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JAMES GLIMM
ARTHUR JAFFE

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Section A :
Physique théorique.

The entropy principle for vertex functions in quantum field models

by

James GLIMM (*)
Courant Institute. New York University. New York 10012

and

Arthur JAFFE (**)
Harvard University. Cambridge, Massachusetts 02138

ABSTRACT. — We obtain the (amputated, one particle irreducible) vertex functions for weakly coupled Euclidean $\mathscr{P}(\Phi)_2$ models. The generating function $\Gamma_2 \{ A \}$ for the vertex functions is jointly analytic in A and in the bare coupling constants λ_j. The generating function $G \{ J \}$ for connected Green's functions has a convergent tree-graph expansion in terms of vertex functions and propagators.

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1. THE VERTEX FUNCTIONS

1.1. The partition function

The partition function in Euclidean $\mathcal{P}(\Phi)_2$ models,

$$Z\{J\} = \int e^{\hat{\Phi}(J)}dq = \int e^{\int(\Phi(x)-\langle\Phi(x)\rangle)J(x)dx}dq = \int e^{\Phi(J)}dq/e^{\langle\Phi\rangle\int J(x)dx}$$

is the generating function for the Schwinger functions

$$S^{(n)}(x_1, \ldots, x_n) = \int \hat{\Phi}(x_1) \ldots \hat{\Phi}(x_n)dq.$$

of the translated field $\hat{\Phi}(x) = \Phi(x) - \langle \Phi(x) \rangle$.

Here dq is the Euclidean measure on $\mathcal{S}'(\mathrm{R}^2)$, associated with the $\mathcal{P}(\Phi)_2$ quantum field. Indicating derivatives by subscripts,

$$S^{(n)}(x_1, \ldots, x_n) = Z_{x_1,\ldots,x_n}\{0\}.$$

The connected parts of the Schwinger functions $G^{(n)}(x_1, \ldots, x_n)$ have the effects of disconnected processes removed. (They have no zero particle intermediate states.) The $G^{(n)}$ functions cluster exponentially, with the exponential decay rate equal to the physical mass m. The generating function

$$G\{J\} = \ln Z\{J\}$$

yields these (connected) Green's functions,

$$G^{(n)}(x_1, \ldots, x_n) = G_{x_1,\ldots,x_n}\{0\}.$$

The vertex functions $\Gamma^{(n)}(x_1, \ldots, x_n)$ are the amputated, one particle irreducible parts of the $G^{(n)}(x_1, \ldots, x_n)$. They have the effects of one particle intermediate states removed, and are believed to cluster with a decay rate $> m$. Thus the vertex functions display the « upper mass gap » in the energy momentum spectrum. Furthermore, the physical charge is defined in terms of a vertex function, so understanding the vertex functions is a step toward understanding charge renormalization.

Construction of the vertex functions is the beginning of Symanzik's program of structure analysis for Green's functions [9]; he proposed construction of n-particle irreducible parts. See also [*11*]. Related to Symanzik's proposed analysis of Green's functions, is our analysis of the n-particle structure of the energy spectrum [3]. Consider the subspace $\mathcal{H}_E \subset \mathcal{H}$ for the spectral interval [0, E] of the Hamiltonian $H = H^*$. In weak $\mathcal{P}(\Phi)_2$ models, cluster properties yield an explicit construction of $\mathcal{H}_E$. Furthermore, this analysis exhibits n-particle structure, since states of energy $\leq (n + 1)m_0(1 - \varepsilon)$ are spanned by polynomials of degree n in the Euclidean field Φ. This result, for $n = 0, 1$, is used to prove the existence of isolated one particle states [3].

In this paper we construct the vertex functions $\Gamma^{(n)}$ and their generating function $\Gamma\{A\}$,

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \Gamma_{x_1,\ldots,x_n}\{A = 0\},$$

for $\mathscr{P}(\Phi)_2$ models with weak coupling. We establish analyticity of $\Gamma\{A\}$ in A, as well as in the bare parameters. In a subsequent paper we establish the Callan-Symanzik equations, giving the change $d\Gamma/d\sigma$ for a mass perturbation $\sigma \int :\Phi^2: d\vec{x}$.

In the definition of the vertex functions, amputation means that in momentum space $G^{(n)}$ is divided by the product

$$\prod_{j=1}^{n} \int G^{(2)}(x, 0) e^{ip_j x} dx$$

of propagators in each momentum variable. In perturbation theory language, amputation removes all mass subdiagrams attached to external legs of diagrams contributing to $G^{(n)}$.

One particle irreducible (1PI) graphs are those which cannot be disconnected by removal of a single line. We use a Legendre transformation (variational or entropy principle) to obtain $\Gamma\{A\}$ from $G\{J\}$. The thermodynamic parameters are identified as follows. J is a chemical potential or magnetic field, and so its conjugate variable has the role of N (= number of particles) or magnetization, M. With $\ln Z = PV$, we see that Γ defined by (1.2.1) is essentially the negative of the Helmholtz free energy $N - PV$. Entropy is the variable conjugate to temperature, and $kT = \beta^{-1}$ is the inverse coefficient of the energy in dq. Below we introduce a coefficient λ_j for each homogeneous contribution to the energy. Conjugate to each λ_j is an « entropy » S_j associated with the j^{th} Legendre transform. The case $j = 2$ is discussed in [11] and leads to two particle irreducible graphs. Suppression of the factor $e^{-\langle \mathscr{S}(J) \rangle}$ in Z leads to graphs which are one particle irreducible in a stronger sense than the graphs considered here, in that tadpoles (i. e. one particle reducible parts with no external legs) are also removed, see [11].

This method goes back to De Dominicis and Martin in statistical mechanics [2] and to Jona-Lasinio in quantum field theory [7]. We follow work of Symanzik [10]. Our contribution is to obtain bounds on $Z\{J\}$, $G\{J\}$ and $G^{(2)}$ which permit the analysis to be carried out rigorously. We also obtain bounds on the vertex functions $\Gamma^{(n)}$. The n-dependence of these bounds ensures convergence of the expansion of $G\{J\}$ (and also $Z\{J\}$ in terms of « tree-like graphs » with vertex kernels $\Gamma^{(r)}$ and propagators $G^{(2)}$. In this manner we exhibit graphically the 1PI property of the vertex functions.

Let H_p denote the complex Sobolev space $\mathscr{D}((-\Delta+1)^{p/2})$, with norm

$$\| f \|_p^2 = \int_{x\in R^2} |((-\Delta+1)^{p/2}f)(x)|^2 dx .$$

Also let

$$H_{p,\varepsilon} = \{ f : f \in H_p, \| f \|_p < \varepsilon \} .$$

We define analytic functions and multilinear forms $\mathscr{ML}_n$ in the Appendix.

The partition function $Z\{J\}$ is a limit of finite volume partition functions, at least for $J \in C_0^\infty$. The finite volume partition functions are jointly analytic in $J \in H_{-1}$ and in the bare parameters of the theory in any « positive but weak coupling » region of the bare parameter space (See Chapter 2). By definition, in such a region each coupling constant λ_j, the coefficient of $:\Phi^j:$ in the interaction Hamiltonian, is sufficiently small with respect to the mass m_0 in the free Hamiltonian. We also require $|\lambda_j| \leq O(1)|\mathrm{Re}\,\lambda_l|$, and $\mathrm{Re}\,\lambda_l > 0$ where λ_l is the coupling constant of the highest degree term, and of course l is even. In this region, with $\| J \|_{-1}$ small the finite volume partition functions are uniformly bounded. Hence by Proposition A3, the infinite volume limit $Z\{J\}$ is analytic for $\| J \|_{-1}$ small and for λ_j in the positive but weak coupling region. { We remark that this argument does not yield derivatives of $Z\{J\}$ with respect to λ_l at $\lambda_l = 0$, but Dimock [*1*] has shown that the Schwinger functions are C^∞ in the λ's in a region including $\lambda_l = 0^+$. } We establish our basic estimates for $Z\{J\}$ in Chapter 2. Fröhlich [*6*] previously used properties of $Z\{J\}$ to study the Schwinger functions in $\mathscr{P}(\Phi)_2$ models. Feldman [*5*] has results for the finite volume Φ_3^4 model.

We assume throughout this paper that the coupling constants λ_j lie in a weak coupling region and that $J \in H_{-1,\varepsilon}$.

