ANNALES DE L'I. H. P., SECTION A

G. 'T HOOFT
M. VELTMAN

One-loop divergencies in the theory of gravitation

Annales de l'I. H. P., section A, tome 20, nº 1 (1974), p. 69-94
<http://www.numdam.org/item?id=AIHPA_1974__20_1_69_0>

Ann. Inst. Henri Poincaré,
Vol. XX, n° 1, 1974, p. 69-94.

Section A :
Physique théorique.

One-loop divergencies in the theory of gravitation

par

G. 't HOOFT (*) and M. VELTMAN (*)
C. E. R. N., Geneva.

ABSTRACT. — All one-loop divergencies of pure gravity and all those of gravitation interacting with a scalar particle are calculated. In the case of pure gravity, no physically relevant divergencies remain; they can all be absorbed in a field renormalization. In case of gravitation interacting with scalar particles, divergencies in physical quantities remain, even when employing the socalled improved energy-momentum tensor.

1. INTRODUCTION

The recent advances in the understanding of gauge theories make a fresh approach to the quantum theory of gravitation possible. First, we now know precisely how to obtain Feynman rules for a gauge theory [*1*]; secondly, the dimensional regularization scheme provides a powerful tool to handle divergencies [*2*]. In fact, several authors have already published work using these methods [*3*], [*4*].

One may ask why one would be interested in quantum gravity. The foremost reason is that gravitation undeniably exists; but in addition we may hope that study of this gauge theory, apparantly realized in nature, gives insight that can be useful in other areas of field theory. Of course, one may entertain all kinds of speculative ideas about the role of gravitation in elementary particle physics, and several authors have amused themselves imagining elementary particles as little black holes etc. It may well be true that gravitation functions as a cut-off for other interactions; in view of the fact that it seems possible to formulate all known

(*) On leave from the University of Utrecht, Netherlands.

interactions in terms of field-theoretical models that show only logarithmic divergencies, the smallness of the gravitational coupling constant need not be an obstacle. For the time being no reasonable or convincing analysis of this type of possibilities has been presented, and in this paper we have no ambitions in that direction. Mainly, we consider the present work as a kind of finger exercise without really any further underlying motive.

Our starting point is the linearized theory of gravitation. Of course, much work has been reported already in the literature [5], in particular we mention the work of B. S. Dewitt [6]. For the sake of clarity and completeness we will rederive several equations that can be found in his work. It may be noted that he also arrives at the conclusion that for pure gravitation the counterterms for one closed loop are of the form R^2 or $R_{\mu\nu}R^{\mu\nu}$; this really follows from invariance considerations and an identity derived by him. This latter identity is demonstrated in a somewhat easier way in appendix B of this paper.

Within the formalism of gauge theory developed in ref. 7, we must first establish a gauge that shows clearly the unitarity of the theory. This is done in section 2. The work of ref. 7, that on purpose has been formulated such as to encompass quantum gravity, assures us that the S-matrix remains invariant under a change of gauge.

In section 3 we consider the one loop divergencies when the gravitational field is treated as an external field. This calculation necessitates a slight generalization of the algorithms recently reported by one of us [8]. From the result one may read off the known fact that there are fewer divergencies if one employs the so-called improved energy-momentum tensor [9]. Symanzik's criticism [10] applies to higher order results, see ref. 11. In the one loop approximation we indeed find the results of Callan *et al.* [9].

Next we consider the quantum theory of gravity using the method of the background field [6], [12]. In ref. 8 it has already been shown how this method can be used fruitfully within this context. In sections 4, 5 and 6 we apply this to the case of gravitation interacting with a scalar field in the conventional way. The counter Lagrangian for pure gravity can be deduced immediately.

In section 7 finally we consider the use of the « improved » energy-momentum tensor. Some results are quoted, the full answer being unprintable.

Appendix A quotes notations and conventions. Appendix B gives the derivation of a well-known [6] but for us very important result. An (also well-known) side results is the fact that the Einstein-Hilbert Lagrangian is meaningless in two dimensions. This shows up in the form of factors $1/(n-2)$ in the graviton propagator, as noted by Neveu [13] and Capper *et al.* [4].

2. UNITARITY

One of the remarkable aspects of gravitation is the freedom of choice in the fundamental fields. In conventional renormalizable field theory the choice of the fields is usually such as to produce the smoothest possible Green's functions. In the case of quantum gravity there seems to be no clear choice based on such a criterium. For instance, one may use as basic field the metric tensor $g_{\mu\nu}$, or its inverse $g^{\mu\nu}$, or any other function of the $g_{\mu\nu}$ involving, say, the Riemann tensor. From the point of view of the S-matrix many choices give the same result, the Jacobian of the transformation being one (provided dimensional regularization is applied).

We chose as basic fields the $h_{\mu\nu}$, related to the $g_{\mu\nu}$ by $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. This is of course the conventional choice. The Lagrangian that we start from is the Einstein-Hilbert Lagrangian, viz:

$$\mathscr{L} = - \sqrt{g}\mathrm{R} \tag{2.1}$$

(see appendix A for symbols and notations). This Lagrangian is invariant under the infinitesimal gauge transformation (or rather it changes by a total derivative)

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\eta^\alpha + g_{\mu\alpha}\partial_\nu\eta^\alpha + \eta^\alpha\partial_\alpha g_{\mu\nu}, \tag{2.2}$$

or

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \mathrm{D}_\mu\eta_\nu + \mathrm{D}_\nu\eta_\mu. \tag{2.3}$$

In here the η_μ are four independent infinitesimal functions of space-time. The D_μ are the usual covariant derivatives.

In order to define Feynman rules we must supplement the Lagrangian (2.1) with a gauge breaking term $-\frac{1}{2}\mathrm{C}_\mu^2$ and a Faddeev-Popov ghost Lagrangian (historically, the name Feynman-DeWitt ghost Lagrangian would be more correct). In order to check unitarity and positivity of the theory we first consider the (non-covariant) Prentki gauge which is much like the Coulomb gauge in quantum-electrodynamics:

$$\sum_{i=1}^{3} \partial_i h_{i\mu} = 0, \qquad \mu = 1, \ldots, 4. \tag{2.4}$$

In the language of ref. 7 we take correspondingly:

$$\mathrm{C}_\mu = b \sum_{i=1}^{3} \partial_i h_{i\mu}, \qquad b \rightarrow \infty. \tag{2.5}$$

With this choice for C the part quadratic in the $h_{\mu\nu}$ is (comma denotes differentiation):

$$-\frac{1}{4}h_{\alpha\beta,\mu}h_{\alpha\beta,\mu}+\frac{1}{4}h_{\alpha\alpha,\mu}h_{\beta\beta,\mu}-\frac{1}{2}h_{\alpha\alpha,\beta}h_{\beta\mu,\mu}$$
$$+\frac{1}{2}h_{\beta\mu,\alpha}h_{\beta\alpha,\mu}-\frac{1}{2}b^2h_{i\mu,i}h_{j\mu,j}. \quad (2.6)$$

This can be written as $\frac{1}{2}h_{\alpha\beta}V_{\alpha\beta\mu\nu}h_{\mu\nu}$. The Fourier transform of V is:

$$V_{\alpha\beta\mu\nu}=\frac{1}{2}k^2(\delta_{\alpha\beta}\delta_{\mu\nu}-\delta_{\alpha\mu}\delta_{\beta\nu})-k_\mu k_\nu\delta_{\alpha\beta}+k_\beta k_\nu\delta_{\alpha\mu}-b^2\vec{k}_\beta\vec{k}_\nu\delta_{\alpha\mu}. \quad (2.7)$$

Calculations in the theory of gravitation are as a rule cumbersome, and the calculation of the graviton propagator gives a foretaste of what is to come. In principle things may be done as follows. First, symmetrize V with respect to $\alpha \leftrightarrow \beta$, $\mu \leftrightarrow \nu$ and $\alpha\beta \leftrightarrow \mu\nu$ interchange. Then find the propagator P from the equation $V.P = -\mathbb{1}$, where

$$\mathbb{1}_{\alpha\beta,\mu\nu}=\frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}). \quad (2.8)$$

Subsequently the limit $b^2 \to \infty$ must be taken. It is of advantage to go in the coordinate system where $k_1 = k_2 = 0$. Alternatively, write $\pi_1 = h_{11}$, $\pi_2 = h_{22}$, $\pi_3 = h_{33}$, $\pi_4 = h_{44}$, $\pi_5 = h_{34}$, $\pi_6 = h_{12}$, $\pi_7 = h_{13}$, $\pi_8 = h_{14}$, $\pi_9 = h_{23}$, $\pi_{10} = h_{24}$. Then V can be rewritten as a rather simple 10×10 matrix, that subsequently must be symmetrized. In the limit $b^2 \to \infty$ one may in a row or column containing a b^2 neglect all elements, except of course the b^2 term itself. Inversion of that matrix is trivial, and providing a minus sign the result is the propagator in the Prentki gauge:

$$P_{\mu\nu,\alpha\beta}(k)=\frac{1}{k^2}\left(\bar{\delta}_{\mu\alpha}\bar{\delta}_{\nu\beta}+\bar{\delta}_{\mu\beta}\bar{\delta}_{\nu\alpha}-\frac{2}{n-2}\bar{\delta}_{\mu\nu}\bar{\delta}_{\alpha\beta}\right)$$
$$+\frac{1}{\vec{k}^2}\Big(\bar{\delta}_{\mu\alpha}\delta_{\nu\beta4}+\bar{\delta}_{\mu\beta}\delta_{\nu\alpha4}+\bar{\delta}_{\nu\alpha}\delta_{\mu\beta4}+\bar{\delta}_{\nu\beta}\delta_{\mu\alpha4}$$
$$-\frac{2}{n-2}\bar{\delta}_{\mu\nu}\delta_{\alpha\beta4}-\frac{2}{n-2}\bar{\delta}_{\alpha\beta}\delta_{\mu\nu4}\Big)+\frac{2n-6}{n-2}\frac{k^2}{\vec{k}^4}\delta_{\mu\nu\alpha\beta4}. \quad (2.9)$$

In here n is the dimensionality of space-time. Further

$$\vec{k}^2=k^2-k_4^2,\qquad \bar{\delta}_{\mu\nu}=(1-\delta_{\mu4})(1-\delta_{\nu4})\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{\vec{k}^2}\right),$$
$$\delta_{\mu\nu4}=\delta_{\mu4}\delta_{\nu4},\qquad \delta_{\mu\nu\alpha\beta4}=\delta_{\mu4}\delta_{\nu4}\delta_{\alpha4}\delta_{\beta4}. \quad (2.10)$$

The first part of eq. (2.9) corresponds in 4 dimensions (i. e. $n = 4$) to the propagation of two polarization states of a mass zero spin 2 particle (see ref. 14, in particular section 3). The second and third part have no pole, they are non-local in space but simultaneous (or « local ») in time. They describe the $1/r$ behaviour of the potential in this gauge. In any case, they do not contribute to the absorptive part of the S-matrix.

In addition to the above we must also consider the Faddeev-Popov ghost. Subjecting C_μ of eq. (2.5) to the gauge transformation (2.3) and working in zero'th order of the field $h_{\mu\nu}$ we obtain the quadratic part of the F-P ghost Lagrangian:

$$\mathscr{L}_{\mathrm{FP}} = \varphi_\mu^*(\delta_{\mu\nu}\vec{\partial}^2 + \vec{\partial}_\nu\partial_\mu)\varphi_\nu \tag{2.11}$$

where the arrow indicates the 3-dimensional derivative. The propagator resulting from this has no pole, therefore does not contribute to the absorptive part of the S-matrix.

The above may be formulated in a somewhat neater way by means of the introduction of a fixed vector with zero space components. For instance:

$$\begin{aligned} \partial_i h_{i\mu} &= \partial_\nu h_{\nu\mu} + \vartheta_\alpha \vartheta_\beta \partial_\alpha h_{\beta\mu}, \\ \vartheta_\alpha &= (0, 0, 0, i). \end{aligned} \tag{2.12}$$

The continuous dimension regularization method can now be applied without further difficulty.

