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Section A :
Physique théorique.

Local existence for finitely predictive two-body interactions

by

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ABSTRACT. — The conditions of finite predictivity for classical relativistic two-body systems are investigated by means of the Cauchy-Kowalewsky theorem. Local existence theorems are given permitting the construction of solutions by Cauchy data on surfaces in phase space. The asymmetric electromagnetic interaction through the advanced field of a particle and the retarded field of the other one is recovered as a particular case.

1. INTRODUCTION

In classical relativistic dynamics, equations of motion for interacting point particles are generally difference-differential equations and very little is known about their solutions.

For instance, Feynman-Wheeler electrodynamics yields such equations for a couple of charged particles. In this example, although configuration space is of finite dimension, there is no evidence for a finite number of degrees of freedom, because phase space (i. e. the space of initial data determining the motion) has most likely infinite dimensions. However, it is also possible to consider interactions of a simpler kind, requiring that

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the equations of motion take the form of an ordinary differential system.

In this case, phase space has only a finite number of dimensions and we shall say that the dynamical system is *finitely predictive.*

It is clear that finitely predictive mechanics is a special case of Action-at-a-distance theories. Even if too simple with respect to the actual interactions present in the Nature, finitely predictive dynamics is at least mathematically tractable and is to be mastered first, before one attacks the general and very intricate case of infinitely dimensional phase spaces. Indeed, finitely predictive dynamics developed slowly and the early literature devoted to this subject has a non covariant form [*1*] [*2*] [*3*].

Standing for manifest covariance we suggested in previous papers [*4*] an equal use of all proper-times. Thus we are dealing with multi-time systems.

In the present article we are mainly concerned with the two-particle case, but the general framework of our formulation is valid in the N-particle case in an obvious way. For two particles, finitely predictive equations of motion have the form of a differential system as follows:

$$\frac{dx}{d\tau} = v \qquad \frac{dv}{d\tau} = \xi(x, v, x', v')$$
$$\frac{dx'}{d\tau'} = v' \qquad \frac{dv'}{d\tau'} = \xi'(x, v, x', v') \tag{1.1}$$

where it is required that the solutions have the form

$$x = x(\tau), \qquad x' = x'(\tau') \tag{1.2}$$

in order to finally find the world lines (^{2}).

Things can be put into geometrical form by saying that solving (1.1) is equivalent to finding in the 16-dimensional space (where coordinates are x, v, x', v') the family of 2-dimensional surfaces tangent to both vector fields

$$X = v.\partial + \xi.\partial/\partial v$$
$$X' = v'.\partial' + \xi'.\partial/\partial v'$$

Conditions (1.2) add that each such integral surface is obtained by cartesian product of the curve $x = x(\tau)$, $v = v(\tau)$, by the curve $x' = x'(\tau')$, $v' = v'(\tau')$, these curves being the lifts of the world lines into the individual phase spaces useful for one-particle physics.

The well-known Frobenius theorem states that (1.1) is integrable, iff X

(^{2}) The momenta v, v', are not constrained. The masses defined by $m^2 = v \cdot v$, $m'^2 = v' \cdot v'$ will be constant provided $v \cdot v$ and $v' \cdot v'$ are first integrals. In this case the canonical parameters τ, τ' are related to the true proper-times through $\tau = s/m$, $\tau' = s'/m'$. When possible the indices are dropped, for instance $v \cdot v$ stands for $v^\alpha v_\alpha$, etc. Greek indices $= 0, 1, 2, 3$, and $c = 1$.

and X′ form a two dimensional LIE algebra. Therefore, we must have, in terms of LIE brackets

(1.3) $$[X, X'] = \alpha X + \beta X'.$$

In order that condition (1.2) holds a stronger condition arises:

(1.4) $$[X, X'] = 0.$$

Proof. — Consider X and X′ as differential operators acting on phase functions. Then we have obviously

$$Xx^\alpha = v^\alpha \qquad Xx^{\alpha'} = 0$$
$$X'v^\alpha = 0 \qquad X'x^\alpha = 0$$

Hence

$$[X, X']x^\mu = 0$$

But from (1.3) we also have

$$[X, X']x^\mu = \alpha v^\mu$$

Thus $\alpha = 0$. A similar argument shows that β also vanishes. Hence we get condition (1.4) that we derived by a different approach in [*4*].