PROPOSITION 1.1.1. — The partition function $Z\{J\}$ is jointly analytic in λ_j and $J \in H_{-1,\varepsilon}$, for $\varepsilon > 0$ sufficiently small.

We postpone the proof to Chapter 2. By Proposition A5 we have.

COROLLARY 1.1.2. — Let $J \in H_{-1,\varepsilon}$, for ε sufficiently small. Then

$$Z_{x_1,\dots,x_n}\{J\} \in \mathscr{ML}_n(H_{-1})$$

with norms bounded by $O(1)L^n n!$

Remark. — Fröhlich obtained a bound $(n!)^{1/2}$ by a more careful analysis [*6*], but with a different norm.

PROPOSITION 1.1.3. — $Z\{J\} \neq 0$ and $Z\{J\}$ is bounded away from zero for $\| J \|_{-1}$ small.

Proof. — For real λ_j and J, the Schwarz inequality yields

$$1 = Z\{0\} = \int dq \leq \left[\int e^{\Phi(J)} dq \int e^{-\Phi(J)} dq \right]^{1/2} = Z\{J\}^{1/2} Z\{-J\}^{1/2}.$$

By Proposition 1.1.1, $Z\{-J\}$ is bounded, so $Z\{J\}^{-1}$ is also bounded. For the general (complex) case, $Z\{0\} = 1$ and $Z\{J\} \neq 0$ by continuity.

Recall $G\{J\} = \ln Z\{J\}$. By Proposition 1.1.3, $G\{J\}$ exists.

PROPOSITION 1.1.4 [*10*]. — For λ_j, J real, $G\{J\}$ is a convex function of J.

Proof. — Let $J = \alpha_1 J_1 + \alpha_2 J_2$ be a convex sum with J_i real. By Hölder's inequality with $p_i = \alpha_i^{-1}$,

$$\begin{aligned} G\{J\} &= \ln Z\{\alpha_1 J_1 + \alpha_2 J_2\} \\ &\leq \ln Z\{J_1\}^{\alpha_1} Z\{J_2\}^{\alpha_2} = \alpha_1 G\{J_1\} + \alpha_2 G\{J_2\}. \end{aligned}$$

By Definition A1, we have

PROPOSITION 1.1.5. — $G\{J\}$ is analytic in H_{-1}, and in the λ_j. The Taylor series coefficients

$$G_{x_1,\dots,x_n}\{J\} \in \mathcal{M}\mathcal{L}_n(H_{-1})$$

have norms $O(1)L^n n!$. Here L is uniformly bounded for λ_j and J as above.

PROPOSITION 1.1.6. — For complex coupling constants in a weak coupling region, or for real coupling with positive physical mass, $G^{(2)}(x, y)$ is the integral kernel of a bounded operator $g^{(2)}$ from H_{-1} to H_1 with a bounded inverse from H_1 to H_{-1}.

We prove this result in Chapter 2.

By analyticity in λ_j, $dG^{(2)}(x, y)/d\lambda_j$ is also bounded, from H_{-1} to H_1.

We define the self energy part Π of $g^{(2)}$ by the resolvent equation,

$$g^{(2)} = c + g^{(2)}\Pi c, \tag{1.1.1}$$

or

$$\Pi = c^{-1} - \Gamma = -\Delta + m_0^2 - \Gamma.$$

Here $c = (-\Delta + m_0^2)^{-1}$ Let $c' = (-\Delta + m_0^2 + 2\lambda_2)^{-1}$.

PROPOSITION 1.1.7. — With λ_j in a weak coupling region, the self energy part has the form

$$\Pi = -2\lambda_2 + \Pi', \tag{1.1.2}$$

where Π' is a bounded transformation from H_1 to H_{-1} with norm $\sum_{j=3}^{l}\sum_{k=1}^{l} O(|\lambda_j \lambda_k|)$. Thus for sufficiently weak coupling,

$$g^{(2)} = c + c\sum_{n=1}^{\infty}(\Pi c)^n \tag{1.1.3}$$
$$= (-\Delta + m_0^2 - \Pi)^{-1}$$
$$= (-\Delta + m_0^2 + 2\lambda_2 - \Pi')^{-1}$$
$$= c' + c'\sum_{n=1}^{\infty}(\Pi' c')^n .$$

Remarks. — We give the proof in Chapter 2. The equation (1.1.4) shows how Π or Π' determines the mass renormalization in the propagator. For a pure quadratic interaction, by (1.1.3), $\Pi' = 0$, and

$$g^{(2)} = (-\Delta + m_0^2 + 2\lambda_2)^{-1} .$$

1.2. **The Legendre transformation** $G\{J\} \leftrightarrow \Gamma\{A\}$

The generating function $\Gamma\{A\}$ for the vertex functions is related to the generating function $G\{J\}$ for Green's functions by a Legendre transformation. We give two formulations [*2*] [*7*] [*10*]: the first formulation involves a variational principle, while the second formulation relies on solving a nonlinear functional equation. Both formulations are defined and agree for real λ_j, A, J, while the second formulation generalizes to the complex case, see Corollaries 1.2.4-1.2.5. Throughout this section we assume that $A \in H_{1,\delta}$. We obtain $\Gamma\{A\}$ and find that it is analytic in A, for δ sufficiently small.

DEFINITION 1.2.1. — Let A, λ_j be real. Define

$$\Gamma\{A\} = \inf_J [-A(J) + G\{J\}], \tag{1.2.1}$$

where the infimum runs over real $J \in H_{-1,\varepsilon}$. We take δ small enough so that the infimum is not attained on the boundary.

Given $J \in H_{-1,\varepsilon}$, the expression

$$A(x) = G_x\{J\} \tag{1.2.2}$$

defines a bounded transformation $J \to A$ from $H_{-1,\varepsilon}$ into H_1. The inverse mapping $A \to J$ from $H_{1,\delta}$ to H_{-1} is obtained by solving (1.2.2) as an equation for J.

PROPOSITION 1.2.2. — Given A, λ_j real, the following conditions are equivalent, for $J \in H_{-1,\varepsilon}$:

(*i*) J minimizes (1.2.1),

(*ii*) J satisfies equation (1.2.2).

Proof. — Since (*ii*) is the variational equation associated with (1.2.1), it follows that (*i*) $\Rightarrow$ (*ii*). We assume (*ii*), and let $M \in H_{-1}$. The function

$$f(\alpha) = -A(J + \alpha M) + G\{J + \alpha M\}$$

is convex (Proposition 1.1.4) and has derivative 0 at $\alpha = 0$, by (*ii*). Since convexity implies f' is monotone, $f' \geq 0$ for $\alpha > 0$ and $f' \leq 0$ for $\alpha < 0$. Since M is arbitrary, J is a minimum, implying (*i*).

A direct proof of the existence and uniqueness of a minimizing J in (1.2.1) can be given. However, we study here the solution to (1.2.2). In both approaches, our proofs require $\| A \|_1$ to be small.

THEOREM 1.2.3. — Consider the complex case and let $\Gamma A \in H_{-1,\varepsilon}$ for ε sufficiently small. Then the equation (1.2.2) has a unique solution $J_A \in H_{-1,2\varepsilon}$. The solution J_A is jointly analytic in A and in λ_j.

Remark. — As before, Γ denotes the linear operator inverse to $g^{(2)}$, the operator with kernel $G^{(2)}(x, y) = G_{x,y}\{0\}$.

Two immediate corollaries are:

COROLLARY 1.2.4. — Let A, λ_j be real, $A \in H_{1,\delta}$. Then (1.2.1) has a unique minimizing function $J = J_A$, and

$$\Gamma\{A\} = -A(J_A) + G\{J_A\}. \tag{1.2.3}$$

COROLLARY 1.2.5. — The generating function $\Gamma\{A\}$ is analytic in $A \in H_{1,\delta}$. In other words, for A, $A + B \in H_{1,\delta}$,

$$\Gamma\{A + B\} = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma^{(n)}_{B,B,\dots,B}\{A\}.$$

The Taylor series coefficients $\Gamma^{(n)}_{x_1,\dots,x_n}\{A\}$ are elements of $\mathscr{ML}_n(H_1)$, with norms $O(L^n n!)$, and are analytic functions of A and the λ_j.

