase-space a subspace of $T(M)$. The second condition leads to consider forces as forms which are power vanishing and homogeneous of degree 2. It would be possible to restrict ourselves only to the analysis of normalized motions (which have direct physical meaning), but in such a case it is not easy to obtain simple equations. I preferred, for this reason, to work in a open phase-space ($C^+(M)$ and not $C^{1/2}(M)$), after having characterized forces and motions with respect to the normalized situation. This is however a choice not imposed by the problem, hence subject to criticism.

The arrow of time, in the classical case, refers to orientation of \mathbb{R} , but, in the relativistic case, it requires the assumption of time orientation of M .

Because of the nature of time, another difference arises concerning many-body-dynamics. In fact, in the classical case, there is a unique time for all particles. Hence, a system of p particles may be represented by a particle moving in a phase-space which is completely analogous to the phase-space for one body. In the relativistic case, the many-body phase-

space is not interpretable as a one body phase-space and the presence of many particles plays an essential role.

I share the opinion of Abraham (see [7]) about variational and symplectic approaches to Mechanics and I have followed the second one. Apart differences of space-time, etc., previously discussed, the starting point is the classical law of motion. A deep analysis of the concept of acceleration and of the several structures involved, lead naturally to the subsequent developments. In fact, to speak of velocity, differential structure is enough, but to speak of acceleration, we need a connection.

The plan of the paper is the following. First we summarize the calculus on the tangent space of a manifold, adding specific results that shall be needed later. Next, in discussing the meaning of acceleration, a connection is characterized by a tensor on the tangent space. Relations between metric tensor, metric function (classically, the kinetic energy) and symplectic structure are analyzed. Collapse between Riemannian structure and connection suggests how to obtain from the law of motion a second order differential equation, which locally is the Lagrange or Hamilton equation. Signature of the metric is studied and if it is $(+, -, \dots, -)$, then time-orientability (this hypothesis has a cosmological meaning) and phase-space are introduced. Forces are then defined on phase-space. All these statements are extended to the product $M = M_1 \times \dots \times M_p$. At this point we are in the position of formulating the law and the equation of motion for one and many bodies, on the tangent and the cotangent space.

The consequences of the equation of motion can, in the scope of our discussion, be derived as in Classical Mechanics (see [6] and [7]). Observe that the condition of power vanishing for the forces excludes conservative Mechanics and connected features.

Our results are entirely valid for Special Relativity. In this case, moreover, it is possible to add some further assumption.

Effective applicability of the theory, especially for many bodies, in physical concrete and relevant problems, is not large. In fact it is difficult in Relativity to give *a priori* forces, because there is no action at distance. For example, the classical many-body problem (see [7]), translated in general relativistic context, involves a much deeper situation, which escapes the present scheme. Nevertheless, I think that the present formulation may have theoretical interest.

I. MATHEMATICAL STRUCTURES

1. Calculus on the tangent space.

Even if not explicitly mentioned, we consider C^∞ differentiable manifolds and maps. Manifolds are considered para-compact.

We denote by T the tangent functor. If M is a manifold, we denote by

$$\tau(M) = (TM, p_M, M)$$

and by

$$\tau^*(M) = (T^*M, q_M, M)$$

the tangent and cotangent fibre bundles of M .

$\mathcal{F}(M)$ is the module of sections of $\tau(M)$ and $\Lambda(M)$ is the algebra of forms of M .

If (x^i) is a chart on $U \subset M$, then

$$(q^i; \dot{q}^j) = (x^i \circ p_M; dx^j)$$

and

$$(q^i; p_j) = \left(x^i \circ q_M; \frac{\partial}{\partial x^j} \right)$$

are charts on TU and on T^*U .

V is the Liouville field on TM (see [6]), given locally by

$$V = \dot{q}^i \frac{\partial}{\partial \dot{q}^i},$$

which generates the group of homothetic transformations of TM , defined by

$$h_t : u \mapsto e^t u, \quad \forall u \in TM.$$

λ is the Liouville form on T^*M (see [6]), given locally by:

$$\lambda = p_j dq^j.$$

s is the canonical involution of TTM , given locally by

$$s : (q^i, \dot{q}^i, dq^i, d\dot{q}^i) \mapsto (q^i, dq^i, \dot{q}^i, d\dot{q}^i).$$

v is the vertical endomorphism both on TTM and $\mathcal{F}(TM)$ (see [6]). v and v^* are given locally by

$$v : a^i \frac{\partial}{\partial q^i} + b^j \frac{\partial}{\partial \dot{q}^j} \mapsto a^i \frac{\partial}{\partial \dot{q}^i},$$

$$v^* : a_i dq^i + b_j d\dot{q}^j \mapsto b_j dq^j.$$

Furthermore, if X is a vector field on TM , we have:

$$i_X v^* = v^* i_{vX}.$$

The vertical derivation i_v is a derivation of degree 0 of $\Lambda(TM)$ and is characterized locally (see [6]) by:

$$\begin{aligned} i_v f &= 0, \\ i_v dq^i &= 0, \\ i_v d\dot{q}^i &= d\dot{q}^i. \end{aligned}$$

The vertical differentiation $d_v = [i_v, d] = i_v d - d i_v$ is an antiderivation of degree 1 of $\Lambda(TM)$ and is characterized locally (see [6]) by:

$$d_v f = \frac{\partial f}{\partial \dot{q}^i} dq^i$$

$$d_v dq^i = d_v d\dot{q}^i = 0.$$

We will use often the following brackets (see [6]).

1. PROPOSITION. — We have:

$$[i_X, i_v] = i_X i_v - i_v i_X = i_{vX}$$

$$[i_v, L_V] = i_v L_V - L_V i_v = i_v$$

$$[i_v, d_v] = i_v d_v - d_v i_v = 0$$

$$[d, d_v] = d d_v + d_v d = 0$$

$$[i_v, d_v] = i_v d_v + d_v i_v = i_v$$

$$[d_v, L_V] = d_v L_V - L_V d_v = d_{v\perp}$$

We denote with $B(TM)$ the algebra of semibasic forms (see [6]):

$$B(TM) = v^* \Lambda(TM).$$

j is the antiderivation of degree 1 of $B(TM)$, given by

$$j v^* = v^* i_v.$$

It is characterized locally by:

$$j f = 0$$

$$j dq^i = \dot{q}^i.$$

We have the following brackets on $B(TM)$ (the proof is left to reader).

2. PROPOSITION. — Let deg be the derivation defined by

$$\text{deg } F = pF,$$

where p is the degree of the form F .

Then, we have:

$$[L_V, j] = L_V j - j L_V = j$$

$$[j, d_v] = j d_v + d_v j = \text{deg} + L_{v\perp}$$

A vector field X on TM is a second order differential equation (s. o. d. e.) if and only if $vX = V$. Concerning s. o. d. e. we will need the following statements.

3. LEMMA. — Let $vX = V$ and $g : TM \rightarrow R$. Then the 1-form

$$F = i_X dd_v g + d(V \cdot g - g)$$

is semibasic.

Proof. — By means of (1), we get:

$$\begin{aligned} v^*F &= i_v F = i_v i_X dd_v g + i_v dL_v g - i_v dg \\ &= i_X i_v dd_v g - i_{vX} dd_v g + d_v L_v g - d_v g \\ &= i_X 0 - i_v dd_v g + L_v d_v g = 0. \end{aligned}$$

4. PROPOSITION. — Let $F \in B^1(TM)$ and $g : TM \rightarrow \mathbb{R}$ be such that $dd_v g$ is a symplectic form. Then the vector field X , given by

$$i_X dd_v g = d(g - V.g) + F,$$

is a s. o. d. e.

