

ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 19, n° 3 (1973), p. 211-295

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The role of locality in perturbation theory

by

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ABSTRACT. — It is shown how an inductive construction of the renormalized perturbation series of quantum field theory automatically yields, at each order, finite terms satisfying the requirements of locality. This method whose result is equivalent to the Bogoliubov-Parasiuk-Hepp prescriptions, also establishes the usual classification between renormalizable and non-renormalizable theories.

RÉSUMÉ. — On montre qu'une construction récurrente de la série des perturbations renormalisée, en théorie quantique des champs, fournit automatiquement, à chaque ordre, des termes finis satisfaisant aux conditions de localité. Cette méthode, dont le résultat équivaut aux prescriptions de Bogoliubov-Parasiuk-Hepp, établit également la classification habituelle des théories renormalisables et non renormalisables.

INTRODUCTION

The theory of renormalization in perturbative Lagrangian quantum field theory [1]-[6] ⁽¹⁾ has been brought by recent investigations ([7]-[11]) to a high degree of elegance and mathematical rigour. However, it does not seem to have been proved, so far, that the renormalized series, as a formal series, satisfies the two requirements of microcausality (or local

⁽¹⁾ It is impossible to quote all the original papers about the theory of renormalization. Many of them are reprinted in [1] while [2]-[6] give a sample of works not contained in [1].

commutativity) and of unitarity ⁽²⁾. The first requirement is equivalent to the condition that renormalized Feynman amplitudes be analytic in the domain predicted by axiomatic field theory. In this paper we shall indicate how a method advocated already long ago [6], [12]-[14], but perhaps not sufficiently thoroughly exploited, can simultaneously provide a simple proof of finiteness at every order (including the classification of renormalizable and non-renormalizable theories) and exhibit manifest microcausality. It will be shown in a later paper that it also leads to a proof of unitarity. This question appears in our approach as one aspect of the problems connected with the adiabatic limit (unitarity is trivially satisfied before the adiabatic limit is taken). Only scalar fields occur in this paper, but the reader will easily convince himself that only notational complications appear in the case of higher spins. A more serious omission is the subject of infrared divergences. We hope to examine some of these questions in future publications.

The method presented here is summarized in [15]. (This reference is perhaps better suited for the hurried reader). The same method, but with the emphasis on the generalized retarded products, has been very briefly summarized in [16].

1. FORMULATION OF THE PROBLEM

1.1. Some heuristic considerations.

In our treatment of perturbation theory, we shall follow the formulation due to Bogoliubov *et al.* [6] [17] because we find that this point of view best allows to keep apart the different difficulties encountered in the solution of the problem. We shall first summarize the Bogoliubov « axioms » in a form (slightly different from the original version) which seems more adequate for our later purposes. In this paper we shall confine ourselves to the case of one neutral scalar field. This inessential restriction will greatly simplify the notation without real loss of generality.

Given an interaction Lagrangian density

$$\mathcal{L}'(x) = g(x)\mathcal{L}(x) + Q(x)A(x)$$

let us suppose that, for every pair of real valued functions $(g, Q) \equiv (g_1, g_2) \equiv g$ belonging to some smooth class—it will always be the Schwartz test function space $\mathcal{S}(\mathbb{R}^4)$ in this paper—there exists a unitary S matrix $S(g, Q) = S(g)$,

⁽²⁾ In spite of the fact that many treatments [6] use these requirements as necessary conditions in order to derive the series.

derived in some way or other from the (rather symbolic!) Schwinger equation of motion

$$i \frac{\delta \mathcal{U}}{\delta \sigma(x)} = - \mathcal{L}'(x) \mathcal{U}(\sigma, \sigma_0),$$

$$\mathcal{U}(\sigma_0, \sigma_0) = 1, \quad S = \mathcal{U}(+\infty, -\infty).$$

Here $A(x) \equiv A_{in}(x)$ is a free neutral scalar field of mass m , \mathcal{L} the essential part of the interaction [for example $\mathcal{L}(x) = : A(x)^{\nu} :$] and the underlying Hilbert space \mathcal{F} is the usual Fock space of A . (The usefulness of a systematic study of the dependance of the S matrix on free parameters such as Q was demonstrated by Schwinger [18]). We shall actually find it useful to take the interaction Lagrangian density in the more general form

$$\mathcal{L}'(x) = \sum_{j=1}^p g_j(x) \mathcal{L}_j(x), \quad g_j = \text{Re } g_j \in \mathcal{S}(\mathbb{R}^4)$$

and use the vector notation

$$\underline{g}(x) = (g_1(x), g_2(x), \dots, g_p(x)).$$

$\mathcal{L}_1(x)$ is to be thought of as the true interaction Lagrangian density, $g_1(x)$ will eventually tend to a constant (the true coupling constant) whereas the other \mathcal{L}_j and g_j serve for generating Heisenberg interacting fields which, in zeroth order in g_j , reduce to the expressions $\mathcal{L}_j(x)$ [e. g. : $A^\lambda(x)$]. The g_j with $j \neq 1$ will eventually be set equal to zero.

From the reality of \underline{g} , the hermiticity of the \mathcal{L}_j , their invariance (more exactly covariance) under space-time translations and Lorentz transformations, and finally the formal multiplicative propagation character of $\mathcal{U}(\sigma, \sigma_0)$ from the spacelike hypersurface σ_0 to the spacelike hypersurface σ , one is led to expect that $S(\underline{g})$ should « satisfy » the following list of properties.