PROPOSITION 1.2.6 [*10*]. — In the real case, $\Gamma\{A\}$ is concave: for real A, $B \in H_{1,\delta}$ and $0 \leq \alpha \leq 1$,

$$\Gamma\{\alpha A + (1-\alpha)B\} \geq \alpha\Gamma\{A\} + (1-\alpha)\Gamma\{B\}.$$

Proof. — By Definition 1.2.1,

$$\begin{aligned}\Gamma\{\alpha A + (1-\alpha)B\} &\\ &= \inf_J [\alpha(-A(J) + G\{J\}) + (1-\alpha)(-B(J) + G\{J\})]\\ &\geq \alpha \inf_J [-A(J) + G\{J\}] + (1-\alpha)\inf_J [-B(J) + G\{J\}]\\ &= \alpha\Gamma\{A\} + (1-\alpha)\Gamma\{B\}.\end{aligned}$$

Remark. — Corollary 1.2.5 holds, in particular, for A = 0. Then $\Gamma^{(n)}(x_1, \ldots, x_n) = \Gamma_{x_1,\ldots,x_n}\{0\}$ and we obtain bounds on the vertex functions for weak $\mathscr{P}(\Phi)_2$ models. We note $\Gamma^{(1)}(x) = \Gamma^{(1)}_x\{0\} = 0$, and $-\Gamma^{(2)}(x, y) = -\Gamma^{(2)}_{x,y}\{0\}$ is the kernel of the linear operator $\Gamma = [g^{(2)}]^{-1}$.

Remark. — By Proposition 1.1.6, the operators $\Gamma : H_1 \to H_{-1}$ and $\Gamma^{-1} : H_{-1} \to H_1$ are both bounded. Thus equation (1.2.2) is equivalent to the equation

$$J = \Gamma A + \Gamma K\{J\}, \tag{1.2.4}$$

where K is the nonlinear operator defined by

$$K\{J\}(x) = -G_x\{J\} + G^{(2)}(x, J).$$

Proof of Theorem 1.2.3. — Explicitly,

$$K\{J\}(x_1) = -\sum_{n=3}^{\infty} \frac{1}{(n-1)!} \int dx_2 \ldots dx_n G^{(n)}(x_1, \ldots, x_n) \cdot J(x_2) \ldots J(x_n). \tag{1.2.5}$$

By Propositions 1.1.5 and 1.1.6, the operator ΓK is a strict contraction on $H_{-1,2\varepsilon}$, for ε sufficiently small. In fact,

$$\begin{aligned} \|\Gamma K\{f\}\|_{-1} &\leq O(1) \sum_{n=3}^{\infty} \frac{n!L^n}{(n-1)!} \|f\|_{-1}^{n-1} \\ &\leq O(1)\|f\|_{-1}^2 \leq \frac{1}{2}\|f\|_{-1}. \end{aligned} \tag{1.2.6}$$

Thus for $\Gamma A \in H_{-1,\varepsilon}$, the operator

$$T_A : f \mapsto \Gamma A + \Gamma K\{f\}$$

maps into itself the sphere in H_{-1} centered at ΓA, and with radius ε. $T_A f$ is analytic in A. Likewise

$$\|T_A^n f\|_{-1} < 2\varepsilon, \tag{1.2.7}$$

and $T_A^n f$ is analytic in A.

Define $h_n(A) = T_A^n f$, with $\Gamma A \in H_{-1,\varepsilon}$. By (1.2.7), the norms $\|h_n(A)\|_{-1}$ are uniformly bounded. Also

$$\|h_{n+1}(A) - h_n(A)\|_{-1} = \|\Gamma K\{h_n(A)\} - \Gamma K\{h_{n-1}(A)\}\|_{-1} \tag{1.2.8}$$

We bound (1.2.8) by using (1.2.5) and Proposition 1.1.5, in a fashion similar to the derivation of (1.2.6). We obtain for $f \in H_{-1,\varepsilon_1}$ (with $\varepsilon_1 \leq \varepsilon$ sufficiently small),

$$\|h_{n+1}(A) - h_n(A)\|_{-1} \leq \frac{1}{2}\|h_n(A) - h_{n-1}(A)\|_{-1}.$$

By iteration,

$$\| h_n(A) - h_m(A) \|_{-1} \leq \sum_{j=m}^{n-1} 2^{-j+1} \leq O(2^{-m}).$$

By Proposition A3, there exists an analytic function $h : H_{1,\varepsilon} \to H_{-1,2\varepsilon}$ such that

$$\| h_n(A) - h(A) \|_{-1} \to 0 .$$

In particular,

$$J_A \equiv h(A) = \lim_{n\to\infty} T_A^n f \in H_{-1,2\varepsilon}$$

exists and is a fixed point of T_A, i. e. a solution to (1.2.4) or (1.2.2). Uniqueness of the solution J_A follows. For two solutions $J_1, J_2 \in H_{-1,2\varepsilon}$, we obtain as above, for ε sufficiently small,

$$\begin{aligned} \| J_1 - J_2 \|_{-1} &= \| \Gamma K \{ J_1 \} - \Gamma K \{ J_2 \} \|_{-1} \\ &\leq O(1) \| K \{ J_1 \} - K \{ J_2 \} \|_{-1} \\ &\leq O(\varepsilon) \| J_1 - J_2 \|_{-1} . \end{aligned}$$

Choosing ε sufficiently small, $O(\varepsilon) < 1$, so $J_1 = J_2$.

The solution J_A is analytic in A, as follows by the analyticity of $h(A)$. Analyticity in λ_j follows similarly.

We now study the inverse Legendre transformation $\Gamma \{ A \} \to G \{ J \}$. Our treatment follows that for $G \{ J \} \to \Gamma \{ A \}$ above.

PROPOSITION 1.2.7. — In the real case, with $\| J \|_{-1}$ small,

$$G \{ J \} = \sup_A [J(A) + \Gamma \{ A \}], \tag{1.2.9}$$

where the supremum runs over $A \in H_{1,\delta}$. A maximizing A always exists and is given by (1.2.2).

Proof. — By Definition 1.2.1, $G \{ J \} \geq \Gamma \{ A \} + J(A)$, with equality for A defined by (1.2.2). This completes the proof.

By Corollary 1.2.5, the expression

$$J(x) = - \Gamma_x \{ A \} \tag{1.2.10}$$

defines a bounded transformation $A \mapsto J$, from $H_{1,\varepsilon}$ to H_{-1}. The inverse mapping $J \mapsto A$ is obtained by solving (1.2.10). From the concavity of $\Gamma \{ A \}$, Proposition 1.2.6, we obtain analogously to Proposition 1.2.2:

PROPOSITION 1.2.8. — Consider the real case. Given J, the following conditions are equivalent, for $A \in H_{1,\varepsilon}$:

(*i*) A maximizes (1.2.9),
(*ii*) A satisfies (1.2.10).

THEOREM 1.2.9. — Consider the complex case and let $\Gamma^{-1} J \in H_{1,\varepsilon}$, with ε sufficiently small. Then the equation (1.2.10) has a unique solution $A_J \in H_{1,2\varepsilon}$. The function A_J is jointly analytic in J and in λ_j, and is the inverse function to J_A of Theorem 1.2.3, i. e., $J_{A_J} = J$.

Proof. — Recall that the operator Γ has the kernel $-\Gamma^{(2)}(x, y)$. We then rewrite (1.2.10) as

$$A = \Gamma^{-1}J + \Gamma^{-1}\mathscr{F}\{A\} \equiv \mathscr{M}_J\{A\}, \tag{1.2.11}$$

where

$$\mathscr{F}\{A\}(x) = \Gamma_x\{A\} - \Gamma A = \sum_{n=3}^{\infty} \frac{1}{(n-1)!}\Gamma^{(n)}(x, A, \ldots, A). \tag{1.2.12}$$

We follow the proof of Theorem 1.2.3 to obtain the existence, uniqueness and analyticity of $A_J = \lim_n \mathscr{M}_J^n \Gamma^{-1} J$. Differentiating (1.2.3) and using (1.2.2) yields (1.2.10). Thus A_J and J_A are inverse functions.

COROLLARY 1.2.10. — In the real case, (1.2.9) has a unique maximizing function $A = A_J$, for $J \in H_{-1,\varepsilon}$ and

$$G\{J\} = J(A_J) + \Gamma\{A_J\}. \tag{1.2.13}$$

Legendre transforms have been considered from a general point of view, with $H_{\pm 1}$ replaced by topological vector spaces X and Y put in duality by a bilinear form, see [*8*]. The formula $J_A = J$ and Propositions 1.2.2 and 1.2.8 are valid in this general context.