Proof. — By means of (1), we get:

$$\begin{aligned} i_{vX} dd_v g &= i_X i_v dd_v g - i_v i_X dd_v g = i_X 0 - i_v (d(g - V.g) + F) \\ &= -d_v g + d_v L_v g + 0 = L_v d_v g = i_v dd_v g \end{aligned}$$

hence

$$vX = V.$$

The local expression of the above vector field X is (see [6]):

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + b^j \frac{\partial}{\partial \dot{q}^j},$$

where b^j is given by

$$\frac{\partial^2 g}{\partial \dot{q}^i \partial \dot{q}^j} b^j = -\frac{\partial^2 g}{\partial q^i \partial \dot{q}^i} \dot{q}^i + \frac{\partial g}{\partial q^i} + F_i.$$

Hence, integral curves of X are locally the solutions of the Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial g}{\partial \dot{q}^i} \right) - \frac{\partial g}{\partial q^i} = F_i.$$

We say that a s. o. d. e. X is a « spray » if (see [6])

$$[V, X] = X.$$

We say that a form F on TM is « homogeneous » of degree k if (see [6])

$$L_v F = kF.$$

The preceding proposition gives:

5. COROLLARY. — Let g be homogeneous of degree 2. Then X is a spray if and only if F is homogeneous of degree 2.

Proof. — We have:

$$\begin{aligned} i_{[V, X]} dd_v g &= L_v i_X dd_v g - i_X L_v dd_v g \\ &= -L_v dg + L_v F - i_X dd_v g \\ &= -2dg + L_v F + dg - F = -dg + L_v F - F. \end{aligned}$$

Furthermore, we also have:

$$i_X dd_v g = - dg + F.$$

Therefore, $[V, X] = X$ if and only if $L_V F = 2F$.

For the definition of force, the following results will be useful.

6. PROPOSITION. — Let $vX = V$ and let $F \in B(TM)$. Then

$$i_X F = jF.$$

Proof. — In fact,

$$i_X v^* = v^* i_{vX} = v^* i_V = jv^*.$$

7. PROPOSITION. — Let $F \in B^1(TM)$ and $jF = 0$.

(a) If $dF = 0$, then $F = 0$.

(b) If $L_V F = kF$ and $k \neq -1$, then

$$F = jF',$$

where F' is the semibasic 2-form, homogeneous of degree $k - 1$, given by

$$F' = \frac{1}{k + 1} d_v F.$$

(c) If there exists a 1-form F' on M , such that $F = p_M^* F'$, then

$$F = 0.$$

Proof. — By (2) we obtain respectively:

(a) $F = F + i_V dF + di_V F = [j, d_v]F = jd_v F = -jdv^* F = -jd0 = 0$;

(b) $jd_v F = [j, d_v]F = F + L_V F = (k + 1)F$;

(c) F is homogeneous of degree 0 and dF is semibasic; hence

$$F = jd_v F = j(i_V dF - di_V F) = j(0 - 0) = 0.$$

We next consider the product of manifolds.

Let M_1, \dots, M_p be p manifolds and let M be their product

$$M = M_1 \times \dots \times M_p.$$

Let Π_1, \dots, Π_p be their projections

$$\Pi_1 : M \rightarrow M_1, \dots, \Pi_p : M \rightarrow M_p.$$

These extend to the projections

$$T\Pi_1 : TM \rightarrow TM_1, \dots, T\Pi_p : TM \rightarrow TM_p$$

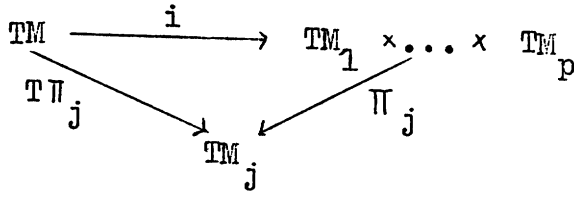
and

$$TT\Pi_1 : TTM \rightarrow TTM_1, \dots, TT\Pi_p : TTM \rightarrow TTM_p.$$

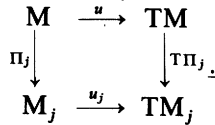
There is a unique isomorphism

$$i : TM \rightarrow TM_1 \times \dots \times TM_p,$$

such that, for each $j = 1, \dots, p$, the following diagram is commutative:



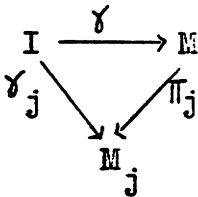
8. PROPOSITION. — If $u_1 \in \mathcal{F}(M_1), \dots, u_p \in \mathcal{F}(M_p)$, then there exists a unique $u \in \mathcal{F}(M)$, such that, for each $j = 1, \dots, p$, the following diagram is commutative:



We denote such a field by $u = \rho(u_1, \dots, u_p)$.
 Locally, we have

$$u = u_1^{i_1} \frac{\partial}{\partial q_1^{i_1}} + \dots + u_p^{i_p} \frac{\partial}{\partial q_p^{i_p}}.$$

9. PROPOSITION. — If $\gamma_1 : I \rightarrow M_1, \dots, \gamma_p : I \rightarrow M_p$ are curves, there is a unique curve $\gamma : I \rightarrow M$, such that, for each $j = 1, \dots, p$, the following diagram is commutative:



We denote such a curve by $\gamma = \rho(\gamma_1, \dots, \gamma_p)$.
 Hence, if $\gamma : I \rightarrow M$ is a curve

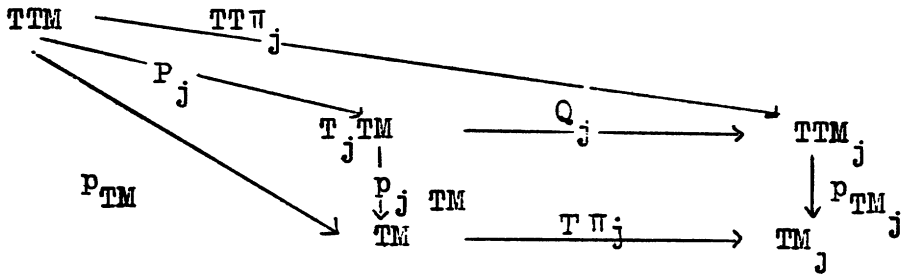
and $\gamma_1 = \Pi_1 \circ \gamma, \dots, \gamma_p = \Pi_p \circ \gamma$,
 then $\gamma = \rho(\gamma_1, \dots, \gamma_p)$.

For $j = 1, \dots, p$, we denote by

$$\tau_j(TM) = (T_j TM, p_{jTM}, TM)$$

the fibre bundle $(T\Pi_j)^*(\tau(TM_j))$ and by $(T\Pi_j, Q_j)$ the homomorphism $\tau_j(TM) \rightarrow \tau(TM_j)$.

Furthermore there exists (see [6]) a map $P_j : TTM \rightarrow T_j TM$, such that the following diagram is commutative:



Notice that Q_j is an isomorphism on fibres. Hence, if $u, v \in T_x TM$, then

$$(T\Pi_j) \circ u = (T\Pi_j) \circ v \Leftrightarrow P_j \circ u = P_j \circ v.$$

We then infer the following proposition.

10. PROPOSITION. — For each $j = 1, \dots, p$, $\tau_j(TM)$ is a subbundle of $\tau(TM)$, the later being equivalent to the direct sum

$$\tau(TM) = \tau_1(TM) \oplus \dots \oplus \tau_p(TM).$$

We can write, therefore

$$\begin{aligned} \mathcal{F}(TM) &= P_1 \mathcal{F}(TM) \oplus \dots \oplus P_p \mathcal{F}(TM), \\ \mathcal{F}^*(TM) &= P_1 \mathcal{F}^*(TM) \oplus \dots \oplus P_p \mathcal{F}^*(TM) \end{aligned}$$

(where, for semplicity, we use a unique notation P_j for the projections).

Now, if V_1, \dots, V_p are the vector fields on TM , given by

$$V_1 = \rho(V, 0, \dots, 0), \dots, V_p = \rho(0, \dots, 0, V),$$

we obtain the antiderivations j_1, \dots, j_p on $B(TM)$, defined by

$$j_1 v^* = v^* i_{V_1}, \dots, j_p v^* = v^* i_{V_p}.$$

The following proposition holds.

11. PROPOSITION. — Let V be the Liouville field on TM . Then

$$V = V_1 + \dots + V_p.$$

Hence, for any $F \in B^1(TM)$, $jF = (j_1 + \dots + j_p)F$ and

$$jP_1 F = j_1 F, \dots, jP_p F = j_p F.$$

These results will be used later in the many-body dynamics.