As already shown, the explicit form of (1.4) is

(1.5) $$\begin{aligned} &(v\cdot\partial + \xi\cdot\partial/v)\xi^{\mu'} = 0 \\ &(v'\cdot\partial' + \xi'\cdot\partial/\partial v')\xi^\mu = 0 \end{aligned}$$

Equation (1.5) is trivially satisfied by any system involving no interaction. In this case ξ depends on x, v only, while ξ' depends on x', v' only.

The explicit construction of non-trivial systems is uneasy because of the non-linearity of (1.5).

Our present goal is to solve the Cauchy problem in order to exhibit local solutions [*5*]. These solutions will not be trivial provided the initial data involve interaction. Of course, not all the solutions of (1.5) satisfy the additional condition of mass constancy:

(1.6) $$\xi\cdot v = \xi'\cdot v' = 0.$$

For simplicity, we shall discuss (1.5) alone. Then the discussion will be generalized in order to deal with (1.6) also.

2. THE CAUCHY PROBLEM

There is no compelling need to require the full Poincaré symmetry from the beginning ([3]), but we shall look for translation invariant solutions,

([3]) The natural representation of the Poincaré group acting on the 2-body phase space has the following generators:

$$P = \partial + \partial', \qquad N = M + M'$$

where

$$M = x \wedge \partial + v \wedge \partial/\partial v, \qquad M' = x' \wedge \partial' + v' \wedge \partial/\partial v'$$

just for convenience, the assumption of translational invariance giving rise to drastic simplifications.

Accordingly, ξ and ξ' will depend on r, v, v' only, with $r = x - x'$ (4).

For the Cauchy problem, phase space is therefore replaced by the 12-dimensional space E of the triplets (r, v, v').

Cauchy problem. — Assume that ξ and ξ' are given and analytic on some 11-dimensional surface S. Determine the eight unknown quantities $\xi^\alpha, \xi^{\alpha'}$ outside (S) within some neighborhood. Since for any differentiable f.

$$\partial_\alpha f = \partial f/\partial r^\alpha, \qquad \partial_{\alpha'} f = -\partial f/\partial r^\alpha$$

The restriction of Equation (1.5) to E is

$$(2.1) \qquad \begin{aligned} &(v\cdot\partial/\partial r + \xi\cdot\partial/\partial v)\xi^{\mu'} = 0 \\ &(-v'\cdot\partial/\partial r + \xi'\cdot\partial/\partial v')\xi^{\mu} = 0 \end{aligned}$$

The most naive attack is to take the hyperplane $r^0 = 0$ as surface (S). Then, considering the evolution in terms of the preferred coordinate r^0, one is led to a straight-forward conclusion of local existence. We do not give nore details because the plane $r^0 = 0$ is not invariant under the Poincaré group. We prefer to start on the surface

$$r\cdot r = \mu^2$$

μ^2 being possibly negative as well.

Keeping in mind that our initial surface in E is physically meaningful as a set of possible initial elements defining the two-body motion, we are led to discard space-like velocities by setting

$$v\cdot v \geq 0 \quad \text{and} \quad v'\cdot v' \geq 0.$$

Moreover, we consider v and v' future oriented. Thus we are now dealing with the surface

$$(2.2) \qquad (\Sigma) = \mathrm{H}_{(\mu)} \times \Omega \times \Omega$$

with the following notations:

$\mathrm{H}_{(\mu)}$ is the hyperboloid (possibly degenerated into a cone) of mass μ. $\mathrm{H}^+_{(\mu)}$ (resp. $\mathrm{H}^-_{(\mu)}$)is the sheet of future (resp. past) oriented r. Ω is the domain of future no -space-like velocities. Naturally, when $\mu \geq 0$ we have to choose a sheet of $\mathrm{H}_{(\mu)}$. It is convenient to choose as new coordinates

$$\rho,\ r^i,\ v^\alpha,\ v^{\alpha'}$$

(where $\rho = r\cdot r$) instead of $r^0, r^i, v^\alpha, v^{\alpha'}$. Note that the transformation from r^α to ρ, r^i is singular on the hyperplane $r^0 = 0$. Provided we keep

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(4) Indices being dropped as often as possible, r is not to be mistaken for $|r\cdot r|^{1/2}$.

off a neighborhood of this hyperplane, Equation (2.1) can be transformed in terms of the new coordinates.

