ANNALES DE L'I. H. P., SECTION A

F. M. KUNI

T. YU. NOVOZHILOVA

L. C. ADJEMJAN

Mori's memory function formalism in nonlinear statistical hydrodynamics

Annales de l'I. H. P., section A, tome 19, nº 4 (1973), p. 375-386

<http://www.numdam.org/item?id=AIHPA_1973__19_4_375_0>

Ann. Inst. Henri Poincaré,
Vol. XIX, n° 4, 1973, p. 375-386.

Section A :
Physique théorique.

Mori's memory function formalism in nonlinear statistical hydrodynamics

by

F. M. KUNI, T. Yu. NOVOZHILOVA and L. C. ADJEMJAN
Department of Theoretical Physics, University of Leningrad, Leningrad

ABSTRACT. — The Mori's method of projection operators is generalized so as to make it applicable to the case of nonlinear statistical hydrodynamics. We use the Kawasaki's trick of introducing additional quadratic combinations into the Mori's equations, but contrary to the Kawasaki's result, we get equations compatible with the usual linear theory.

We consider also the inclusion of boundary conditions into the equations for averaged quantities and show that these equations are now translationally invariant in time variable.

INTRODUCTION

We consider the class of nonequilibrium processes under which the conserved quantities (or to be precise, their local values) rather slowly relaxate to their equilibrium values. Our aim is to study a hydrodynamical stage of these processes. The problem arising in the way of consistent investigation consists in bringing the equations of motions to such a form where the secular contributions connected with slow motions are completely included into the conserved quantities. The most successful solution of the problem has been obtained through the application of the projection operator technique developed by Mori [*1*]. In this approach, the secular components of the currents are subtracted from the currents appearing in the equations of motion. In an another approach [*2*] the secular contributions are separated by means of the Enskog-Chapman procedure,

i. e. by the step by step elimination of all the time derivatives of the thermodynamical parameters.

The Mori's approach is simpler than the Dufty's one [2] not only because it does not make use of the complicated Enskog-Chapman procedure, but also because it is dealing with the equations of motion (which contain only the longuitudinal current components).

At present, the Mori's technique of projection operators is well developed for the linear processes. In an attempt to generalize his method to nonlinear hydrodynamical processes, Mori proposed [3] to write equation not for the conserved quantity but for a random force which constitutes the inhomogeneous term in the general Mori's equation. This proposal however contradicts to the original Mori's idea of eliminating the conserved quantities from all the expressions of microscopic character. In an attempt to overcome difficulties of nonlinear theories, Kawasaki [4] included supplementary conserved quantities (which are quadratic in hydrodynamic variables) into the Mori's scheme. But the Kawasaki's approximations led him to unreliable results; even the linear theory was not sufficiently accurate.

In the present paper we treat a nonlinear statistical hydrodynamics by means of the Mori's projection operator technique. We use the Kawasaki trick of introducing additional quadratic combinations into the Mori's equations, but our way of doing it is in accordance with the usual linear theory. We consider also the inclusion of boundary conditions into the equations for averaged quantities and demonstrate that t these equations are translationally invariant in the time variable.

Since quantum and classical cases are easily transcribable from one to the other, we shall use the classical language. Our compact matrix expressions allow us to consider the most general case of none equilibrium thermodynamical parameters. The system may be multicomponent.

1. MORI'S EQUATION

Let us consider a Hilbert space of phase functions $\hat{G}$ (depending on the phase variables of the system) and denote the scalar product of two phase functions $\hat{F}$ and $\hat{G}$ by the parenthesis $(\hat{F}, \hat{G}^+)$. The scalar product has the usual properties

$$(1.1) \qquad (\hat{G}, \hat{F}^+) = (\hat{F}, \hat{G}^+)^+, \qquad (\hat{F}, \hat{G}^+) \geqslant 0,$$

$$(1.2) \qquad \Big(\sum_j c_j \hat{G}_j\,, \hat{F}^+\Big) = \sum_j c_j(\hat{G}_j\,, \hat{F}^+).$$

