ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 19, nº 4 (1973), p. 341-356
<http://www.numdam.org/item?id=AIHPA_1973__19_4_341_0>

Ann. Inst. Henri Poincaré,
Vol. XIX, n° 4, 1973, p. 341-356.

Section A :
Physique théorique.

A theory of relativistic unstable particles (*)

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ABSTRACT. — The construction of a class of indecomposable representations of the Poincaré group, using the Fell's formalism, is given. The application of these representations for a description of unstable particles of arbitrary spin is considered and various properties of unstable particles are analysed.

INTRODUCTION

There exist several approaches for a description of unstable particles based mainly on dynamical models ([*1*]-[*5*]). There is, however, a conviction among many particle physicists that a proper tool for describing relativistic unstable particles would be nonunitary representations of the Poincaré group.

One would first consider representations of the Poincaré group in an arbitrary topological vectors space. This problem was analyzed by Flato and Sternheimer [*6*]. They showed in particular that any irreducible representation of the Poincaré group P corresponding to a positive mass m, realized in a topological vector space is equivalent to a unitary irreducible representation. Since representations determined by a negative or zero mass are naturally excluded as possible candidates for a description of an unstable particle the Flato and Sternheimer's theorem implies that only admissible candidates for unstable particles may be representations of P associated with a complex mass M. Now, if a nonunitary representation is realized in a Hilbert space H then evidently the scalar pro-

(*) Supported by the National Science Foundation, U. S. A.

duct (u, w), $u, w \in \mathrm{H}$ is not conserved, i. e., $(\mathrm{T}_g u, \mathrm{T}_g w) \neq (u, w)$. On the other hand, for a probabilistic interpretation we need a sesquilinear (i. e. linear-antilinear) form which is conserved under the action of group representation, i. e. by a simultaneous transformation of a physical system w and a measuring device u (*see* Section II).

The relativistic covariance of the theory of unstable particles is of fundamental importance in our approach: first it compels us to remove the concept of the Hilbert space from our framework; second, it suggests the introduction of a pair of topological spaces $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$: the complex linear topological space $\overset{2}{\Phi}$ contains all physically admissible wave functions of a physical system corresponding to an unstable particle: the complex linear topological space $\overset{1}{\Phi}$ contains all possible measuring devices of a physical system. The theory will be relativistically covariant if one is able to introduce a sesquilinear form $(.,.)$ in $\overset{1}{\Phi} \times \overset{2}{\Phi}$ such that a representation of $g \to \mathrm{T}_g = \langle \overset{1}{\mathrm{T}}_g, \overset{2}{\mathrm{T}}_g \rangle$ satisfies

$$(1) \qquad (\overset{1}{\mathrm{T}}_g u, \overset{2}{\mathrm{T}}_g w) = (u, w) \qquad \text{for all} \qquad u \in \overset{1}{\Phi},\ \mathrm{W} \in \overset{2}{\Phi} \text{ and } g \in \mathrm{P}.$$

A representation $g \to \langle \overset{1}{\mathrm{T}}_g, \overset{2}{\mathrm{T}}_g \rangle$ in $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ satisfying the condition (1) is called sesquilinear system (SLS) representation.

In this work we propose a group-theoretic approach based on the application of indecomposable representations of the Poincaré group. In Section I we give a general construction of induced representations of the Poincaré group using the Fell's formalism of SLS [7]. Next, in Section II we construct a class of indecomposable representations of the Poincaré group. In Section III we apply these representations for a description of unstable particles of arbitrary spin. Finally, in Section IV we present a discussion of our results and their correlations with other approaches. For the convenience of the reader, an account of Fell's theory of SLS representations is given in Appendix A.

I. NONUNITARY INDUCED REPRESENTATIONS OF THE POINCARÉ GROUP

We shall now construct a class of nonunitary representations of the Poincaré group which might correspond to unstable particles. We shall use the Fell-Wigner-Mackey technique of sesquilinear system of representations. For the convenience of the reader we exhibit the basic properties of SLS representations in the Appendix A. Let G be the Poincaré group $\mathrm{G} = \mathrm{P} = \mathrm{T}^4 \boxed{\mathrm{S}} \mathrm{SL}(2, \mathrm{C})$, and let K be the closed subgroup of P such that G/K has an invariant measure. Let $k \to \mathrm{L}_k = \langle \overset{1}{\mathrm{L}}_k, \overset{2}{\mathrm{L}}_k \rangle$ be a finite-dimensional SLS representation of K in the vector space $\tilde{\Phi} = \langle \overset{1}{\tilde{\Phi}}, \overset{2}{\tilde{\Phi}} \rangle$. If $\langle u, w \rangle \in \tilde{\Phi}$ then the sesquilinear form $(.,.)$ can be written as

$$(\mathrm{I}.1) \qquad (u, w) = \bar{u}_s w_s \qquad s = 1, 2, \ldots, \dim \mathrm{L}.$$

and the SLS representation L_k, by definition, satisfies

$$(\mathrm{I}.2) \qquad (\overset{1}{\mathrm{L}}_k u, \overset{2}{\mathrm{L}}_k w) = (u, w) \qquad \text{for all} \qquad \langle u, w \rangle \in \tilde{\Phi} \text{ and } k \in \mathrm{K}.$$