The identity

$$\partial/\partial r^{\alpha} = 2r_{\alpha}\partial/\partial\rho + \delta^{i}_{\alpha}\partial/\partial r^{i} \tag{2.3}$$

implies

$$\begin{aligned} v\cdot\partial/\partial r &= 2v\cdot r\partial/\partial\rho + v^{i}\partial/\partial r^{i} \\ v'\cdot\partial/\partial r &= 2v'\cdot r\partial/\partial\rho + v^{i'}\partial/\partial r^{i} \end{aligned} \tag{2.4}$$

Inserting (2.4) into (2.1) we get

$$\begin{aligned} 2v\cdot r\partial\xi^{\mu'}/\partial\rho &\sim 0 \\ 2v'\cdot r\partial\xi^{\mu}/\partial\rho &\sim 0 \end{aligned}$$

modulo the Cauchy Data (C. D.) which are the ξ^{μ}, $\xi^{\mu'}$ and their derivatives with respect to r^{i}, v^{α}, $v^{\alpha'}$.

Thus $\partial\xi^{\mu}/\partial\rho$ and $\partial\xi^{\mu'}/\partial\rho$ are determined in terms of the C. D. except on the points of E where either $v\cdot r$ or $v'\cdot r$ vanish.

i) Case $\mu^2 < 0$.

Besides the fact that $H_{(\mu)}$ intersects the hyperplane $r^0 = 0$, (Σ) is characteristic on infinitely many points, because r, being space-like, can be orthogonal to either v or v' for some choices of the velocities in Ω. The trouble is that we cannot discard all these points. For instance, the case of v and v' parallel to each other and both orthogonal to a space-like r, represents two particles mutually at rest when their initial positions are considered. Finally, there is no obvious answer to the Cauchy problem when $\mu^2 < 0$.

ii) Case $\mu^2 > 0$.

From (Σ) let us consider only one sheet, for instance, $(\Sigma^-) = H^-_{(\mu)} \times \Omega \times \Omega$ which corresponds to choosing x anterior to x'. First, (Σ^-) does not intersect the hyperplane $r^0 = 0$. Moreover, (Σ^-) is nowhere characteristic, because on (Σ^-) the vector r is always time-like, hence can be orthogonal neither to v nor to v'. Then Cauchy-Kowalewsky theorem applies.

If ξ and ξ' are given and analytic on (Σ^-) we have an analytic solution of (2.1) in some neighborhood of (Σ^-) [*6*].

iii) Case $\mu = 0$.

Now $(\Sigma) = H_{(0)} \times \Omega \times \Omega$ and H(0) is the null cone. Let us choose its sheet $H^-_{(0)}$, for instance. Then, we consider, instead of (Σ), the one-sheet surface $H^-_{(0)} \times \Omega \times \Omega$. To avoid singularity we must now subtract from $H^-_{(0)}$ a neighborhood of the vertex, say ω. We must also avoid having either v or v' null and colinear to r, because we cannot let $v\cdot r$ or $v'\cdot r$ vanish. So we require

$$v\cdot v \geq \zeta \quad \text{and} \quad v'\cdot v' \geq \zeta$$

for some fixed ζ, arbitrarily small but anyway positive. By all these restrictions (Σ) is finally replaced by a surface (Σ_*^-) such that: (Σ_*^-) has only one sheet.

The transformation from ρ, r^i to r^0, r^i is not singular on it. (Σ_*^-) has everywhere a normal and is never characteristic, because v and v', being strictly time-like, cannot be orthogonal to r which is null. Thus, for ξ and ξ' analytically given on (Σ_*^-) we have a unique analytic solution of (2.1) in some neighborhood of (Σ_*^-) [6].