2. Connection and curvature of a curve.

Let M be a vector space and let $\gamma : I \rightarrow M$ be a curve. Then

$$TM = M \times M$$

and

$$TTM = M \times M \times M \times M.$$

The first and second derivatives of γ are the curves

$$\begin{aligned} D\gamma : I &\rightarrow L(R, M) = M, \\ D^2\gamma : I &\rightarrow L(R, M) = M. \end{aligned}$$

We can also consider the curves $\gamma' : I \rightarrow TM$ and $\gamma'' : I \rightarrow TTM$, given by

$$\begin{aligned} \gamma'(t) &= T\gamma(t, 1) = (\gamma(t), D\gamma(t)) \in M \times M, \\ \gamma''(t) &= T\gamma'(t, 1) = (\gamma(t), D\gamma(t), D^2\gamma(t)) \in M \times M \times M \times M. \end{aligned}$$

Hence, if we know γ' and γ'' , we can obtain canonically $D\gamma$ and $D^2\gamma$, these being respectively the tangent vector and the curvature vector applied along the curve. This is the situation occurring in classical free point Mechanics.

The situation is quite different if $\tau(M)$ is not trivial. In this case we can define γ' and γ'' by means of the tangent functor, but $D\gamma$ and $D^2\gamma$ have no sense. Moreover, $\gamma'(t)$ is an applied vector on M , whereas $\gamma''(t)$ is an applied vector on TM and it is not possible, by means of the differentiable structure alone, to get a reasonable vector on M .

Taking into account the group action on the fibre of $\tau(TM)$ (see [6]), we see that there is a canonical isomorphism

$$vT_y TM \rightarrow T_x M, \quad \forall x \in M, \quad y \in p_M^{-1}(x).$$

But there is not a canonical projection

$$T_y TM \rightarrow vT_y TM,$$

as in the case where M is a vector space.

There is a natural injective homomorphism (see [6]) of the fibre bundle tangent to the fibres of $\tau(M)$, denoted by

$$p_M^*(\tau(M)) = (p_M^*(TM), \pi, TM),$$

into $\tau(TM)$.

Such a fibre bundle may then be considered as subbundle of $\tau(TM)$, by means of a mapping that, locally, is written as follows

$$(q^i, q^i, dq^i) \mapsto (q^i, \dot{q}^i, 0, dq^i).$$

There is also a homomorphism

$$\pi' : p_M^*(TM) \rightarrow TM,$$

such that the following diagram is commutative

$$\begin{array}{ccc} p_M^*(TM) & \xrightarrow{\pi'} & TM \\ \pi \downarrow & & \downarrow p_M \\ TM & \xrightarrow{p_M} & M \end{array}$$

and whose local expression is

$$(q^i, \dot{q}^i, dq^i) \mapsto (q^i, dq^i).$$

Hence, vertical vectors on TM may be viewed as vectors of the fibre bundle $p_M^*(TM)$ and these are sent by π' onto TM . This is precisely the way to obtain the isomorphism between vertical fibres of $\tau(TM)$ and fibres of $\tau(M)$.

Locally we can write:

$$(q^i, \dot{q}^i, dq^i, d\dot{q}^i) \xrightarrow{v} (q^i, \dot{q}^i, 0, dq^i) \xrightarrow{\pi'} (q^i, dq^i).$$

Hence the problem of extending the definition of the curvature of γ , to the

case where M is a manifold, requires the assignment of such a projection, that is, it requires a new structure. We want to show, now, that this additional structure is actually a connection.

A (torsion free) connection on M is given (see [4]) by a pairing

$$\mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathcal{F}(M),$$

denoted by

$$(v, u) \mapsto \nabla_v u,$$

such that, $\forall v, v', u, u' \in \mathcal{F}(M), f \in \mathcal{D}(M)$,

C.1
$$\nabla_{v+v'} u = \nabla_v u + \nabla_{v'} u,$$

C.2
$$\nabla_{fv} u = f \nabla_v u,$$

C.3
$$\nabla_v(u + u') = \nabla_v u + \nabla_v u',$$

C.4
$$\nabla_v(fu) = f \nabla_v u + (v \cdot f)u,$$

C.5 the operator ∇_v is a local operator,

C.6
$$\nabla_u v - \nabla_v u = [u, v].$$

Any $u \in \mathcal{F}(M)$ defines, by means of the tangent functor T and the canonical involution s of TTM (see [6]), a natural vector field $r(u)$ on TM ; we set

$$r(u) = sTu.$$

Locally we have:

$$r(u) = u^i \frac{\partial}{\partial q^i} + \frac{\partial u^i}{\partial q^j} q^j \frac{\partial}{\partial q^i}.$$

It is then reasonable to try to obtain a connection ∇ by using the tangent functor T and therefore the lifting r . This will lead us, according to the discussion above, to the decomposition of each fibre of $\tau(\text{TM})$ in the direct sum of the vertical subspace and of a fixed complementary subspace. Before stating the next theorem, we need the notion of connection tensor.

1. DÉFINITION. — A « connection tensor » on TM is a vector bundle endomorphism $\Gamma : \text{TTM} \rightarrow \text{TTM}$, such that

C.T.1
$$\Gamma(\text{TTM}) \subset v(\text{TTM}),$$

C.T.2
$$\Gamma \circ v = v,$$

C.T.3
$$\Gamma \circ s = \Gamma.$$

Connection tensors are characterized by the local expression

$$\Gamma = (\Gamma_{ij}^k \dot{q}^i dq^j + d\dot{q}^k) \otimes \frac{\partial}{\partial \dot{q}^k}, \quad (\partial \Gamma_{ij}^k / \partial \dot{q}^h) = 0,$$

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

2. THEOREM. — (a) Let $\Gamma : \text{TTM} \rightarrow \text{TTM}$ be a vector bundle morphism. Then, the map

$$c : (u, v) \mapsto \pi' \circ \Gamma \circ (ru) \circ v, \quad \forall u, v \in \mathcal{F}(M),$$

is a connection on M if and only if Γ is a connection tensor.

(b) Furthermore, there is a unique bijection between connections ∇ and connection tensors Γ , characterized by

$$\nabla_v u = c(u, v), \quad \forall u, v \in \mathcal{F}(M).$$

Proof. — (a) If c gives a connection, we see that Γ is a connection tensor by its local expression, taking into account C.5. If Γ is a connection tensor, then one proves, locally, that c gives a connection and, by a partition of unity on M , we get a global connection on M , taking into account C.5.

(b) The bijection is obtained, locally, by means of the $(\Gamma_{ij}^k)_\perp$.

We are now in the position of stating the definition of the curvature of a curve. In fact, $\Gamma \circ \gamma''$ is a curve

$$\Gamma \circ \gamma'' : I \rightarrow \text{TTM}$$

and satisfies

$$\Gamma \circ \gamma''(I) \in v\text{TTM},$$

hence $\pi' \circ \Gamma \circ \gamma''$ is a curve

$$\pi' \circ \Gamma \circ \gamma'' : I \rightarrow \text{TM}$$

and this settles our problem. But, for our purpose, it is immaterial to use explicitly the projection π' . In fact, it suffices to know that $\Gamma \circ \gamma''$ consists of vectors which are projectable on M , this property being essential. It is therefore natural to give the following definition.

3. DÉFINITION. — Let $\gamma : I \rightarrow M$ be a curve and let Γ be a connection tensor on TM . We call « curvature » of γ the curve

$$\bar{a}_\gamma : I \rightarrow \text{TTM},$$

given by

$$\bar{a}_\gamma = \Gamma \circ \gamma''_\perp$$

Locally we have:

$$\bar{a}_{\gamma}(t) = (\gamma^i(t), D\gamma^i(t), 0, D^2\gamma^i(t) + (\Gamma_{jk}^i \circ \gamma')(t)D\gamma^j(t)D\gamma^k(t)).$$

Observe that

$$\pi' a_\gamma(t) = (\gamma^i(t), D^2\gamma^i(t) + (\Gamma_{jk}^i \circ \gamma')(t)D\gamma^j(t)D\gamma^k(t))$$

hence the usual acceleration $D^2\gamma^i$ is « compensated » by the term

$$(\Gamma_{jk}^i \circ \gamma')(t)D\gamma^j(t)D\gamma^k(t)$$

of the connection, in order to make it intrinsic.

Observe also that if we had not assumed a torsion free connection, then its antisymmetric part would give no contribution to acceleration.

In order to express the law of motion by an equation, we shall need the following proposition.

4. PROPOSITION. — The curve γ is a solution of the s. o. d. X ($X \circ \gamma' = \gamma''$) if and only if

$$\Gamma \circ X \circ \gamma' = \Gamma \circ \gamma''.$$

Proof. — (a) If $X \circ \gamma' = \gamma''$, then $\Gamma \circ X \circ \gamma' = \Gamma \circ \gamma''$.