Now let D(G) be the vector space of functions $\langle u(g), w(g) \rangle$ on P with values in $\tilde{\Phi}$, such each component $u_i(g)$ or $w_s(g)$, $i, s = 1, 2, \ldots, \dim \mathrm{L}$, is an element of the Schwartz space of infinitely differentiable functions with compact support. Denote by $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ the vector space of all functions $\langle u(g), w(g) \rangle \in \mathrm{D(G)}$ such that

$$(\mathrm{I}.3) \qquad w(gk) = \mathrm{L}_k^{-1} w(g)$$

and

$$(\mathrm{I}.4) \qquad u(gk) = (\mathrm{L}_k^{-1})^{\mathrm{C}} u(g).$$

The symbol « C » denotes the operation of taking the contragradient [i. e. for a bounded operator X in $\overset{2}{\Phi} : \mathrm{X}^{\mathrm{C}} \equiv (\mathrm{X}^*)^{-1}$]. The vector space of functions satisfying the conditions (I.3) and (I.4) can be easily constructed. Indeed, if $\langle u(g), w(g) \rangle \in \mathrm{D(G)}$ then

$$(\mathrm{I}.5) \qquad \hat{w}(g) = \int_{\mathrm{K}} \mathrm{L}_k w(gk) dk$$

and

$$(\mathrm{I}.6) \qquad \hat{u}(g) = \int_{\mathrm{K}} \mathrm{L}_k^{\mathrm{C}} u(gk) dk$$

satisfy the conditions (I.3) and (I.4), respectively. It is evident from Equation (I.5) [resp. (I.6)] that $\hat{w}(g) = 0$ [resp. $\hat{u}(g) = 0$] if $g \notin \mathrm{SK}$ where S is the compact support of the function $w(g)$ [resp. $u(g)$]. Hence if

$$\langle u(g), w(g) \rangle \in \mathrm{D(G)} \qquad \text{then} \qquad \langle \hat{u}(g), \hat{w}(g) \rangle$$

has a compact support on G/K. Consequently the sesquilinear form

$$(\mathrm{I}.7) \qquad (\hat{u}, \hat{w}) \equiv \int_{\mathrm{G/K}} \bar{\hat{u}}_s(g) \hat{w}_s(g) d\mu(\dot{g}), \quad \dot{g} \equiv g\mathrm{K},$$

is well defined.

The action of the SLS representation $\mathrm{T^L} = \langle \overset{1}{\mathrm{T}}{}^{\mathrm{L}}, \overset{2}{\mathrm{T}}{}^{\mathrm{L}} \rangle$ of P in the space $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ is given by the left translation

$$(\mathrm{I}.8) \qquad \overset{2}{\mathrm{T}}{}^{\mathrm{L}}_{g_0} w(g) = w(g_0^{-1} g),$$

$$(\mathrm{I}.9) \qquad \overset{1}{\mathrm{T}}{}^{\mathrm{L}}_{g_0} u(g) = u(g_0^{-1} g).$$

The sesquilinear form (I.7) is conserved by the representation $g \to T_g^L$. Indeed using Mackey decomposition $g = b_g k$, where b_g belongs to the Borel set $B \subset G(B \sim G/K)$ and $k \in K$ one obtains ($\dot{g} \equiv x_g$, $G/K \equiv X$):

$$\text{(I.10)} \quad (T_{g_0}u, T_{g_0}w) = \int_X \bar{u}_s(g_0^{-1}g)w_s(g_0^{-1}g)d\mu(\dot{g})$$

$$= \int_X \bar{u}_s(b_{g_0^{-1}g}k)w_s(b_{g_0^{-1}g}k)d\mu(x_g) = \text{by virtue of (I.3) and (I.4)}$$

$$= \int_X \bar{u}_s(g_0^{-1}x_g)w_s(g_0^{-1}x_g)d\mu(x_g)$$

$$= \int_X \bar{u}_s(x')w_s(x')d\mu(x') = (u, w).$$

Because $u(g), w(g) \in C_0(G/K)$ the map $g \to (u, T_g^L w)$ is continuous. Consequently the map $g \to T_g^L = \langle \overset{1}{T_g^L}, \overset{2}{T_g^L} \rangle$ is an SLS representation of P in the space $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ induced by the SLS representation $k \to L_k = \langle \overset{2}{L_k}, \overset{2}{L_k} \rangle$.

The formulas (I.8) and (I.9) give the action of induced representation $\langle \overset{1}{T}, \overset{2}{T} \rangle$ in the space $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ of functions defined on the group manifold. In many applications it is more convenient to have a realization directly on the function space on the homogeneous space $X = G/K$. This can be easily calculated; in fact, using Mackey decomposition $g = b_g k_g$ and the condition (I.3) one obtains that the map $w(g) \to L_{k_g}w(g)$ represent the map from space of functions defined on group manifold to the space of functions defined on the coset space $X = G/K$. The transformed function $(\overset{2}{T_{q_0}^L}W)(q)$ is mapped onto

$$L_{k_g}(\overset{2}{T}_{g_0}w)(g) = L_{k_g}L^{-1}_{k_{g_0^{-1}g}}w(x_{g_0^{-1}g}) = L^{-1}_{k_{g^{-1}b_g}}w(g_0^{-1}x_g).$$