3. EXTERNAL GRAVITATIONAL FIELD

In this section we assume that the reader is acquainted with the work of ref. 8. The principal result is the following. Let there be given a Lagrangian

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\varphi_i\partial_\mu\varphi_i + \varphi_i \mathrm{N}_{ij}^\mu \partial_\mu \varphi_j + \frac{1}{2}\varphi_i \mathrm{M}_{ij}\varphi_j \tag{3.1}$$

where N and M are functions of external fields etc., but do not depend on the quantum fields φ_i. The counter-Lagrangian $\Delta\mathscr{L}$ that eliminates all one loop divergencies is

$$\Delta\mathscr{L} = \frac{1}{\varepsilon}\,\mathrm{Tr}\left(\frac{1}{4}\mathrm{X}^2 + \frac{1}{24}\mathrm{Y}_{\mu\nu}\mathrm{Y}_{\mu\nu}\right). \tag{3.2}$$

The trace is with respect to the indices $i, j, \ldots$ Further

$$\begin{aligned} \mathrm{X} &= \mathrm{M} - \mathrm{N}^\mu \mathrm{N}^\mu, \\ \mathrm{Y}_{\mu\nu} &= \partial_\mu \mathrm{N}^\nu - \partial_\nu \mathrm{N}^\mu + \mathrm{N}^\mu \mathrm{N}^\nu - \mathrm{N}^\nu \mathrm{N}^\mu, \\ \frac{1}{\varepsilon} &= \frac{1}{8\pi^2(n-4)}. \end{aligned} \tag{3.3}$$

It is assumed that N^μ is antisymmetric and M symmetric in the indices i, j.

For our purposes it is somewhat easier to work with complex fields. Given the Lagrangian

$$\mathscr{L} = - \partial_\mu \varphi_i^* \partial_\mu \varphi_i + 2\varphi_i^* \mathscr{N}_{ij}^\mu \partial_\mu \varphi_j + \varphi_i^* \mathscr{M}_{ij} \varphi_j \tag{3.4}$$

one easily derives

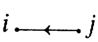
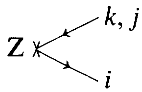
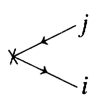
$$\begin{aligned} \Delta\mathscr{L} &= \frac{1}{\varepsilon} \operatorname{Tr} \left(\frac{1}{2} \mathscr{X}^2 + \frac{1}{12} \mathscr{Y}_{\mu\nu} \mathscr{Y}_{\mu\nu} \right), \\ \mathscr{X} &= \mathscr{M} - \mathscr{N}^\mu \mathscr{N}^\mu - \partial_\mu \mathscr{N}^\mu \\ \mathscr{Y}_{\mu\nu} &= \partial_\mu \mathscr{N}^\nu - \partial_\nu \mathscr{N}^\mu + \mathscr{N}^\mu \mathscr{N}^\nu - \mathscr{N}^\nu \mathscr{N}^\mu. \end{aligned} \tag{3.5}$$

Writing $\varphi = (A + iB)/\sqrt{2}$ with real fields A and B, it is seen that (3.2) is contained twice in (3.5) provided $\mathscr{M} - \partial_\mu \mathscr{N}_\mu$ and $\mathscr{N}_\mu$ are symmetric and antisymmetric respectively. Eq. (3.5) is valid independently of these symmetry properties.

There is now a little theorem that says that the counter-Lagrangian remains unchanged if in the original Lagrangian, eq. (3.4), everywhere φ_i^* is replaced by $\varphi_k^* Z_{ki}$ where Z is a (possibly space-time dependent) matrix. To see that we consider the following simplified case:

$$\mathscr{L} = \varphi_i^* \partial^2 \varphi_i + \varphi_i^* Z_{ij} \partial^2 \varphi_j + \varphi_i^* \mathscr{M}_{ij} \varphi_j. \tag{3.6}$$

The Feynman rules are

	$\dfrac{\delta_{ij}}{k^2 - i\varepsilon}$	φ-propagator,
	$-k^2 Z_{ij}$	Z-vertex,
	$\mathscr{M}_{ij}$	$\mathscr{M}$-vertex.

Any $\mathscr{M}$ vertex may be followed by 0, 1, 2, ... vertices of the Z-type. Now it is seen that the Z-vertex contains a factor that precisely cancels the propagator attached at one side. So one obtains a geometric series of the form

$$\mathscr{M} - Z\mathscr{M} + Z^2\mathscr{M} - Z^3\mathscr{M} \ldots = \frac{1}{1+Z}\mathscr{M}.$$

Clearly, the results are identical in case we had started with the Lagrangian

$$\mathscr{L} = \varphi_i^* \partial^2 \varphi_i + \varphi^* \left(\frac{1}{1+Z} \right) \mathscr{M} \varphi, \tag{3.7}$$

which is related to the above Lagrangian by the replacement

$$\varphi^* \rightarrow \varphi^* \frac{1}{1+Z}.$$

In the following we need the generalization of the above to the case that the scalar product contains the metric tensor $g_{\mu\nu}$. This tensor, in the applications to come, is a function of space-time, but not of the fields φ. Thus we are interested in the Lagrangian:

$$\mathscr{L} = \sqrt{g}(-\partial_\mu\varphi^* g^{\mu\nu}\partial_\nu\varphi + 2\varphi^*\mathscr{N}^\mu\partial_\mu\varphi + \varphi^*\mathscr{M}\varphi). \tag{3.8}$$

This Lagrangian is invariant under general coordinate transformations, and so will be our counter-Lagrangian. This counter-Lagrangian is given in eq. (3.35), and we will sketch the derivation.

Taking into account that $\Delta\mathscr{L}$ will contain terms of a certain dimensionality only, we find as most general form:

$$\begin{aligned}\Delta\mathscr{L} = \sqrt{g}(&a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + b_1\mathscr{M}R + b_2 D_\mu D^\mu\mathscr{M}\\ &+ b_3\mathscr{M}\mathscr{N}^\mu\mathscr{N}_\mu + b_4\mathscr{M}D_\mu\mathscr{N}^\mu + b_5\mathscr{M}^2\\ &+ c_1 D_\mu\mathscr{N}^\mu R + c_2\mathscr{N}^\mu\mathscr{N}_\mu R + c_3\mathscr{N}^\mu\mathscr{N}^\nu R_{\mu\nu}\\ &+ c_4 D_\mu\mathscr{N}^\nu D^\mu\mathscr{N}_\nu + c_5 D_\mu\mathscr{N}^\nu D_\nu\mathscr{N}^\mu + c_6\mathscr{N}^\mu\mathscr{N}_\mu D_\nu\mathscr{N}^\nu\\ &+ c_7\mathscr{N}^\nu\mathscr{N}_\mu D_\nu\mathscr{N}^\mu + c_8\mathscr{N}^\mu\mathscr{N}_\mu\mathscr{N}^\nu\mathscr{N}_\nu + c_9\mathscr{N}^\mu\mathscr{N}^\nu\mathscr{N}_\mu\mathscr{N}_\nu).\end{aligned} \tag{3.9}$$

The term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ need not be considered; see Appendix B. As usual, $\mathscr{N}_\mu = g_{\mu\nu}\mathscr{N}^\nu$ etc. Several coefficients can readily be determined by comparison with the special case $g_{\mu\nu} = \delta_{\mu\nu}$:

$$\begin{gathered} b_2 = 0,\ b_3 = -1,\ b_4 = -1,\ b_5 = \frac{1}{2}. \\ c_4 = \frac{1}{6},\ c_5 = \frac{1}{3},\ c_6 = \frac{2}{3},\ c_7 = -\frac{2}{3},\ c_8 = \frac{1}{3},\ c_9 = \frac{1}{6}. \end{gathered} \tag{3.10}$$

The remaining coefficients are determined in two steps. First, we take the special case

$$g^{\mu\nu} = \frac{\delta^{\mu\nu}}{F},\ g_{\mu\nu} = \delta_{\mu\nu}F,\ \sqrt{g} = F^2,\ F = 1 - f. \tag{3.11}$$

In here f is an arbitrary function of space-time. The Lagrangian (3.8) becomes:

$$\mathscr{L} = \varphi^* F\partial^2\varphi + 2\varphi^*\left(F^2\mathscr{N}^\mu + \frac{1}{2}\partial_\mu F\right)\partial_\mu\varphi + \varphi^* F^2\mathscr{M}\varphi. \tag{3.12}$$

The replacement $\varphi^* \to \varphi^* F^{-1}$ leaves the counter-Lagrangian unchanged; we have thus the equivalent Lagrangian

$$\mathscr{L} = \varphi^*\partial^2\varphi + 2\varphi^*\left(F\mathscr{N}^\mu + \frac{1}{2}F^{-1}\partial_\mu F\right)\partial_\mu\varphi + \varphi^* F\mathscr{M}\varphi. \tag{3.13}$$

This is precisely of the form studied before; the answer is as given in eq. (3.5), but now with the replacements

$$\mathcal{M} \to \mathrm{F}\mathcal{M},$$
$$\mathcal{N}^{\mu} \to \mathrm{F}\mathcal{N}^{\mu} + \frac{1}{2}\mathrm{F}^{-1}\partial_{\mu}\mathrm{F} = \mathcal{N}_{\mu} + \frac{1}{2}\mathrm{F}^{-1}\partial_{\mu}\mathrm{F}. \tag{3.14}$$

To proceed, it is necessary to compute a number of quantities for this choice of $g_{\mu\nu}$. Using the notations

$$f_{\alpha} = \partial_{\alpha} f, \qquad f_{\alpha\beta} = \partial_{\alpha}\partial_{\beta} f,$$

we have (on the right hand side there is no difference between upper and lower indices):

$$\Gamma^{\alpha}_{\mu\nu} = -\frac{1}{2}\mathrm{F}^{-1}(\delta^{\alpha}_{\mu}f_{\nu} + \delta^{\alpha}_{\nu}f_{\mu} - \delta_{\mu\nu}f^{\alpha}). \tag{3.15}$$

For any contravariant vector Z^{α}:

$$\begin{aligned} \mathrm{D}_{\mu}\mathrm{Z}^{\alpha} &= \partial_{\mu}\mathrm{Z}^{\alpha} + \Gamma^{\alpha}_{\mu\nu}\mathrm{Z}^{\nu} \\ &= \partial_{\mu}\mathrm{Z}^{\alpha} - \frac{1}{2}\mathrm{F}^{-1}(\delta^{\alpha}_{\mu}f_{\nu}\mathrm{Z}^{\nu} + f_{\mu}\mathrm{Z}^{\alpha} - f^{\alpha}\mathrm{Z}_{\mu}). \\ \mathrm{D}_{\mu}\mathrm{Z}^{\mu} &= \partial_{\mu}\mathrm{Z}^{\mu} - 2\mathrm{F}^{-1}f_{\mu}\mathrm{Z}^{\mu}. \end{aligned}$$

$$\begin{aligned} \mathrm{R}^{\mu}_{\nu\alpha\beta} = &-\frac{1}{2}\mathrm{F}^{-1}(\delta^{\mu}_{\beta}f_{\alpha\nu} - \delta^{\mu}_{\alpha}f_{\beta\nu} - \delta_{\nu\beta}f^{\mu}_{\alpha} + \delta_{\nu\alpha}f^{\mu}_{\beta}) \\ &-\frac{1}{4}\mathrm{F}^{-2}(3\delta^{\mu}_{\beta}f_{\alpha}f_{\nu} - 3\delta^{\mu}_{\alpha}f_{\beta}f_{\nu} - 3\delta_{\nu\beta}f_{\alpha}f^{\mu} \\ &\qquad + 3\delta_{\nu\alpha}f_{\beta}f^{\mu} + \delta_{\beta\nu}\delta^{\mu}_{\alpha}f^{\gamma}f_{\gamma} - \delta_{\alpha\nu}\delta^{\mu}_{\beta}f^{\gamma}f_{\gamma}). \end{aligned} \tag{3.16}$$

$$\mathrm{R}_{\nu\alpha} = -\frac{1}{2}\mathrm{F}^{-1}(2f_{\alpha\nu} + \delta_{\alpha\nu}f_{\gamma\gamma}) - \frac{3}{2}\mathrm{F}^{-2}f_{\alpha}f_{\nu}. \tag{3.17}$$

$$\mathrm{R} = -3\mathrm{F}^{-2}f_{\gamma\gamma} - \frac{3}{2}\mathrm{F}^{-3}f_{\gamma}f_{\gamma}. \tag{3.18}$$

$$\begin{aligned} \mathrm{R}_{\nu\alpha}\mathrm{R}^{\nu\alpha} = &\ \mathrm{F}^{-4}(f_{\alpha\nu}f_{\alpha\nu} + 2f_{\alpha\alpha}f_{\nu\nu}) \\ &+ \mathrm{F}^{-5}\left(3f_{\alpha\nu}f_{\alpha}f_{\nu} + \frac{3}{2}f_{\alpha\alpha}f_{\nu}f_{\nu}\right) + \frac{9}{4}\mathrm{F}^{-6}f_{\alpha}f_{\alpha}f_{\nu}f_{\nu}. \end{aligned} \tag{3.19}$$

$$\mathrm{R}^{2} = 9\mathrm{F}^{-4}f_{\alpha\alpha}f_{\nu\nu} + 9\mathrm{F}^{-5}f_{\nu\nu}f_{\alpha}f_{\alpha} + \frac{9}{4}\mathrm{F}^{-6}f_{\alpha}f_{\alpha}f_{\nu}f_{\nu}. \tag{3.20}$$