1.3. **Tree graphs**

The Legendre transformation

$$G\{J\} \to \Gamma\{A\} \tag{1.3.1}$$

and its inverse

$$\Gamma\{A\} \to G\{J\} \tag{1.3.2}$$

generate convergent expansions for $G\{J\}$ and $\Gamma\{A\}$ in terms of tree-like graphs. For instance, in the series for $G\{J\}$, each vertex in a tree graph corresponds to a kernel $\Gamma^{(r)}(x_1, \ldots, x_r)$ and each line corresponds to a kernel $G^{(2)}(x, y)$ (propagator). In perturbation theory, such graph expansions are used to define the vertex functions.

We start from the identities

$$G_x\{J\} = + A_J(x) \tag{1.3.3}$$

and

$$\Gamma_x\{A\} = - J_A(x) \tag{1.3.4}$$

obtained from (1.2.2) and (1.2.10), along with Propositions 1.2.2 and

1.2.8. These identities relate $G\{J\}$ and $\Gamma\{A\}$ directly to A_J and J_A, and provide our tree graph expansions. For instance, the power series for (1.3.3),

$$(1.3.5) \qquad A_J(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} G^{(n)}(x, J, \ldots, J)$$

has an alternative expression from iterating (1.2.11) to obtain A_J. We identify the coefficients of order $n-1$ in J in these two expressions, and thereby obtain an expression for $G^{(n)}(x_1, \ldots, x_n)$. This expression is a finite sum, and each term is an integral that we represent by a graph. The graphs contributing to $G^{(n)}(x_1, \ldots, x_n)$ have n external lines labelled by $x_1, \ldots, x_n$. The graphs have a « root », the external line labelled by x_1, and branch in one direction away from the root. Each vertex with r legs has the kernel $\Gamma^{(r)}(y_1, \ldots, y_r)$ and each line has the kernel $G^{(2)}(x, y)$, the kernel of Γ^{-1}. For instance, the tree-like graph of Figure 1 contributes to $G^{(5)}$ and corresponds (up to a combinatoric factor) to the integral

$$(1.3.6) \qquad \int G^{(2)}(x_1, y_1) G^{(2)}(x_2, y_2) G^{(2)}(y_3, z_1) G^{(2)}(x_3, z_2) G^{(2)}(x_4, z_3) G^{(2)}(x_5, z_4) \Gamma^{(3)}(y_1, y_2, y_3) \Gamma^{(4)}(z_1, z_2, z_3, z_4)\, dy\, dz\,.$$

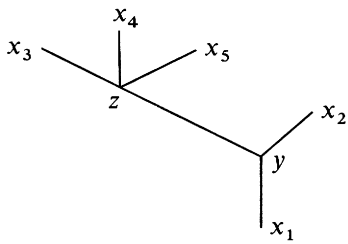

FIG. 1. — A tree graph contributing to $G^{(5)}$.

Retaining only the linear term $A = \Gamma^{-1}J$ in (1.2.11), we obtain the single vertex contribution to $G^{(n)}(x_1, \ldots, x_n)$, namely its one particle irreducible part. Each external leg has the operator Γ^{-1} (propagator) which replaces the mass subdiagrams removed by amputation. For $n = 3$, this is the only contribution to $G^{(3)}$ so

$$(1.3.7) \quad G^{(3)}(x_1, x_2, x_3) = \int dy\, G^{(2)}(x_1, y_1) G^{(2)}(x_2, y_2) G^{(2)}(x_3, y_3) \Gamma^{(3)}(y_1, y_2, y_3)\,.$$

Substituting at least one nonlinear term from $\mathcal{F}\{A\}$ in (1.2.12) into (1.2.11), we obtain the one particle reducible part of $G^{(n)}(x_1, \ldots, x_n)$,

$G^{(4)}(1, \ldots, 4) =$

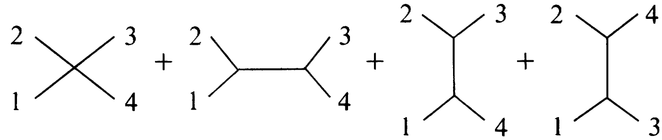

FIG. 2. — In the tree graph expansion of $G^{(n)}$, vertices have kernels $\Gamma^{(r)}$ and lines are propagators $G^{(2)}(x, y)$.

represented by tree graphs with more than one vertex. For example, $G^{(4)}(x_1, \ldots, x_4)$ has the expansion given in Figure 2. Here we abbreviate x_j by j.

For fixed n, the number of tree graphs contributing to $G^{(n)}$ is finite. This number diverges as $n \to \infty$, but nevertheless the power series expansion of $G\{J\}$ in powers of J is convergent, since $G_x\{J\} = A_J(x)$ and A_J is analytic in J. Thus we have proved,

PROPOSITION 1.3.1. — $G\{J\}$ has a convergent tree graph expansion in terms of vertices $\Gamma^{(r)}$ and propagators $G^{(2)}$.

Likewise, we use (1.3.4) and (1.2.4) to generate a tree graph expansion for $\Gamma\{A\}$. In this case, the vertices have kernels equal to the amputated Green's functions,

$$G_a^{(r)}(x_1, \ldots, x_r) = \left(\prod_{i=1}^{r} \Gamma(i)\right) G^{(r)}(x_1, \ldots, x_r),$$

where $\Gamma(i)$ denotes the inverse propagator acting on the variable x_i. Also $r \geq 3$. The lines have kernels given by the propagator $G^{(2)}(x, y)$, as before. In addition, because of the -1 in (1.2.5), there is a factor -1 for each vertex as well as an overall factor -1 from (1.3.4). The sign of a term is therefore $+1$ for an odd number of vertices and -1 for an even number.

Retaining the linear factor $J = \Gamma A$ in (1.2.4), we obtain the single vertex contribution $G_a^{(n)}(x_1, \ldots, x_n)$ to $\Gamma^{(n)}(x_1, \ldots, x_n)$. For $n = 3$, this is the only contribution, and $\Gamma^{(3)}(x_1, x_2, x_3) = G_a^{(3)}(x_1, x_2, x_3)$, which is another form of (1.3.7).

Substituting at least one nonlinear term from (1.2.5) into (1.2.4) we obtain the remaining contributions to $\Gamma^{(n)}$, represented by tree graphs as before. For example, $\Gamma^{(4)}(x_1, \ldots, x_4)$ has the expansion given in Figure 3.

$\Gamma^{(4)}(1, 2, 3, 4) =$

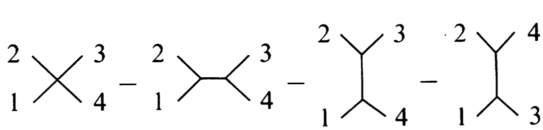

FIG. 3. — In the tree graph expansion of $\Gamma^{(n)}$, vertices have kernels $G_a^{(r)}$ and lines are propagators $G^{(2)}(x, y)$. A minus sign occurs for terms with an even number of vertices.

PROPOSITION 1.3.2. — The analytic function $\Gamma\{A\}$ has a convergent tree graph expansion in terms of vertices $G_a^{(r)}$, of propagators $G^{(2)}$, and with a minus sign for each graph having an even number of vertices.

2. ESTIMATES

In this chapter we establish the estimates stated in Chapter 1. One new feature of our estimate on $Z\{J\}$ is the global bound for $J \in H_{-1}$, as opposed to the local bounds for $J \in L_2$, suppt. J compact, established previously. It is clear that if $\langle \Phi \rangle \neq 0$,

$$\exp \langle \Phi(J) \rangle = \exp \left[\langle \Phi(\cdot) \rangle \int J(x)\, dx \right] \tag{2.1}$$

is bounded in L_1, but not in H_{-1}. This is the reason we substract $\langle \Phi \rangle$ from Φ in the definition of Z. It ensures that no contribution in ln Z is linear in J, and it eliminates the parts of Z which are unbounded in H_{-1}.