(I) (initial condition)

$$S(0) = 1.$$

(U) (Unitarity)

$$S(\underline{g})^* S(\underline{g}) = S(\underline{g}) S(\underline{g})^* = 1$$

for all real $\underline{g} \in \mathcal{S}$.

(Triv) (translational invariance)

$$U(a, 1) S(\underline{g}) U(a, 1)^{-1} = S(\underline{g}_a)$$

for all $a \in \mathbb{R}^4$ where $\underline{g}_a(x) = \underline{g}(x-a)$.

(Linv) (Lorentz invariance; optional)

$$U(0, \Lambda) S(\underline{g}) U(0, \Lambda)^{-1} = S(\underline{g}_\Lambda)$$

for all $\Lambda \in L_+^\uparrow$ where $\underline{g}_\Lambda(x) = \underline{g}(\Lambda^{-1}x)$.

Here, $(a, \Lambda) \rightarrow U(a, \Lambda)$ is the usual representation of the Poincaré group \mathcal{P}_4^\dagger in Fock space. We have separated the two conditions (Trinv) and (Linv) in order to stress the fact that the whole of perturbation theory (including the correct analyticity properties in momentum space) can be worked out without requiring (Linv).

(Caus) The causality condition will be stated in four equivalent forms, all useful for later purposes. If we define

$$V(\underline{g}, \underline{h}) = S(\underline{g})^{-1} S(\underline{g} + \underline{h})$$

this unitary operator is required to be « causally additive » in \underline{h} for each fixed \underline{g} :

$$(C. A.) \quad V(\underline{g}, \underline{h}_1 + \underline{h}_2) = V(\underline{g}, \underline{h}_1) V(\underline{g}, \underline{h}_2)$$

if $\text{supp } \underline{h}_1 \gtrsim \text{supp } \underline{h}_2$.

The latter condition means that the two closed subsets of \mathbb{R}^4 (support of \underline{h}_1) and (support of \underline{h}_2) can be separated by a spacelike surface. More precisely, let X and Y be two subsets of \mathbb{R}^4 . We denote

$$X \gtrsim Y \quad \text{or} \quad Y \lesssim X$$

the condition:

$$X \cap (Y + \bar{V}^-) = \emptyset$$

which is equivalent to

$$Y \cap (X + \bar{V}^+) = \emptyset$$

or: X does not intersect the past causal shadow of Y ;

$$Y \gg \gg \gg \gg \text{future} \gg \gg \gg X.$$

(Here

$$V^+ = -V^- = \{x \in \mathbb{R}^4 : x^0 > |\bar{x}|\}, \quad \bar{V}^+ = -\bar{V}^- = \{x \in \mathbb{R}^4 : x^0 \geq |\bar{x}|\}.)$$

We shall also use the notation ([6]) $X \sim Y$ (X and Y spacelike separated) to mean that $X \gtrsim Y$ and $X \lesssim Y$.

An equivalent form of (C. A) is

$$(C. C.) \quad V(\underline{g} + \underline{h}_1, \underline{h}_2) = V(\underline{g}, \underline{h}_2) \quad \text{if} \quad \text{supp } \underline{h}_1 \gtrsim \text{supp } \underline{h}_2.$$

We shall call it the « causality condition » proper. It says that $V(\underline{g}, \underline{h})$ can only depend on the values taken by \underline{g} in the past causal shadow of $\text{supp } \underline{h}$. It is this form which is most readily inferred from the propagation properties of $\mathcal{U}(\sigma, \sigma_0)$.

We can exchange the role of past and future by introducing

$$W(\underline{g}, \underline{h}) = S(\underline{g} + \underline{h}) S(\underline{g})^{-1} = S(\underline{g}) V(\underline{g}, \underline{h}) S(\underline{g})^{-1}.$$

The last expression of $W(\underline{g}, \underline{h})$ makes it clear that W must satisfy

$$(C. A') \quad W(\underline{g}, \underline{h}_1 + \underline{h}_2) = W(\underline{g}, \underline{h}_1) W(\underline{g}, \underline{h}_2)$$

for $\text{supp } \underline{h}_1 \supseteq \text{supp } \underline{h}_2$. Moreover:

(A. C.)
$$W(\underline{g} + \underline{h}_2, \underline{h}_1) = W(\underline{g}, \underline{h}_1)$$

if $\text{supp } \underline{h}_1 \supseteq \text{supp } \underline{h}_2$ (anticausality condition).

We wish to mention explicitly an important and immediate consequence of (C. A.) and (C. A'):

(L. C.) (local commutativity)

$$[V(\underline{g}, \underline{h}_1), V(\underline{g}, \underline{h}_2)] = [W(\underline{g}, \underline{h}_1), W(\underline{g}, \underline{h}_2)] = 0$$

if $\text{supp } \underline{h}_1 \sim \text{supp } \underline{h}_2$.