We leave it to the reader to verify that

$$\sqrt{g}(\mathrm{R}_{\mu\nu\alpha\beta}\mathrm{R}^{\mu\nu\alpha\beta} - 4\mathrm{R}_{\nu\alpha}\mathrm{R}^{\nu\alpha} + \mathrm{R}^{2}) = \text{total derivative}. \tag{3.21}$$

In this particular case, unfortunately, also another identity holds:

$$\sqrt{g}\left(\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} - \frac{1}{3}\mathrm{R}^{2}\right) = \text{total derivative}. \tag{3.22}$$

We must now try to write the counter-Lagrangian in terms of covariant objects. First:

$$\left(\mathrm{F}\mathcal{N}^\mu + \frac{1}{2}\mathrm{F}^{-1}\partial_\mu\mathrm{F}\right)^2 + \partial_\mu\left(\mathrm{F}\mathcal{N}^\mu + \frac{1}{2}\mathrm{F}^{-1}\partial_\mu\mathrm{F}\right)$$
$$= \mathrm{F}\mathcal{N}^\mu\mathcal{N}_\mu - \mathcal{N}^\mu f_\mu + \frac{1}{4}\mathrm{F}^{-2}f_\mu f_\mu + \mathrm{F}\partial_\mu\mathcal{N}^\mu$$
$$- f_\mu\mathcal{N}^\mu - \frac{1}{2}\mathrm{F}^{-2}f_\mu f_\mu - \frac{1}{2}\mathrm{F}^{-1}f_{\mu\mu}$$
$$= \mathrm{F}\mathcal{N}^\mu\mathcal{N}_\mu + \mathrm{F}\mathrm{D}_\mu\mathcal{N}^\mu + \frac{1}{6}\mathrm{F}\mathrm{R}. \qquad (3.23)$$

Similarly:

$$\partial_\mu\left(\mathcal{N}_\nu + \frac{1}{2}\mathrm{F}^{-1}\partial_\nu\mathrm{F}\right) - \partial_\nu\left(\mathcal{N}_\mu + \frac{1}{2}\mathrm{F}^{-1}\partial_\mu\mathrm{F}\right)$$
$$+ \left(\mathcal{N}_\mu + \frac{1}{2}\mathrm{F}^{-1}\partial_\mu\mathrm{F}\right)\left(\mathcal{N}_\nu + \frac{1}{2}\mathrm{F}^{-1}\partial_\nu\mathrm{F}\right)$$
$$- \left(\mathcal{N}_\nu + \frac{1}{2}\mathrm{F}^{-1}\partial_\nu\mathrm{F}\right)\left(\mathcal{N}_\mu + \frac{1}{2}\mathrm{F}^{-1}\partial_\mu\mathrm{F}\right)$$
$$= \mathrm{D}_\mu\mathcal{N}_\nu - \mathrm{D}_\nu\mathcal{N}_\mu + \mathcal{N}_\mu\mathcal{N}_\nu - \mathcal{N}_\nu\mathcal{N}_\mu. \qquad (3.24)$$

This equation looks more complicated then it is; one has

$$\mathrm{D}_\mu\mathcal{N}_\nu = \partial_\mu\mathcal{N}_\nu - \Gamma^\alpha_{\mu\nu}\mathcal{N}_\alpha.$$

Now Γ is symmetrical in the two lower indices, and therefore

$$\mathrm{D}_\mu\mathcal{N}_\nu - \mathrm{D}_\nu\mathcal{N}_\mu = \partial_\mu\mathcal{N}_\nu - \partial_\nu\mathcal{N}_\mu.$$

The various F dependent terms all cancel out.

The result for the special case $g_{\mu\nu} = \delta_{\mu\nu}(1 - f)$ is:

$$\Delta\mathcal{L} = \frac{\sqrt{g}}{\varepsilon}\,\mathrm{Tr}\left\{\frac{1}{12}\mathcal{Y}^{\mu\nu}\mathcal{Y}_{\mu\nu} + \frac{1}{2}\left(\mathcal{M} - \mathcal{N}^\mu\mathcal{N}_\mu - \mathrm{D}_\mu\mathcal{N}^\mu - \frac{1}{6}\mathrm{R}\right)^2\right\}. \qquad (3.25)$$

Inspecting the general form eq. (3.9) and remembering the identity (3.22) we see that we have determined $\Delta\mathcal{L}$ up to a term

$$\sqrt{g}a_0\left(\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} - \frac{1}{3}\mathrm{R}^2\right). \qquad (3.26)$$

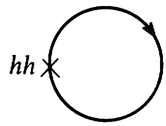
To determine the coefficient a_0 we consider the special case

$$\mathcal{L} = \sqrt{g}(-\partial_\mu\varphi^* g^{\mu\nu}\partial_\nu\varphi). \qquad (3.27)$$

With $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ we need to expand up to first order in h

$$\mathcal{L} = -\partial_\mu\varphi^*\partial_\mu\varphi + \partial_\mu\varphi^*\left(h^{\mu\nu} - \frac{1}{2}\delta^{\mu\nu}h^{\alpha\alpha}\right)\partial_\nu\varphi. \qquad (3.28)$$

With $s_{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\delta^{\mu\nu}h^{\alpha\alpha}$ we need only compute the selfenergy type of graph with two s vertices. Note that the terms of second order in h in the Lagrangian do not contribute because they give rise to a vertex with two h, and they can only contribute to $\Delta\mathscr{L}$ in order h^2 by closing that vertex into itself:

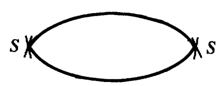


These tadpole type diagrams give integrals of the form

$$\int d_n p \frac{p_\mu p_\nu}{p^2},$$

and these are zero in the continuous dimension method.

The computation of the pole part of the graph



is not particularly difficult. The result is:

$$\Delta\mathscr{L} = \frac{1}{\varepsilon}\left[\frac{1}{480}\partial^2 s_{\mu\mu}\partial^2 s_{\alpha\alpha} + \frac{1}{240}\partial^2 s_{\mu\alpha}\partial^2 s_{\mu\alpha} + \frac{1}{60}\partial^2 s_{\mu\mu}\partial_\alpha\partial_\beta s_{\alpha\beta} \right.$$
$$\left. - \frac{1}{120}\partial_\alpha\partial_\nu s_{\mu\nu}\partial_\alpha\partial_\beta s_{\mu\beta} + \frac{1}{60}\partial_\mu\partial_\nu s_{\mu\nu}\partial_\alpha\partial_\beta s_{\alpha\beta}\right], \quad (3.29)$$
$$s_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h_{\alpha\alpha}.$$

Working to second order in h one has

$$R_{\nu\alpha} = \frac{1}{2}(\partial_\alpha\partial_\nu h_{\mu\mu} - \partial_\nu\partial_\mu h_{\alpha\mu} - \partial_\alpha\partial_\mu h_{\nu\mu} + \partial^2 h_{\nu\alpha}). \quad (3.30)$$

$$R_{\alpha\beta}R^{\alpha\beta} = \frac{1}{4}(\partial^2 h_{\mu\mu}\partial^2 h_{\alpha\alpha} + \partial^2 h_{\mu\alpha}\partial^2 h_{\mu\alpha} - 2\partial^2 h_{\mu\mu}\partial_\alpha\partial_\beta h_{\alpha\beta}$$
$$- 2\partial_\alpha\partial_\nu h_{\mu\nu}\partial_\alpha\partial_\beta h_{\mu\beta} + 2\partial_\mu\partial_\nu h_{\mu\nu}\partial_\alpha\partial_\beta h_{\alpha\beta}). \quad (3.31)$$

$$R = \partial^2 h_{\mu\mu} - \partial_\alpha\partial_\mu h_{\alpha\mu}. \quad (3.32)$$

$$R^2 = \partial^2 h_{\mu\mu}\partial^2 h_{\alpha\alpha} - 2\partial^2 h_{\mu\mu}\partial_\alpha\partial_\beta h_{\alpha\beta} + \partial_\alpha\partial_\beta h_{\alpha\beta}\partial_\mu\partial_\nu h_{\mu\nu}. \quad (3.33)$$

The result is:

$$\Delta\mathscr{L} = \frac{\sqrt{g}}{\varepsilon}\left[\frac{1}{72}R^2 + \frac{1}{60}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right]. \quad (3.34)$$

The first term has been found before (see eq. 3.25).

All coefficients have now been determined and we can write the final result. To the Lagrangian (3.8) corresponds the counter-Lagrangian:

$$\Delta\mathscr{L} = \frac{\sqrt{g}}{\varepsilon}\,\mathrm{Tr}\left\{ \frac{1}{12}\mathscr{Y}^{\mu\nu}\mathscr{Y}_{\mu\nu} + \frac{1}{2}\left(\mathscr{M} - \mathscr{N}^{\mu}\mathscr{N}_{\mu} - \mathrm{D}_{\mu}\mathscr{N}^{\mu} - \frac{1}{6}\mathrm{R}\right)^{2} + \frac{1}{60}\left(\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} - \frac{1}{3}\mathrm{R}^{2}\right)\right\}. \tag{3.35}$$

Note that a trace is to be taken; the last term has as factor the unit matrix. As before

$$\mathscr{Y}_{\mu\nu} = \mathrm{D}_{\mu}\mathscr{N}_{\nu} - \mathrm{D}_{\nu}\mathscr{N}_{\mu} + \mathscr{N}_{\mu}\mathscr{N}_{\nu} - \mathscr{N}_{\nu}\mathscr{N}_{\mu}. \tag{3.36}$$

To obtain the result for real fields, write

$$2\varphi^{*}\mathscr{N}^{\mu}\partial_{\mu}\varphi = \varphi^{*}\mathscr{N}^{\mu}\partial_{\mu}\varphi - \partial_{\mu}\varphi^{*}\mathscr{N}^{\mu}\varphi - \varphi^{*}\partial_{\mu}\mathscr{N}^{\mu}\varphi, \tag{3.37}$$

and substitute

$$\varphi = \frac{1}{\sqrt{2}}(\mathrm{A} + i\mathrm{B}), \qquad \varphi^{*} = \frac{1}{\sqrt{2}}(\mathrm{A} - i\mathrm{B}). \tag{3.38}$$

The result is then as follows. To the Lagrangian

$$\mathscr{L} = \sqrt{g}\left(- \frac{1}{2}\partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\varphi + \varphi\mathrm{N}^{\mu}\partial_{\mu}\varphi + \frac{1}{2}\varphi\mathrm{M}\varphi\right), \tag{3.39}$$

corresponds the counter-Lagrangian

$$\Delta\mathscr{L} = \frac{\sqrt{g}}{\varepsilon}\,\mathrm{Tr}\left\{ \frac{1}{24}\mathrm{Y}^{\mu\nu}\mathrm{Y}_{\mu\nu} + \frac{1}{4}\left(\mathrm{M} - \mathrm{N}^{\mu}\mathrm{N}_{\mu} - \frac{1}{6}\mathrm{R}\right)^{2} + \frac{1}{120}\left(\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} - \frac{1}{3}\mathrm{R}^{2}\right)\right\}, \tag{3.40}$$

$$\mathrm{Y}_{\mu\nu} = \mathrm{D}_{\mu}\mathrm{N}_{\nu} - \mathrm{D}_{\nu}\mathrm{N}_{\mu} + \mathrm{N}_{\mu}\mathrm{N}_{\nu} - \mathrm{N}_{\nu}\mathrm{N}_{\mu}.$$

Eq. (3.40) contains a well known result. The gravitational field enters through $g_{\mu\nu}$ and the Lagrangian (3.39) describes the interaction of bosons with gravitation, whereby gravity is treated in the tree approximation. If one adds now to the Lagrangian (3.39) the term

$$\frac{1}{12}\mathrm{R}\varphi\varphi,$$

then the « unrenormalizable » counterterms of the form MR and $\mathrm{N}^{\mu}\mathrm{N}_{\mu}\mathrm{R}$ disappear. Indeed, the energy-momentum tensor of the expression

$$\sqrt{g}\left(- \frac{1}{2}\partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\varphi + \frac{1}{12}\mathrm{R}\varphi^{2}\right),$$

is precisely the « improved » energy-momentum tensor of ref. 9. We see also that closed loops of bosons introduce nasty divergencies quadratic in the Riemann tensor. This unpleasant fact remains if we allow also for closed loops of gravitons. This is the subject of the following sections.