3. MASS CONSTANCY

As previously mentioned, the mass constancy condition (1.6) is not contained in Equation (1.5). But if we require that ξ and ξ' have the form

$$\xi^\alpha = v^\beta \phi^\alpha_\beta, \qquad \xi^{\alpha'} = v^{\beta'} \phi'^\alpha_\beta, \tag{3.1}$$

where $\phi_{\alpha\beta}$ and $\phi'_{\alpha\beta}$ are skew symmetric in the indices, then (1.6) is automatically satisfied. Therefore, let us assume that ξ and ξ' have this form and take ϕ and ϕ' as unknown quantities to be determined in such a way that (1.5) should be satisfied. Inserting (3.1) into (1.5) we get

$$\begin{aligned} &v^{\alpha'}(v\cdot\partial/\partial r + (v\cdot\phi)\cdot\partial/\partial v)\phi'_{\alpha\beta} = 0 \\ &v^\alpha(-\, v'\cdot\partial/\partial r + (v'\cdot\phi')\cdot\partial/\partial v')\phi_{\alpha\beta} = 0 \end{aligned} \tag{3.2}$$

as the condition to be satisfied by ϕ and ϕ'.

Since all we want now is to exhibit some solutions, let us consider the *sufficient* condition

$$\begin{aligned} &(v\cdot\partial/\partial r + (v\cdot\phi)\cdot\partial/\partial v)\phi'_{\alpha\beta} = 0 \\ &(-\, v'\cdot\partial/\partial r + (v'\cdot\phi')\partial/\partial v')\phi_{\alpha\beta} = 0 \end{aligned} \tag{3.3}$$

Again we can mak use of (2.4) and it turns out that the Cauchy Problem for ϕ and ϕ' with Equation (3.3) is the same as for ξ and ξ' with Equation (2.1). The discussion carried out in Section 2, is still valid for the « fields » ϕ, ϕ'. Therefore, (3.3) has local solutions on the domains discussed above.

COROLLARY. — If ξ and ξ' have the form (3.1) on the initial surface (Σ^-) or (Σ_*^-), this form is preserved by evolution, in some neighborhood.

This follows immediately from unicity.

4. THE ASYMMETRIC SOLUTION

The most general kind of electromagnetic interaction is not likely to exhibit finite predictivity [7]. Nevertheless, it is reasonable to ask whether some special case of motion under electromagnetic forces is finitely predictive, some connection between electromagnetic interactions and finitely predictive systems being possible after all.

Schild's solution [8] and Driver's one dimensional motion support this possibility. In the present approach we look for the C. D. appropriate to electromagnetic interpretation. Due to the propagation and invariance properties of electromagnetism, it is convenient to take a part of the surface $(x - x')^2 = 0$ as initial surface. In a perturbative treatment of Equation (1.5) L. Bel, A. Salas, and J. M. Sánchez [9] determine series solutions by requiring that, when $(x - x')^2 = 0$ the expressions of ξ and ξ' coincide with those given by Lienard-Wiechert formulae. We somehow follow this line in the context of Cauchy Problem, but we insist that particular attention is needed here, because the surface $(x - x')^2 = 0$ has *two* sheets. Only one sheet at once can be considered in order to apply the Cauchy-Kowalewsky theorem. Let us for instance, take the surface Σ_*^- described in Section 2. On Σ_*^-, x' always lies in the future of x.

On this surface, x' always lies in the future of x (i. e., $x - x'$ is past oriented). The possibility of inserting Lienard-Wiechert expressions into the Cauchy data arises as follows:

Consider the two world lines occuring in the two-body problem, pick up x on the one and x' on the other. As usual, introduce retarded and advanced positions $x_{\text{ret.}}$, etc. In general $x_{\text{ret.}}$ *is not* a function on phase space, because it depends on x' and instead of simply x, on the world-line passing through x. However, in the special case where $x - x'$ is null and *past-oriented*, then $x_{\text{ret.}}$ coincides with x. Moreover, in this case, $x'_{\text{adv.}}$ coincides with x'.

As a result, all we can do is to use the *retarded* field generated by a charge at x and acting on x', with the *advanced* field generated by a charge at x' and acting on x. This *asymmetric combination* of Lienard-Wiechert formulae is unusual and unsatisfactory in many respects. Actually, such a combination has been already introduced by Rudd an Hill [10] in the non-covariant formalism, and also Staruszkiewicz within a covariant one parameter framework very specific of the model [11]. We are going to incorporate this Rudd-Hill-Staruszkiewicz (R. H. S.) solution into the multi-time formalism, not explicitly, but as defined by initial data.