(b) Let us assume that $\Gamma \circ X \circ \gamma' = \Gamma \circ \gamma''$, where the local expression of X is

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + b^j \frac{\partial}{\partial \dot{q}^j}.$$

Then, locally

$$\Gamma \circ X \circ \gamma'(t) = ((\Gamma_{jk}^i \circ \gamma')(t) D\gamma^j(t) D\gamma^k(t) + (b^i \circ \gamma')(t)) \frac{\partial}{\partial \dot{q}^i},$$

$$\Gamma \circ \gamma''(t) = ((\Gamma_{jk}^i \circ \gamma')(t) D\gamma^j(t) D\gamma^k(t) + D^2\gamma^i(t)) \frac{\partial}{\partial \dot{q}^i}.$$

Hence $b^j \circ \gamma' = D^2\gamma^j$ and so $X \circ \gamma' = \gamma''$.

Let us next introduce the notion of product connection.

5. PROPOSITION. — Let $\nabla_1, \dots, \nabla_p$ be connections on M_1, \dots, M_p , respectively. There is a unique connection ∇ on $M = M_1 \times \dots \times M_p$, such that

$$\nabla_v u = \rho(\nabla_{1v_1} u_1, \dots, \nabla_{pv_p} u_p),$$

for any $u_1, v_1 \in \mathcal{F}(M_1), \dots, u_p, v_p \in \mathcal{F}(M_p)$, with

$$u = \rho(u_1, \dots, u_p),$$

$$v = \rho(v_1, \dots, v_p).$$

Proof. — This connection is given locally by

$$\Gamma = (\Gamma_{1i,j_1}^{k_1} \dot{q}_1^{i_1} dq_1^{j_1} + d\dot{q}_1^{k_1}) \otimes \frac{\partial}{\partial \dot{q}_1^{k_1}} + \dots + (\Gamma_{pi,j_p}^{k_p} \dot{q}_p^{i_p} dq_p^{j_p} + d\dot{q}_p^{k_p}) \otimes \frac{\partial}{\partial \dot{q}_p^{k_p}}.$$

6. DÉFINITION. — We call such a connection ∇ the « product » of $\nabla_1, \dots, \nabla_p$.

7. COROLLARY. — Let $\gamma_1 : I \rightarrow M_1, \dots, \gamma_p : I \rightarrow M_p$ be curves. Then, for each $j = 1, \dots, p$,

$$T\Pi_j \circ \bar{a}_\gamma = \bar{a}_{\gamma_j}$$

and

$$\rho(a_{\gamma_1}, \dots, a_{\gamma_p}) = \bar{a}_\gamma.$$

These results will be used in the many-body dynamics.

3. Riemannian structure.

We need several facts concerning Riemannian structures, that shall be stated in the following theorem.

We denote by $n(M)$ the null section of $\tau(M)$ and by $B_1^*(TM)$ the module of semi-basic 1-forms $g : TM \rightarrow T^*TM$, which are linear on fibres of $\tau(M)$.

1. THEOREM. — Let us consider the following objects:

- (a) g a non-degenerate symmetric tensor of type (0, 2) on M ;
 (b) $g : TM \rightarrow \mathbb{R}$ a function, quadratic on fibres and such that $d_v g = 0$ only on $n(M)$;
 (c) $\underline{g} \in B_1^1(TM)$ a form such that $d_v \underline{g} = 0$ and $d\underline{g}$ is a symplectic form;
 (d) $\hat{g} : \tau(M) \rightarrow \tau^*(M)$ a symmetric isomorphism on M .

Then the following correspondences determine bijections between such objects.

(a \rightarrow b) We put $g(u) = (1/2)\underline{g}_x(u, u)$, $\forall x \in M$, $\forall u \in T_x M$.

(b \rightarrow a) We put $\underline{g}_x(u, v) = g(u+v) - g(u) - g(v)$, $\forall x \in M$, $\forall u, v \in T_x M$.

(b \rightarrow c) We put $\underline{g} = d_v g$.

(c \rightarrow b) We put $g = (1/2)j\underline{g}$.

(c \rightarrow d) We call \hat{g} the unique isomorphism $TM \rightarrow T^*M$, such that

$$q_M \circ \hat{g} = p_M, \quad \hat{g}^* \lambda = g.$$

(d \rightarrow c) We put $\underline{g} = \hat{g}^* \lambda$.

(d \rightarrow a) We put $\underline{g}_x(u) = \hat{g}(u) \in T_x^* M$, $\forall x \in M$, $\forall u \in T_x M$.

(a \rightarrow d) We call \hat{g} the unique isomorphism $\tau(M) \rightarrow \tau^*(M)$, such that

$$q_M \circ \hat{g} = p_M, \\ \hat{g}(u) = \underline{g}_x(u) \in T_x^* M, \quad \forall x \in M, \quad \forall u \in T_x M.$$

Proof. — Differentiability can be proved locally.

Then non-degeneracy of \underline{g} is equivalent to the condition:

$$d_v g = 0 \quad \text{only on } n(M).$$

Furthermore, non-degeneracy of \underline{g} is equivalent to the condition: $d\underline{g}$ is a symplectic form.

$d_v \underline{g} = 0$ is the condition of integrability of \underline{g} :

$$\exists g \in \mathcal{D}(TM), \quad \text{such that } \underline{g} = d_v g.$$

Furthermore $d_v \underline{g} = 0$ is equivalent to the symmetry of \underline{g} .

The local expressions of \underline{g} , g , \underline{g} , \hat{g} , $d\underline{g}$, $(d\underline{g})^n$ are:

$$\underline{g} = g_{ij} dx^i \otimes dx^j,$$

$$g = (1/2)g_{ij} \dot{q}^i \dot{q}^j,$$

$$\underline{g} = g_{ij} \dot{q}^i dq^j,$$

$$p_j \circ \hat{g} = g_{ij} \dot{q}^i,$$

$$d\underline{g} = \frac{\partial g_{kj}}{\partial q^i} \dot{q}^k dq^i \wedge dq^j + g_{ij} d\dot{q}^i \wedge dq^j,$$

$$(d\underline{g})^n = (-1)^{n(n-1)/2} n! \det(g_{ij}) d\dot{q}^1 \wedge \dots \wedge d\dot{q}^n \wedge dq^1 \wedge \dots \wedge dq^n.$$

We call « Riemannian structure » on M any one of the preceding objects,

which are called, respectively, « metric » tensor, function, form and isomorphism.

Let us notice that the symplectic form $dd_v g$ gives an isomorphism

$$\omega : v\mathcal{T}(\text{TM}) \rightarrow \text{B}^1(\text{TM}),$$

such that the following diagram is commutative:

$$\begin{array}{ccc} v\mathcal{T}(\text{TM}) & \xrightarrow{\omega} & \text{B}^1(\text{TM}) \\ \uparrow v \circ r & & \uparrow p^*_M \\ \mathcal{T}(\text{M}) & \xrightarrow{g} & \Lambda^1(\text{M}) \end{array}$$

Locally we have

$$\omega\left(b^j \frac{\partial}{\partial \dot{q}^j}\right) = g_{ij} b^j dq^i.$$

We will use the following notations:

$$\omega(\bar{\text{F}}) = \underline{\text{F}}, \quad \omega^{-1}(\underline{\text{F}}) = \bar{\text{F}}.$$

Now, let ∇ be the Riemannian connection on M (see [4]) and let $\gamma : \text{I} \rightarrow \text{M}$ be a curve. We can define the « co-curvature » of γ as

$$\underline{a}_\gamma = \omega(\bar{a}_\gamma).$$

Locally we have:

$$\underline{a}_\gamma(t) = ((g_{ij} \circ \gamma')(t) D\gamma^j(t) + (\Gamma_{i,jk} \circ \gamma')(t) D\gamma^j(t) D\gamma^k(t)) dq^i \in \text{T}^*_{\gamma'(t)}(\text{TM}).$$

Let X be a s. o. d. e. and γ a curve on M ; recalling proposition (2.4), we infer that

$$\gamma \text{ is a solution of } \text{X} \text{ if and only if } \underline{a}_\gamma = \omega \Gamma \circ \text{X} \circ \gamma'.$$

This, together with the following result is essential in dynamics.