4. CLOSED LOOPS INCLUDING GRAVITONS

We now undertake the rather formidable task of computing the divergencies of one loop graphs including gravitons. The starting point is the Lagrangian

$$\mathscr{L} = \sqrt{\bar{g}}\left(- \bar{\mathrm{R}} - \frac{1}{2}\partial_\mu\bar{\varphi}\bar{g}^{\mu\nu}\partial_\nu\bar{\varphi}\right) \tag{4.1}$$

$\bar{\mathrm{R}}$ is the Riemann scalar constructed from $\bar{g}_{\mu\nu}$. In section 7 we will include other terms, such as $\bar{\mathrm{R}}\bar{\varphi}^2$.

Again using the background field method [6], [12] we write

$$\begin{aligned} \bar{\varphi} &= \tilde{\varphi} + \varphi, \\ \bar{g}_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu}. \end{aligned} \tag{4.2}$$

If we take the c-number quantities $\tilde{\varphi}$ and $g_{\mu\nu}$ such that they obey the classical equations of motion then the part of $\mathscr{L}$ linear in the quantum fields φ and $h_{\mu\nu}$ is zero. The part quadratic in these quantities determines the one loop diagrams. We have:

$$\mathscr{L} = \mathscr{L}^{\mathrm{cl}} + \underline{\mathscr{L}} + \underline{\underline{\mathscr{L}}} + \mathscr{L}^{\mathrm{rest}}. \tag{4.3}$$

$\underline{\mathscr{L}}$ and $\underline{\underline{\mathscr{L}}}$ are linear and quatratic in the quantum fields φ and h respectively. The higher order terms contained in $\mathscr{L}^{\mathrm{rest}}$ play a role only in multiloop diagrams.

At this point we may perhaps clarify our notations. In the following we will meet quantities like the Riemann tensor $\mathrm{R}_{\mu\nu}$. This is then the tensor made up from the classical field $g_{\mu\nu}$. In the end we will use the classical equations of motion for this tensor. All divergencies that are physically irrelevant will then disappear. In fact, using these classical equations of motion is like putting the external lines of the one loop diagrams on mass-shell, with physical polarizations. Note that we still allow for trees connected to the loop. Only the very last branches of the trees must be physical.

To obtain $\underline{\mathscr{L}}$ and $\underline{\underline{\mathscr{L}}}$ we must expand the various quantities in eq. (4.1) up to second order in the quantum fields. We list here a number of subresults. Note that

$$h^\alpha_\beta = g^{\alpha\gamma}h_{\gamma\beta}. \tag{4.4}$$

Thus indices are raised and lowered by means of the classical field $g_{\mu\nu}$.

The following equations hold up to terms of third and higher order in h and φ:

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} = g_{\mu\alpha}(\delta^\alpha_\nu + h^\alpha_\nu). \tag{4.3}$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^\mu_\alpha h^{\alpha\nu}. \tag{4.4}$$

Using

$$\sqrt{\bar{g}} = \sqrt{\det(\bar{g})} = \exp\left(\frac{1}{2}\operatorname{Tr}\ln \bar{g}\right)$$
$$= \sqrt{g}\exp\left(\frac{1}{2}\operatorname{Tr}\ln(\delta^\alpha_\nu + h^\alpha_\nu)\right)$$
$$= \sqrt{g}\exp\left(\frac{1}{2}\operatorname{Tr}\left(h^\alpha_\nu - \frac{1}{2}h^\alpha_\beta h^\beta_\nu\right)\right)$$
$$= \sqrt{g}\exp\left(\frac{1}{2}h^\alpha_\alpha - \frac{1}{4}h^\alpha_\beta h^\beta_\alpha\right),$$

we find

$$\sqrt{\bar{g}} = \sqrt{g}\left(1 + \frac{1}{2}h^\alpha_\alpha - \frac{1}{4}h^\alpha_\beta h^\beta_\alpha + \frac{1}{8}h^{\alpha 2}_\alpha\right). \tag{4.5}$$

Further

$$\bar{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \underline{\Gamma}^\alpha_{\mu\nu} + \underline{\underline{\Gamma}}^\alpha_{\mu\nu}. \tag{4.6}$$

$$\underline{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2}(h^\alpha_{\nu,\mu} + h^\alpha_{\mu,\nu} - h_{\mu\nu}{}^{;\alpha}). \tag{4.7}$$

$$\underline{\underline{\Gamma}}^\alpha_{\mu\nu} = -\frac{1}{2}h^{\alpha\gamma}(h_{\gamma\nu,\mu} + h_{\mu\gamma,\nu} - h_{\mu\nu,\gamma}). \tag{4.8}$$

$$\underline{\Gamma}^\alpha_{\mu\alpha} = \frac{1}{2}h^\alpha_{\alpha,\mu}, \qquad \underline{\underline{\Gamma}}^\alpha_{\mu\alpha} = -\frac{1}{2}h^\alpha_\beta h^\beta_{\alpha,\mu}. \tag{4.9}$$

We used here the fact that the co- or contra-variant derivative of $g_{\mu\nu}$ is zero; therefore

$$h^\alpha_{\mu,\alpha} = D_\alpha h^\alpha_\mu = g^{\alpha\beta}D_\alpha h_{\mu\beta} = D^\alpha h_{\mu\alpha} = h_{\mu\alpha}{}^{;\alpha}. \tag{4.10}$$

Note that we employ the standard notation to denote the co- and contra-variant derivatives:

$$h^\alpha_{\nu,\mu} = D_\mu h^\alpha_\nu, \qquad h_{\nu\beta,}{}^{;\mu}{}_\alpha = D_\alpha D^\mu h_{\nu\beta}, \text{ etc.} \tag{4.11}$$

Observe that the order of differentiation is relevant. The D symbol involves the Christoffel symbol Γ made up from the classical field $g_{\mu\nu}$.

$$\bar{R}^\mu_{\nu\alpha\beta} = R^\mu_{\nu\alpha\beta} + \underline{R}^\mu_{\nu\alpha\beta} + \underline{\underline{R}}^\mu_{\nu\alpha\beta}. \tag{4.12}$$

$$\underline{R}^\mu_{\nu\alpha\beta} = D_\alpha\underline{\Gamma}^\mu_{\nu\beta} - D_\beta\underline{\Gamma}^\mu_{\nu\alpha} = \frac{1}{2}(h^\mu_{\beta,\nu\alpha} - h_{\nu\beta,}{}^{;\mu}{}_\alpha - h^\mu_{\alpha,\nu\beta} + h_{\nu\alpha,}{}^{;\mu}{}_\beta)$$
$$+ \frac{1}{2}R^\mu_{\gamma\alpha\beta}h^\gamma_\nu + \frac{1}{2}R^\gamma_{\nu\beta\alpha}h^\mu_\gamma. \tag{4.13}$$

In the derivation we used the identity

$$\mathrm{D}_{\alpha}\mathrm{D}_{\beta}h^{\mu}_{\nu} - \mathrm{D}_{\beta}\mathrm{D}_{\alpha}h^{\mu}_{\nu} = \mathrm{R}^{\mu}_{\gamma\alpha\beta}h^{\gamma}_{\nu} + \mathrm{R}^{\gamma}_{\nu\beta\alpha}h^{\mu}_{\gamma}. \qquad (4.14)$$

Further:

$$\underline{\underline{\mathrm{R}}}^{\mu}_{\nu\alpha\beta} = \mathrm{D}_{\alpha}\underline{\underline{\Gamma}}^{\mu}_{\nu\beta} - \mathrm{D}_{\beta}\underline{\underline{\Gamma}}^{\mu}_{\nu\alpha} + \underline{\Gamma}^{\gamma}_{\beta\nu}\underline{\Gamma}^{\mu}_{\gamma\alpha} - \underline{\Gamma}^{\gamma}_{\alpha\nu}\underline{\Gamma}^{\mu}_{\gamma\beta}. \qquad (4.15)$$

$$\bar{\mathrm{R}}_{\nu\alpha} = \mathrm{R}_{\nu\alpha} + \underline{\mathrm{R}}_{\nu\alpha} + \underline{\underline{\mathrm{R}}}_{\nu\alpha}. \qquad (4.16)$$

$$\underline{\mathrm{R}}_{\nu\alpha} = \frac{1}{2}\left(h^{\beta}_{\beta,\nu\alpha} - h^{\beta}_{\nu,\beta\alpha} - h^{\beta}_{\alpha,\nu\beta} + h_{\nu\alpha,}{}^{,\beta}{}_{\beta}\right) + \frac{1}{2}h^{\beta}_{\nu,\beta\alpha} - \frac{1}{2}h^{\beta}_{\nu,\alpha\beta}$$

$$= \frac{1}{2}\left(h^{\beta}_{\beta,\nu\alpha} - h^{\beta}_{\nu,\alpha\beta} - h^{\beta}_{\alpha,\nu\beta} + h_{\nu\alpha,}{}^{,\beta}{}_{\beta}\right). \qquad (4.17)$$

$$\underline{\underline{\mathrm{R}}}_{\nu\alpha} = -\frac{1}{2}\mathrm{D}_{\alpha}\left(h^{\beta}_{\mu}h^{\mu}_{\beta,\nu}\right) + \frac{1}{2}\mathrm{D}_{\beta}\left\{ h^{\beta}_{\gamma}\left(h^{\gamma}_{\nu,\alpha} + h^{\gamma}_{\alpha,\nu} - h^{,\gamma}_{\alpha\nu}\right) \right\}$$

$$+ \frac{1}{4}\left(h^{\gamma}_{\beta,\nu} + h^{\gamma}_{\nu,\beta} - h^{,\gamma}_{\beta\nu}\right)\left(h^{\beta}_{\gamma,\alpha} + h^{\beta}_{\alpha,\gamma} - h^{,\beta}_{\gamma\alpha}\right)$$

$$- \frac{1}{4}\left(h^{\gamma}_{\alpha,\nu} + h^{\gamma}_{\nu,\alpha} - h^{,\gamma}_{\nu\alpha}\right)h^{\beta}_{\beta,\gamma}. \qquad (4.18)$$

$$\bar{\mathrm{R}} = \mathrm{R} + \underline{\mathrm{R}} + \underline{\underline{\mathrm{R}}} \qquad (4.19)$$

$$\underline{\mathrm{R}} = h^{\beta,\alpha}_{\beta,\alpha} - h^{\beta,\alpha}_{\alpha,\ \beta} - \mathrm{R}^{\alpha}_{\nu}h^{\nu}_{\alpha}. \qquad (4.20)$$

$$\underline{\underline{\mathrm{R}}} = -\frac{1}{2}\mathrm{D}_{\alpha}\left(h^{\beta}_{\mu}h^{\mu,\alpha}_{\beta}\right) + \frac{1}{2}\mathrm{D}_{\beta}\left\{ h^{\beta}_{\nu}\left(2h^{\nu\alpha}_{,\alpha} - h^{\alpha,\nu}_{\alpha}\right)\right\}$$

$$+ \frac{1}{4}\left(h^{\nu}_{\beta,\alpha} + h^{\nu}_{\alpha,\beta} - h^{,\nu}_{\beta\alpha}\right)\left(h^{\beta,\alpha}_{\nu} + h^{\beta\alpha}_{,\nu} - h^{\alpha,\beta}_{\nu}\right)$$

$$- \frac{1}{4}\left(2h^{\nu\alpha}_{,\alpha} - h^{\alpha,\nu}_{\alpha}\right)h^{\beta}_{\beta,\nu} - \frac{1}{2}h^{\nu\alpha}h^{\beta}_{\beta,\nu\alpha}$$

$$+ \frac{1}{2}h^{\nu}_{\alpha}\mathrm{D}_{\beta}\left(h^{\beta,\alpha}_{\nu} + h^{\beta\alpha}_{,\nu} - h^{\alpha,\beta}_{\nu}\right) + h^{\nu}_{\beta}h^{\beta}_{\alpha}\mathrm{R}^{\alpha}_{\nu}. \qquad (4.21)$$