Suppose that in our space of phase functions some system of basic vectors, which will be represented by the column vector $\hat{\alpha}$, is chosen. The time development of the basic vectors is described by an equation

$$\frac{\partial \hat{\alpha}(t)}{\partial t} = i\mathrm{M}\hat{\alpha}(t)\,, \tag{1.3}$$

which is equivalent to

$$\hat{\alpha}(t) = \exp(it\mathrm{M}) \cdot \hat{\alpha}(0), \tag{1.4}$$

where M is a time-independent linear operator which we require to be Hermitian:

$$(\mathrm{M}\hat{\mathrm{G}}, \hat{\mathrm{Q}}^+) = (\hat{\mathrm{G}}, (\mathrm{M}\hat{\mathrm{Q}})^+). \tag{1.5}$$

Now we transform (1.3) to the equivalent Mori's equation. The projection of a vector $\hat{\mathrm{G}}$ onto $\hat{\alpha}$ is given by

$$\mathscr{P}_\alpha \hat{\mathrm{G}} \equiv (\hat{\mathrm{G}}, \hat{\alpha}^+)(\hat{\alpha}, \hat{\alpha}^+)^{-1}\hat{\alpha}. \tag{1.6}$$

This equation defines a linear Hermitian operator $\mathscr{P}_\alpha$ in the Hilbert space.

Let us split $i\mathrm{M}\hat{\alpha}$ into the longuitudinal and transversal components:

$$i\sigma\hat{\alpha} \equiv \mathscr{P}_\alpha i\mathrm{M}\hat{\alpha}, \tag{1.7}$$

$$\hat{v} \equiv (1 - \mathscr{P}_\alpha) i\mathrm{M}\hat{\alpha}, \tag{1.8}$$

where the frequency matrix σ is defined by

$$i\sigma = (i\mathrm{M}\hat{\alpha}, \hat{\alpha}^+)(\hat{\alpha}, \hat{\alpha}^+)^{-1}. \tag{1.9}$$

Then the equation (1.3) can be rewritten in the form

$$\frac{\partial \hat{\alpha}(t)}{\partial t} = i\sigma\hat{\alpha}(t) + \exp(it\mathrm{M}) \cdot \hat{v}. \tag{1.10}$$

Using the operator identity

$$e^{it\mathrm{M}} = e^{it(\mathrm{M}-\Delta\mathrm{M})} + \int_0^t ds\, e^{i(t-s)\mathrm{M}} i\Delta\mathrm{M} e^{is(\mathrm{M}-\Delta\mathrm{M})},$$

where we put $\Delta\mathrm{M} = \mathscr{P}_\alpha \mathrm{M}$, we obtain

$$e^{it\mathrm{M}}\hat{v} = \hat{v}[t] + \int_0^t ds\, e^{i(t-s)\mathrm{M}} i\mathscr{P}_\alpha \mathrm{M}\hat{v}[s]. \tag{1.11}$$

Here

$$\hat{v}[t] = \exp[it(1 - \mathscr{P}_\alpha)\mathrm{M}] \cdot \hat{v} \tag{1.12}$$

is so called random force, which is orthogonal to $\hat{\alpha}$:

$$(\hat{v}[t], \hat{\alpha}^+) = 0. \tag{1.13}$$

It follows from (1.5), (1.6) and (1.13) that

$$i\mathscr{P}_\alpha \mathrm{M}\hat{v}[s] = -\lambda(s)\hat{\alpha}, \tag{1.14}$$

where

$$\lambda(s) = (\hat{v}[s], \hat{v}^+)(\hat{\alpha}, \hat{\alpha}^+)^{-1} \tag{1.15}$$

is the Mori's memory function.