Continuity: It was assumed up to now that $S(\underline{g})$ was a unitary-operator-valued function defined over the whole of $\mathcal{S}(\mathbb{R}^4)^p$. If exact (non-perturbative) solutions of these « axioms » were to be sought, this would probably turn out to be too restrictive a condition. The restriction of the domain of definition of $S(\underline{g})$ to a sufficiently small subset of $\mathcal{S}(\mathbb{R}^4)^p$ seems physically more plausible. Also properties of continuity of $S(\underline{g})$ with respect to \underline{g} should be postulated [$\mathcal{S}(\mathbb{R}^4)^p$ is supposed to carry the usual Schwartz topology]. As in the theory of unitary representations of Lie groups, weak continuity of S at point \underline{g} implies strong continuity at \underline{g} . Therefore it seems appropriate to require this kind of continuity. Unfortunately, here the analogy with the theory of group representation seems to end; the authors do not see an argument to show, e. g., that continuity at 0 implies it elsewhere, or an argument leading to the existence of analytic vectors. However, the purpose of this article is to show that a general solution to the preceding « axioms » in the sense of formal power series in \underline{g} does exist, in a sense to be made precise later. It is, to the best of our knowledge, also the first complete proof. The point where—again in the sense of perturbation theory—only fragmentary results are obtained in this paper, is the last of our general conditions.

The adiabatic limit: We are ultimately interested in a theory in which the « true » coupling function $g_1(x)$ is replaced by a constant λ . Let us now put in evidence the special role played by g_1 and denote

$$\underline{g} = (g, \underline{Q}), \quad g \equiv g_1, \quad \underline{Q} = \{g_2, \dots, g_p\}.$$

We are led to postulate that the strong limit

(A)
$$s\text{-}\lim_{g(x) \rightarrow \lambda} S(\underline{g}, \underline{Q}) = S_\lambda(\underline{Q}) \quad \text{for any } \underline{Q} \in \mathcal{S}(\mathbb{R}^4)^{p-1}$$

should exist. The meaning of $g(x) \rightarrow \lambda$ has to be precisely specified. If we only wish to guarantee the unitarity of $S_\lambda(\underline{Q})$ it is sufficient to require, e. g., that the above limit exists for a sequence of test functions $g_{(n)} \in \mathcal{S}(\mathbb{R}^4)$ tending uniformly to the constant λ on any compact subset of \mathbb{R}^4 as $n \rightarrow \infty$. Since we wish to preserve all the properties (I) to (C. C.) we require that, for $n \rightarrow \infty$ the limit (A) should also exist for all the Poincare transforms of the sequence $\{g_{(n)}\}$, $\{g_{(n),L} = Lg_{(n)}\}$, and be independent of $L \in \mathcal{P}_+^\uparrow$.

1.2. Perturbation theory.

The term « Perturbation Theory » will be understood here as the problem of finding a *formal power series* in $\underline{g}[\underline{g} = (g_1, \dots, g_p) \in (\mathcal{S}(\mathbb{R}^4))^p]$:

$$(1) \quad \begin{aligned} S(\underline{g}) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \sum_{j_1, \dots, j_n} T_{j_1 \dots j_n}(x_1, \dots, x_n) g_{j_1}(x_1) \dots g_{j_n}(x_n) dx_1 \dots dx_n \\ &\equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \int T(x_1, \dots, x_n) \underline{g}(x_1) \dots \underline{g}(x_n) dx_1 \dots dx_n \end{aligned}$$

which satisfies the postulates (I) to (Caus) in the sense of *formal power series* in \underline{g} , and, under certain conditions, the most general such series. These conditions concern the domain of definition of the n th order term of the series as an operator in Fock space: this operator is necessarily unbounded; since we wish expressions of the form $S(\underline{g}_1) \dots S(\underline{g}_p)$ to be well-defined formal series in $\underline{g}_1, \dots, \underline{g}_p$, we must require the n th term of (1) to be defined on a dense subspace D_1 of \mathcal{F} , independent of n and \underline{g} which it should map into itself. Under this condition $S(\underline{g})$ (indeed any formal series starting with 1) will have an inverse which we shall denote

$$(2) \quad S(\underline{g})^{-1} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \bar{T}(x_1, \dots, x_n) \underline{g}(x_1) \dots \underline{g}(x_n) dx_1 \dots dx_n.$$

\bar{T} can be simply expressed in terms of T : see below. By their very definition, the operator-valued tempered distributions T and \bar{T} should be symmetric under permutations:

$$(3) \quad T_{j_1 \dots j_n}(x_1, \dots, x_n) = T_{j_{\pi_1} \dots j_{\pi_n}}(x_{\pi_1}, \dots, x_{\pi_n}),$$

$$(4) \quad \bar{T}_{j_1 \dots j_n}(x_1, \dots, x_n) = \bar{T}_{j_{\pi_1} \dots j_{\pi_n}}(x_{\pi_1}, \dots, x_{\pi_n}).$$

This will enable us to use the following abbreviations. If $J = \{u_1, \dots, u_q\}$ is a set of q distinct integers, we shall write

$$\begin{aligned} T(J) \text{ to denote: } & T_{j_{u_1} \dots j_{u_q}}(x_{u_1}, \dots, x_{u_q}), \\ \bar{T}(J) \gg \gg : & \bar{T}_{j_{u_1} \dots j_{u_q}}(x_{u_1}, \dots, x_{u_q}). \end{aligned}$$