Inserting the various quantities in eq. (4.1), we find

$$\underline{\mathscr{L}} = \sqrt{g}\Big(-\frac{1}{2}h^{\alpha}_{\alpha}\mathrm{R} - \frac{1}{4}\partial_{\mu}\tilde{\varphi}g^{\mu\nu}\partial_{\nu}\tilde{\varphi}h^{\alpha}_{\alpha} - h^{\beta,\alpha}_{\beta,\alpha} + h^{\beta\alpha}_{,\alpha\beta}$$

$$+ \mathrm{R}^{\alpha}_{\beta}h^{\beta}_{\alpha} - \partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\tilde{\varphi} + \frac{1}{2}\partial_{\mu}\tilde{\varphi}\partial_{\nu}\tilde{\varphi}h^{\mu\nu}\Big). \qquad (4.22)$$

This expression will eventually supply us with the equations of motion for the classical fields $\tilde{\varphi}$ and $g_{\mu\nu}$. Allowing partial (co- and contra-variant) integration and omitting total derivatives:

$$\underline{\mathscr{L}} = \sqrt{g}\Big(-\frac{1}{2}h^{\alpha}_{\alpha}\mathrm{R} - \frac{1}{4}h^{\alpha}_{\alpha}\partial_{\mu}\tilde{\varphi}g^{\mu\nu}\partial_{\nu}\tilde{\varphi} + h^{\beta}_{\alpha}\mathrm{R}^{\alpha}_{\beta}$$

$$+ \frac{1}{2}h^{\mu\nu}\partial_{\mu}\tilde{\varphi}\partial_{\nu}\tilde{\varphi} - \partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\tilde{\varphi}\Big). \qquad (4.23)$$

Further:

$$
\begin{aligned}
\underline{\underline{\mathscr{L}}} = \sqrt{g}\Big[& -\frac{1}{2}\partial_\mu\varphi\partial_\nu\tilde{\varphi}(g^{\mu\nu}h^\alpha_\alpha - 2h^{\mu\nu}) - \frac{1}{2}\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi}\left(h^\mu_\alpha h^{\alpha\nu} - \frac{1}{2}h^\alpha_\alpha h^{\mu\nu}\right) \\
& - \frac{1}{2}\partial_\mu\varphi g^{\mu\nu}\partial_\nu\varphi - \left(\frac{1}{8}(h^\alpha_\alpha)^2 - \frac{1}{4}h^\alpha_\beta h^\beta_\alpha\right)\left(\mathrm{R} + \frac{1}{2}\partial_\mu\tilde{\varphi}g^{\mu\nu}\partial_\nu\tilde{\varphi}\right) \\
& - \frac{1}{2}h^\alpha_\alpha(h^{\beta,\mu}_{\beta,\mu} - h^{\beta,\mu}_{\mu,\ \beta} - \mathrm{R}^\beta_\nu h^\nu_\beta) + \frac{1}{2}\mathrm{D}_\alpha(h^\nu_\mu h^{\mu,\alpha}_\nu) \\
& - \frac{1}{2}\mathrm{D}_\beta\left\{ h^\beta_\nu(2h^{\nu\alpha}_{,\alpha} - h^{\alpha,\nu}_\alpha) \right\} \\
& - \frac{1}{4}(h^\nu_{\beta,\alpha} + h^\nu_{\alpha,\beta} - h^{,\nu}_{\beta\alpha})(h^{\beta,\alpha}_\nu + h^{\beta\alpha}_{,\nu} - h^{\alpha,\beta}_\nu) \\
& + \frac{1}{4}(2h^{\nu\alpha}_{,\alpha} - h^{\alpha,\nu}_\alpha)h^\beta_{\beta,\nu} + \frac{1}{2}h^{\nu\alpha}h^\beta_{\beta,\nu\alpha} - \frac{1}{2}h^\nu_\alpha\mathrm{D}_\beta(h^{\beta,\alpha}_\nu + h^{\beta\alpha}_{,\nu} - h^{\alpha,\beta}_\nu) \\
& - h^\nu_\beta h^\beta_\alpha \mathrm{R}^\alpha_\nu\Big].
\end{aligned}
\tag{4.24}
$$

Performing partial integration and omitting total derivatives:

$$
\begin{aligned}
\underline{\underline{\mathscr{L}}} = \sqrt{g}\Big[& -\frac{1}{2}\partial_\mu\varphi\partial_\nu\tilde{\varphi}(g^{\mu\nu}h^\alpha_\alpha - 2h^{\mu\nu}) - \frac{1}{2}\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi}\left(h^\mu_\alpha h^{\alpha\nu} - \frac{1}{2}h^\alpha_\alpha h^{\mu\nu}\right) \\
& - \frac{1}{2}\partial_\mu\varphi g^{\mu\nu}\partial_\nu\varphi - \left(\frac{1}{8}(h^\alpha_\alpha)^2 - \frac{1}{4}h^\alpha_\beta h^\beta_\alpha\right)\left(\mathrm{R} + \frac{1}{2}\partial_\mu\tilde{\varphi}g^{\mu\nu}\partial_\nu\tilde{\varphi}\right) \\
& - h^\nu_\beta h^\beta_\alpha \mathrm{R}^\alpha_\nu + \frac{1}{2}h^\alpha_\alpha h^\nu_\beta \mathrm{R}^\beta_\nu - \frac{1}{4}h^\beta_{\alpha,\nu}h^{\alpha,\nu}_\beta + \frac{1}{4}h^\alpha_{\alpha,\mu}h^{\beta,\mu}_\beta \\
& - \frac{1}{2}h^\alpha_{\alpha,\beta}h^{\beta,\mu}_\mu + \frac{1}{2}h^{\nu,\alpha}_\beta h^\beta_{\alpha,\nu}\Big]
\end{aligned}
\tag{4.25}
$$

The Lagrangian $\mathscr{L}$ is invariant to the gauge transformation of φ and

$$h_{\mu\nu} \to h_{\mu\nu} + (g_{\alpha\nu} + h_{\alpha\nu})\mathrm{D}_\mu\eta^\alpha + (g_{\mu\alpha} + h_{\mu\alpha})\mathrm{D}_\nu\eta^\alpha + \eta^\alpha\mathrm{D}_\alpha h_{\mu\nu}. \tag{4.26}$$

To have Feynman rules we must supplement the Lagrangian eq. (4.25) with a gauge fixing part $-\frac{1}{2}(\mathrm{C}_\mu)^2$ and a Faddeev-Popov ghost Lagrangian. In section 2 we have shown that there exists a gauge that allows easy verification of unitarity; the work of ref. 7 tells us that other choices of C are physically equivalent, and describe therefore also a unitary theory.

We will employ the following C:

$$\mathrm{C}_\alpha = \sqrt[4]{g}\left(h^\nu_{\mu,\nu} - \frac{1}{2}h^\nu_{\nu,\mu} - \varphi\partial_\mu\tilde{\varphi}\right)t^{\mu\alpha}. \tag{4.27}$$

The quantity $t^{\mu\alpha}$ is the root of the tensor $g^{\mu\nu}$:

$$t^{\mu\alpha}t^{\alpha\nu} = g^{\mu\nu}. \tag{4.28}$$

It has what one could call « mid-indices », but it will not play any substantial role. Using (4.27) we find:

$$-\frac{1}{2}C_\mu^2 = -\frac{1}{2}\sqrt{g}\left(h^\nu_{\mu,\nu} - \frac{1}{2}h^\nu_{\nu,\mu}\right)\left(h^{\alpha\mu}_{,\alpha} - \frac{1}{2}h^{\alpha,\mu}_\alpha\right) + \sqrt{g}\left(h^{\nu\mu}_{,\nu} - \frac{1}{2}h^{\nu,\mu}_\nu\right)\varphi\partial_\mu\tilde{\varphi} - \frac{1}{2}\sqrt{g}\varphi^2\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi}g^{\mu\nu}. \quad (4.29)$$

With this choice for C we obtain:

$$\underline{\underline{\mathscr{L}}} - \frac{1}{2}C_\mu^2 = \sqrt{g}\left(-\frac{1}{4}h^\beta_{\alpha,\nu}h^{\alpha,\nu}_\beta + \frac{1}{8}h^\alpha_{\alpha,\nu}h^{\beta,\nu}_\beta - \frac{1}{2}\partial_\mu\varphi g^{\mu\nu}\partial_\nu\varphi + \frac{1}{2}h^\alpha_\beta X^{\beta\mu}_{\alpha\nu}h^\nu_\mu + \varphi Y^\alpha_\beta h^\beta_\alpha + \frac{1}{2}\varphi Z\varphi\right), \quad (4.30)$$

with

$$X^{\beta\mu}_{\alpha\nu} = 2\left(-\frac{1}{2}\delta^\beta_\nu D^\mu\tilde{\varphi}D_\alpha\tilde{\varphi} + \frac{1}{4}\delta^\beta_\alpha D^\mu\tilde{\varphi}D_\nu\tilde{\varphi} - \frac{1}{16}\delta^\beta_\alpha\delta^\mu_\nu D_\gamma\tilde{\varphi}D^\gamma\tilde{\varphi} + \frac{1}{8}\delta^\beta_\nu\delta^\mu_\alpha D_\gamma\tilde{\varphi}D^\gamma\tilde{\varphi} - \frac{1}{8}\delta^\beta_\alpha\delta^\mu_\nu R + \frac{1}{4}\delta^\beta_\nu\delta^\mu_\alpha R - \frac{1}{2}\delta^\beta_\nu R^\mu_\alpha + \frac{1}{2}\delta^\beta_\alpha R^\mu_\nu + \frac{1}{2}R^{\beta\mu}{}_{\alpha\nu}\right), \quad (4.31)$$

$$Y^\alpha_\beta = \frac{1}{2}\delta^\alpha_\beta D_\nu D^\nu\tilde{\varphi} - D_\beta D^\alpha\tilde{\varphi}, \quad (4.32)$$

$$Z = -D_\mu\tilde{\varphi}D^\mu\tilde{\varphi}. \quad (4.33)$$

The Lagrangian eq. (4.30) is formally of the same forms as considered in the previous section, with fields φ_i written for the h^β_α. Even if the result of the previous section was very simple it still takes a considerable amount of work to evaluate the counter Lagrangian. This will be done in the next section. There also the ghost Lagrangian will be written down.

5. EVALUATION OF THE COUNTER LAGRANGIAN

To evaluate the counter Lagrangian we employ first the doubling trick. In addition to the fields h and φ we introduce fields h' and φ' that interact with one another in the identical way as the h and φ. That is, to the expression (4.30) we add the identical expression but with h' and φ' instead of h and φ. Obviously our counter Lagrangian will double, because in addition to any closed loop with h and φ particles we will have the same closed loop but with h' and φ' particles.