By virtue of (1.14), (1.11), (1.4) and (1.10) we get

$$\frac{\partial \hat{\alpha}(t)}{\partial t} = i\sigma\hat{\alpha}(t) - \int_0^t ds\lambda(s)\hat{\alpha}(t-s) + \hat{v}[t]. \tag{1.16}$$

Equation (1.16) is the Mori's equation for $\hat{\alpha}(t)$. Generalizing this equation we obtain instead of (1.16):

$$\hat{\beta}(t) = i\tau\hat{\alpha}(t) - \int_0^t ds\eta(s)\hat{\alpha}(t-s) + \hat{w}[t] \tag{1.17}$$

for an arbitrary $\hat{\beta}(t) \equiv \exp(it\mathrm{M}) \cdot \hat{\beta}$ developing in time by the same propagator $\exp(it\mathrm{M})$ as for $\hat{\alpha}(t)$. Here we denote

$$i\tau = (\hat{\beta}, \hat{\alpha}^+)(\hat{\alpha}, \hat{\alpha}^+)^{-1}, \tag{1.18}$$

$$\eta(s) = (\hat{w}[s], \hat{v}^+)(\hat{\alpha}, \hat{\alpha}^+)^{-1}, \tag{1.19}$$

where

$$\hat{w}[t] = \exp[it(1 - \mathcal{P}_\alpha)\mathrm{M}] \cdot (1 - \mathcal{P}_\alpha)\hat{\beta} \tag{1.20}$$

has an obvious property

$$(\hat{w}[t], \hat{\alpha}^+) = 0. \tag{1.21}$$

2. SYSTEM OF EQUATIONS FOR CONSERVED QUANTITIES

In hydrodynamics, the most interesting problem to study is the time evolution of the conserved quantities. As usual, the word « conserved quantity » is an abbreviated form for « local value of the conserved quantity ». Therefore, such a conserved quantity is not an integral of motion necessarily.

Let the components of a volumn-vector $\hat{a}(t)$ correspond to the densities (e. g. densities of energy, particle numbers, momentum taken at various spatial points of the system). Since $\hat{a}$ is a dynamical variable, the development of $\hat{a}(t)$ in time is defined by the equation

$$\frac{\partial \hat{a}(t)}{\partial t} = i\mathrm{L}\hat{a}(t) \tag{2.1}$$

or

$$\hat{a}(t) = \exp(it\mathrm{L}) \cdot \hat{a},$$

where L is the Liouville operator. From (1.16) for $\mathrm{M} \equiv \mathrm{L}$, $\hat{\alpha} \equiv \hat{a}$ we obtain

$$\frac{\partial \hat{a}(t)}{\partial t} = i\omega\hat{a}(t) - \int_0^t ds\varphi(s)\hat{a}(t-s) + \hat{f}[t], \tag{2.2}$$

where

$$i\omega = (iL\hat{a}, \hat{a}^{+})(\hat{a}, \hat{a}^{+})^{-1}, \tag{2.3}$$

$$\hat{f}[t] = \exp\,[it(1 - \mathscr{P}_a)L]\cdot(1 - \mathscr{P}_a)iL\hat{a}, \tag{2.4}$$

$$\varphi(s) = (\hat{f}[s], \hat{f}^{+})(\hat{a}, \hat{a}^{+})^{-1}, \tag{2.5}$$

$\mathscr{P}_a$ is the projection operator onto $\hat{a}$, so that

$$(\hat{f}[t], \hat{a}^{+}) = 0. \tag{2.6}$$

The Mori's equation (2.2) for $\hat{a}(t)$ is not closed, because it contains the random force $\hat{f}[t]$, and therefore an additional relation is needed. To get such a relation Mori [3] conjectured that $\hat{f}[t]$ also satisfies the equation of the same type as (1.16).

However, the random force $\hat{f}[t]$ is not a conserved quantity. The more consistent procedure is to write equations of the (1.16) type for the additional conserved quantities constructed out of $\hat{a}$-components (quadratic, cubic ... combinations) and to treat these additional quantities as a system of basic vectors complimentary to $\hat{a}$. The random force $\hat{f}[t]$ should be then expressed in terms of these new basic vectors by means of the relation similar to (1.17).

It is more convenient to deal with the conserved quantities in the Mori's equation because of two reasons. First of all, for the conserved quantities the random force (1.12) and the memory function (1.15) are small for small deviations from homogeneity. Secondly, in this case due to the orthogonality relation (1.13) the random force (1.12) will not contain (in our approximation) the secular conserved quantities. Then the memory function (1.15) will contain only nonsecular contributions and, in accordance with the condition of waekening of correlations, it will decrease for large values of s. As to secular terms, they all will be inclused into the conserved quantities, for which there exist the Mori's equations.