We note that if two formal power series $U(\underline{g})$ and $V(\underline{g})$ are given by

$$\begin{aligned} U(\underline{g}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int U(x_1, \dots, x_n) \underline{g}(x_1) \dots \underline{g}(x_n) dx_1 \dots dx_n, \\ V(\underline{g}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int V(x_1, \dots, x_n) \underline{g}(x_1) \dots \underline{g}(x_n) dx_1 \dots dx_n \end{aligned}$$

their product $U(\underline{g})V(\underline{g})$ is the formal power series

$$W(\underline{g}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int W(x_1, \dots, x_n) \underline{g}(x_1) \dots \underline{g}(x_n) dx_1 \dots dx_n,$$

where

$$(5) \quad W(X) = \sum_{I \subset X} U(I)V(X \setminus I)$$

the summation running over all subsets I of X , including $I = \emptyset$ (empty subset) and $I = X$. The notation $X \setminus I$ means the set theoretical difference, i. e., the set of those elements of X which do not belong to I . Of course, (5) makes sense only if domain assumptions of the type previously mentioned have been made for U and V . Under similar assumptions it is easy to compute the inverse of a formal power series of the type $1 + K(\underline{g})$ with $K(0) = 0$:

$$(1 + K(\underline{g}))^{-1} = \sum_{n=0}^{\infty} (-K(\underline{g}))^n = \sum_{m=0}^{\infty} \frac{1}{m!} \int H(x_1, \dots, x_m) \underline{g}(x_1) \dots \underline{g}(x_m) dx_1 \dots dx_m$$

and

$$H(X) = \sum_{n=0}^{|X|} (-1)^n \sum_{\substack{I_1, \dots, I_n \\ I_1 \cup \dots \cup I_n = X \\ I_j \cap I_k = \emptyset \text{ for } j \neq k \\ I_j \neq \emptyset \text{ for all } j}} K(I_1) \dots K(I_n)$$

($|X|$ denotes the number of elements in X).

The operators $T(X)$ and $\bar{T}(X)$ will be determined inductively on $|X|$ by the requirement that $S(\underline{g})$ should satisfy the postulates, the principal roles being played by causality and translational invariance. The constraints placed on the $T(X)$ by these requirements will be best stated by formulating our induction hypothesis.

2. INDUCTION HYPOTHESIS

We assume that, for every $v \leq n - 1$, a set of operator-valued tempered distributions $T_{i_1 \dots i_v}(x_1, \dots, x_v)$ has been constructed with the following properties.

(1) *Domain*

For every $\underline{f} = \{f_{j_1 \dots j_v}\} \in [\mathcal{S}(\mathbb{R}^{4v})]^{p_v}$,

$$\int T(x_1, \dots, x_v) \underline{f}(x_1, \dots, x_v) d^4x_1 \dots d^4x_v$$

is an operator defined on a dense subspace D_1 of \mathcal{F} and maps D_1 into itself. It carries any fixed vector of D_1 into a vector which depends continuously on f (in the weak topology of \mathcal{F}). The subspace D_1 is independent of v and f . It contains the vacuum Ω and

$$U(a, \Lambda)D_1 = D_1 \quad \text{for all} \quad (a, \Lambda) \in \mathcal{P}_+^\dagger.$$

Furthermore, we assume that expressions of the form

$$\int T(x_1, \dots, x_{v_1})T(x_{v_1+1}, \dots, x_{v_2}) \dots T(x_{v_{r-1}+1}, \dots, x_{v_r}) \underline{f}(x_1, \dots, x_{v_r}) dx_1 \dots dx_{v_r}$$

(for any integer r and $v_1, \dots, v_r \leq n - 1$ and for any $f \in [\mathcal{L}(\mathbb{R}^{4v_r})]^{p_{v_r}}$) are defined as operators on D_1 and map D_1 into itself (the nuclear theorem shows that they are defined as bilinear forms on D_1 and depend continuously on f).

For $v = 1$ we require:

$$(6) \quad T_j(x) = \mathcal{L}_j(x).$$

The $\mathcal{L}_j(x)$ are the given Lagrangian densities already mentioned, expressed in terms of the free field $A(x)$, and we recall that $\mathcal{L}_p(x)$ has been chosen equal to $A(x)$ (up to a numerical constant factor). From this and the preceding assumptions, it follows that D_1 must contain all vectors of the form

$$\int A(x_1) \dots A(x_N) f(x_1, \dots, x_N) dx_1 \dots dx_N \Omega$$

where $f \in \mathcal{L}$. Thus:

$$(7) \quad D_0 \subset D_1$$

(it will be shown later that we can require $D_1 = D_0$ without loss of generality).

When using the notation $T(X)$ we shall set, by definition, $T(\emptyset) = 1$ ($\emptyset =$ empty set).

(2) *Translational invariance (Trinv)*

For every $v \leq n - 1$, and every $a \in \mathbb{R}^4$,

$$(8) \quad U(a, 1)T_{j_1 \dots j_v}(x_1, \dots, x_v)U(a, 1)^{-1} = T_{j_1 \dots j_v}(x_1 + a, \dots, x_v + a).$$

This must hold in the sense of operator-valued-tempered distributions acting on D_1 .

(3) *Causality*

(Caus. 1) If $|X| \leq n - 1$, $X = P \cup Q$, $P \cap Q = \emptyset$, then $T(X) = T(P)T(Q)$ in the region

$$(9) \quad \{x \in \mathbb{R}^{4|X|} : \{x_j\}_{j \in P} \gtrsim \{x_j\}_{j \in Q}\}$$

(in the sense of distributions and on D_1).

We shall use the notation $\{x\}_P$ to denote the subset of \mathbb{R}^4 formed by the points $x_j, j \in P$.

Clearly, if the inductive construction is to be carried out so as to extend the condition (Caus. 1) to the case $|X| \geq n$, we have to include in the induction hypothesis the condition:

(Caus. 2) If $|X| \leq n - 1, |Y| \leq n - 1, X \cap Y = \emptyset$, then

$$(10) \quad [T(X), T(Y)] = 0 \quad \text{in the region} \quad \{x\}_X \gtrsim \{x\}_Y.$$

(again in the sense of distributions and on D_1).