2. LEMMA. — Let X be a s. o. d. e. Then

$$\omega(\Gamma \circ \text{X}) = i_{\text{X}} dd_v g + dg.$$

Proof. — If the local expression of X is $\text{X} = \dot{q}^i \frac{\partial}{\partial q^i} + b^j \frac{\partial}{\partial \dot{q}^j}$, then the local expression of $i_{\text{X}} dd_v g + dg$ is

$$i_{\text{X}} dd_v g + dg = (g_{ij} b^j + \Gamma_{i,jk} \dot{q}^j \dot{q}^k) dq^i.$$

3. THEOREM. — Let $\gamma : \text{I} \rightarrow \text{M}$ be a curve, $\underline{\text{F}} \in \text{B}^1(\text{TM})$ and X the s. o. d. e. given by

$$i_{\text{X}} dd_v g = -dg + \underline{\text{F}}.$$

Then the following conditions are equivalent:

- (a) $\bar{a}_\gamma = \bar{\text{F}} \circ \gamma'.$
- (a') $\underline{a}_\gamma = \underline{\text{F}} \circ \gamma'.$
- (b) $\text{X} \circ \gamma' = \gamma''$

(That is, γ is a solution of X).

Proof. — Recalling (2.4), we have:

$$\begin{aligned} \underline{a}_\gamma &= \underline{F} \circ \gamma' \Leftrightarrow \omega \Gamma \circ \gamma'' = (i_X dd_v g + dg) \circ \gamma' \Leftrightarrow \omega \Gamma \circ \gamma'' = \omega \Gamma \circ X \circ \gamma' \\ &\Leftrightarrow \Gamma \circ \gamma'' = \Gamma \circ X \circ \gamma' \Leftrightarrow \gamma'' = X \circ \gamma' \end{aligned}$$

Hence, γ satisfies condition (a) if and only if it is locally a solution of Lagrange equations

$$\frac{d}{dt} \frac{\partial g}{\partial \dot{q}^i} - \frac{\partial g}{\partial q^i} = F_i.$$

4. COROLLARY. — Let γ satisfy condition (a). Then

$$D(g \circ \gamma') = 0$$

if and only if

$$j\underline{F} = 0.$$

Proof. — We have by proposition (1.6)

$$0 = D(g \circ \gamma') = X. g = i_X dg = -i_X i_X dd_v g + i_X \underline{F} = j\underline{F}.$$

Let us now introduce the notion of product of Riemannian structures.

5. PROPOSITION. — Let $(M_1, g_1), \dots, (M_p, g_p)$ be Riemannian structures, let $M = M_1 \times \dots \times M_p$ and let g be the function

$$g = (T\Pi_1)^* g_1 + \dots + (T\Pi_p)^* g_p.$$

Then (M, g) is a Riemannian structure.

6. DÉFINITION. — We call such a Riemannian structure (M, g) the « product » of $(M_1, g_1), \dots, (M_p, g_p)$.

We can prove the following proposition, by means of local charts.

7. PROPOSITION. — (a) If $u_1 \in \mathcal{T}(M_1), \dots, u_p \in \mathcal{T}(M_p)$, then

$$g(\rho(u_1, \dots, u_p)) = \Pi_1^* g_1(u_1) + \dots + \Pi_p^* g_p(u_p).$$

(b) The Riemannian connection of M is the product connection.

(c) Let $\gamma_1 : I \rightarrow M_1, \dots, \gamma_p : I \rightarrow M_p$ be curves and let be

$$\gamma = \rho(\gamma_1, \dots, \gamma_p).$$

Then

$$\begin{aligned} g \circ \gamma &= g_1 \circ \gamma_1 + \dots + g_p \circ \gamma_p, \\ \underline{a}_\gamma(t) &= (T\Pi_1)_{\gamma_1'(t)}^* \underline{a}_{\gamma_1}(t) + \dots + (T\Pi_p)_{\gamma_p'(t)}^* \underline{a}_{\gamma_p}(t), \quad \forall t \in I. \end{aligned}$$

We can extend some of the preceding results to a product situation.

8. THEOREM. — Let $\gamma : I \rightarrow M$ be a curve, $\underline{F} \in B^1(TM)$ and X the s. o. d. e. given by

$$i_X dd_v g = -dg + \underline{F}.$$

Then the following conditions are equivalent:

(a) $\bar{a}_\gamma = \bar{F} \circ \gamma'.$

- (b) $T\Pi_j \bar{a}_\gamma = T\Pi_j \bar{F} \circ \gamma', \quad \text{for } j = 1, \dots, p.$
- (b') $P_j \circ \bar{a}_\gamma = P_j \circ \bar{F} \circ \gamma', \quad \text{for } j = 1, \dots, p.$
- (a') $\underline{a}_\gamma = \underline{F} \circ \gamma'.$
- (c) $X \circ \gamma' = \gamma''.$

Notice that, if X is a s. o. d. e., the following conditions are equivalent:

$$i_X dd_v g = - dg + \underline{F};$$

$$i_X dd_v (T\Pi_j)^* g_j = - d(T\Pi_j)^* g_j + P_j \circ \underline{F}, \quad \text{for } j = 1, \dots, p.$$

But any single of the preceding p relations does not determine X, since $dd_v (T\Pi_j)^* g_j$ is not a symplectic form.

Furthermore we can not get a reasonable form on $T(M_j)$ by means of every semibasic 1-form \underline{F} on TM and therefore we can not get Lagrange equations on TM_j .

9. COROLLARY. — Let γ satisfy condition (a). Then, for $i = 1, \dots, p$

if and only if

$$D(g_i \circ (T\Pi_i) \circ \gamma) = 0$$

$$j_i \underline{F} = 0.$$

These results will be essential for the many-body dynamics.

4. Signature of Riemannian structures.

The preceding results were stated regardless of the signature of the metric. We shall now give some statements concerning the signature, in order to introduce the notion of space-time.

1. PROPOSITION. — Let (M, g) be a Riemannian structure. There exists an open covering $(U_i)_{i \in I}$ of M, such that, for each $i \in I, U_i$ is parallelizable by an orthonormal basis of $\mathcal{F}(U_i)$.

Proof. — By non-degeneracy and continuity of g , we know that, for each $x \in M$, there exist a parallelizable neighborhood U_x of x and a vector field v_1 on U_x , such that $g \circ v_1$ has constant sign (and $g \circ v_1 \neq 0$).

If

$$V_1 = (fv_1; f \in \mathcal{D}(U_x))$$

and

$$V_1^\perp = (v \in \mathcal{F}(U_x); \underline{g}(v_1, v) = 0),$$

then

$$\mathcal{F}(U_x) = V_1 \oplus V_1^\perp.$$

In fact,

$$v = fv_1 + (v - fv_1), \quad \forall v \in \mathcal{F}(U_x),$$

where

$$f = \underline{g}(v, v_1) / \underline{g}(v_1, v_1).$$

We can then get the result by induction, restricting, step by step, if necessary, the neighborhood U_x , in order to obtain new vector fields v_k such that $g \circ v_k$ has constant sign.

2. COROLLARY. — If M is connected, then g_x has, for each $x \in M$, constant index of positivity (and hence of negativity) (see [1]).

Proof. — Let $i \in I$. Then g_x has constant index of positivity, for $x \in U_i$.

Let $I_q = \{i \in I; \text{index of positivity of } g/U_i \text{ is } q\}$, $q = 0, 1, \dots, n$, and set $A_q = \bigcup_{i \in I_q} U_i$. Then $M = \bigcup_{q=0} A_q$ and, since M is connected (see [2]), for each $q = 0, 1, \dots, n$, A_q is either empty or has a point in common with some other $A_{q'}$. But the index of positivity of g_x cannot be both q and q' ;

hence I_q or $I_{q'}$ is empty. It follows that all the I_0, \dots, I_n are empty except one.

This corollary implies that, for each $x, x' \in M$, there exists an isomorphism

$$h : T_x M \rightarrow T_{x'} M,$$

such that

$$g_x = g_{x'} \circ (h \otimes h).$$

Therefore, algebraically speaking, we are dealing with a unique bilinear form $g \in R^{n*} \otimes R^{n*}$. Let H be the subgroup of $GL(n, R)$ composed of the automorphisms of g (see [1]). The covering $(U_i)_{i \in I}$ gives a reduction to H of the principal bundle of frames of M (see [4]).