The introduction of a new quantity $\hat{A}$ (quadratic in $\hat{a}$) providing a supplementary system of basic vectors has been considered by Kawasaki [4]. However, the time evolution of $\hat{A}$ was defined by

$$\hat{A}(t) = \exp\,(itL)\cdot\hat{A},$$

with the same propagator as in the case of dynamical quantity $\hat{a}(t)$. As a result the orthogonality of $\hat{A}$ and $\hat{a}$, which may be always achieved at a given moment $t = 0$ (by introducing linear in $\hat{a}$ terms into $\hat{A}$), is lost for the other moments. Therefore one should consider both $\hat{a}(t)$ and $\hat{A}(t)$ as a single entity and write a single Mori's equation for them.

We propose the following law for the time evolution of $\hat{A}$:

$$A[t] = \exp\,[it(1 - \mathscr{P}_a)L]\cdot\hat{A}. \tag{2.6'}$$

Under the time evolution (2.6) the orthogonality at $t = 0$ will be preserved also for $t \neq 0$:

$$(\hat{A}[t], a^+) = 0. \tag{2.7}$$

The equation for $\hat{A}[t]$ is then obtained by substituting $\hat{\alpha} = \hat{A}$, $M = (1 - \mathscr{P}_a)L$ into (1.16):

$$\frac{\partial \hat{A}[t]}{\partial t} = i\Omega\hat{A}[t] - \int_0^t ds\Phi(s)A[t-s] + \hat{\mathscr{F}}\{t\}, \tag{2.8}$$

where

$$i\Omega = ((1 - \mathscr{P}_a)iL\hat{A}, \hat{A}^+)(\hat{A}, \hat{A}^+)^{-1}, \tag{2.9}$$

$$\hat{\mathscr{F}}\{t\} = \exp[it(1 - \mathscr{P}_A)(1 - \mathscr{P}_a)L]\cdot(1 - \mathscr{P}_A)(1 - \mathscr{P}_a)iL\hat{A} \tag{2.10}$$

$$\Phi(s) = (\hat{\mathscr{F}}\{s\}, \mathscr{F}^+)(\hat{A}, \hat{A}^+)^{-1}, \tag{2.11}$$

$\mathscr{P}_A$ is a projection operator onto $\hat{A}$ axis.

The new equation (2.8) for $\hat{A}[t]$ is not related directly to equation (2.2) for $\hat{a}(t)$. Indeed, besides the orthogonality relation (2.7) we have also the orthogonality relation for $\hat{\mathscr{F}}\{t\}$ and $\hat{a}$

$$(\hat{\mathscr{F}}\{t\}, \hat{a}^+) = 0. \tag{2.12}$$

Apart from (2.12) the random force $\hat{\mathscr{F}}\{t\}$ satisfies

$$(\hat{\mathscr{F}}\{t\}, \hat{A}^+) = 0. \tag{2.13}$$

To find an equation connecting $\hat{f}[t]$ and $\hat{A}[t]$ we can use now (1.17) with

$$\hat{\beta} = \hat{f}, \qquad \hat{\alpha} = \hat{A}, \qquad M = (1 - \mathscr{P}_a)L.$$

It follows from (2.4) and (2.6) that

$$\hat{f}[t] = iv\hat{A}[t] - \int_0^t ds\zeta(s)\hat{A}[t-s] + \hat{W}\{t\}, \tag{2.14}$$

where

$$iv = (\hat{f}, \hat{A}^+)(\hat{A}, \hat{A}^+)^{-1}, \tag{2.15}$$

$$W\{t\} = \exp[it(1 - \mathscr{P}_A)(1 - \mathscr{P}_a)L]\cdot(1 - \mathscr{P}_A)\hat{f}, \tag{2.16}$$

$$\zeta(s) = (\hat{W}\{s\}, \hat{\mathscr{F}}^+)(\hat{A}, \hat{A}^+)^{-1}. \tag{2.17}$$

It is evident that the new random force $\hat{W}\{t\}$ satisfies the same orthogonality relations as $\hat{\mathscr{F}}\{t\}$:

$$(\hat{W}\{t\}, \hat{a}^+) = 0, \qquad (\hat{W}\{t\}, \hat{A}^+) = 0. \tag{2.18}$$

Thus, the new equation (2.14) does not engage directly the equation (2.2).