An intermediate consequence of (10) is that, in particular

$$[\mathcal{L}_j(x), \mathcal{L}_k(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0.$$

must hold (on D_1) for every j and k . Since $\mathcal{L}_p(x) = cA(x)$, the $\mathcal{L}_j(x)$ must be in the Borchers class of the free field $A(x)$, i. e., they must be Wick polynomials in $A(x)$ and its derivatives [19]:

$$\mathcal{L}_j(x) = : P_j(A(x), \partial_\mu A(x), \dots, \partial_{\mu_1 \dots \mu_p} A(x)) :$$

P_j being some polynomial of a finite number of variables.

[The polynomial character of the interactions forced by our formalism is due to our allowing g_j to be any element of $\mathcal{S}(\mathbb{R}^4)$. By restricting the possible choice of the g_j to smaller functional spaces (Jaffe's class for example), one could investigate theories where the \mathcal{L}_j are certain entire functions. This will not be attempted in the present paper.]

Before adding to these conditions the requirements of Lorentz invariance and of unitarity, it is useful to investigate the mechanics of the inductive construction. It will be seen that the conditions 2.1 to 2.3 are the really essential ones.

We note that the assumptions 2.1 to 2.3 allows us to define, for all

$$(11) \quad \bar{T}(X) = \sum_{r=1}^{|X|} (-1)^{|X|+r} \sum_{\substack{I_1, \dots, I_r \neq \emptyset \\ I_j \cap I_k = \emptyset \text{ for all } j \neq k \\ I_1 \cup \dots \cup I_r = X}} T(I_1) \dots T(I_r).$$

This yields operator-valued-tempered distributions acting on D_1 and mapping D_1 into itself. It is easy to check [setting, by definition, $\bar{T}(\emptyset) = 1$], the two identities

$$(12) \quad \left\{ \begin{array}{l} \sum_{J \in X} (-1)^{|J|} T(J) \bar{T}(X \setminus J) = 0 \\ \sum_{J \in X} (-1)^{|J|} T(X \setminus J) \bar{T}(J) = 0 \end{array} \right\} \quad \begin{array}{l} \text{for any} \quad X \neq \emptyset \\ \text{with} \quad |X| \leq n - 1 \end{array}$$

(valid on D_1).

Clearly, it follows from the definition and (Caus. 2) that, if

$$|X| \leq n - 1, |Y| \leq n - 1 \quad \text{and} \quad X \cap Y = \emptyset,$$

$$(13) \quad [T(X), \bar{T}(Y)] = [\bar{T}(X), T(Y)] = 0 \quad \text{in the region} \quad \{x\}_X \sim \{x\}_Y$$

(on D_1).

Let us now verify that, if $1 \leq |X| \leq n - 1$, and $X = P \cup Q, P \cap Q = \emptyset$, then

$$(14) \quad \bar{T}(X) = \bar{T}(Q)\bar{T}(P) \quad \text{in the region} \quad \{x\}_P \gtrsim \{x\}_Q.$$

Assume this to be true for all X with $|X| \leq v - 1$ and let $|X| = v \geq 2$. We can write:

$$(15) \quad (-1)^{|X|}\bar{T}(X) = - \sum_{\substack{J \subset X \\ J \neq X \\ J' = X \setminus J}} (-1)^{|J|}\bar{T}(J)T(J').$$

In the region $\{x\}_P \gtrsim \{x\}_Q$, this coincides with

$$\begin{aligned} (-1)^{|X|}\bar{T}(X) &= - \sum_{\substack{J \cup J' = X \\ J \cap J' = \emptyset \\ J' \neq \emptyset}} (-1)^{|J|}\bar{T}(Q \cap J)\bar{T}(P \cap J)T(P \cap J')T(Q \cap J') \\ &= - \sum_{\substack{K \cup K' = P, K \cap K' = \emptyset \\ L \cup L' = Q, L \cap L' = \emptyset \\ L' \neq \emptyset}} (-1)^{|K|+|L|}\bar{T}(L)\bar{T}(K)T(K')T(L') \\ &\quad - \sum_{\substack{K \cup K' = P \\ K \cap K' = \emptyset \\ K' \neq \emptyset}} (-1)^{|Q|+|K|}\bar{T}(Q)\bar{T}(K)T(K'). \end{aligned}$$

If $P \neq \emptyset$, only the second sum contributes and it is equal to

$$(-1)^{|X|}\bar{T}(Q)\bar{T}(P).$$

Besides the antichronological product \bar{T} , it is also useful to consider the totally advanced and totally retarded products. Let $|Y| \leq n - 2, j \notin Y$. We set:

$$(16) \quad A(Y; j) = \sum_{r=0}^{|Y|+1} (-1)^{r-1} \cdot \sum_{\substack{I_1, \dots, I_r \neq \emptyset \\ I_1 \cup \dots \cup I_r = Y \cup \{j\} \\ I_i \cap I_k = \emptyset \text{ for all } i \neq k \\ j \in I_1}} T(I_1) \dots T(I_r).$$

$$(17) \quad A(Y; j) = \sum_{\substack{I \cup I' = Y \\ I \cap I' = \emptyset}} (-1)^{|I|} T(I', j)\bar{T}(I).$$

We also define

$$(18) \quad A'(Y; j) = \sum_{\substack{I \cup I' = Y \\ I \cap I' = \emptyset \\ I \neq \emptyset}} (-1)^{|I|} T(I', j) \bar{T}(I) \\ = A(Y; j) - T(Y, j);$$