In order to estimate $\Phi - \langle \Phi \rangle$, it is convenient to introduce a new symmetry by a tensor product (doubling) procedure, introduced by Ginibre in statistical mechanics. Let $\mathcal{S}'(R^2)^\sim$, $dq^\sim$ be an isomorphic copy of $\mathcal{S}'(R^2)$, dq and let

$$\hat{\mathcal{S}}' = \mathcal{S}'(R^2) \otimes \mathcal{S}'(R^2)^\sim$$
$$d\hat{q} = dq \otimes dq^\sim .$$

For any function A on $\mathcal{S}'(R^2)$, let $A^\sim$ be its isomorphic image, a function on $\mathcal{S}'(R^2)^\sim$, and define an odd function $A_0 = A - A^\sim$ or in tensor product notation

$$A_0 = A \otimes I - I \otimes A^\sim .$$

Then

$$\langle AB \rangle - \langle A \rangle \langle B \rangle = \frac{1}{2} \langle A_0 B_0 \rangle , \tag{2.2}$$

which gives a natural way to perform the subtraction of $\langle \Phi \rangle$ as it occurs in the expansion.

Before getting into the detailed proof of Proposition 1.1.1, we give the proof of Propositions 1.1.6 and 1.1.7. In the case of real coupling constants, let

$$G^{(2)}(p)^\sim = \int_0^\infty \frac{d\rho(a)}{p^2 + a} \tag{2.3}$$

be the spectral representation for the Fourier transform of the connected two point Euclidean Green's function (We assume here Lorentz invariance together with the bound $\int (1 + a)^{-1} d\rho(a) < \infty$). The positivity condition ensures $d\rho(a)$ is a positive measure.

THEOREM 2.1 (*a*). — For real coupling constants $G^{(2)}: H_{-1} \to H_1$ is continuous if and only if

$$\int_0^\infty d\rho(a) + \int_0^\infty a^{-1} d\rho(a) < \infty . \tag{2.4}$$

(*b*) $\Gamma = G^{(2)^{-1}}: H_1 \to H_{-1}$ is continuous.

Proof. — (*a*) Note that $G^{(2)\sim}$ is the Fourier transform of $G^{(2)}(x, 0)$. Then $G^{(2)}: H_{-1} \to H_1$ is a bounded operator, if and only if $(1+p^2)G^{(2)\sim}(p)$ is a bounded function of p. Assume $(1 + p^2)G^{(2)\sim}(p)$ is bounded. Since $d\rho(a)$ is positive,

$$\int (a^{-1} + 1) d\rho(a) = G^{(2)\sim}(0) + \sup p^2 G^{(2)\sim}(p) < \infty .$$

Conversely, assuming (2.4), we have

$$\begin{aligned}\sup_p (1 + p^2)G^{(2)\sim}(p) &\leq \sup \int (p^2 + a)^{-1} d\rho(a) + \sup \int p^2(p^2 + a^2)^{-1} d\rho(a) \\ &= \int a^{-1} d\rho(a) + \int d\rho(a) < \infty .\end{aligned}$$

(*b*) $\Gamma: H_1 \to H_{-1}$ is a bounded operator if and only if $[G^{(2)\sim}(p)(p^2 + 1)]^{-1}$ is a bounded function of p. There is an interval $[0, \beta]$, such that $\int_0^\beta d\rho(a)$ is nonzero. Then

$$\begin{aligned}\inf_p (p^2 + 1)G^{(2)\sim}(p) &\geq \inf_p \int_0^\beta d\rho(a)(p^2 + 1)(p^2 + a)^{-1} \\ &\geq \min \{ 1, \beta^{-1} \} \int_0^\beta d\rho(a) > 0 ,\end{aligned}$$

and Γ is bounded.

Remarks. — 1. The hypotheses hold in weakly coupled $\mathscr{P}(\Phi)_2$ models. Hence both $G^{(2)}: H_{-1} \to H_1$ and its inverse $\Gamma: H_1 \to H_{-1}$ are bounded operators. In fact, all $\mathscr{P}(\Phi)_2$ quantum field models obey the canonical commutation relations (before field strength renormalization) so for these models

$$\int d\rho(a) = 1 .$$

Weakly coupled $\mathscr{P}(\Phi)_2$ models have isolated one particle states, so $\int a^{-1} d\rho(a) < \infty$ and Proposition 1.1.6 follows for λ_j real.

2. More generally, (*a*) holds for $\mathscr{P}(\Phi)_2$ models except at the critical points (points of zero mass). It should also hold for Φ_3^4, or for other models with a strictly positive mass and a finite field strength renormalization

constant $Z^{-1} = \int d\rho(a)$. Furthermore, (*b*) should hold in any Wightman field theory.

Proof of Proposition 1.1.7. — Let $c = (-\Delta + 1)^{-1}$ have the kernel $C(x - y)$. We integrate by parts on q-space to obtain

$$S^{(2)}(x - y) = C(x - y) - \int dx' C(x - x') \int V'(\Phi(x'))\Phi(y) dq ,$$

where V′ is the derivative of the interaction polynomial V. Similarly,

$$G^{(2)}(\bar{f}, f) - \| f \|_{-1}^2 = - \langle V'(c\bar{f})\Phi(f) \rangle_T ,$$

where T denotes the connected (truncated) part,

$$\langle V'\Phi \rangle_T = \langle V'\Phi \rangle - \langle V' \rangle \langle \Phi \rangle .$$

Let $K(x, y) = \langle V'(\Phi(x))\Phi(y) \rangle_T$. By translation invariance of the measure dq, $K(x, y) = \zeta(x - y)$. Hence the operator k with kernel $K(x, y)$ commutes with c, and k is bounded on H_p if and only if k is bounded on H_0, as we also see by

(2.5) $\quad | K(f, g) | = | \langle f, kg \rangle | = | \langle c^{p/2}f, kc^{-p/2}g \rangle | \leq \| k \| \ \ \| f \|_{-p} \| g \|_p .$

Here $\| k \|$ denotes the operator norm on H_0 (hence on H_p).

We bound $\| k \|$ using estimates from the cluster expansion [3]. Locally, $K(x, y)$ is L_2 and hence L_1. It follows that $\zeta(x)$ is locally L_1. As a consequence of the spectral condition and the mass gap for the relativistic weak coupling $\mathscr{P}(\Phi)_2$ model with real coupling constants [3], $\zeta(x)$ is analytic for $x \neq 0$ and for $| x |$ bounded away from zero,

(2.6) $$| \zeta(x) | \leq O(\sup_j | \lambda_j |) e^{-m|x|} .$$

Thus $\zeta(x)$ is L_1 and

(2.6′) $$\| k \| \leq \| \zeta \|_{L_1} < O(\sup_j | \lambda_j |) .$$

It follows by (2.5) that k is a bounded operator on H_1, and that

(2.7) $$\Pi = - \Gamma k$$

is a bounded operator from H_1 to H_{-1}.

We next isolate the first order part of Π, by writing

$$V'(\Phi(x)) = \lambda_1 + 2\lambda_2 \Phi(x) + W(x) ,$$

with

$$W(x) = \sum_{n=3}^{l} n\lambda_n : \Phi(x)^{n-1} : .$$

By definition,

$$K(x, y) = 2\lambda_2 G^{(2)}(x, y) + K_1(x, y),$$

where $K_1(x, y) = \langle W(x)\Phi(y) \rangle_T$ is the integral kernel of the operator k_1. Thus

$$\Pi = -2\lambda_2 - \Gamma k_1, \tag{2.8}$$

and as above, Γk_1 is bounded from H_1 to H_{-1}. Since deg $W \geq 2$, integrating by parts produces at least one more V′ vertex in the evaluation of $\langle W\Phi \rangle_T$. This proves that $\| k_1 \|$ is at least second order in the λ_j, and at least first order in some λ_j, $j \geq 3$. The remaining assertions immediately follow for real coupling.

For complex coupling, we use the cluster expansion [*4*] to replace (2.6) by the estimate

$$\int_{|x| \leq 1} | \zeta(x + y) | \, dx \leq O(1) e^{-(m-\varepsilon)|y|}.$$

From (2.6′) we deduce Proposition 1.1.6 in the case of complex coupling constants, and then the remainder of the proof follows as before.

We begin the proof of Proposition 1.1.1 with two preliminary estimates. We use the notation of [*4*].

LEMMA 2.2. — The operator $(-\Delta + 1)^{1/2}$ maps $L_2(R^2)$ boundedly into $L_p(R^2)$ for all $p \in [2, \infty)$.