The equations (2.2), (2.8) and (2.14) form the system of equations for conserved quantities. It is easy now to continue our investigation introducing new combinations of conserved quantities. Writing down the independent equations for the new combinations we can find the random force which is the inhomogeneous term in the fundamental equation (2.2).

Let us make a remark about the difference between our method and the Kawasaki's one [4]. In fact, this difference stems from our using $\hat{A}[t]$ instead of Kawasaki's $\hat{A}(t)$. From (1.11) for $\hat{\alpha}=\hat{a}$, $\hat{v}=\hat{A}$, $M=L$ and taking account of (2.1), (2.7) and the definition (2.4) for $t = 0$ we obtain

$$\hat{A}(t) = \hat{A}[t] - \int_0^t ds\chi(s)\hat{a}(t - s), \tag{2.19}$$

where

$$\chi(s) = (\hat{A}[s],\hat{f}^+)(\hat{a}, \hat{a}^+)^{-1}. \tag{2.20}$$

As to Kawasaki, he introduced $A(t)$ into the Mori's equation for $\hat{a}(t)$ but he omitted the equation for $\hat{A}(t)$, which is necessary for the consistent calculation of quadratic contributions. Then he made an approximation where $\hat{A}(t)$ was quadratic in $\hat{a}(t)$ but he ignored the second term in the right hand side of (2.19) (which is linear in $\hat{a}$). As a result he obtained the equation in which even the linear terms were calculated approximately.

3. CLOSING OF EQUATIONS SYSTEM AND THE DEFINITION OF SCALAR PRODUCT

For the correct evaluation of our approximation we shall close the set of equations (2.2), (2.8) and (2.14) after going over to the averaged quantities. As different phase quantities develop in time with different propagators, it is convenient to use the Heisenberg representation.

Let us represent an average of a phase quantity $\hat{G}$ at time t as

$$\langle \hat{G}(t - t_0) \rangle_l^{t_0},$$

where the angular brackets denote the average over the quasiequilibrium ensemble at the initial time t_0:

$$\hat{\rho}_l(t_0) = \Xi^{-1}(t_0) \exp [- \hat{a}^+F(t_0)]. \tag{3.1}$$

Here $F(t_0)$ is a column vector, its components being local values of thermodynamical parameters (at time t_0), conjugated to $\hat{a}$; Ξ^{-1} is a normalization factor.

Let $\hat{\rho}_0$ be the canonical distribution function corresponding to the total equilibrium to which our system relaxates. We assume that the average of $\hat{a}$ over the canonical equilibrium ensemble $\hat{\rho}_0$ is equal to zero:

$$\langle \hat{a} \rangle_0 = 0, \tag{3.2}$$

where $\langle \ \rangle_0$ denotes the average over the canonical ensemble.

If the deviation of the $\hat{\rho}_l(t_0)$ from the $\hat{\rho}_0$ is small, we can expand $\hat{\rho}_l(t_0)$ in a Taylor series in terms of deviations $\Delta F(t_0)$ of the thermodynamic parameters at initial time t_0 from their equilibrium values. In this expansion we can express $\Delta F(t_0)$ in terms of initial average values $a(t_0)$ of $\hat{a}$ over quasiequilibrium ensemble

$$a(t_0) = \langle \hat{a} \rangle_l^{t_0}.$$

In the second order in $a(t_0)$ we get

$$(3.3)\quad \hat{\rho}_l(t_0) = \hat{\rho}_0[1 + \hat{a}^+ \langle \hat{a}\hat{a}^+ \rangle_0^{-1} a(t_0) + \frac{1}{2} \hat{B}^+ \langle \hat{a}\hat{a}^+ \rangle_0^{-1} \langle \hat{a}\hat{a}^+ \rangle_0^{-1} a(t_0)a(t_0)],$$

where $\hat{B}$ is a quadratic polynomial in $\hat{a}$:

$$(3.4)\qquad \hat{B} = \hat{a}\hat{a} - \langle \hat{a}\hat{a} \rangle_0 - \langle \hat{a}\hat{a}\hat{a}^+ \rangle_0 \langle \hat{a}\hat{a}^+ \rangle_0^{-1} \hat{a}.$$