Hence,

$$\text{(I.11)} \qquad \overset{2}{T^L_{g_0}}w(x_g) = L^{-1}_{k_{g_0^{-1}b_g}}w(g_0^{-1}x_g).$$

Consequently selecting a definite stability subgroup K of G and its arbitrary representation $k \to L_k$ one obtains an explicit realization of the induced representation $\overset{2}{T^L}$ of G by formula (I.11). Similarly we have

$$\text{(I.12)} \qquad \overset{1}{T^L_{g_0}}u(x) = (L^{-1}_{k_{g_0^{-1}b_g}})^C u(g_0^{-1}x), \quad u \in \overset{1}{\Phi}(X).$$

Consequently, an SLS representation $g \to T_g^L = \langle \overset{1}{T_g^{L^c}}, \overset{2}{T_g^L} \rangle$ of G induced by a representation $k \to L_k$ of a closed subgroup K of P is realized in the space $\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$ by formula (I.11) and (I.12). The sesquilinear form $(.,.)$ on $\overset{1}{\Phi}(X) \times \overset{2}{\Phi}(X)$ is given by the formula

$$(I.13) \qquad (u, w) = \int_X \bar{u}_s(x) w_s(x) d\mu(x), \quad X = P/K,$$

where $u \in \overset{1}{\Phi}(X)$, $w \in \overset{2}{\Phi}(X)$, and $d\mu(x)$ is an invariant measure on $X = P/K$.

The whole formalism can be directly applied to an arbitrary locally compact topological group G. In the general case it is only necessary to put in front of formulas (I.11) and (I.12) the factor $\left(\frac{d\mu(g_0^{-1}x)}{d\mu(x)}\right)^{1/2}$ representing the square root of the Radon-Nikodym derivative of $d\mu$ on X.

II. DESCRIPTION OF UNSTABLE PARTICLES

There is so far no satisfactory definition of an unstable particle. Hence it seems most reasonable to use as a guide a phenomenological description. An unstable particle is experimentally determined as an object with the following properties:

(*i*) It has a definite spin J and a definite space parity P.

(*ii*) It has a mass distribution or equivalently it has a definite decay law.

The decay law is for most particles exponential i. e. $p(t) \sim e^{-\Gamma t}$. However, it was suggested in some cases, as for instance for the A_2 meson, that a decay law might be an algebraic-exponential of the form $p(t) = (a + bt + ct^2)e^{-\Gamma t}$. In what follows we mean by an isolated unstable particle one which is under the influence of forces causing the decay only.

We shall construct in this section a class of nonunitary representations of the Poincaré group P by means of which we can reproduce all properties possessed by the phenomenological unstable particle.

We begin with a determination of the stability subgroup K of the Poincaré group P. We showed in the Introduction that by virtue of the Theorem of Flato and Sternheimer an unstable particle might be only determined by a complex mass M. A complex mass determines a complex orbit $\mathcal{O}$ in the space of complex momenta $p = k + iq$ for which $p^2 = M^2$. The stability subgroup G_p of a vector $p \in \mathcal{O}$ is the subgroup $T^4 \boxed{S} G_k \cap G_q$. Putting k in the rest system and setting $q = (q_0, 00, q_3)$ by a proper rotation, we conclude that in general $G_p = T^4 \boxed{S} U(1)$. Since we want to have a definite spin J as a quantum number characterizing an unstable particle we must have $G_k \cap G_q = SU(2)$: this is only possible if $q = \lambda k$. Hence $p = \lambda q + iq$. It is convenient to write $p = Mv$ where $M = M_0 - i(\Gamma/2)$ and $v = (v_0, \vec{v})$ is the relativistic four-velocity $(v_\mu v^\mu = 1)$.

We usually consider in particle physics the irreducible representations of P. It seems however that for the description of an unstable particle or a composite system a reducible representation of P is more appropriate. Hence we now give a general construction of nonunitary representations T^L of P induced by an arbitrary nonunitary reducible representation L of $K = T^4 \boxed{S} SU(2)$.

Let $k \to L_k = \langle \overset{1}{L}_k, \overset{2}{L}_k \rangle$ be an SLS representation of K in $\tilde{\Phi} = \langle \overset{1}{\tilde{\Phi}}, \overset{2}{\tilde{\Phi}} \rangle$:

(II.1) $\quad k = (a, r) \to \overset{2}{L}_k = N_a D^J(r), \quad \overset{1}{L} = L^c, \quad a \in T^4, \quad \gamma \in SU(2)$

where $a \to N_a$ is a reducible representation of the translation group T^4 and $r \to D^J(r)$ is an irreducible representation of SU(2), characterized by an integer or half-integer number J. The composition law in K:

(II.2) $$(a, r)(a', r') = (a + ra', rr')$$

implies that N_a must be of the form $N_{(a,\mathring{v})}$ where $\mathring{v} = (v_0, 0, 0, 0)$ is a time-like vector and $(a, \mathring{v})$ is the Minkowski scalar product. The sesquilinear form $(u, w)_L$ in $\tilde{\Phi} = \langle \overset{1}{\tilde{\Phi}}, \overset{2}{\tilde{\Phi}} \rangle$ has now the form

(II.3) $$(u, w)_L = \bar{u}_{i\mu} w_{i\mu}$$

where $i = 1, 2, \ldots, \dim N_{(a\mathring{v})}$, and $\mu = -J, -J, -J+1, \ldots, J-1, J$, is the spin index.