Proof. — This follows by Hausdorff-Young and Hölder inequalities. We consider the operators

$$\partial^b C_\gamma = \partial^b(-\Delta_\gamma + m_0^2)^{-1} = (-\Delta_\gamma + m_0^2)^{-1} - (\Delta_{\gamma \cup b} + m_0^2)^{-1}$$

and

$$\partial^\Gamma C_\gamma = \partial^{b_1} \ldots \partial^{b_m} C_\gamma$$

where b and b_i are lattice segments and $\Gamma = \{ b_1, \ldots, b_m \}$. Let

$$d(\alpha, \beta, \Gamma) = \sup_{\substack{i=\alpha,\beta \\ b \in \Gamma}} \operatorname{dist}(\Delta_i, b).$$

Recall that Δ_i, Δ_α and Δ_β are lattice squares, subsets of R^2, while Δ_γ, $\Delta_{\gamma \cup b}$, are Laplace operators with zero Dirichlet boundary data on lattice curves γ, $\gamma \cup b$, ...

LEMMA 2.3. — Let q be given, $1 \leq q < \infty$, and let m_0 be sufficiently large. The operator

$$(-\Delta + 1)^{1/2} \partial^\Gamma C_\gamma (-\Delta + 1)^{1/2} : L_2(\Delta_\alpha) \to L_2(\Delta_\beta)$$

has a norm bounded by

$$m_0^{O(1) - |\Gamma|/2q} K(\Gamma) \exp[-m_0 d(\alpha, \beta, \Gamma)/2]$$

where the constant $K(\Gamma)$ is independent of α, β, γ, Γ, m_0. Furthermore, the combinatoric estimate of [*4*], Proposition 8.2 holds.

The proof of the combinatoric estimates follows as in [*4*]. The analytic estimate is based on

LEMMA 2.4. — Let $(\Delta_\gamma - m_0^2)u = 0$ on some subdomain $\mathscr{D}$, and let $u \in L_2$. Then $\nabla u \in L_2$ on any interior subdomain $\mathscr{E}$ which is bounded and bounded away from $\partial\mathscr{D}$. Also

$$\| \nabla u \|_{L_2(\mathscr{E})} \leq \text{const.} \| u \|_{L_2(\mathscr{D})} .$$

Proof. — Away from γ, standard regularity estimates for elliptic operators apply. Thus $u \in C^\infty$ and the Sobolev norms are dominated by $\| u \|_{L_2(\mathscr{D})}$. Thus we consider u up to γ, which consists of lattice line segments. Along the interior of a line segment, u is regular by a reflection principle. For instance, consider the data u given on a semicircle. Extend the data to the opposite semicircle by (-1) times the reflection about the diameter. This determines a harmonic function, zero along the diameter, and hence regular up to the diameter. In the case of an interior corner, a similar reflection argument gives regularity up to γ.

It remains to consider u in a neighborhood of an exterior corner or endpoint ξ of γ. Let C be a circle about ξ, and let $v = \zeta u$, with ζ a smooth function, defined on the interior of C, equal to zero near ξ and equal to one near C. Then v, ∇v and $f = \Delta_\gamma v$ are bounded by const. $\| u \|_{L_2(\mathscr{D})}$. However, $\Delta_{\gamma \cup C}(v - u) = f$, so

$$\nabla u = \nabla v - \nabla(\Delta_{\gamma \cup C})^{-1} f .$$

Since $\nabla(\Delta_{\gamma \cup C})^{-1}$ is a bounded operator, ∇u is bounded in L_2 norm on a neighborhood of ξ.

This completes the proof of Lemma 2.4.

Away from Γ, the kernel of $\partial^\Gamma C_\gamma$ is a solution of the equations

$$(-\Delta_{\gamma,x} + m_0^2)\partial^\Gamma C_\gamma(x, y) = 0 = (-\Delta_{\gamma,y} + m_0^2)\partial^\Gamma C_\gamma(x, y) .$$

Thus by Lemma 2.4, away from Γ, L_2 bounds on $\partial^\Gamma C_\gamma$ [4, Proposition 8.1] imply corresponding L_2 bounds on $\nabla_x \nabla_y \partial^\Gamma C_\gamma(x, y)$. For x or y near Γ, we consider several cases. If either x or y is near $\Gamma_0 \subset \Gamma$, but bounded away from $\Gamma \sim \Gamma_0$, then L_2 bounds on

$$\partial^{\Gamma \sim \Gamma_0} C_{\gamma \cup \Gamma_{00}}(x, y) , \qquad \Gamma_{00} \subset \Gamma_0$$

imply L_2 bounds on

$$\nabla_x \nabla_y \partial^{\Gamma \sim \Gamma_0} C_{\gamma \cup \Gamma_{00}}(x, y) , \qquad \Gamma_{00} \subset \Gamma_0 .$$

The latter give (by linear combination and the definition $\partial^\Gamma = \partial^{\Gamma_0} \partial^{\Gamma \sim \Gamma_0}$ above) L_2 bounds on

$$\nabla_x \nabla_y \partial^\Gamma C_\gamma(x, y) ,$$

with the same decay as the bounds on $\partial^\Gamma C_\gamma$ of [4, Proposition 8.1]. Here we assume $\Gamma_0 \neq \Gamma$ and use the distance $d(\alpha, \beta, \Gamma \sim \Gamma_0)$ to give the decay. If $\Gamma_0 = \Gamma$, then the decay comes only from the separation between x and y. For x bounded away from y

$$(-\Delta_y + m_0^2)C_\gamma(x, y) = 0 = (-\Delta_x + m_0^2)C_\gamma(x, y)$$

and as above we find that

$$\nabla_x \nabla_y C_{\gamma \cup \Gamma_{00}}(x, y), \qquad \Gamma_{00} \subset \Gamma_0 = \Gamma$$

and

$$\nabla_x \nabla_y \partial^\Gamma C_\gamma(x, y)$$

have L_2 bounds with the same decay as $\partial^\Gamma C_\gamma$.

Finally for x, y and Γ all near each other, so $|\Gamma| \leq O(1)$, we use the fact that

$$C_\gamma \leq C_\phi = (-\Delta + m_0^2)^{-1}$$

to see that

$$\| \nabla_x \nabla_y C_\gamma \|$$

is bounded, where $\| \cdot \|$ denotes the L_2 operator norm. In this case, we do not obtain the factor $m_0^{-(2/q)|\Gamma|} = m_0^{-O(1)}$. Combining all cases gives the proof of Lemma 2.3.

The cluster expansion of [4] is based on localizations of the interaction occurring in exponents such as $\exp(\Phi(J))$. With $J \in H_{-1}$, we cannot localize by multiplying J by a characteristic function χ_α of a unit lattice square Δ_α. Instead we define

$$J_\alpha = (-\Delta + 1)^{1/2}\chi_\alpha(-\Delta + 1)^{-1/2}J.$$

Thus $J_\alpha \in H_{-1}$ and

$$J = \sum_\alpha J_\alpha, \qquad \| J \|_{-1}^2 = \sum_\alpha \| J_\alpha \|_{-1}^2.$$

We let γ denote a union of lattice segments in R^2, and let Δ_γ be the Laplace operator with Dirichlet boundary conditions on γ. We let $d\Phi_\gamma$ denote the Gaussian measure on $\mathcal{S}'$ with mean zero and covariance $(-\Delta_\gamma + m_0^2)^{-1} = C_\gamma$. We may take the infinite volume limit for the interacting theory with the Dirichlet contour γ and define for $h \in C_0^\infty$,

$$dq_\gamma = e^{-V(h)} d\Phi_\gamma.$$

We note that dq_γ is not normalized, and refers to a finite volume approximation to the infinite volume measure. We let $J \in C_0^\infty$ and define

$$Z\{J, \gamma\} = \int e^{\Phi(J)} dq_\gamma / e^{\int \langle \Phi(x) \rangle J(x) dx}$$

$$Z_0\{J, \gamma\} = Z\{J, \gamma\}\, e^{\int \langle \Phi(x) \rangle J(x) dx} = \int e^{\Phi(J)} dq_\gamma.$$

Our estimates will be uniform for $\| J \|_{-1}$ small. The contour γ does not occur in the expectation $\langle \Phi \rangle$ in the exponent. In case $R^2 \sim \gamma$ has two components—Ext γ and Int γ—the measure dq_γ factors into an interior and an exterior measure. In an obvious notation

$$Z_0 \{ J, \text{Int } \gamma \} = \int \exp \left[\int_{\text{Int } \gamma} \Phi(x) J(x) dx \right] dq_\gamma ,$$
$$Z \{ J, \gamma \} = Z \{ J, \text{Ext } \gamma \} Z_0 \{ J, \text{Int } \gamma \} .$$

Observe that the normalization differs $\left(\text{by the factor } \int dq \right)$ from that of § 1.1. The statement of the following proposition is independent of the normalization, since it concerns a ratio of partition functions. In the proof we use the above unnormalized measures.