Thus the averages can be written as

$$(3.5)\quad \langle \hat{G}(t - t_0) \rangle_l^{t_0} = \langle G(t - t_0) \rangle_0 + \langle \hat{G}(t - t_0)\hat{a}^+ \rangle_0 \langle \hat{a}\hat{a}^+ \rangle_0^{-1} a(t_0) + \frac{1}{2} \langle \hat{G}(t - t_0)\hat{B}^+ \rangle_0 \langle \hat{a}\hat{a}^+ \rangle_0^{-1} \langle \hat{a}\hat{a}^+ \rangle_0^{-1} a(t_0)a(t_0).$$

Now we can turn over to the equations (2.2), (2.8) and (2.13). After averaging we obtain the following system

$$(3.6)\quad \frac{\partial}{\partial t} \langle \hat{a}(t - t_0) \rangle_l^{t_0} = i\omega \langle \hat{a}(t - t_0) \rangle_l^{t_0} - \int_0^{t-t_0} ds\varphi(s) \langle \hat{a}(t - t_0 - s) \rangle_l^{t_0} + \langle \hat{f}[t - t_0] \rangle_l^{t_0},$$

$$(3.7)\quad \frac{\partial}{\partial t} \langle \hat{A}[t - t_0] \rangle_l^{t_0} = i\Omega \langle \hat{A}[t - t_0] \rangle_l^{t_0} - \int_0^{t-t_0} ds\Phi(s) \langle \hat{A}[t - t_0 - s] \rangle_l^{t_0} + \langle \hat{\mathscr{F}} \{ t - t_0 \} \rangle_l^{t_0},$$

$$(3.8)\quad \langle \hat{f}[t - t_0] \rangle_l^{t_0} = iv \langle \hat{A}[t - t_0] \rangle_l^{t_0} - \int_0^{t-t_0} ds\zeta(s) \langle \hat{A}[t - t_0 - s] \rangle_l^{t_0} + \langle \hat{W} \{ t - t_0 \} \langle_l^{t_0} .$$

Up to this point our derivation was formal and rigorous. We note that we have not used any specific definition of the scalar product in the phase functions space. We shall choose the scalar product in such a way as to make the average values of random forces

$$\langle \hat{\mathscr{F}} \{ t - t_0 \} \rangle_l^{t_0} \qquad \text{and} \qquad \langle \hat{W} \{ t - t_0 \} \rangle_l^{t_0}$$

as small as possible. Then these random forces averages will be neglected in order to close the system (3.6), (3.7) and (3.8).

Remark. — All the phase quantities can be treated as having vanishing average values. Indeed, because of (3.4) and (3.2) the average of $\hat{B}$ is equal to zero automatically. As to $\hat{A}$, it is possible w without loss of generality to impose the condition $\langle \hat{A} \rangle_0 = 0$. Then we have $\langle \mathscr{P}_a \hat{G} \rangle_0 = 0$, $\langle \mathscr{P}_A \hat{G} \rangle_0 = 0$ for an arbitrary G. It is evident also that $\langle LG \rangle_0 = 0$.

Let us now define the scalar product by the following formulæ

$$(\hat{G}, \hat{Q}^+) \equiv \langle \hat{G}\hat{Q}^+ \rangle_0 . \tag{3.9}$$

This definition coincides with that of Mori [1] which, however was given only for the linear theory. The usefulness of the definition (3.9) for the nonlinear theory is related to the fact that with (3.9) the averages

$$\langle \hat{\mathscr{F}} \{ t - t_0 \} \rangle_0 \qquad \text{and} \qquad \langle \hat{W} \{ t - t_0 \} \rangle_0$$

do not contain terms which are linear and quadratic in $a(t_0)$. By virtue of (3.9) the polynomial $\hat{B}$ is orthogonal to $\hat{a}: (\hat{B}, \hat{a}^+) = 0$ and

$$\hat{B} = \hat{A}. \tag{3.10}$$

As to $\hat{f}[t]$ and $\hat{A}[t]$, their averages $\langle \hat{f}[t] \rangle_t$ and $\langle \hat{A}[t] \rangle_t$ vanish in the linear approximation [with respect to $a(t_0)$]. Of course, in the expansion of $\langle \hat{a}(t) \rangle_t$ the linear term $a(t_0)$ is present.