In this work the indecomposable representations $a \to N_{(av)}$ of T^4 play an important role. The simplest example of such a representation is given by the formula

(II.4) $$T^4 \ni a \to N_{(av)} = e^{-iM(av)} \begin{vmatrix} 1 & \gamma(av) \\ 0 & 1 \end{vmatrix}, \qquad \gamma \in C^1.$$

Using the induction method one may find that an n-dimensional indecomposable representation of T^4 may be taken to be in the form

(II.5) $\quad T^4 \ni a \to N_{(av)} =$

$$= e^{-iM(av)} \left\| \begin{array}{cccccc} 1 & \gamma_{n-1}(av) & \gamma_{n-2}\gamma_{n-1}(av)^{2/2}, \ldots, & & & \dfrac{\gamma_1 \cdots \gamma_{n-1}}{(n-1)!}(av)^{n-1} \\ 0 & 1 & \gamma_{n-2}(av) & , \ldots, & & \dfrac{\gamma_1 \cdots \gamma_{n-2}}{(n-2)!}(av)^{n-2} \\ & & \ddots & \cdots & \cdots & \cdots \\ & & & 1 & \gamma_2(av) & \dfrac{\gamma_1\gamma_2}{2}(av)^2 \\ & 0 & & & 1 & \gamma_1(av) \\ & & & & 0 & 1 \end{array} \right\|$$

One may construct also other classes of indecomposable representations of T^4. However, the representations (II.5) are most important for us, since they provide algebraic-exponential decay law [cf. Equation (II.17)].

We now give the explicit form of the representation T^L_g of P induced by a generally reducible representation $k \to L_k$ of the subgroup $K = T^4 \boxed{S} SU(2)$.

PROPOSITION 1. — *Let $k \to L_k$ be a representation of the subgroup $K = T^4 \boxed{S} SU(2)$ given by Equation* (II.1) *and let $\overset{i}{\Phi}(X)$, $i = 1, 2$ be Schwartz's spaces* D(X), *where* $X = G/K$. *Then the* SLS *representation*

$$g \to T^L_g = \langle \overset{1}{T^L_g}, \overset{2}{T^L_g} \rangle$$

of the Poincaré group P *is given in the space*

$$\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$$

by the formulas

$$\text{(II.6)} \qquad \overset{2}{T}{}^L_{\{a,\Lambda\}} w(v) = N_{(a,v)} D^J(r_\Lambda) w(L_\Lambda^{-1} v), \qquad w \in \overset{2}{\Phi}(X),$$

and

$$\text{(II.7)} \qquad \overset{1}{T}{}^L_{\{a,\Lambda\}} u(v) = N^C_{(a,v)} (D^J)^C(r_\Lambda) u(L_\Lambda^{-1} v), \qquad u \in \overset{1}{\Phi}(X),$$

where $X = G/K$ *is the velocity hyperboloid* ($X \ni x \sim \{ v_\mu \}, v_\mu v^\mu = 1, v_0 > 0$), *and* $r_\Lambda = \Lambda_v \Lambda \, \Lambda_{L_\Lambda^{-1} v}$ *is the Wigner rotation,* Λ_v *is the Lorentz transformation implied by the Mackey decomposition*

$$\text{(II.8)} \quad \Lambda = \Lambda_v r, \quad \Lambda_v = \left\| \begin{matrix} \lambda & z \\ 0 & \lambda^{-1} \end{matrix} \right\|, \qquad \lambda \in R^1, \quad z \in C^1, \quad r \in SU(2)$$

of SL(2, C) *and* $L_v \in SO(3, 1)$ *is the Lorentz transformation in* T^4 *implied by the element* $\Lambda_v \in SL(2, C)$.

The sesquilinear form (., .) *in* $\Phi(X)$ *is given now by the formula*

$$\text{(II.9)} \qquad (u, w) = \int \frac{d_3 v}{v_0} \bar{u}_{i\mu}(v) w_{i\mu}(v)$$

where $i = 1, 2, \ldots, \dim N_{(av)}$ *and* $\mu = -J + 1, \ldots, J - 1, J$.

Proof. — *See* Appendix B.

Remark. — Clearly we may take instead of Schwartz's D(X) space any other nuclear space of $C^\infty(X)$ functions (e. g. S-space) for which the sesquilinear form (II.9) converges.

We now show that various nonunitary representations $T^L = \langle \overset{1}{T^L}, \overset{2}{T^L} \rangle$ realized in the space $\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$ might provide a description of unstable particles.

Let us first give a physical interpretation for the pair of spaces $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$. According to the basic concept of Quantum Mechanics a measurement is an operation which prescribes a number to every wave function, w; consequently a measurement is in fact a functional on the space of wave functions. This suggests to consider in our case the space $\overset{2}{\Phi}$ as the space of wave functions and the space $\overset{1}{\Phi}$ as the space of measuring devices. The probability amplitude in the measurement of a state $w \in \overset{2}{\Phi}$ by a measuring device being in the state $u \in \overset{1}{\Phi}$ is then given by the sesquilinear form (II.9). Clearly by virtue of Equations (I.10) this probability amplitude is invariant with respect to simultaneous transformations of the state w and the measuring device u.