After doubling of eq. (4.30) and some trivial manipulations we obtain:

$$\underline{\underline{\mathscr{L}}}_d = \sqrt{g} \left\{ h^*_{\alpha\beta} \mathrm{P}^{\alpha\beta\mu\nu} \mathrm{D}_\gamma \mathrm{D}^\gamma h_{\mu\nu} + \frac{1}{2} h^*_{\alpha\beta} (\mathrm{X}^{\alpha\beta\mu\nu} + \mathrm{X}^{\mu\nu\alpha\beta}) h_{\mu\nu} \right.$$
$$\left. + \varphi^* \mathrm{D}_\mu \mathrm{D}^\mu \varphi + \varphi^* \mathrm{Y}^{\alpha\beta} h_{\alpha\beta} + h^*_{\alpha\beta} \mathrm{Y}^{\alpha\beta} \varphi + \varphi^* \mathrm{Z} \varphi \right\}, \quad (5.1)$$

$$\mathrm{P}^{\alpha\beta\mu\nu} = \frac{1}{2} g^{\alpha\mu} g^{\beta\nu} - \frac{1}{4} g^{\alpha\beta} g^{\mu\nu}. \quad (5.2)$$

The counter Lagrangian is invariant to the replacement

$$h^*_{\alpha\beta} \rightarrow h^*_{\mu\nu} \mathrm{P}^{-1}_{\mu\nu\alpha\beta}, \quad (5.3)$$

with (compare eq. 2.8):

$$\mathrm{P}^{-1}_{\mu\nu\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - g_{\mu\nu} g_{\alpha\beta}. \quad (5.4)$$

It is to be noted that the replacement (5.3) is not a covariant replacement. But at this point the transformation properties of the $h_{\mu\nu}$ are no more relevant, they are treated simply as certain fields φ_i as occurring in the equations of section 3.

We so arrive at a Lagrangian of which the part containing two derivatives (with respect to the fields h and φ) is of the form

$$\sqrt{g} \varphi^*_i \mathrm{D}_\mu \mathrm{D}^\mu \varphi_i = - \sqrt{g} \partial_\mu \varphi^*_i g^{\mu\nu} \partial_\nu \varphi_i. \quad (5.5)$$

Note that $\mathrm{D}_\mu \mathrm{D}^\mu h_{\alpha\beta}$ is *not* the same thing as $\mathrm{D}_\mu \mathrm{D}^\mu \varphi_i$, treating the φ_i as scalars. We must rewrite $\mathrm{D}_\mu \mathrm{D}^\mu h_{\alpha\beta}$ in terms of derivatives $\bar{\mathrm{D}}$ that do not work on the indices α, β. We have:

$$\mathrm{D}_\mu \mathrm{D}_\nu h_{\alpha\beta} = \frac{1}{2} \bar{\mathrm{D}}_\mu \bar{\mathrm{D}}_\nu h_{\alpha\beta} - 2\Gamma^\gamma_{\mu\alpha} \partial_\nu h_{\gamma\beta} - (\partial_\mu \Gamma^\gamma_{\nu\alpha}) \delta^\pi_\beta h_{\gamma\pi}$$
$$+ \Gamma^\pi_{\mu\nu} \Gamma^\gamma_{\pi\alpha} \delta^\vartheta_\beta h_{\gamma\vartheta} + \Gamma^\pi_{\mu\alpha} \Gamma^\gamma_{\nu\pi} \delta^\vartheta_\beta h_{\gamma\vartheta} + \Gamma^\pi_{\mu\beta} \Gamma^\gamma_{\nu\alpha} h_{\gamma\pi}$$
$$+ (\text{same, but with } \alpha \leftrightarrow \beta). \quad (5.6)$$

In this way one obtains

$$\sqrt{g} h^*_{\alpha\beta} \mathrm{D}_\mu \mathrm{D}^\mu h_{\alpha\beta} = \sqrt{g} (h^*_{\alpha\beta} \bar{\mathrm{D}}_\mu \bar{\mathrm{D}}^\mu h_{\alpha\beta} + 2 h^*_{\alpha\beta} \mathscr{N}^{\mu\gamma\nu}_{\alpha\beta} \bar{\mathrm{D}}_\mu h_{\gamma\nu} + h^*_{\alpha\beta} \mathscr{T}^{\mu\nu}_{\alpha\beta} h_{\mu\nu}), \quad (5.7)$$

with

$$\mathscr{N}^{\mu\nu\gamma}_{\alpha\beta} = (-2 g^{\mu\pi} \Gamma^\nu_{\pi\alpha} \delta^\gamma_\beta)_{\text{symm } \alpha,\beta;\, \nu,\gamma}. \quad (5.8)$$

We have written for symplicity only one term, the subscript « symm » denotes that only the part symmetrical with respect to α, β exchange, as well as ν, γ exchange is to be taken. Further:

$$\mathscr{T}^{\mu\nu}_{\alpha\beta} = \mathrm{D}_\gamma \mathscr{N}^{\gamma\mu\nu}_{\alpha\beta} + \Gamma^\gamma_{\gamma\pi} \mathscr{N}^{\pi\mu\nu}_{\alpha\beta} + \mathscr{N}^{\pi\delta\upsilon}_{\alpha\beta} \mathscr{N}^{\gamma\mu\nu}_{\delta\upsilon} g_{\pi\gamma}, \quad (5.9)$$

or symbolically

$$\mathscr{T} = \mathrm{D}_\mu \mathscr{N}^\mu + \mathscr{N}_\mu \mathscr{N}^\mu, \quad (5.10)$$

where the covariant derivative « sees » only the index explicity written.

We can now apply the equations of section 3. The fields h_{11}, h_{22}, etc. may be renamed $\varphi_1, \ldots, \varphi_{10}$, the remaining fields φ as φ_{11} etc. The matrix $\mathcal{N}$ of section 3 can be identified with eq. (5.8), that is non-zero only in the first 10×10 submatrix. The matrix $\mathcal{M}$ is of the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{T} + \mathrm{P}^{-1}\bar{\mathrm{X}} & \mathrm{P}^{-1}\bar{\mathrm{Y}} \\ \bar{\mathrm{Y}} & \mathrm{Z} \end{pmatrix}. \tag{5.11}$$

Here

$$\bar{\mathrm{X}}^{\alpha\beta\mu\nu} = \frac{1}{2} g^{\alpha\gamma} g^{\nu\pi} \mathrm{X}^{\beta\mu}_{\gamma\pi} + \frac{1}{2} g^{\beta\gamma} g^{\mu\pi} \mathrm{X}^{\alpha\nu}_{\gamma\pi}. \tag{5.12}$$

$$\bar{\mathrm{Y}}^{\alpha\beta} = g^{\alpha\nu} \mathrm{Y}^{\beta}_{\nu}, \tag{5.13}$$

with X, Y, Z given in eqs. (4.31-33) and P^{-1} in eq. (5.4). The counter Lagrangian due to all this is given in eq. (3.35). Note that the $\mathcal{T}$ in eq. (5.11) cancels out (see eq. 5.10). A calculation of a few lines gives:

$$\begin{aligned} \mathrm{Tr}\,(\mathcal{Y}^{\mu\nu}\mathcal{Y}_{\mu\nu}) &= 6 g^{\mu\gamma} g^{\nu\beta} \mathrm{R}^{\pi}_{\alpha\mu\nu} \mathrm{R}^{\alpha}_{\pi\gamma\beta} \\ &= -\,6 \mathrm{R}_{\alpha\beta\mu\nu} \mathrm{R}^{\alpha\beta\mu\nu} = 6\mathrm{R}^2 - 24 \mathrm{R}_{\alpha\beta} \mathrm{R}^{\alpha\beta}. \end{aligned} \tag{5.14}$$

See appendix B for the last equality (apart from total derivatives). From eq. (3.35) we see that we must evaluate

$$\left(\mathcal{M} - \mathcal{N}^{\mu}\mathcal{N}_{\mu} - \mathrm{D}_{\mu}\mathcal{N}^{\mu} - \frac{1}{6}\mathrm{R} \right)^2.$$

The various pieces contributing to this are

$$\mathrm{Tr}\,(\mathrm{P}^{-1}\bar{\mathrm{X}}\mathrm{P}^{-1}\bar{\mathrm{X}}) = 2\mathrm{R}^2 + 6\mathrm{R}_{\alpha\beta}\mathrm{R}^{\alpha\beta} + 3(\partial_\mu\tilde{\varphi} g^{\mu\nu} \partial_\nu\tilde{\varphi})^2. \tag{5.15}$$

$$\mathrm{Tr}\left(-\frac{1}{3}\mathrm{P}^{-1}\bar{\mathrm{X}}\mathrm{R} + \frac{1}{36}\mathrm{R}^2 \right) = -\frac{31}{18}\mathrm{R}^2. \tag{5.16}$$

In evaluating terms like $\mathrm{Tr}(\mathrm{R}^2)$ remember that one takes the trace of a 10×10 matrix.

$$2\mathrm{Tr}(\bar{\mathrm{Y}}\mathrm{P}^{-1}\bar{\mathrm{Y}}) = 2(\mathrm{D}_\mu \mathrm{D}^\mu \tilde{\varphi})^2 + 4\mathrm{R}^{\alpha\beta}(\partial_\alpha\tilde{\varphi}\partial_\beta\tilde{\varphi}). \tag{5.17}$$

$$\mathrm{Tr}\left(\mathrm{Z} - \frac{1}{6}\,\mathrm{R}^2 \right) = \frac{1}{3}\mathrm{R}(\partial_\nu\tilde{\varphi} g^{\mu\nu} \partial_\mu\tilde{\varphi}) + \frac{1}{36}\mathrm{R}^2 + (\partial_\mu\tilde{\varphi} g^{\mu\nu} \partial_\nu\tilde{\varphi})^2. \tag{5.18}$$

As a final step we must compute the contribution due to the Faddeev-Popov ghost. The Faddeev-Popov ghost Lagrangian is obtained by subjecting C_μ to a gauge transformation. Without any difficulty we find (note $\varphi \rightarrow \varphi + \eta^\alpha \mathrm{D}^\alpha(\tilde{\varphi} + \varphi)$):

$$\mathcal{L}_{\text{ghost}} = \sqrt{g}\eta^*_\mu \left\{ \eta_{\mu;\alpha}^{\;;\alpha} - \mathrm{R}_{\alpha\mu}\eta^\alpha - (\partial_\alpha\tilde{\varphi}\partial_\mu\tilde{\varphi})\eta^\alpha \right\}. \tag{5.19}$$

Terms containing h or φ can be dropped, because we are not splitting up η in classical and quantum part. The Faddeev-Popov ghost is never external. In deriving (5.19) we used an equation like (4.14), and transformed a factor $\sqrt[4]{gt}$ away by means of an η^* substitution. Again, it is necessary to work out the covariant derivatives; the contribution due to that is:

$$\mathrm{Tr}\,(\mathscr{Y}^{\mu\nu}\mathscr{Y}_{\mu\nu})_{\text{ghost}} = -\,R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta}. \tag{5.20}$$

The evaluation of the rest is not particularly difficult; the total result is:

$$(\Delta\mathscr{L})_{\text{ghost}} = -\frac{\sqrt{g}}{\varepsilon}\left\{\frac{1}{6}R(\partial_\mu\tilde{\varphi}g^{\mu\nu}\partial_\nu\tilde{\varphi}) + \frac{17}{60}R^2 + \frac{7}{30}R_{\alpha\beta}R^{\alpha\beta}\right.$$
$$\left. + R^{\alpha\beta}(\partial_\alpha\tilde{\varphi}\partial_\beta\tilde{\varphi}) + \frac{1}{2}(\partial_\mu\tilde{\varphi}g^{\mu\nu}\partial_\nu\tilde{\varphi})^2\right\}. \tag{5.21}$$

Notice the minus sign that is to be associated with F-P ghost loops.