Similarly, we denote

$$(19) \quad \left\{ \begin{aligned} R(Y; j) &= \sum_{r=0}^{|Y|+1} (-1)^{r-1} \sum_{\substack{I_1, \dots, I_r \neq \emptyset \\ I_1 \cup \dots \cup I_r = Y \cup \{j\} \\ I_l \cap I_k = \emptyset \text{ for } l \neq k \\ j \in I_1}} T(I_r) T(I_{r-1}) \dots T(I_1) \\ &= \sum_{\substack{I \cup I' = Y \\ I \cap I' = \emptyset}} (-1)^{|I|} T(I) T(I', j) \end{aligned} \right.$$

$$(20) \quad \equiv T(Y, j) + R'(Y; j).$$

The support of $A(Y; j)$ is

$$(21) \quad \{x : x_k - x_j \in \bar{V}^+, \quad \forall k \in Y\}$$

and that of $R(Y; j)$ is

$$(22) \quad \{x : x_k - x_j \in \bar{V}^-, \quad \forall k \in Y\}.$$

To see this, consider two subsets P and Q of Y such that

$$P \neq \emptyset, \quad P \cup Q = Y, \quad P \cap Q = \emptyset.$$

In the region defined by

$$\{x\}_Q \cup \{x_j\} \succeq \{x\}_P$$

we find

$$A'(Y; j) = \sum_{\substack{I \cup I' = Y \\ I \cap I' = \emptyset \\ I \neq \emptyset}} (-1)^{|I|} T(I' \cap Q, j) T(I' \cap P) \bar{T}(I \cap P) \bar{T}(I \cap Q) \\ = \sum_{\substack{K \cup K' = Q, K \cap K' = \emptyset \\ J \cup J' = P, J \cap J' = \emptyset \\ K \cup J \neq \emptyset}} (-1)^{|K|+|J|} T(K', j) T(J') \bar{T}(J) \bar{T}(K)$$

The terms for which $K \neq \emptyset$ yield zero, so that *in this region*,

$$(23) \quad A'(Y; j) = T(Q, j) \sum_{\substack{J \cup J' = P \\ J \cap J' = \emptyset \\ J \neq \emptyset}} (-1)^{|J|} T(J') \bar{T}(J) \\ = -T(Q, j) T(P).$$

In this region we have therefore

$$A(Y; j) = T(Y, j) - T(Q, j)T(P) = 0$$

and the conclusion is that the support of $A(Y; j)$ is (21).

Similarly, in the region

$$\{x\}_Q \cup \{x_j\} \lesssim \{x\}_P$$

we find

$$(24) \quad R'(Y; j) = -T(P)T(Q, j)$$

so that $R(Y; j)$ vanishes in this region; hence the support of $R(Y; j)$ is (22).

The induction hypothesis will be supplemented later by other conditions, but we shall first examine how to go from $n - 1$ to n .

3. GOING FROM $n - 1$ TO n

If the passage from $n - 1$ to n can be accomplished, i. e., if an operator-valued-tempered distribution $T(1, \dots, n)$ can be found so that the conditions 2.1 to 2.3 hold with $n - 1$ replaced by n , we shall be able to define

$$\begin{aligned} A(1, 2, \dots, n - 1; n) &= T(1, \dots, n) + A'(1, \dots, n - 1; n), \\ R(1, 2, \dots, n - 1; n) &= T(1, \dots, n) + R'(1, \dots, n - 1; n), \end{aligned}$$

with

$$(25) \quad A'(1, \dots, n - 1; n) = \sum_{\substack{I \cup I' = \{1, \dots, n-1\} \\ I \cap I' = \emptyset \\ I \neq \emptyset}} (-1)^{|I|} T(I', n) \bar{T}(I)$$

and

$$(26) \quad R'(1, \dots, n - 1; n) = \sum_{\substack{I \cup I' = \{1, \dots, n-1\} \\ I \cap I' = \emptyset \\ I \neq \emptyset}} (-1)^{|I|} \bar{T}(I) T(I', n)$$

and $A(1, 2, \dots, n - 1; n)$ and $R(1, \dots, n - 1; n)$ will have their supports in Γ^+ and Γ^- , respectively,

$$(27) \quad \Gamma^+ = \{x = (x_1, \dots, x_n) : x_j - x_n \in \bar{V}^+ \text{ for all } j\} = -\Gamma^-.$$

But we note that the formulae (25) and (26) give R' and A' in terms of quantities which are supposed to be known according to the induction hypothesis. Hence, for the problem to be soluble, the known quantity

$$(28) \quad D(1, \dots, n - 1; n) = A'(1, \dots, n - 1; n) - R'(1, \dots, n - 1; n)$$

should have its support in $\Gamma^+ \cup \Gamma^-$. To verify that this is indeed guaranteed by the induction hypothesis, we note that the calculations leading to (23) and (24) are still valid for $|Y| = n - 1$.