Let (M, g) be a Riemannian structure. We define, for any $k \in R$,

$$C^k M = g^{-1}(k) \subset TM,$$

$$C^\pm M = g^{-1}(R^\pm) \subset TM$$

and, similarly, for $x \in M$,

$$C^k_x = C^k M \cap T_x M,$$

$$C^\pm_x = C^\pm M \cap T_x M.$$

We call g « hyperbolic normal » if the index of positivity of g is 1.

A vector $u \in TM$ is called « time-like », « space-like » or « light-like », according whether $u \in C^+ M$, $u \in C^- M$ or $u \in C^0 M$.

In order to give the notion of phase-space, we study $C^+ M$. Moreover, we assume for the remainder of this section, that (M, g) is a hyperbolic normal structure.

3. PROPOSITION. — Let $x \in M$. Then $C^+ x$ has two connected components $C^+_1 x$ and $C^+_2 x$. Moreover there is a diffeomorphism

$$C^+ x \rightarrow R^n - R^{n-1} = (R^+ \cup R^-) \times R^{n-1}.$$

Proof. — There exists an orthonormal basis e_0, e_1, \dots, e_{n-1} , such that $e_0^2 = 1, e_1^2 = \dots = e_{n-1}^2 = -1$; hence $v = v^i e_i \neq 0$ belongs to $C^+ x$ if and only if $|v^0| > ((v^1)^2 + \dots + (v^{n-1})^2)^{1/2}$ and therefore $C^+ x$ is open in $T_x M$. Furthermore, the mapping defined, for such n -tuples, by

$$(v^0, v^1, \dots, v^{n-1}) \mapsto (v^0, v^1/(v^0 - r), \dots, v^{n-1}/(v^0 - r)),$$

where

$$r = ((v^1)^2 + \dots + (v^{n-1})^2)^{1/2},$$

gives the desired diffeomorphism.

In the same way, we can prove the following proposition.

4. PROPOSITION. — Let $U \subset M$ be connected and parallelizable. The C^+U has two connected components

$$C_1^+U = \bigcup_{x \in U} C_1^+x,$$

$$C_2^+U = \bigcup_{x \in U} C_2^+x.$$

Moreover there is a diffeomorphism

$$C^+U \rightarrow U \times (\mathbb{R}^n - \mathbb{R}^{n-1}) = U((\mathbb{R}^+ \cup \mathbb{R}^-) \times \mathbb{R}^{n-1}).$$

In general it is not possible to extend the above situation to any connected open subset U and hence to a connected manifold M (see [5]). Therefore, if we want to introduce the notion of phase-space, we shall need a further hypothesis.

5. The phase-space.

We assume that M is connected and that g is a hyperbolic normal metric.

1. THEOREM. — Let us suppose that C^+M has two connected components C_1^+M and C_2^+M . Let $(U_i)_{i \in I}$ be an open covering of M , such that each U_i is parallelizable by an orthonormal basis. Then:

(a) for any $i \in I$, each C_1^+M and C_2^+M is contained in one of the two sets $C_1^+U_i, C_2^+U_i$;

(b) if $C_1^+U_i \subset C_1^+M$, then $C_2^+U_i \subset C_2^+M$ and *vice versa*.

Proof. — (a) Let $i \in I$. Then $C_1^+U_i$ has common points at least with one of the two sets C_1^+M, C_2^+M . However, if $C_1^+U_i$ had common points with both, then the set

$$C^+M = C_1^+M \cup C_2^+M \cup C_1^+U_i$$

would be connected (see [2]), since C_1^+M, C_2^+M and $C_1^+U_i$ are connected.

The same argument holds for $C_2^+U_i$.

(b) Let (I_1, I_2, I_0) be the partition of I , given by

$$I_1 = \{i \in I; C_1^+U_i, C_2^+U_i \subset C_1^+M\},$$

$$I_2 = \{i \in I; C_1^+U_i, C_2^+U_i \subset C_2^+M\},$$

$$I_0 = \{i \in I; C_1^+U_i \subset C_1^+M \text{ and } C_2^+U_i \subset C_2^+M, \text{ or } \textit{vice versa}\}.$$

We want to prove that $I_1 = I_2 = \emptyset$. Let

$$V_m = \bigcup_{i \in I_m} U_i, \quad \text{for } m = 1, 2, 0.$$

If $V_1 \cap V_0 \neq \emptyset$, there exists $x \in M$ and $i_1 \in I_1, i_0 \in I_0$, such that

$$x \in U_{i_1}, \quad x \in U_{i_0}.$$

Then, since $x \in U_{i_1}$, we have

$$C_1^+ x, \quad C_2^+ x \subset C_1^+ M.$$

But, we also have

$$C_1^+ x \subset C_1^+ U_{i_0}, \quad C_2^+ x \subset C_2^+ U_{i_0}$$

hence both $C_1^+ U_{i_0}, C_2^+ U_{i_0}$ have common points with $C_1^+ M$. Therefore, by the same argument as in (a), we obtain

$$C_1^+ U_{i_0}, \quad C_2^+ U_{i_0} \subset C_1^+ M.$$

But this is not possible, by our assumption on I_0 .

It follows that $V_1 \cap V_0 = \emptyset$ and in the same way we can prove that $V_2 \cap V_0 = \emptyset$.

Furthermore, $V_1 \subset C_1^+ M$ and $V_2 \subset C_2^+ M$ hence $V_1 \cap V_2 = \emptyset$. But the set $M = V_1 \cup V_2 \cup V_0$ is connected and therefore $V_1 = V_2 = \emptyset$. This proves that $I_1 = I_2 = \emptyset$.

2. COROLLARY. — The mapping

$$f : u \mapsto -u, \quad \forall x \in M, \quad \forall u \in T_x M,$$

gives a natural diffeomorphism

$$C_1^+ M \rightarrow C_2^+ M.$$

Proof. — By means of the covering $(U_i)_{i \in I}$, we see that f is differentiable and that, if $u \in C_1^+ M$, then $f(u) \in C_2^+ M$ and *vice versa*. Furthermore

$$\begin{aligned} f/C_2^+ M \circ f/C_1^+ M &= id_{C_1^+ M}, \\ f/C_1^+ M \circ f/C_2^+ M &= id_{C_2^+ M}. \end{aligned}$$

Now we are in the position of giving the following definition.

3. DÉFINITION. — Let M be connected and g a hyperbolic normal metric. We say that M is « time-orientable » if $C^+ M$ has two connected components. One of these two equivalent components is called the « future » component and the other the « past » component. A choice of the future component is a « time-orientation » of M .

If M is time-oriented, the future component is also called the « phase-space » PM .

Using the isomorphism $\hat{g} : TM \rightarrow T^*M$, we can transpose all these considerations to T^*M .

We thus get the « cophase-space » P^*M , defined by

$$P^*M = \hat{g}(PM).$$

PM is not a vector bundle but it is open in TM . Hence, we can extend to PM the calculus of TM , given by the local operators d, i_v, d_v, j, \dots .

In particular, the results of preceding sections give valid restrictions to PM. This will be assumed in the sequel, without any further reference.

Let us conclude by examining the phase-space of a product. Let $(M_1, g_1), \dots, (M_p, g_p)$ be hyperbolic normal structures and let (M, g) be their product. Then g is not hyperbolic normal. Furthermore, the set of all vectors $v \in TM$, such that

$$(\Pi_i)v \in C^+M_i, \quad \text{for all } i = 1, \dots, p,$$

is properly contained in C^+M .

We shall define the phase-space of M in a way that involves essentially the projections Π_1, \dots, Π_p . Hence PM will depend on the product structure and not solely on (M, g) .

4. DÉFINITION. — The « phase-space » of the product $M = M_1 \times \dots \times M_p$ is the set of all vectors $v \in TM$, such that

$$(\Pi_i)v \in PM_i, \quad \text{for all } i = 1, \dots, p.$$

Recalling the natural diffeomorphism (see section (1))

$$TM \rightarrow TM_1 \times \dots \times TM_p,$$

we see that there is a natural diffeomorphism

$$PM \rightarrow PM_1 \times \dots \times PM_p.$$

Hence PM is open in TM and it is connected.

II. DYNAMICS

1. Forces.

Let (M, g) be a hyperbolic normal structure and $F \in B^1(PM)$. For each $x \in M$ and $u \in P_xM$, $F(u)$ may be considered as a vector of T_xM , which, in general, is not orthogonal to u . The orthogonality condition holds if and only if $jF = 0$ and will be referred to as « motion orthogonality ». In this case, F will be said « power vanishing ». The metric g defines a natural projection of the forms of $B^1(PM)$ into their motion orthogonal components as shown in the following proposition.