Consider now various special cases:

A. Scalar unstable particle.

Consider first the case of a one-dimensional nonunitary representation of the translation group

$$a \to \mathrm{N}_{(a\overset{\circ}{v})} = e^{-i\mathrm{M}a\overset{\circ}{v}}, \qquad \text{where} \qquad \mathrm{M} = \mathrm{M}_0 - \frac{i\Gamma}{2}.$$

Let $u(v) \in \overset{1}{\Phi}$ and $w(v) \in \overset{2}{\Phi}$ be the states of the measuring device and of the unstable particle, respectively, at $t = 0$. The time evolution of function is given by the formula (II.6) i. e., $w(t; v) = \mathrm{N}_{tv_0}w(v)$. By virtue of Equation (II.9) the probability of measuring the state $w(t, v)$ by a measuring device in a state u is given by the formula

$$p(t) = |(u(t = 0), w(t))|^2. \tag{II.10}$$

This is the probability that an unstable particle has not decayed at time t. To obtain an expression for $p(t)$ in the rest frame of the unstable particle we assume a measuring device in the state $u(t = 0, v)$ in the form $u(t = 0; v) = \delta_\varepsilon(v)$, where $\delta_\varepsilon(v)$ is an ε-model with compact support of the Dirac δ-function. By virtue of Equation (II.10) we obtain the following formula for $p(t)$:

$$p(t) = |(u(t = 0), \; w(t))|^2 = \left| \int \frac{d_3v}{v_0} \delta_\varepsilon(v) w(v) e^{-i\mathrm{M}v_0 t} \right|^2 \underset{\varepsilon \to 0}{\longrightarrow} |w(0)|^2 e^{-\Gamma t}. \tag{II.11}$$

This formula agrees with the conventional expression for the time-dependence of probability resulting in Weisskopf-Wigner formalism.

B. Consider now the case of a scalar particle (J = 0) whose wave function $w_i(v)$, $i = 1, 2, \ldots, \dim \mathrm{N}_{(a\overset{\circ}{v})}$, transforms according to a reducible representation of the translation group $a \to \mathrm{N}_{(a,\overset{\circ}{v})}$, $\dim \mathrm{N}_{(a,\overset{\circ}{v})} > 1$. Assuming now the state of the measuring device to be in the form $u_i(v) = \delta_\varepsilon(v)\alpha_i$ where $\{ \alpha_i \}$ is a vector in $\overset{1}{\Phi}$ one obtains the following expression for the probability amplitude

$$(u(t = 0), w(t)) = \int \frac{d_3 v}{v_0} \delta_\varepsilon(v)\bar{\alpha}_i(\mathrm{N}_{tv_0})_{ik} w_k(v). \tag{II.12}$$

In the limit $\varepsilon \to 0$ this gives

$$p(t) = |(u(t = 0), w(t)|^2 \to |(\alpha, \mathrm{N}_{tv_0}\beta)|^2 \qquad \beta \equiv w(t = 0; v = 0). \tag{II.13}$$

The explicit form of the time-dependence of $p(t)$ depends now on the form of the representation of the translation group. In particular if we take the two-dimensional representation (II.4) then one obtains the following expression:

$$p(t) = (a + bt + ct^2)e^{-\Gamma t} \tag{II.14}$$

where

$$\begin{cases} a = |(\alpha, \beta)|^2, \\ b = 2 \operatorname{Re} (\alpha, \beta) \operatorname{Re} (\gamma \; \bar{\alpha}_1\beta_2) + 2\mathrm{J}m(\alpha, \beta)\cdot \mathrm{J}m(\gamma \; \bar{\alpha}_1\beta_2), \\ c = |\gamma \; \bar{\alpha}_1\beta_2|^2. \end{cases} \tag{II.15}$$

It is interesting that this type of time-dependence was suggested on the basis of experimental results for the decay of A_2 meson. Notice that the states of the form

$$w(v) = \begin{pmatrix} w_1(v) \\ 0 \end{pmatrix} \tag{II.16}$$

form an invariant subspace $\check{\Phi}$ in $\overset{2}{\Phi}$ for the representation $\overset{2}{\mathrm{T}}{}^{\mathrm{L}}$; for $w \in \check{\Phi}$ by virtue of Equation (II.15) the probability $p(t)$ has the form $p(t) = a\, e^{-\Gamma t}$. This provides an illustration of a phenomenon that the decay law $p(t)$ depends on production and on detection arrangements [cf. Equations (II.13) and (II.15)]. This fact was observed in certain dynamical models by Bell and Goebel [8] and in a different context by Khalfin [9].

In general taking an n-dimensional representation $a \to \mathrm{N}_{(av)}$ of the translation group T^4 given by Equation (II.5) one obtains the decay law $p(t)$ in the form

$$p(t) = e^{-\Gamma t} \sum_{k=0}^{2(n-1)} a_k t^k \tag{II.17}$$

where a_k, $k = 0, 1, \ldots, 2(n - 1)$, in formula (II.17) depend on detection and production arrangements [cf. Equation (II.13)].

It is interesting that Equation (II.17) for a decay law coincides with the expression obtained in S-matrix theory on the assumption that a scattering amplitude has an n-fold complex pole (cf. [*10*] Equation (4.9)). This result shows that group theory may provide a description of unstable particle, which is equivalent to a dynamical one.