Let K denote the set of points $x = (x_1, \dots, x_n)$ of \mathbb{R}^{4n} such that, in some Lorentz frame (possibly dependent on x),

$$(29) \quad \begin{cases} x_j^0 - x_n^0 > 0 & \text{for all } j \in P_1 \neq \emptyset, \\ x_k^0 - x_n^0 < 0 & \text{for all } k \in Q_1 \neq \emptyset, \\ x_h^0 - x_n^0 = 0 & \text{for all } h \in S \end{cases}$$

(with, of course, $P_1 \cup Q_1 \cup \{n\} = \{1, \dots, n\}$). Then D vanishes in a neighbourhood of K . For, if x satisfies (29), in a neighbourhood of x we have

$$R'(1, \dots, n - 1; n) = -T(P_1)T(Q_1 \cup S)$$

and (since $|Q_1 \cup S| \leq n - 1$) this is equal to $-T(P_1)T(S)T(Q_1)$. Similarly, in a neighbourhood of x ,

$$A'(1, \dots, n - 1; n) = -T(P_1 \cup S)T(Q_1) = -T(P_1)T(S)T(Q_1).$$

Let now $x = (x_1, \dots, x_n) \notin \Gamma^+ \cup \Gamma^-$. This may happen in two ways.

a. One of the points x_j (for example x_1) is such that

$$x_1 - x_n \in \bar{V}^+, \quad x_1 - x_n \neq 0$$

and another x_j (for example x_2) is such that

$$x_2 - x_n \in \bar{V}^-, \quad x_2 - x_n \neq 0.$$

Then x is in K and D vanishes in its neighbourhood.

b. One of the points x_j (for example x_1) satisfies

$$(x_1 - x_n)^2 < 0.$$

Let us choose a Lorentz frame in which $x_1^0 - x_n^0 = 0$. If $x_j^0 - x_n^0 > 0$ for some j and $x_k^0 - x_n^0 < 0$ for some k , x is in K and D vanishes in its neighbourhood. Assume therefore $x_j^0 - x_n^0 \geq 0$ for all j (a similar argument holds if $x_j^0 - x_n^0 \leq 0, \forall j$). If $x_j^0 - x_n^0 > 0$ for some j (for example $x_2^0 - x_n^0 > 0$), we can perform a small Lorentz transformation which makes $x_1^0 - x_n^0 < 0$ while preserving $x_2^0 - x_n^0 > 0$, hence $x \in K$. The only remaining possibility is $x_j^0 - x_n^0 = 0$ for all j . In this case, let

$$P = \{j : x_j = x_1\} \quad \text{and} \quad Q = \{1, \dots, n\} \setminus P.$$

We have $\{x\}_P \sim \{x\}_Q$, hence

$$\begin{aligned} A'(1, \dots, n - 1; n) &= -T(P)T(Q), \\ R'(1, \dots, n - 1; n) &= -T(Q)T(P) \end{aligned}$$

in a neighbourhood of x . But these two expressions coincide there by (Caus. 2); hence D vanishes near x .

We have proved:

$$(30) \quad \text{supp. } D(1, \dots, n - 1; n) = \Gamma^+ \cup \Gamma^-.$$

Let us suppose that we have found an operator-valued-tempered distribution $A''(1, \dots, n-1; n)$ with the following properties.

A. When smeared with a test function $f \in [\mathcal{S}(\mathbb{R}^{4n})]^{p^n}$ it is a well-defined operator on D_1 , maps D_1 into itself and depends continuously on f ; similarly for finite products such as

$$T(I_1) \dots T(I_q)A''(1, \dots, n-1; n)T(I_{q+1}) \dots T(I_r)A''(j_1, \dots, j_{n-1}; j_n) \dots$$

B.

$$U(a, 1)A''(x_1, \dots, x_{n-1}; x_n)U(a, 1)^{-1} = A''(x_1 + a, \dots, x_{n-1} + a; x_n + a)$$

for all $a \in \mathbb{R}^4$.

C. It is local with respect to itself and to already constructed $T(J)$, i. e.:

$$[A''(x_1, \dots, x_{n-1}; x_n), A''(y_1, \dots, y_{n-1}; y_n)] = 0 \quad (\text{on } D_1)$$

in $\{x, y: (x_j - y_k)^2 < 0 \text{ for all } j \text{ and } k\}$, and, for all $v \leq n-1$,

$$[A''(x_1, \dots, x_{n-1}; x_n), T(y_1, \dots, (y_v))] = 0 \quad (\text{on } D_1)$$

in $\{x, y: (x_j - y_k)^2 < 0 \text{ for all } j \text{ and } k\}$.

D.

$$\text{supp. } A''(1, \dots, n-1; n) \subset \Gamma^+$$

and $A''(1, \dots, n-1; n)$ coincides with $D(1, \dots, n-1; n)$ in the complement of Γ^- .

Define $T'(1, \dots, n)$ and $R''(1, \dots, n-1; n)$ by

$$(31) \quad T'(1, \dots, n) = A''(1, \dots, n-1; n) - A'(1, \dots, n-1; n)$$

$$(32) \quad = R''(1, \dots, n-1; n) - R'(1, \dots, n-1; n).$$

Then $A''(1, \dots, n-1; n) - R''(1, \dots, n-1; n) = D(1, \dots, n-1; n)$ so that $\text{supp } R''(1, \dots, n-1; n) = \Gamma^-$ and R'' coincides with D in the complement of Γ^+ . The operator-valued-tempered distribution T' is correctly transformed by translations, is local with respect to itself and to $T(J)$ for $|J| \leq n-1$.

Suppose $\{1, \dots, n\} = P \cup Q$, $P \cap Q = \emptyset$, P and $Q \neq \emptyset$. In the region $\{x\}_P > \{x\}_Q$ we have

$$T'(1, \dots, n) = T(P)T(Q).$$

Indeed if $n \in Q$, we use (32) and the fact that $R' = -T(P)T(Q)$ in this region, where $R''(1, \dots, n-1; n)$ vanishes. If $n \in P$ we use (31) and the fact that $A' = -T(P)T(Q)$ in this region.