We suppose that the smallness of the random forces averages [averaged over the initial amplitudes $a(t_0)$] means also that these averages are rapidly decreasing in time. Then we can put

$$\langle \hat{\mathscr{F}} \{ t - t_0 \} \rangle_t^{t_0} = 0, \qquad \langle \hat{W} \{ t - t_0 \} \rangle_t^{t_0} = 0 \tag{3.11}$$

for all moments t. The condition (3.11) enables us to close the system (3.6), (3.7) and (3.8). These equations [together with (3.11)] describe now the theory in which the quantities of the third and higher orders in $a(t_0)$ are neglected.

It is easy to verify that the definition of the scalar product (3.9) meets all the necessary requirements (1.1) and (1.2). The Liouville operator is Hermitian; the operator $(1 - \mathscr{P}_a)L$ is Hermitian in the subspace orthogonal to $\hat{a}$. These conditions are sufficient for the validity of the formalism developed in the section 2.

4. INCLUSION OF BOUNDARY RETARDED CONDITIONS

The equations (3.6)-(3.8) obtained in the section 2 contain the initial time t_0 and are not time translationally invariant. To eliminate this drawback we shall perform (following Zubarev [5]) a limiting transition $t_0 \to -\infty$.

We suppose that the averaged phase quantities in the equations (3.6)-(3.8) have the definite values at $t_0 \to -\infty$ — at least in the Abel sense:

$$\lim_{t' \to -\infty} g(t') = \varepsilon \int_{-\infty}^{0} dt' e^{\varepsilon t'} g(t'), \tag{4.1}$$

where $\varepsilon \to +0$ (after thermodynamical limiting transition is done).

Thus, we define

$$a(t) = \lim_{t_0 \to -\infty} \langle \hat{a}(t - t_0) \rangle_l^{t_0} \tag{4.2}$$

$$f(t) = \lim_{t_0 \to -\infty} \langle \hat{f}(t - t_0) \rangle_l^{t_0} \tag{4.3}$$

$$\mathrm{A}[t] = \lim_{t_0 \to -\infty} \langle \hat{\mathrm{A}}[t - t_0] \rangle_l^{t_0}, \tag{4.4}$$

where all limits are to be understood in the Abel sense.

Let us remind the physical meaning of the limiting procedure $t_0 \to -\infty$. This transition $t_0 \to -\infty$ has been used by Zubarev [5] in the construction of the nonequilibrium statistical operator $\hat{\rho}(t)$ which is the solution of the Liouville equation with the initial condition $\hat{\rho}(t_0) = \hat{\rho}_l(t_0)$ at $t_0 \to -\infty$. Then, taking into account (4.1) we obtain

$$\hat{\rho}(t) = \varepsilon \int_{-\infty}^{0} dt' e^{\varepsilon t'} e^{it'\mathrm{L}} \hat{\rho}_l(t + t'). \tag{4.5}$$

It is important to note that under the $t_0 \to -\infty$ transition not only the beginning of the dynamical development of the system is changed (i. e. goes to infinite past), but the thermodynamical parameters (i. e. conditions of the contact of the system with a thermostat) are also changed. Both the circumstances are essential, for example, in the Bogolubov kinetic theory, as it was shown in [6]. In this case the correlations at finite time do arise even if these correlations were absent at infinite past.