C. Unstable particle with spin.

The wave function of an unstable particle with spin J is a vector function $w_{i\mu}(v), i = 1, 2, \ldots, \dim \mathrm{N}_{(a\mathring{v})}, \mu = -\mathrm{J}, -\mathrm{J}+1, \ldots, \mathrm{J}-1, \mathrm{J}$, on the velocity hyperboloid. Using the same arguments as above we obtain the following expression for the probability amplitude

$$(\text{II.18}) \qquad (u(t=0), \mathrm{N}_{tv_0}w(t=0)) = \int \frac{d_3v}{v_0}\delta_\varepsilon(v)\bar{\alpha}_{i\mu}(\mathrm{N}_{tv_0})_{ik}w_{k\mu}(v).$$

In the limit $\varepsilon \to 0$ this gives

$$(\text{II.19}) \quad p(t) = |(u, w(t))|^2 = |\bar{\alpha}_\mu \mathrm{N}_{tv_0}\beta_\mu|^2, \qquad \beta_{i\mu} \equiv w_{i\mu}(t=0; v=0).$$

This shows that the decay law for an unstable particle with an arbitrary spin J is determined in fact by the representation $a \to \mathrm{N}_{(av)}$ of the translation group T^4.

III. PHYSICAL INTERPRETATION

We shall now discuss various physical aspects of the theory presented in previous sections.

(*i*) We shall first discuss properties of observables in this formulation. The global representation (II.6) allows us to calculate the explicit form of generators in the space $\overset{2}{\Phi}(\mathrm{X})$ of wave functions. A generator $\overset{2}{\mathrm{X}}_k$ of a one-parameter subgroup $g(s_k)$ is defined by the formula

$$(\text{III.1}) \qquad \overset{2}{\mathrm{X}}_k w = \lim_{s_k \to 0} is_k^{-1}(\mathrm{T}^{\mathrm{L}}_{g(s_k)} - \mathrm{I})w, \qquad w \in \overset{2}{\Phi}$$

where the limit is taken in the topology of $\overset{2}{\Phi}$. Clearly in the present case any element of $\overset{2}{\Phi}(\mathrm{X})$ is in the domain of X_k, $k = 1, 2, \ldots, \dim \mathrm{G}$.

If $\dim \mathrm{N}_{(av)} = 1$ then momenta P_μ are expressed in terms of the following operators in $\overset{2}{\Phi}$:

$$(\text{III.2}) \qquad \mathrm{P}_\mu = \mathrm{M}v_\mu, \qquad \mathrm{M} = \mathrm{M}_0 - \frac{i}{2}\Gamma.$$

The fact that momenta are complex is not surprising: this only reflects the fact that part of unstable particles in a beam decay and therefore there is a « leakage » of the total momentum.

In the general case of reducible representation (II.6) and (II.5) momenta P_μ are finite-dimensional matrices of the form

$$\text{(III.3)}\quad P_\mu = Mv_\mu I + v_\mu \left\| \begin{matrix} 0 & \gamma_1 & 0 & 0 & ,\dots\dots, & 0 \\ & 0 & \gamma_2 & 0 & ,\dots\dots, & 0 \\ & & \dots & \dots & \dots & \\ & & & \dots & \dots & \\ 0 & & & 0 & \gamma_{n-2} & 0 \\ & & & & 0 & \gamma_{n-1} \\ & & & & & 0 \end{matrix} \right\|, \quad M = M_0 - \frac{i}{2}\Gamma.$$

Consequently the mass operator $\hat{M}^2 = P_\mu P^\mu$ is in general not diagonal in the carrier space $\overset{2}{\Phi}$. For instance in the case of two-dimensional representation (II.4), $\hat{M}^2$ has the form

$$\text{(III.4)}\qquad \hat{M}^2 = \left\| \begin{matrix} M^2, & 2\gamma M \\ 0\,, & M^2 \end{matrix} \right\|.$$

In this case the invariant subspace in $\overset{2}{\Phi}$ consisting of elements

$$w = \begin{pmatrix} w_1 \\ 0 \end{pmatrix}$$

is an eigenspace for $\hat{M}^2$ (the corresponding states have exponential decay law $\sim e^{-\Gamma t}$, *cf.* II B): the remaining elements in $\overset{2}{\Phi}$ are not eigenstates of $\hat{M}^2$. This is another illustration of the fact that properties of unstable particles depend on production and detection arrangements (*cf.* II B).

The generators of the Lorentz group have the same expression as in the theory of stable particles.

The matrix elements or expectation values of a tensor operator T^μ acting in the space $\overset{2}{\Phi}$ of wave functions are defined in the following manner

$$\text{(III.5)}\qquad t^\mu = (u, T^\mu w), \qquad u \in \overset{2}{\Phi}, \qquad w \in \overset{2}{\Phi}.$$

Because by definition $\overset{2}{T}{}_g^{-1} T^\mu \overset{2}{T}_g = L^\mu{}_\nu T^\nu$, $g = \{a, \Lambda\}$, by virtue of equality $\overset{1}{T}{}^* = (\overset{2}{T})^{-1}$ one obtains the following transformation law

$$\text{(III.6)}\qquad t^\mu = (\overset{1}{T}{}^L_g u,\ T^\mu \overset{2}{T}{}^L_g w) = (u,\ \overset{1}{T}{}^{L*}_g T^\mu \overset{2}{T}{}^L_g w) = L^\mu{}_\nu t^\nu.$$

i. e., the relativistic covariance persists. In particular if T is a scalar operator then its expectation values are constants of motion.