1. PROPOSITION. — Let $F \in B^1(PM)$. Then the semibasic 1-form

$$F^\perp = F - d_v g \frac{jF}{2g}$$

is power vanishing.

Proof. — g being homogeneous of degree 2, we have:

$$2g = L_v g = [j, d_v]g = j d_v g.$$

Hence

$$j\mathbb{F}^\perp = j\mathbb{F} - 2g \frac{j\mathbb{F}}{2g} = 0.$$

The following results are useful in the physical interpretation of the theory.

2. PROPOSITION. — If $\mathbb{F} \in B^1(\text{PM})$ is homogeneous of degree k , then \mathbb{F} is determined by its value on $C^{1/2}\text{M} \cap \text{PM}$.

Proof. — We have (see [6]):

$$h_t^* \mathbb{F} = e^{kt} \mathbb{F}, \quad \forall t \in \mathbb{R}.$$

If $u \in \text{PM}$, we can write $u = e^c v$, where $c = (-1/2) \log(2g(u))$ and where $v \in C^{1/2}\text{M} \cap \text{PM}$.

If X is a vector field on PM , then, since $h_c^* \circ h_{-c}^* = id$, we have:

$$\langle \mathbb{F}, X \rangle(u) = e^{kc} \langle h_{-c}^* \mathbb{F}, X \rangle(e^c v) = e^{kc} \langle \mathbb{F}, \text{Th}_{-c} X \rangle(e^c v).$$

Furthermore $(\text{Th}_{-c} X)(e^c v) \in T_v \text{TM}$, being $h_{-c}(e^c v) = v$. Hence $\mathbb{F}(u)$ is determined by $\mathbb{F}(v)$.

3. COROLLARY. — Let $k \neq k'$. If $\mathbb{F}' \in B^1(\text{PM})$ is homogeneous of degree k' , there exists a unique form \mathbb{F} on PM , homogeneous of degree k , such that $\mathbb{F} = \mathbb{F}'$ on $C^{1/2}\text{M} \cap \text{PM}$. Namely, we have:

$$\mathbb{F} = (2g)^{(k-k')/2} \mathbb{F}'.$$

Proof. — The form \mathbb{F} is homogeneous of degree k . Uniqueness is ensured by the preceding proposition.

We can now introduce the notion of force.

4. DÉFINITION. — A « (relativistic) force » is a form $\mathbb{F} \in B^1(\text{PM})$, which is power vanishing and homogeneous of degree 2.

The meaning of these two conditions will be clarified by the results of next section. However, both depend on the fact that, during a relativistic motion, g is constant (see also (I.3.4)). The second condition (\mathbb{F} homogeneous of degree 2) is not strictly requested, but it seems physically unburdensome and geometrically advantageous.

5. EXAMPLE. — Let F be a 2-form on M , which represents, physically speaking, the electromagnetic field. The « Lorentz form » is the semi-basic 1-form on PM , homogeneous of degree 1, given by

$$\mathbb{F} = j p_M^*(F).$$

Then, the « Lorentz force » is the semibasic 1-form, homogeneous of degree 2, given by

$$\mathbb{F}' = (2g)^{1/2} j p_M^*(F).$$

Notice that, if $\gamma': I \rightarrow \text{PM}$ is a curve such that $g \circ \gamma' = 1/2$, then

$$\mathbb{F}' \circ \gamma' = \mathbb{F} \circ \gamma'.$$

If we abandon the second condition on forces, then we can take, as Lorentz force, the Lorentz form.

Let us now extend the notion of force to a product.

6. PROPOSITION. — If $\underline{F}_1, \dots, \underline{F}_p$ are forces on PM_1, \dots, PM_p , then

$$\underline{F} = (\Pi_1) * \underline{F}_1 + \dots + (\Pi_p) * \underline{F}_p$$

is such that

$$j\underline{F} = 0$$

and

$$L_v \underline{F} = 2\underline{F}_\perp$$

From this fact and in view of results to be obtained in the last section, we extend to PM the definition (4) of force.

7. DÉFINITION. — A « (relativistic) force » is a form $\underline{F} \in B^1(PM)$, which is homogeneous of degree 2 and such that $\omega(P_i \circ \bar{F})$ is power vanishing, for each $i = 1, \dots, p$.

We say also that the force \underline{F} is « without interaction » if \underline{F} admits a decomposition

$$\underline{F} = (\Pi_1) * \underline{F}_1 + \dots + (\Pi_p) * \underline{F}_p,$$

where $\underline{F}_1, \dots, \underline{F}_p$ are forces on PM_1, \dots, PM_p .

Notice that proposition (2) and corollary (3) can be extended naturally to a product situation.

2. One-body dynamics.

In this section we suppose that (M, g^0) is a time-oriented hyperbolic normal structure. It is, physically speaking, the « space-time ». In the context of the present discussion, the hypothesis that $\dim M = 4$ is not relevant.

Let m be a positive number, which, physically speaking, is the « (rest) mass » of a particle. Then we call « motion space-time » of the particle with mass m , the new structure (M, g) , where

$$g = mg^0.$$

This procedure, which will eliminate the mass in the law of motion, is not important for one-body dynamics, but its generalization to many-body dynamics will be essential. In fact we can prove, by local charts, the following proposition.

1. PROPOSITION. — Let $\gamma : I \rightarrow M$ be a curve. Let \bar{a}_γ^0 and \bar{a}_γ be the curvatures with respect to g^0 and to g and ω^0 and ω be the symplectic isomorphisms induced by g^0 and g . Then

$$\begin{aligned} \bar{a}_\gamma &= \bar{a}_\gamma^0, \\ \underline{a}_\gamma &= \omega \circ \bar{a}_\gamma = m\underline{a}_\gamma^0 = m\omega^0 \circ \bar{a}_\gamma^0. \end{aligned}$$

Henceforth, in this section, we will refer to the structure (M, g) . The mass of the particle will be thus incorporated in g .

We state now the definition of motion.

2. DÉFINITION. — A « motion » on M is a curve $\gamma : I \rightarrow M$, such that

$$\gamma'(I) \subset PM$$

and such that

$$D(g \circ \gamma') = 0.$$

If $2g \circ \gamma' = m$, we say that γ is « normalized ».

Physically speaking, if the motion is normalized, then the parameter t is the « standard proper time » of the motion (see [8]). If the motion is not normalized, then it may be interpreted as a change of the unity of measure for the time.

We can normalize a motion $\gamma : I \rightarrow M$, considering the new normalized motion

$$\tilde{\gamma} = \gamma \circ (1/k) : kI \rightarrow M,$$

where

$$k = \left(\frac{2g}{m} \circ \gamma'(t) \right)^{1/2} \in \mathbb{R}^+, \quad \text{with } t \in I.$$

Only normalized motions are physically relevant, but it seems to be advantageous to allow a change of unity of measure, in order to get a s. o. d. e. from the law of motion. This situation is clarified by the following theorem.

3. THEOREM. — Let X be a s. o. d. e. on PM , such that $X.g = 0$. Then, the following conditions are equivalent:

(a) X is a spray;

(b) the motion γ is a solution of X if and only if its normalization $\tilde{\gamma}$ is a solution of X .

Proof (see [3] and [6]). — Let (U, f) be a maximal local group of diffeomorphisms of PM generated by X . Then the following conditions are equivalent:

(a) X is a spray;

(c) $(st, v) \in U$ if and only if $(t, sv) \in U$ and, in this case,

$$p_M f_{sv}(t) = p_M f_v(st).$$

But conditions (b) and (c) are equivalent. In fact, if the motion γ is a solution of X , then, for each $t \in I$,

$$(t, \gamma'(0)) \in U$$

and

$$\gamma(t) = p_M f_{\gamma'(0)}(t).$$

Furthermore

$$\gamma'(0) = \left(\frac{2g}{m} \circ \gamma'(0) \right)^{1/2} \tilde{\gamma}'(0).$$

Hence we get the equivalence by putting

$$v = \tilde{\gamma}'(0), \quad s = \left(\frac{2g}{m} \circ \gamma'(0) \right)^{1/2} \underline{\quad}$$

Finally we can state the law of motion, recalling theorem (I.3.3), corollary (I.3.4), corollary (I.1.5) and definition (II.1.4).