After the limit $t_0 \to -\infty$ is performed the equations (3.6)-(3.8) lead to

$$\frac{\partial a(t)}{\partial t} = i\omega a(t) - \int_{-\infty}^{0} dt' e^{\varepsilon t'} \varphi(-t') a(t + t') + f[t], \tag{4.6}$$

$$\frac{\partial \mathrm{A}[t]}{\partial t} = i\Omega \mathrm{A}[t] - \int_{-\infty}^{0} dt' e^{\varepsilon t'} \Phi(-t') \mathrm{A}[t + t'], \tag{4.7}$$

$$f[t] = i\nu \mathrm{A}[t] - \int_{-\infty}^{0} dt' e^{\varepsilon t'} \zeta(-t') \mathrm{A}[t + t']. \tag{4.8}$$

Thus, the Mori's equations become independent of initial moment chosen and, hence, translationally invariant in time.

If it is possible to neglect retardation effects, the equations (4.6)-(4.8) may be simplified

$$\frac{\partial a(t)}{\partial t} = \gamma a(t) + f[t], \tag{4.9}$$

$$\frac{\partial \mathrm{A}[t]}{\partial t} = \Gamma \mathrm{A}[t], \tag{4.10}$$

$$f[t] = \Lambda \mathrm{A}[t], \tag{4.11}$$

where

$$\gamma = i\omega - \varphi, \qquad \Gamma = i\Omega - \Phi, \qquad \Lambda = iv - \zeta,$$

$$\varphi = \int_{-\infty}^{0} dt' e^{\varepsilon t'} \langle \hat{f}[-t'] \hat{f}^+ \rangle_0 \langle \hat{a}\hat{a}^+ \rangle_0^{-1},$$

$$i\omega = \langle i\mathrm{L}\hat{a}\hat{a}^+ \rangle_0 \langle \hat{a}\hat{a}^+ \rangle_0^{-1},$$

$$\Phi = \int_{-\infty}^{0} dt' e^{\varepsilon t'} \langle \hat{\mathscr{F}} \{ -t' \} \hat{\mathscr{F}}^+ \rangle_0 \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0^{-1},$$

$$i\Omega = \langle (1 - \mathscr{P}_a) i\mathrm{L}\hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0 \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0^{-1},$$

$$\zeta = \int_{-\infty}^{0} dt' e^{\varepsilon t'} \langle \hat{\mathrm{W}} \{ -t' \} \hat{\mathscr{F}}^+ \rangle_0 \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0^{-1},$$

$$iv = \langle \hat{f}\hat{\mathrm{A}}^+ \rangle_0 \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0,$$

and operators are defined with the scalar product (3.9).

The matrices γ, Γ, Λ might be called kinetic coefficients. The matrices $i\omega$, $i\Omega$, iv represent statical contributions which reduce to the single-time correlation functions. φ, Φ, ζ represent dynamical contributions which are connected with the two-time correlation functions.

The homothetic transformation

$$\mathrm{A}'[t] = \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0^{-1} \mathrm{A}[t],$$

might be useful. Here $\mathrm{A}'[t]$ denotes the average value of

$$\hat{\mathrm{A}}'[t] = \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0^{-1} \hat{\mathrm{A}}[t].$$

In terms of these averages the equations (4.9)-(4.11) take the form

$$\frac{\partial \mathrm{A}'[t]}{\partial t} = \Gamma' \mathrm{A}'[t],$$

$$f[t] = \Lambda' \mathrm{A}'[t],$$

where

$$\Gamma' = \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0^{-1} \Gamma \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0, \qquad \Lambda' = \Lambda \langle \hat{\mathrm{A}}\hat{\mathrm{A}}^+ \rangle_0 .$$

ACKNOWLEDGEMENTS

We wish to thank D. N. Zubarev for useful remarks and discussion and B. Storonkin for the stimulating remarks.

REFERENCES

[1] H. Mori, *Progr. Theor. Phys.*, t. **33**, 1965, p. 423.
[2] J. Dufty, *Phys. Rev.*, t. **176**, 1968, p. 398.

[3] H. Mori, *Progr. Theor. Phys.*, t. **34**, 1965, p. 399.
[4] K. Kawasaki, *Ann. Phys.* (N. Y.), t. **61**, 1970, p. 1.
[5] D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics* (in Russian), Nauka, Moscow, 1971.
[6] D. N. Zubarev and M. Y. Novikov, *Theoret. Matemat. Fizika*, Moscow, t. **13**, 1972, p. 406.

(Manuscrit reçu le 6 juillet 1973).