The law of retarded action from x, with charge e, to x' with charge e', yields

$$\xi^{\alpha'} = e' v^{\sigma'} \mathrm{F}'^{\alpha}_{\sigma\,\text{ret.}} \tag{4.1}$$

and the advanced action from x' to x yields

$$\xi^{\alpha} = e v^{\sigma} \mathrm{F}^{\alpha}_{\sigma\,\text{adv.}} \tag{4.2}$$

where the general Lienard-Wiechert formulae give $\mathrm{F}'_{\text{ret.}}$ and $\mathrm{F}_{\text{adv.}}$. Actually, these formulae give F' and F as some expressions of r, the charges, the velocities and the accelerations, to be taken when $r^2 = 0$.

Let us consider, for instance, the field F which depends on e', r, u' and also the acceleration of the charge e'. Writing explicitly u'^2 instead of the unity at some place in the conventional formula makes it homogeneous, that, is, invariant when u' is replaced by $\lambda u'$, while the acceleration of charge e' is multiplied by λ^2. Now one can as well write v' instead of u' and ξ' instead of the acceleration. Finally one gets

$$\begin{aligned} F_{\text{ret. adv.}} = \pm e' \{ (r\cdot v')^{-3}(v'^2 - r\cdot\xi')r \wedge v' \\ + (r\cdot v')^{-2} r \wedge \xi' \} \end{aligned} \tag{4.3}$$

where it is understood that $r^2 = 0$, and a similar formula for F′ say (4.3′). Therefore, on the surface Σ_*^- we have

$$\begin{aligned} F_{\text{adv.}} &= P(r, v', \xi') \\ F'_{\text{ret.}} &= Q(r, v, \xi) \end{aligned} \tag{4.4}$$

P and Q being defined by (4.3) (4.3′). Inserting (4.4) into (4.1) (4.2) yields

$$\begin{aligned} \xi &= R(r, v, v') \\ \xi' &= S(r, v, v') \end{aligned} \tag{4.5}$$

where R and S are defined by solving (4.1) (4.2) with respect to the unknown ξ and ξ'. Without solving explicitly, which is possible but involved, one can assert that R and S exist uniquely at least when the coupling constant $e\cdot e'$ is not too large. Indeed, (4.1) (4.2) is a linear system in ξ, ξ'. Its determinant Δ, is a polynomial in ee', and whatever $r, v. v'$ are, $\Delta = 1$ when $ee' = 0$. Thus the absolute values of its roots are always strictly positive. For any given *compact* subset $\tilde{\Sigma}^-$ of Σ_*^- these functions have an inferior bound, hence one can find $\alpha > 0$ such that Δ never vanishes for $|ee'| < \alpha$, r, v, v' being taken on $\tilde{\Sigma}^-$.

Taking (4.5) as initial values on $\tilde{\Sigma}^-$ and applying the results of Section 2, now provides a unique solution of (1.5). Note that working on the other sheet Σ_*^+ (where x' is anterior to x) compells us to use $F'_{\text{adv.}}$ and $F_{\text{ret.}}$ and yields the same system up to the notations, that is labelling x' the absorber and x the emettor instead of the reverse.

So long as one considers the whole system in phase space, the two bodies are discernable only by the property of either receiving or emitting photons. Exchanging the charges is irrelevant and the masses are not *a priori* fixed, being first integrals.

It is clear that the R. H. S. solution is all we can afford as electromagnetic by means of the Cauchy theorem. Both the fully retarded law and the Feynman-Wheeler combination would imply that we consider the case « x anterior to x' » and the reverse case together, thus setting the Cauchy problem on a two-sheet surface, which is far out of the range of conventional methods.

5. CONCLUSION

When we have specified CAUCHY DATA on a part of the surface $(x-x')^2=0$, the physical interpretation was in terms of photons emitted by one particle and absorbed by the other.

It is clear that, instead of electromagnetic interaction, one can invoke other processes describing the motion of an emettor and an absorber of mass-less particles. (In particular, since neutrinos have two possible helicities, an asymmetric model involving neutrinos is a little bit more realistic than the one constructed with photons. We may also note that the absorber and the emettor are likely to be particles of different kinds, and not two electrons as it is simply assumed in the R. H. S. solution).

In any case, the need to choose a sheet of the surface $(x-x')^2=\mu^2$ leads to asymmetric models only.

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