PROPOSITION 2.5. — Let $R^2 \sim \gamma$ have two components and let J belong to $H_{-1,\varepsilon}$. In a weak coupling region,

$$| Z \{ J, \text{Ext } \gamma \} / Z \{ J \} | \leq e^{O(|\text{Int } \gamma| + \|J\|_{-1}^2)} .$$

Proof. — The normalizing factor $\exp \left[\int \langle \Phi(x) \rangle J(x) \, dx \right]$ cancels between the numerator and denominator and plays no role in the proof. Let (Int γ)* be the set of lattice lines in Int γ. We write

$$(2.9) \quad \left| \frac{Z \{ J, \text{Ext } \gamma \}}{Z \{ J \}} \right| = \left| \frac{Z \{ J, \gamma \}}{Z \{ J \} Z_0 \{ J, \text{Int } \gamma \}} \right|$$
$$= \left| \frac{Z_0 \{ J, (\text{Int } \gamma)^* \}}{Z_0 \{ J, \text{Int } \gamma \}} \times \frac{1}{Z_0 \{ J, (\text{Int } \gamma)^* \}} \times \frac{Z \{ J, \gamma \}}{Z \{ J \}} \right|$$

and we bound each of these factors by $\exp \left[O(|\text{Int } \gamma| + \| J \|_{-1}^2) \right]$. We first bound $| Z \{ J, \gamma \} / Z \{ J \} |$. This bound is proved in the finite volume approximation, uniformly in the volume, using the Kirkwood-Salsburg equations [*4*, Proposition 5.2]. The same argument, applied within Int γ, bounds the ratio $| Z_0 \{ J, (\text{Int } \gamma)^* \} / Z_0 \{ J, \text{Int } \gamma \} |$. We have only to indicate how the proof of [*4*] is modified to allow estimates depending on $J \in H_{-1}$.

A typical term in the expansion has the form

$$(2.10) \quad \int K(\Phi, J, \gamma') e^{\int (\Phi - \langle \Phi \rangle) J(x) dx} dq_{\gamma'} / Z \{ J \}$$
$$= \int K(\Phi, J, \gamma') e^{\int_{\text{Int } \gamma'} \Phi(x) J(x) dx} dq_{\text{Int } \gamma'} Z \{ J, \text{Ext } \gamma \} / Z \{ J \}$$

where $K(\Phi, J, \gamma')$ denotes some function on path space. The Kirkwood-Salsburg equations for $\rho(\gamma) = Z\{J, \gamma\}/Z\{J\}$ have the form

$$\rho = 1 + \mathscr{K}\rho,$$

see [4, Chapters 3, 6]. These equations have a unique solution $(I - \mathscr{K})^{-1}1 = \rho$ satisfying

$$|\rho(\gamma)| \leq \exp(|\operatorname{Int}\gamma|)$$

if the operator $\mathscr{K}$ is a contraction. Using the Banach space as in [4], we require bounds of order $\exp(-K|\operatorname{Int}\gamma'|)$ on the kernel in (2.10),

$$\int K(\Phi, J, \gamma') e^{\int_{\operatorname{Int}\gamma'}\Phi(x)J(x)dx} dq_{\operatorname{Int}\gamma'} Z_0\{J, (\operatorname{Int}\gamma')^*\}^{-1}. \tag{2.11}$$

By the Schwarz inequality, (2.11) is dominated by

$$\left[\int |K(\Phi, J, \gamma')|^2 d\Phi_{\operatorname{Int}\gamma'}\right]^{1/2} \left[\int dq'_{\operatorname{Int}\gamma'}\right]^{1/4} \times \left[\int \exp\left[4 \operatorname{Re}\int_{\operatorname{Int}\gamma'} \Phi(x)J(x)dx\right] d\Phi_{\operatorname{Int}\gamma'}\right]^{1/4} Z_0\{J, (\operatorname{Int}\gamma')^*\}^{-1}. \tag{2.12}$$

Here the measure dq' has all the parameters λ_j replaced by $4 \operatorname{Re}\lambda_j$. The integral over dq' is independent of J, and by standard estimates is dominated by $\exp(O|\operatorname{Int}\gamma'|)$. The last factor in (2.12) can be calculated explicitly as

$$\exp\left[2\int_{\operatorname{Int}\gamma'} \operatorname{Re} J(x)C_\gamma(x, y) \operatorname{Re} J(y)dx\,dy\right] \leq \exp\left[2\|J\|_{-1}^2\right].$$

In the first integral, only the J-vertices require special treatment. We use the H_{-1} localization $J = \Sigma J_\alpha$ as above. The case of a contraction $\partial^\Gamma C_\gamma$ joining J_α and J_β contributes to K the numerical factor

$$|\langle J_\alpha \partial^\Gamma C_\gamma J_\beta \rangle| \leq \|\chi_{\Delta_\alpha}(-\Delta + 1)^{1/2}\partial^\Gamma C_\gamma(-\Delta + 1)^{1/2}\chi_{\Delta_\beta}\| \ \|J_\alpha\|_{-1} \|J_\beta\|_{-1}.$$

The sum over α, β of these factors is bounded by Lemma 2.3. In the case of a contraction $\partial^\Gamma C_\gamma$ joining J_α to a vertex from dq, the vertex is multiplied by the function $\partial^\Gamma C_\gamma(x, J_\alpha)$. By Lemmas 2.2 and 2.3, this function is L_p for all $p < \infty$, with a norm bounded by the decay rate specified in Lemma 2.3. Hence for $p \in [2, \infty)$,

$$\|\partial^\Gamma C_\gamma(\cdot, J)\|_{L_p} \leq \text{const.}\ \|J\|_{-1}.$$

Multiplication of a vertex from dq by an L_p function does not affect the proofs of [4], and so with the above modifications the proof is that of [4].

Finally, to complete the bound on (2.12), we must show that

$$|Z_0\{J, (\text{Int }\gamma)^*\}|^{-1} \leq e^{O|\text{Int }\gamma|},$$

which then also yields the desired bound on (2.9). We note that

$$Z_0\{J, (\text{Int }\gamma)^*\} = \Pi Z_0\{J, \text{Int }\Delta_\alpha\}$$

where the product ranges over $\Delta_\alpha \subset \text{Int }\gamma$. Thus we must show that

$$(2.13) \qquad |Z_0\{J, \text{Int }\Delta_\alpha\}|^{-1} \leq O(1).$$

Using a Duhamel formula and a Schwartz inequality with respect to the free measure, one shows that

$$|Z_0\{J, \text{Int }\Delta_\alpha\} - Z_0\{0, \text{Int }\Delta_\alpha\}| \leq O(\|J\|_{-1}).$$

Hence for small λ_j, (2.13) holds and the proof of the proposition is complete. In the foregoing, we have for notational convenience suppressed the dependence on the multiparameter s which interpolates between the various measures dq_γ. The factor $m_0^{O(1)}$ from Lemma 2.3 is compensated for by factors $O(|\lambda_j|) \ll 1$.

Proof of Proposition 1.1.1. — We obtain a uniform bound on the finite volume approximations to $Z\{J\}$. The desired result then follows from convergence for $J \in C_0^\infty$ and Proposition A3. We proceed by induction, adding one local piece J_α at each step. Let $\{\zeta_\alpha : \alpha \in Z^2\}$ be a C^∞ partition of unity, invariant under lattice translations, and let $J_\alpha = \zeta_\alpha J$. We will show later that $\Sigma \|J_\alpha\|_{-1}^2 \leq \text{const.} \|J\|_{-1}^2$. Let $B \subset Z^2$ be a finite set, let $\alpha \in Z^2 \sim B$, and define

$$J_B = \sum_{\beta \in B} J_\beta$$

$$J(s) = (1-s)J_B + s(J_\alpha + J_B) = J_B + sJ_\alpha.$$

To state the inductive hypothesis, we define

$$c_{\beta_1\beta_2} = O(1)e^{-(m_0-\varepsilon)|\beta_1-\beta_2|}$$

$$C_{B,\beta} = \sum_{\beta_1 \in B} c_{\beta_1,\beta} \leq C \equiv O(1)\sum_{\beta_1 \in Z^2} c_{\beta_1,\beta}.$$