Adding all pieces together, not forgetting the factor 2 to undo the doubling of the non-ghost part, gives the total result (remember also the last part of eq. 3.35 for the non-ghost part; one must add 11 times that part):

$$\Delta\mathscr{L} = \frac{\sqrt{g}}{\varepsilon}\left\{\frac{9}{720}R^2 + \frac{43}{120}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{2}(\partial_\mu\tilde{\varphi}g^{\mu\nu}\partial_\nu\tilde{\varphi})^2\right.$$
$$\left. - \frac{1}{12}R(\partial_\mu\tilde{\varphi}g^{\mu\nu}\partial_\nu\tilde{\varphi}) + 2(D_\mu D^\mu\tilde{\varphi})^2\right\}. \tag{5.22}$$

The obtain the result for pure gravitation we note that contained in eq. (5.22) are the contributions due to closed loops of φ-particles. But this part is already known, from our calculations concerning a scalar particle in an external gravitational field. It is obtained from eq. (3.40) with $M = N = 0$:

$$\frac{\sqrt{g}}{\varepsilon}\left(\frac{1}{144}R^2 + \frac{1}{120}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{360}R^2\right). \tag{5.23}$$

Subtracting this from eq. (5.22) and setting $\tilde{\varphi}$ equal to zero gives the counter Lagrangian for the case of pure gravity:

$$\Delta\mathscr{L}_{\text{grav}} = \frac{\sqrt{g}}{\varepsilon}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\alpha\beta}R^{\alpha\beta}\right). \tag{5.24}$$

6. EQUATIONS OF MOTION

From eq. (4.23) we can trivially read off the equations of motion that the classical fields must obey in order that the first order part $\underline{\mathscr{L}}$ disappears:

$$D_\mu D^\mu\tilde{\varphi} = 0, \tag{6.1}$$

$$\left(-\frac{1}{2}R - \frac{1}{4}D_\alpha\tilde{\varphi}D^\alpha\tilde{\varphi}\right)\delta^\mu_\nu + R^\mu_\nu + \frac{1}{2}D_\nu\tilde{\varphi}D^\mu\tilde{\varphi} = 0. \tag{6.2}$$

Taking the trace of eq. (6.2) we find

$$R = -\frac{1}{2}(D_\alpha\tilde{\varphi}D^\alpha\tilde{\varphi}).$$

Substituting this back into eq. (6.2) gives us the set:

$$\begin{aligned} D_\mu D^\mu \tilde{\varphi} &= 0, \\ R_{\mu\nu} &= -\frac{1}{2}(D_\mu\tilde{\varphi})(D_\nu\tilde{\varphi}), \\ R &= -\frac{1}{2}(D_\mu\tilde{\varphi})(D^\mu\tilde{\varphi}). \end{aligned} \tag{6.3}$$

For pure gravity we have simply

$$R^\mu_\nu = 0, \qquad R = 0. \tag{6.4}$$

Inserting eq. (6.3) into eq. (5.22) and (5.24) gives:

$$\Delta\mathscr{L}_{\text{grav.scal}} = \frac{\sqrt{g}}{\varepsilon}\frac{203}{80}R^2, \tag{6.5}$$

$$\Delta\mathscr{L}_{\text{grav}} = 0. \tag{6.6}$$

If one were to approach the theory of gravitation just as any other field theory, one recognizes that the counterterm eq. (6.5) is not of a type present in the original Lagrangian eq. (4.1), and is therefore of the non-renormalizable type.

The question arises if the counterterm can be made to disappear by modification of the original Lagrangian. This will be investigated in the next section.

7. THE « IMPROVED » ENERGY-MOMENTUM TENSOR

The Lagrangian eq. (4.1) can be modified by inclusion of two extra terms:

$$\mathscr{L} = \sqrt{g}\left(-R - \frac{1}{2}\partial_\mu\varphi g^{\mu\nu}\partial_\nu\varphi + aR\varphi^2 + bR^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right). \tag{7.1}$$

The last term cannot improve the situation, because it has not the required dimension. So, we have not considered the case $b \neq 0$. Concerning the coefficient a we know already that the choice $a = \frac{1}{12}$ reduces divergencies of diagrams without internal gravitons. These are not present in eq. (7.1) because the φ-field has no non-gravitational interactions of the type $\alpha\varphi^4$, say. Actually this same choice for a seems of some help in the more general case, but it still leaves us with divergencies.

The essential tool in the study of more complicated theories is the Weyl transformation (see for example ref. 15). This concerns the behaviour under the transformation

$$g_{\mu\nu} \rightarrow g_{\mu\nu} f, \tag{7.2}$$

where f is any function of space-time. By straightforward calculation one establishes that under this transformation

$$\begin{aligned} R^{\mu}_{\nu\alpha\beta} \rightarrow R^{\mu}_{\nu\alpha\beta} &- \frac{1}{2}\delta^{\mu}_{\alpha}D_{\beta}s_{\nu} + \frac{1}{2}\delta^{\mu}_{\beta}D_{\alpha}s_{\nu} + \frac{1}{2}D_{\beta}(g_{\nu\alpha}s^{\mu}) \\ &- \frac{1}{2}D_{\alpha}(g_{\nu\beta}s^{\mu}) + \frac{1}{4}\delta^{\mu}_{\alpha}s_{\beta}s_{\nu} - \frac{1}{4}\delta^{\mu}_{\beta}s_{\alpha}s_{\nu} \\ &+ \frac{1}{4}\left(g_{\beta\nu}s_{\alpha}s^{\mu} - g_{\beta\nu}\delta^{\mu}_{\alpha}s^{2} - g_{\alpha\nu}s_{\beta}s^{\mu} + g_{\alpha\nu}\delta^{\mu}_{\beta}s^{2}\right), \end{aligned} \tag{7.3}$$

with

$$s_{\alpha} = \frac{\partial_{\alpha} f}{f} = \partial_{\alpha}(\ln f). \tag{7.4}$$

From this:

$$R_{\nu\alpha} \rightarrow R_{\nu\alpha} + D_{\alpha}s_{\nu} + \frac{1}{2}g_{\nu\alpha}D_{\beta}s^{\beta} - \frac{1}{2}s_{\alpha}s_{\nu} + \frac{1}{2}g_{\alpha\nu}s^{2}, \tag{7.5}$$

and

$$\begin{aligned} R = g^{\nu\alpha}R_{\nu\alpha} \rightarrow \frac{1}{f}R &+ \frac{1}{f}\left(3D_{\alpha}s^{\alpha} + \frac{3}{2}s_{\alpha}s^{\alpha}\right) \\ &= \frac{1}{f}R + \frac{3}{f}D_{\alpha}\frac{f^{\alpha}}{f} + \frac{3}{2f^{3}}f_{\alpha}f^{\alpha}. \end{aligned} \tag{7.6}$$

Conversely

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu}\frac{1}{f}, \\ R &\rightarrow fR - 3fD_{\alpha}\frac{f^{\alpha}}{f} + \frac{3}{2f}f_{\alpha}f^{\alpha}. \end{aligned} \tag{7.7}$$

Consider now the Lagrangian

$$\mathscr{L} = \sqrt{g}\left(-fR - \frac{1}{2}\partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\varphi - \frac{1}{2}m^{2}\varphi^{2}\right), \tag{7.8}$$

with $f = 1 - a\varphi^2$. Now perform the transformation (7.7). We obtain

$$\begin{aligned} \mathscr{L} = \sqrt{g}\Big(-R + 3D_{\alpha}\frac{f^{\alpha}}{f} - \frac{3}{2f^{2}}f_{\alpha}f^{\alpha} &- \frac{2}{2f}\partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\varphi \\ &- \frac{1}{2f^{2}}m^{2}\varphi^{2}\Big). \end{aligned} \tag{7.9}$$

This Lagrangian belongs to the general class

$$\mathscr{L} = \sqrt{g}\left\{-R + \frac{1}{2}\partial_{\mu}\varphi g^{\mu\nu}\partial_{\nu}\varphi f_{1}(\varphi) + f_{2}(\varphi)\right\}. \tag{7.10}$$

It is not too difficult to see the changes with respect to the treatment of the previous sections. However, it becomes quite cumbersome to work out the quantity $(\mathcal{M})^2$, and we have taken recourse to the computer and the Schoonschip program [16]. Roughly speaking the following obtains. The required Lagrangian (7.1) has $f = 1 - a\varphi^2$. As is clear from eq. (7.9) the Lagrangian written in the form eq. (7.10) becomes a power series in the field φ, with coefficients depending on a. The counter Lagrangian becomes also a power series in $\tilde{\varphi}$ with non-trivial coefficients. Putting the coefficient a to $\frac{1}{12}$ is of little help, the final result seems not to be of any simple form.

8. CONCLUSIONS

The one loop divergencies of pure gravitation have been shown to be such that they can be transformed away by a field renormalization. This depends crucially on the well-known identity (see appendix B)

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 = \text{total derivative},$$

which is true in four-dimensional space only.

In case of gravity interacting with scalar particles divergencies of physically meaningful quantities remain. They cannot be absorbed in the parameters of the theory.

Modification of the gravitational interaction, such as would correspond to the use of the improved energy-momentum tensor is of help only with respect to a certain (important) class of divergencies, but unrenormalizable divergencies of second order in the gravitational coupling constant remain. We do not feel that this is the last word on this subject, because the situation as described in section 7 is so complicated that we feel less than sure that there is no way out. A certain exhaustion however prevents us from further investigation, for the time being.

APPENDIX A

NOTATIONS AND CONVENTIONS

Our metric is that corresponding to a purely imaginary time coordinate. In flat space $g_{\mu\nu} = \delta_{\mu\nu}$. Units are such that the gravitational coupling constant is one.

The point of view we take is that gravitation is a gauge theory. Under a gauge transformation scalars, vectors, tensors are assigned the following behaviour under infinitesimal transformations

$$\varphi' = \varphi + \eta^\alpha \partial_\alpha \varphi \qquad \text{scalar}$$
$$A'_\mu = A_\mu + A_\alpha \partial_\mu \eta^\alpha + \eta^\alpha \partial_\alpha A_\mu \qquad \text{covariant vector}$$
$$A'^\mu = A^\mu - A^\alpha \partial_\alpha \eta^\mu + \eta^\alpha \partial_\alpha A^\mu \qquad \text{contravariant vector}$$
$$B'_{\mu\nu} = B_{\mu\nu} + B_{\alpha\nu}\partial_\mu \eta^\alpha + B_{\mu\alpha}\partial_\nu \eta^\alpha + \eta^\alpha \partial_\alpha B_{\mu\nu}. \qquad \text{covariant two-tensor}$$

Note that dot-products such as $A^\mu B_\mu$ are not invariant but behave as a scalar.

Let now $B_{\mu\nu}$ be an arbitrary two-tensor. It can be established that under a gauge transformation

$$\sqrt{\det B'} = \sqrt{\det B} + \partial_\alpha(\eta^\alpha \sqrt{\det B}) \tag{A.1}$$

A Lagrangian of the form

$$\int d_4x \sqrt{\det B}\,.\,\varphi$$

where φ is a scalar (in the sense defined above) is invariant under gauge transformations. One finds:

$$\int d_4x \sqrt{\det B'}\varphi' = \int d_4x \left\{ \sqrt{\det B}\varphi + \partial_\alpha(\eta^\alpha \sqrt{\det B}\varphi) \right\} \tag{A.2}$$

The second term is a total derivative and the integral of that term vanishes (under proper boundary conditions).

Further invariants may be constructed in the usual way:

$$D_\nu \varphi = \partial_\nu \varphi \tag{A.3}$$
$$D_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\alpha_{\nu\mu} A_\alpha, \text{ etc.} \tag{A.4}$$

with

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}). \tag{A.5}$$

In here $g_{\mu\nu}$ may be any symmetric two tensor possessing an inverse $g^{\mu\nu}$, but in practice one encounters here only the metric tensor. The quantities Γ do not transform under gauge transformations as its indices indicate; in fact

$$\Gamma'^\mu_{\nu\alpha} = (\Gamma'^\mu_{\nu\alpha})_{\text{tensor}} + \partial_\nu \partial_\alpha \eta^\mu. \tag{A.6}$$

The quantities $D_\nu A_\mu$ behave under gauge transformations as a covariant two tensor. Similarly

$$D_\mu B^\nu = \partial_\mu B^\nu + \Gamma^\alpha_{\mu\alpha} B^\alpha \tag{A.7}$$

behaves as a mixed two tensor.