4. DÉFINITION. — Let $\gamma : I \rightarrow M$ be a curve, such that $\gamma'(I) \subset PM$. Let F be a force. We say that γ is a motion « under the action » of the force F if one of the following *equivalent* conditions are satisfied:

- (a) $a_{\tilde{\gamma}} = \underline{F} \circ \tilde{\gamma}'$;
- (b) $a_{\gamma} = \underline{F} \circ \gamma'$;
- (c) $\tilde{\gamma}$ is a solution of the spray X on PM ;
- (d) γ is a solution of the spray X on PM , where X is given by

$$i_X dd_v g = - dg + \underline{F} \underline{\quad}$$

Of course, physically speaking, the preceding statement is an axiom, which enables us to make previsions.

In this way, the law of motion has a very strong similarity with the classical one (see [10]). The differences consist in the phase-space (PM is properly contained in TM) and in the possible choice of \underline{F} ($j\underline{F} = 0$ and $L_V \underline{F} = 2\underline{F}$). These express nothing else but the different nature of time, in Relativity, with respect to Classical Physics.

We can also transpose, as in Classical Mechanics (see [6] and [10]), the equation of motion to the co-phase-space P^*M . But we cannot obtain a Hamiltonian system (see [6]), if $F \neq 0$, recalling proposition (I.1.7). In fact, we have the following proposition.

5. PROPOSITION. — The motion γ is under the action of the force \underline{F} if and only if it is a solution of the s. o. d. e.

$$X = (T\hat{g})^{-1} \circ Y \circ \hat{g},$$

on PM , where Y is the vector field on P^*M given by

$$i_Y d\lambda = - dg^* + \underline{F}^*,$$

being

$$g^* = g \circ \hat{g}^{-1}, \quad \underline{F}^* = (\hat{g}^{-1})^* \underline{F}.$$

Proof. — Recalling theorem (I.3.1), we have:

$$i_X dd_v g = i_X \hat{g}^* d\lambda = \hat{g}^* i_Y d\lambda,$$

where

$$Y = T\hat{g} \circ X \circ \hat{g}^{-1} \underline{\quad}$$

Notice that we can, as in Classical Mechanics (see [6]), introduce the Poisson brackets, with respect to the symplectic structure $dd_v g$ on PM ,

or $d\lambda$ on P^*M . But we cannot use classical techniques which involve Hamiltonian systems, since these do not exist, if $F \neq 0$.

6. EXAMPLE. — Let us refer to example (1.5). The spray X is given by

$$i_X dd_v g = -dg + (2g)^{1/2} j p_M^*(F).$$

However it is also possible to abandon the spray condition on X and allow only normalized solutions of X as physical motions. Then X is given by

$$i_X dd_v g = -dg + j p_M^*(F) \quad (*).$$

Furthermore, let F be exact (locally it is always true):

$$F = dA.$$

Then, $\underline{A} = p_M^* A$ is a semibasic 1-form homogeneous of degree 0 and, if X is a s. o. d. e., we have (see (I.1.6) and (I.1.2)):

$$j d\underline{A} = i_X d\underline{A} = i_X dd_v j \underline{A}.$$

In this way, the equation (*) is written as

$$i_X dd_v (g + f) = -dg \equiv d((g + f) - V.(g + f)),$$

where $f \equiv -j\underline{A}$. Hence f characterizes completely the electromagnetic field, which appears here as a modification of the symplectic structure (notice that $g + f$ is not a new metric).

3. Many-body dynamics.

As in the preceding section, we suppose that (M^0, g^0) is a time-oriented hyperbolic normal structure, representing the space-time.

Let m_1, \dots, m_p be positive numbers, that, physically speaking, are the (rest) masses of particles. We put

$$m = m_1 + \dots + m_p.$$

We call « motion space-time » of the particles with masses m_1, \dots, m_p the new structure (M, g) , which is the product of the structures

$$(M_1, g_1), \dots, (M_p, g_p),$$

where $M_1 = \dots = M_p = M^0$ and where $g_i = m_i g^0$, for $i = 1, \dots, p$.

Let us observe that (M, g) is not a hyperbolic normal structure.

We define a motion on M as follows.

1. DÉFINITION. — A « motion » on M is a curve $\gamma : I \rightarrow M$, such that for each $i = 1, \dots, p$, $\gamma_i = \Pi_i \circ \gamma$ is a motion on M_{i-} .

Let $\tilde{\gamma}$ be the « normalization » of γ , given by

$$\tilde{\gamma} = \gamma \circ (1/k) : kI \rightarrow M,$$

where

$$k = \left(\frac{2g}{m} \circ \gamma'(t) \right)^{1/2}, \quad t \in I.$$

Notice that, if $\gamma_1, \dots, \gamma_p$ are motions on M_1, \dots, M_p , then

$$\gamma = \rho(\gamma_1, \dots, \gamma_p)$$

is a motion on M such that (recalling proposition (I.3.7))

$$\begin{aligned} \gamma'(I) &\subset \text{PM}, \\ D(g \circ \gamma') &= 0. \end{aligned}$$

Furthermore, if $\gamma = \rho(\tilde{\gamma}_1, \dots, \tilde{\gamma}_p)$, then we obtain its normalization by

$$\tilde{\gamma} = \gamma \circ p^{-1/2} : p^{1/2}I \rightarrow M.$$

Notice also that γ being normalized does not imply that each γ_i is normalized. But, given γ , we can change the parameter of γ , by a constant factor, such that, for a fixed i , the resulting motion on M_i is normalized. This factor may not be the same for all the indices i . The physical meaning of this fact is that we have not made the restriction of assuming the same unity of measure for the motion of each single point. This point of view, in the definition of motion, is, altogether, in accord with the desire of allowing every solution for the law of motion (see def. 3).

As in the preceding section, we have, with obvious notations

$$m_i \underline{a}^0 \gamma_i = \omega_i \bar{a}_{\gamma_i} = \underline{a}_{\gamma_i} = (\text{TT}\Pi_i) \underline{a}_\gamma = (\text{TT}\Pi_i) \omega \bar{a}_\gamma = m_i (\text{TT}\Pi_i) \omega^0 \bar{a}_\gamma^0.$$

We can also extend in a natural way theorem 3 of the preceding section.

We have further the following result, which suggests the law of motion for many bodies.

2. PROPOSITION. — Let \underline{F} be a force without interaction, that is of the form

$$\underline{F} = (\text{TT}\Pi_1) * \underline{F}_1 + \dots + (\text{TT}\Pi_p) * \underline{F}_p$$

and let $\gamma : I \rightarrow M$ be a curve, such that $\gamma'(I) \subset \text{PM}$. Then the following conditions are equivalent:

(a) for each $i = 1, \dots, p$, $\gamma_i = \Pi_i \gamma$ is a motion on M_i , under the action of the force \underline{F}_i ;

(b)
$$\bar{a}_\gamma = \bar{F} \circ \gamma' \perp$$

We are now in the position of giving the law of motion for a system of p particles with the help of theorem (I.3.8), corollary (I.3.9), corollary (I.1.5) and definition (II.1.7).

3. DÉFINITION. — Let $\gamma : I \rightarrow M$ be a curve, such that $\gamma'(I) \subset \text{PM}$ and

let \underline{F} be a force. We say that γ is a motion « under the action » of the force \underline{F} if one of the following *equivalent* conditions is satisfied:

$$(a) \quad \underline{a}_{\gamma_i} = \underline{F}_i, \quad i = 1, \dots, p;$$

$$(b) \quad \underline{a}_{\tilde{\gamma}} = \underline{F} \circ \tilde{\gamma}';$$

$$(c) \quad \underline{a}_{\gamma} = \underline{F} \circ \gamma';$$

(d) $\tilde{\gamma}$ is a solution of the spray X on PM ;

(e) γ is a solution of the spray X on PM ,

where

$$\gamma_i = \Pi_i \circ \gamma, \quad \bar{F}_i = (T\Pi_i) \circ \bar{F} \circ \gamma'$$

and where X is given by $i_x dd_v g = -dg + \underline{F}_\perp$.

Of course, physically speaking, the preceding statement is an axiom, which, in the case of a non-interacting force, is a consequence of the law of motion for one body. Apart this special case, the axiom is strongly suggested by the previous one.

It is easy to extend to many-body dynamics the last conclusions concerning one body, because the equation of motion is the same.

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