(*ii*) We now show the relativistic transformation law of the lifetime $\tau = 1/\Gamma$. Consider for simplicity the case $J = 0$ and $\dim N_{(av)} = 1$.

Take the measuring device represented by a function $u(v) = \delta_\varepsilon(v)$ in a Lorentz frame moving with a velocity β along χ^1 axis and consider the wave function w shifted to the point $\Delta a = (\Delta t, \Delta\vec{x})$ [i. e. $w(v) = T^2_{(\Delta a v)}w(v)$]. Then the probability $p(\Delta t)$ that an unstable particle has not decayed after time interval Δt has the form

$$p(\Delta t) = |(\overset{1}{T}_{L(\beta)}u,\ \overset{2}{T}_{(\Delta a v)}w)|^2 = \left| \int \frac{d_3 v}{v_0} \delta_\varepsilon(L^{-1}v)e^{-iM\Delta a v}w(v) \right|^2$$
$$= \left| \int \frac{dv}{v_0} \delta_\varepsilon(v)e^{-iML^{-1}\Delta a v}w(Lv) \right|^2.$$

Because we want to measure in a moving frame the time interval $\Delta t'$ at a given space point, $\Delta\vec{a}' = 0$ and we have: $L^{-1}\Delta a = ((1-\beta^2)^{1/2}\Delta t, 0)$ (*cf.* [*12*], p. 21). Hence

$$p(\Delta t) \underset{\varepsilon\to 0}{\longrightarrow} |w(0)|^2 e^{-\Gamma(1-\beta^2)^{1/2}\Delta t}.$$

This implies $\Gamma' = \Gamma(1 - \beta^2)^{1/2}$. Consequently

$$\tau' = \frac{1}{\Gamma'} = \frac{\tau}{(1 - \beta^2)^{1/2}},$$

i. e. the unstable particle in a moving frame lives longer in agreement with the well-known experimental result.

(*iii*) The action (II.6) of the Lorentz group SL(2, C) is the same as in case of stable particles. Hence we can define the parity operator in the carrier space of the unstable particle if we use the reducible representation [e. g. $D^{(J,0)} \oplus D^{(0,J)}$] of the Lorentz group for the construction of inducing representation (II.1) of SU(2).

The above results show that the description of a quantum-dynamical unstable system in terms of nonunitary representations of the Poincaré group realized in a pair $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ of topological spaces has all the physical properties shared by a conventional description for stable particle given in the framawork of Hilbert space. On the other hand the present formalism provides a possibility of analysis of a much larger class of quantum systems; stable systems represent a very special subclass.

IV. DISCUSSIONS

(*i*) The idea of using nonunitary representations of the Poincaré group as a tool for a description of relativistic unstable particles was first explicitly formulated by Zwanzinger [*13*]. In a hidden form this description appears already in Mathews-Salam approach to unstable particles [*5 a*]. Later on this description was analyzed from various points of view by a number of authors [*14*]. However, in all (except Simonius) works the theory was presented in the framework of the one-space formalism, which was a direct extension of Hilbert space formalism. Therefore one could not obtain a covariant probabilistic interpretation of the theory.

(*ii*) We constructed in Section II only a certain class of reducible nonunitary indecomposable representations of the Poincaré group, namely those which might describe unstable particles. The full classification of indecomposable nonunitary representations of the Poincaré group will be published elsewhere. An alternative method of construction of nonunitary representations of the Poincaré group was given by Bertrand and Rideau [*15*].

(*iii*) The formula (II.17) shows that even for a real mass $M(\Gamma = 0)$ we can arbitrarily approximate by polynomials exponential (or any other) decay by taking sufficiently high dimensional reducible representation of the translation group T^4.

ACKNOWLEDGEMENTS

I would like to thank Professors A. O. Barut, J. Bialynicki, M. Flato, D. Sternheimer and S. Woronowicz for interesting discussions and critical remarks. The authors would like to thank also Professors A. O. Barut and W. E. Brittin for the hospitality at the University of Colorado, and the National Science Foundation for financial support.

APPENDIX A

Nonunitary representations of topological groups.

We shall present here an outline of Fell's theory of representations of topological groups [7].

A sesquilinear system, SLS, is a pair $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle \equiv \Phi$ of complex linear spaces $\overset{1}{\Phi}$ and $\overset{2}{\Phi}$ together with a sesquilinear (linear-antilinear) form $(.,.)$ on $\overset{1}{\Phi} \times \overset{2}{\Phi}$ such that

$$(\alpha_i u_i,\ \beta_k w_k) = \bar{\alpha}_i \beta_k (u_k,\ w_k) \tag{A.1}$$

and

$$\begin{cases} (u, \overset{2}{\Phi}) = 0 & \text{iff} \quad u = 0, \\ (\overset{1}{\Phi}, w) = 0 & \text{iff} \quad w = 0. \end{cases} \tag{A.2}$$

An isomorphism F between two SLS Φ and $\tilde{\Phi}$ is a pair $\langle \mathrm{F}_1, \mathrm{F}_2 \rangle$, where F_1 is a linear isomorphism of $\overset{i}{\Phi}$ onto $\overset{i}{\tilde{\Phi}}$, $i = 1, 2$, and $(\mathrm{F}_1 u, \mathrm{F}_2 w) = (u, w)$ for all $u \in \overset{1}{\Phi}$, $w \in \overset{2}{\Phi}$.