Let now $g_{\mu\nu}$ be the tensor used in the definition of the covariant derivatives. Then it is easy to show that

$$D_\alpha g_{\mu\nu} = 0. \tag{A.8}$$

Another useful equation relates the Christoffel symbol Γ and the determinant of the tensor used in its construction:

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{\sqrt{\det g}} \partial_{\mu}\sqrt{\det g}. \tag{A.9}$$

This leads to the important equation

$$\sqrt{\det g}\, D_{\mu}A^{\mu} = \partial_{\mu}(\sqrt{\det g}\, A^{\mu}). \tag{A.10}$$

As a consequence one can perform partial differentiation much like in the usual theories:

$$\int d_4x\sqrt{g}(\varphi D_{\mu}A^{\mu}) = -\int d_4x\sqrt{g}(D_{\mu}\varphi A^{\mu}). \tag{A.11}$$

Given any symmetric two tensor having an inverse one can construct the associated Riemann tensor:

$$R^{\mu}_{\nu\alpha\beta} = -\partial_{\beta}\Gamma^{\mu}_{\nu\alpha} + \partial_{\alpha}\Gamma^{\mu}_{\nu\beta} + \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\gamma}_{\nu\beta} - \Gamma^{\mu}_{\gamma\beta}\Gamma^{\gamma}_{\nu\alpha}. \tag{A.12}$$

We use the convention $R^{\mu}_{\nu\alpha\beta} = R^{\mu}{}_{\nu\alpha\beta}$, which is of importance in connection with the raising and lowering of indices.

The Riemann tensor has a number of symmetry properties. With, as usual

$$R_{\mu\nu\alpha\beta} = g_{\mu\gamma}R^{\gamma}_{\nu\alpha\beta}, \qquad R_{\nu\beta} = R^{\mu}_{\nu\beta\mu}, \qquad R = R_{\nu\beta}g^{\nu\beta} \tag{A.13}$$

one has

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}, \qquad R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}, \qquad R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu},$$
$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0, \qquad R_{\nu\beta} = R_{\beta\nu}, \tag{A.14}$$

The Bianchi identities are

$$D_{\gamma}R_{\alpha\beta\mu\nu} + D_{\nu}R_{\alpha\beta\gamma\mu} + D_{\mu}R_{\alpha\beta\nu\gamma} = 0. \tag{A.15}$$

The generalisation of the completely antisymmetric four-tensor is

$$\eta^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{g}}\varepsilon^{\alpha\beta\mu\nu}, \qquad \eta_{\alpha\beta\mu\nu} = \sqrt{g}\varepsilon_{\alpha\beta\mu\nu}$$
$$\varepsilon^{\alpha\beta\mu\nu} = \varepsilon_{\alpha\beta\mu\nu} = \begin{cases} 1 \quad \text{if} \quad \alpha, \beta, \mu, \nu = 1, 2, 3, 4 \\ \text{antisymmetric under exchange of any two indices.} \end{cases} \tag{A.16}$$

Note that

$$\eta^{\alpha\beta\mu\nu}\eta^{\pi\gamma\delta\vartheta}g_{\alpha\pi}g_{\beta\gamma}g_{\mu\delta}g_{\nu\vartheta} = 1. \tag{A.17}$$

It is easily shown that

$$D_{\gamma}\eta^{\alpha\beta\mu\nu} = 0 \tag{A.18}$$

In the derivation one uses the fact that in four dimensions a totally antisymmetric tensor with five indices is necessarily zero. Thus:

$$\Gamma^{\gamma}_{\pi\delta}\eta^{\alpha\beta\mu\nu} = \Gamma^{\alpha}_{\pi\delta}\eta^{\gamma\beta\mu\nu} + \Gamma^{\beta}_{\pi\delta}\eta^{\alpha\gamma\mu\nu} + \Gamma^{\mu}_{\pi\delta}\eta^{\alpha\beta\gamma\nu} + \Gamma^{\nu}_{\pi\delta}\eta^{\alpha\beta\mu\gamma} \tag{A.19}$$

Finally, the Bianchi identities lead directly to the following equation

$$D_{\gamma}R_{\alpha\beta\mu\nu}\eta^{\gamma\mu\nu\pi} = 0. \tag{A.20}$$

APPENDIX B

PRODUCTS OF RIEMANN TENSORS [6]

Let us consider the following Lagrangian

$$\int d_4x\sqrt{g}R_{\pi\vartheta\alpha\beta}R_{\mu\nu\gamma\delta}\eta^{\pi\vartheta\mu\nu}\eta^{\alpha\beta\gamma\delta}. \tag{B.1}$$

Subjecting $g_{\mu\nu}$ to a small variation

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu},$$

we will show that the integral remains unchanged (the integrand changes by a total derivative).

In section 4 the equations showing the variation of the various objects under a change of the $g_{\mu\nu}$ have been given. One finds:

$$\begin{aligned}\delta(\sqrt{g}RR\eta\eta) = & -\frac{1}{2}g^{\alpha\beta}h_{\alpha\beta}(\sqrt{g}RR\eta\eta)\\ & - 4\sqrt{g}g_{\gamma\mu}(D_\beta\underline{\Gamma}^{\mu}_{\nu\alpha})R_{\rho\vartheta\pi\delta}\eta^{\gamma\nu\pi\delta}\eta^{\rho\vartheta\alpha\beta}\\ & + 2\sqrt{g}h_{\gamma\mu}g^{\mu\pi}R_{\pi\vartheta\alpha\beta}R_{\rho\tau\nu\delta}\eta^{\gamma\vartheta\nu\delta}\eta^{\rho\tau\alpha\beta}.\end{aligned}$$

The last term can be treated by means of the same identity as we used at the end of appendix A, eq. (A.19); i. e. this term is equal to the sum of the four terms obtained by interchanging π with γ, ϑ, ν and δ respectively. The last three terms are equal to minus the original term, and one obtains, after some fiddling with indices:

$$g^{\mu\pi}R_{\pi\vartheta\alpha\beta}R_{\rho\tau\nu\delta}\eta^{\gamma\vartheta\nu\delta}\eta^{\rho\tau\alpha\beta} = \frac{1}{4}g^{\mu\gamma}R_{\pi\vartheta\alpha\beta}R_{\rho\tau\nu\delta}\eta^{\pi\vartheta\nu\delta}\eta^{\rho\tau\alpha\beta}. \tag{B.2}$$

The final result is

$$\delta(\sqrt{g}RR\eta\eta) = -4\sqrt{g}g_{\mu\nu}(D_\beta\underline{\Gamma}^{\mu}_{\vartheta\alpha})R_{\pi\tau\gamma\delta}\eta^{\nu\vartheta\gamma\delta}\eta^{\pi\tau\alpha\beta}. \tag{B.3}$$

Using the Bianchi identities in the form (A.20), and exploiting the fact that the covariant derivatives of $g_{\mu\nu}$ and $\eta^{\mu\nu\alpha\beta}$ are zero we see that

$$\delta(\sqrt{g}RR\eta\eta) = -4\sqrt{g}D_\beta(g_{\nu\mu}\underline{\Gamma}^{\mu}_{\vartheta\alpha}R_{\pi\tau\gamma\delta}\eta^{\nu\vartheta\gamma\delta}\eta^{\pi\tau\alpha\beta}) \tag{B.4}$$

which is the desired result.

The consequence of this work is that in a Lagrangian the expression (B.1) can be omitted. Now (B.1) can be worked out by using the well known identity

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\pi\vartheta\gamma\delta} = \delta^{\mu}_{\pi}\delta^{\nu}_{\vartheta}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\mu}_{\pi}\delta^{\nu}_{\vartheta}\delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta}$$
$$+ \text{(all permutations of the upper indices with the approprate sign).} \tag{B.5}$$

One obtains

$$R_{\pi\vartheta\alpha\beta}R_{\mu\nu\gamma\delta}\eta^{\pi\vartheta\mu\nu}\eta^{\alpha\beta\gamma\delta} = 4(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2). \tag{B.6}$$

The above derivation can be generalized to an arbitrary number of dimensions. The recipe is simple: take two totally antisymmetric objects and saturate them with the Riemann four tensor. The resulting expression is such that its variation is a total derivative. For instance in two dimensions:

$$R_{\alpha\beta\mu\nu}\eta^{\alpha\beta}\eta^{\mu\nu} = R_{\alpha\beta}{}^{\mu\nu}(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}) = -2R. \tag{B.7}$$

Clearly, the Einstein-Hilbert Lagrangian is meaningless in two dimensions. This fact shows up as an n-dependence in the graviton propagator; in the Prentki gauge as shown in eq. (2.9). Also in other gauges factors $1/(n - 2)$ appear, as found by Neveu [*13*], and Capper *et al.* [*4*].

ACKNOWLEDGMENTS

The authors are indebted to Prof. S. Deser, for very useful comments.

REFERENCES

[*1*] G. 'T HOOFT, *Nuclear Phys.*, **35B**, 1971, p. 167.
[*2*] G. 'T HOOFT and M. VELTMAN, *Nuclear Phys.*, **44B**, 1972, p. 189;
C. G. BOLLINI and J. J. GIAMBIAGI, *Il Nuovo Cim.*, **12B**, 1972, p. 20.
[*3*] D. H. CAPPER and G. LEIBBRANDT, *Lett. Nuovo Cim.*, t. **6**, 1973, p. 117;
M. BROWN, *Nuclear Phys.*, **56B**, 1973, p. 194.
[*4*] D. H. CAPPER, G. LEIBBRANDT and M. RAMON MEDRANO, ICTP Trieste preprint 73/76, April 1973.
[*5*] R. ARNOWITT, S. DESER and C. MISNER, *Phys. Rev.*, t. **113**, 1959, p. 745; *Phys. Rev.*, t. **116**, 1959, p. 1322; *Phys. Rev.*, t. **117**, 1960, p. 1595;
R. P. FEYNMAN, *Acta Phys. Polon.*, t. **24**, 1963, p. 697;
S. MANDELSTAM, *Phys. Rev.*, t. **175**, 1968, p. 1580, 1604;
L. D. FADDEEV and V. N. POPOV, *Phys. Lett.*, **B25**, 1967, p. 29;
E. S. FRADKIN and I. V. TYUTIN, *Phys. Rev.*, **2D**, 1970, p. 2841;
C. J. ISHAM, A. SALAM and J. STRATHDEE, *Phys. Rev.*, **3D**, 1971, p. 867; *Lett. Nuovo Cim.*, t. **5**, 1972, p. 969.
[*6*] Relativity, Groups and Topology, Les Houches, 1963;
B. S. DEWITT, *Phys. Rev.*, t. **162**, 1967, p. 1195, 1239.
[*7*] G. 'T HOOFT and M. VELTMAN, *Nuclear Phys.*, **50B**, 1972, p. 318.
[*8*] G. 'T HOOFT, *Nuclear Phys.*, **62 B**, 1973, p. 444.
[*9*] C. G. CALLAN, S. COLEMAN and R. JACKIW, *Annals of Physics*, t. **59**, 1970, p. 42.
[*10*] K. SYMANZIK, private communication cited in ref. 9; *Comm. Nath. Phys.*, t. **18**, 1970, p. 227.
[*11*] J. H. LOWENSTEIN, *Comm. Math. Phys.*, t. **24**, 1971, p. 1;
B. SCHROER, *Lett. Nuovo Cim.*, t. **2**, 1971, p. 867.
[*12*] J. HONERKAMP, *Nuclear Phys.*, **48B**, 1972, p. 269 ; Proc. Marseille Conf., 19-23 June 1972, C. P. Korthals-Altes Ed.
[*13*] A. NEVEU, *Inst. For Adv. Study*, Princeton, preprint, dec. 1972.
[*14*] H. VAN DAM and M. VELTMAN, *Nuclear Phys.*, **22B**, 1970, p. 397.
[*15*] B. ZUMINO, Brandeis Lectures July 1970.
[*16*] SCHOONSCHIP, A CDC 6000 program for symbolic evaluation of algebraic expressions. M. Veltman, CERN preprint 1967. For further information: H. Strubbe, Div. DD, CERN.

(Manuscrit reçu le 4 septembre 1973).