The inductive hypothesis is

$$(2.14) \qquad |Z\{J_B\}| \leq \exp\left[\sum_{\beta \in B} C_{B,\beta}\|J_\beta\|_{-1}^2\right],$$

which yields

$$|Z\{J\}| \leq \exp\left[C\Sigma\|J_\beta\|_{-1}^2\right] \leq \exp\left(C'\|J\|_{-1}^2\right).$$

We have, with $\hat{\Phi} = \Phi - \langle \Phi \rangle$,

$$\begin{aligned} Z\{ J_B + J_\alpha \} - Z\{ J_B \} &= \int_0^1 \frac{d}{ds} Z\{ J(s) \} ds \\ &= \int_0^1 ds \int [(\Phi(x) - \langle \Phi \rangle) J_\alpha(x) dx] e^{\int \hat{\Phi}(J(s))} dq \\ &= \int_0^1 ds [\langle \Phi(J_\alpha) e^{\hat{\Phi}(J(s))} \rangle - \langle \Phi(J_\alpha) \rangle \langle e^{\hat{\Phi}(J(s))} \rangle] . \end{aligned}$$

Assuming the bound (2.14) on $Z\{ J_B \}$,

$$(2.15) \quad | Z\{ J_B + J_\alpha \} | \leq \exp \left[\sum_{\beta \in B} C_{B,\beta} \| J_\beta \|_{-1}^2 \right] + \sup_s | \langle \Phi(J_\alpha) e^{\hat{\Phi}(J(s))} \rangle_T | .$$

We show that

$$(2.16) \quad | \langle \quad \rangle_T | \leq | Z\{ J_B \} | \left[(C_{B,\alpha} + c_{\alpha\alpha}) \| J_\alpha \|_{-1}^2 + \sum_{\beta \in B} c_{\beta\alpha} \| J_\beta \|_{-1}^2 \right] .$$

Substituting in (2.15) yields

$$| Z\{ J_B + J_\alpha \} | \leq \exp \left[\sum_{\beta \in B \cup \alpha\}} C_{B \cup \alpha, \beta} \| J_\beta \|_{-1}^2 \right]$$

and completes the induction.

To bound $\langle \quad \rangle_T$, we choose

$$A = \Phi(J_\alpha) , \qquad B = e^{\hat{\Phi}(J(s))}$$

in (2.2). We apply the cluster expansion to $\langle A_0 B_0 \rangle$, and each expansion term is even under the symmetry $\Phi \leftrightarrow \Phi^{\sim}$, while A_0 and B_0 are individually odd. Using the fact that

$$B_0 = (e^{\hat{\Phi}(J(s))} - I)_0 = (e^{\hat{\Phi}(J(s))} - I) \otimes I - I \otimes (e^{\hat{\Phi}(J(s))} - I) ,$$

we see that every term in $\langle A_0 B_0 \rangle$ is at least quadratic in J:

$$(\text{linear in } J_\alpha) \times (\text{linear in } J(s)) .$$

In the cluster expansion we choose contours γ which contain suppt J_α. Then the region exterior to the contour contributes the factor

$$O(1) | Z\{ J_B, \text{Ext } \gamma \} | \leq e^{O|\text{Int } \gamma| + \|J_B\|_{-1}^2} | Z\{ J_B \} | ,$$

by Proposition 2.5. The region Int γ contributes $O(1) \exp[- K | \text{Int } \gamma |]$ as in [*4*, Proposition 5.2]. We choose K sufficiently large, and the remaining estimates follow as in [*4*].

To complete the proof, we only need to show that

$$\Sigma \| J_\alpha \|_{-1}^2 \leq \text{const.} \| J \|_{-1}^2 .$$

In fact

$$\Sigma \| J_\alpha \|_{-1}^2 = \Sigma \| \zeta_\alpha J \|_{-1}^2 = \| J \|_{-1}^2 + \Sigma \langle \zeta_\alpha J, [(-\Delta + 1)^{-1}, \zeta_\alpha] J \rangle .$$

The second term above is dominated by

$$\begin{aligned}(\Sigma \| J_\alpha \|_{-1}^2)^{1/2}(\Sigma \| (-\Delta + 1)^{-1/2}[-\Delta, \zeta_\alpha](-\Delta + 1)^{-1} J \|_0^2)^{1/2} \\ \leq (\Sigma \| J_\alpha \|_{-1}^2)^{1/2} \| \Sigma - \Delta + 1)^{-1/2}[-\Delta, \zeta_\alpha](-\Delta + 1)^{-1} \\ [-\Delta, \zeta_\alpha](-\Delta + 1)^{-1/2} \|^{1/2} \| J \|_{-1} \\ \leq \text{const.} \| J \|_{-1} (\Sigma \| J_\alpha \|_{-1}^2)^{1/2} ,\end{aligned}$$

from which the desired bound follows.

APPENDIX

ANALYTIC FUNCTIONS

We consider functions f which map a subset Ω of one Banach space E into a Banach space F. The case $E = C^n$ is the case of vector valued analytic functions. Consider two conditions:

(i) Ω is open in the norm topology.

(ii) Suppose that for each finite sequence $A_j \in E$, $z_j \in C$ such that $z_1A_1 + \ldots + z_nA_n \in \Omega$, the function $f(z_1A_1 + \ldots + z_nA_n)$ is analytic in $z = (z_1, \ldots, z_n)$.

DEFINITION A1. — If (i), (ii) are satisfied, the function $f : \Omega \to F$ is *analytic* in Ω. If $A \in \Omega$, we say that $f(\cdot)$ is analytic at A. If $\Omega = E$, then f is *entire*.

DEFINITION A2. — Let $\mathcal{M}\mathcal{L}_n(E, F)$ be the space of bounded multilinear forms of n arguments in E, with values in F. Let $\mathcal{M}\mathcal{L}_n(E, C) = \mathcal{M}\mathcal{L}_n(E)$.

By Vitali's theorem and a 3ε-argument we obtain

PROPOSITION A3. — Let $f_n : \Omega \to F$ be a sequence of functions analytic in Ω, with $\| f_n \|_F$ uniformly bounded on Ω. If $f_n \to f$ pointwise on a dense subset of Ω, then f is analytic in Ω. Also both f_n and the derivatives of $f_n(\Sigma z_i B_i)$ converge pointwise in F norm to f or its derivatives.

PROPOSITION A4. — If $f : \Omega \to F$ is analytic at $A \in \Omega$, then the derivative

$$\frac{d}{dz_1} \cdots \frac{d}{dz_n} f\left(A + \sum_{i=1}^{n} z_i B_i\right)\Bigg|_{z=0} = h_A^{(n)}(B_1, \ldots, B_n)$$

is an element of $\mathcal{M}\mathcal{L}_n(E, F)$ with norm less than $\alpha\beta^n n\,!$, for constants $\alpha = \alpha(A)$ and $\beta = \beta(A)$.

Proof. — Consider $n = 1$. By analyticity

$$\| f(A + z_1(B + C)) - f(A) - z_1 h_A^{(1)}(B + C) \|_F \leq O(| z_1 |^2).$$

Also for $n = 2$,

$$\| f(A + z_1 B + z_2 C) - f(A) - z_1 h_A^{(1)}(B) - z_2 h_A^{(1)}(C) \|_F \leq O(| z_1 |^2 + | z_1 z_2 | + | z_2 |^2).$$

Setting $z_1 = z_2$, we have

$$h_A^{(1)}(B + C) = h_A^{(1)}(B) + h_A^{(1)}(C).$$

The multilinearity of $h_A^{(n)}$ for $n > 1$ follows similarly by induction.

We choose $\| B_i \|_E < 1$. Then by the Cauchy integral formula (with integration over a product of circles of radius $O(n^{-1})$),

$$\| h_A^{(n)}(B_1, \ldots, B_n) \|_F \leq \alpha(A)(\beta n)^n.$$

By multilinearity, for arbitrary $B_i \in E$,

$$\| h_A^{(n)}(B_1, \ldots, B_n) \|_F \leq \alpha(A)(\beta n)^n \prod_{i=1}^{n} \| B_i \|_E,$$

which completes the proof.

PROPOSITION A5. — If f is an entire function, and $\varepsilon > 0$,

$$\| h_A^{(n)}(B_1, \ldots, B_n) \|_F \leq \alpha(A)\varepsilon^n n! \prod_{i=1}^{n} \| B_i \|_E .$$

This proof follows the usual proof for $E = C^n$.

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