Using the sesquilinear form $(.,.)$ on $\overset{1}{\Phi} \times \overset{2}{\Phi}$ one can define the locally convex topology $\tau(\overset{1}{\Phi})$ on $\overset{1}{\Phi}$, generated by functionals $u \to (u, w), u \in \overset{1}{\Phi}$ where w runs over $\overset{2}{\Phi}$: one may define similarly the topology on $\overset{2}{\Phi}$.

A SLS representation T of a locally compact group G on a SLS $\Phi(\mathrm{T}) = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ is a pair $\langle \overset{1}{\mathrm{T}}, \overset{2}{\mathrm{T}} \rangle$, where

1. $\overset{1}{\mathrm{T}}$ (resp. $\overset{2}{\mathrm{T}}$) is a homomorphism of G into the group of invertible linear endomorphisms of $\overset{1}{\Phi}$ (resp. $\overset{2}{\Phi}$).

2. $$(\overset{1}{\mathrm{T}}_g u, \overset{2}{\mathrm{T}}_g w) = (u, w) \qquad \text{for all} \qquad g \in \mathrm{G},\ u \in \overset{1}{\Phi},\ w \in \overset{2}{\Phi}.$$

3. The map $g \to (\overset{1}{\mathrm{T}}_g u, w)$ is continuous on G for each $u \in \overset{1}{\Phi}$, and $w \in \overset{2}{\Phi}$.

If X is a bounded linear operator in $\overset{1}{\Phi}$ then the adjoint X^* is defined by the equality

$$(\mathrm{X}^* u, w) = (u, \mathrm{X} w). \tag{A.3}$$

The condition 2 means that

$$\overset{1}{\mathrm{T}}_g = (\overset{2}{\mathrm{T}}{}_g^{-1})^*, \tag{A.4}$$

i. e. a representation $g \to \overset{1}{\mathrm{T}}_g$ in $\overset{1}{\Phi}$ is contragradient to $g \to \overset{2}{\mathrm{T}}_g$. Clearly $\overset{1}{\mathrm{T}} = \overset{2}{\mathrm{T}}$ if $\overset{2}{\mathrm{T}}$ is unitary.

A representation T is (topologically) irreducible if $\overset{1}{\Phi}$ has non-trivial $\tau(\overset{1}{\Phi})$-closed $\overset{1}{\mathrm{T}}$-stable subspaces. [Clearly this implies that $\overset{2}{\Phi}$ has no non-trivial $\tau(\overset{2}{\Phi})$-closed $\overset{2}{\mathrm{T}}$-stable subspaces.]

APPENDIX B

It is well-known that the homogeneous space $X = G/K$, by virtue of the decomposition (II.8) can be realized as the velocity hyperboloid. The correspondence $\Lambda_v \to v$ is given by the formula

$$\Lambda_v = \begin{Vmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{Vmatrix} \to v = \begin{Vmatrix} v_0 - v_3 & v_2 - iv_1 \\ v_2 + iv_1 & v_0 + v_3 \end{Vmatrix} = \Lambda_v \Lambda_v^* . \tag{B.1}$$

The explicit action of $\overset{2}{T}{}^{L}_{g}$ in $\overset{2}{\Phi}(X)$ can be calculated in the following manner:

$$\begin{aligned} (T_{\{a\Lambda\}}w)(0, \Lambda_v) &= w((a, \Lambda)^{-1}(0, \Lambda_v)) \\ &= w((-L_\Lambda^{-1}a, \Lambda^{-1}))(0, \Lambda_v) = w((-L_\Lambda^{-1}a, \Lambda^{-1}\Lambda_v)) \\ &= w((0, \Lambda_{L_\Lambda^{-1}v})(-L^{-1}_{\Lambda_{L_\Lambda^{-1}v}} L_\Lambda^{-1}a, \Lambda^{-1}_{L_\Lambda^{-1}v}\Lambda^{-1}\Lambda_v)) \\ &= N(L^{-1}_{\Lambda_{L_\Lambda^{-1}v}} L_\Lambda^{-1}a, \mathring{v}) D^{J^{-1}}(\Lambda^{-1}_{L_\Lambda^{-1}v}\Lambda^{-1}\Lambda_v) w(0, \Lambda_{L^{-1}v}) \\ &= N_{(a,v)} D^J(\Lambda_v^{-1}\Lambda\, \Lambda_{L_\Lambda^{-1}v}) w(0, L_\Lambda^{-1}v). \end{aligned} \tag{B.2}$$

The element $\Lambda_v^{-1}\Lambda\ \Lambda_{L_\Lambda^{-1}v}$ transforms $\mathring{v}$ into $\mathring{v}$: consequently it represents a rotation $r_\Lambda \in SU(2)$ (Wigner's rotation). If we use the correspondence $\Lambda_v \sim v$ given by Equation (B.1), then the formula (B.2) can be written in the form

$$(T_{\{a,\Lambda\}}w)(v) = N_{(a,v)} D^J(r_\Lambda) w(L_\Lambda^{-1}v). \tag{B.3}$$

Similarly one obtains formula (II.7).

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(Manuscrit reçu le 4 septembre 1973).