If we now define

$$T(1, \dots, n) = \frac{1}{n!} \sum_{\pi} T'(\pi 1, \dots, \pi n)$$

(where the summation extends over all permutations of $\{1, \dots, n\}$), the

resulting operator-valued distribution satisfies all the conditions necessary to extend the induction hypothesis from $n - 1$ to n .

The problem of defining T has thus been reduced to that of splitting the operator D into two parts with supports Γ^+ and Γ^- , respectively. Suppose this can be done in two different ways:

$$D = A''^{(1)} - R''^{(1)} = A''^{(2)} - R''^{(2)}.$$

Then

$$\begin{aligned} \mathcal{O}(x_1, \dots, x_n) &= A''^{(1)}(x_1, \dots, x_{n-1}; x_n) - A''^{(2)}(x_1, \dots, x_{n-1}; x_n) \\ &= R''^{(1)}(x_1, \dots, x_{n-1}; x_n) - R''^{(2)}(x_1, \dots, x_{n-1}; x_n) \end{aligned}$$

has its support in $\{x: x_1 = x_2 = \dots = x_n\}$. Furthermore, on D_1 ,

$$U(a, 1)\mathcal{O}(x_1, \dots, x_n)U(a, 1)^{-1} = \mathcal{O}(x_1 + a, \dots, x_n + a)$$

for all $a \in \mathbb{R}^4$ and

$$[\mathcal{O}(x_1, \dots, x_n), A(y)] = 0$$

if $(x_j - y)^2 < 0$ for all $j = 1, \dots, n$, hold in the sense of distributions. It follows, by an easy generalization of [19] (see Appendix 2) that, on D_0 , $\mathcal{O}(x_1, \dots, x_n)$ is of the form

$$(33) \quad \mathcal{O}(x_1, \dots, x_n) = \sum_{\substack{\alpha \\ 0 \leq |\alpha| \leq N}} P_\alpha(x_n) D^\alpha \delta(x_1 - x_n) \dots \delta(x_{n-1} - x_n)$$

where α is a multi-index,

$$D^\alpha = \prod_{j=1}^{n-1} \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} = \prod_{j=1}^{n-1} \prod_{\mu=0}^3 \left(\frac{\partial}{\partial x_j^\mu} \right)^{\alpha_{j\mu}}$$

and, for each α , $P_\alpha(x)$ is a Wick polynomial in $A(x)$:

$$(34) \quad P_\alpha(x) = : P_\alpha(A(x), \partial_\mu A(x), \dots, \partial_{\mu_1 \dots \mu_r} A(x)) :$$

It follows, in particular, that, for any $f \in \mathcal{S}$,

$$\int \mathcal{O}(x_1, \dots, x_n) \underline{f}(x_1, \dots, x_n) dx_1 \dots dx_n$$

maps D_0 into itself. Hence if one of the possible solutions for $T(x_1, \dots, x_n)$ is such that $\int T(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1 \dots dx_n$ maps D_0 into itself, the same holds for the most general solution. Furthermore, this encourages us to look for solutions concocted with Wick polynomials. It will be seen in the next section how this leads to the most general solution.

Remark. — The totally advanced and retarded products

$$(35) \quad A(1, \dots, n-1; n) = T(1, \dots, n) + A'(1, \dots, n-1; n),$$

$$(36) \quad R(1, \dots, n-1; n) = T(1, \dots, n) + R'(1, \dots, n-1; n),$$

defined from the $T(1, \dots, n)$ obtained above satisfy the conditions (1) to (4) required from A'' and R'' . In fact,

$$(37) \quad \begin{aligned} A(1, \dots, n-1; n) - A''(1, \dots, n-1; n) \\ &= R(1, \dots, n-1; n) - R''(1, \dots, n-1; n) \\ &= T(1, \dots, n-1; n) - T'(1, \dots, n-1; n) \end{aligned}$$

is an operator $\mathcal{O}(x_1, \dots, x_n)$ of the type discussed above. It is zero if $T'(1, \dots, n)$ happens to be symmetrical.

4. THE MOST GENERAL SOLUTION

If the operators $\mathcal{L}_j(x)$ were continuous functions of x instead of tempered distributions we could construct a solution of our problem by setting

$$(38) \quad T_{i_1 \dots i_n}(x_1, \dots, x_n) = \sum_{\pi} \theta(x_{\pi 1}^0 - x_{\pi 2}^0) \dots \theta(x_{\pi(n-1)}^0 - x_{\pi n}^0) \mathcal{L}_{i_{\pi 1}}(x_{\pi 1}) \dots \mathcal{L}_{i_{\pi n}}(x_{\pi n}).$$

This is indeed the « solution » which yields (in momentum space) the unre-normalized Feynman graphs; it becomes meaningful if the \mathcal{L}_j are replaced by suitably regularized operators. A remarkable property of this formal solution appears if we use Wick's theorem to express a product of Wick polynomials of the type appearing at the end of (38). If we suppose, at first, that the $\mathcal{L}_j(x)$ are simple Wick powers of $A(x)$ (containing no derivatives):

$$\mathcal{L}_j(x) = : A(x)^{v_j} :$$

we can define, for any integer $r \geq 0$,

$$(39) \quad \begin{cases} \mathcal{L}_j^{(r)}(x) = \frac{v_j!}{(v_j - r)!} : A(x)^{v_j - r} : & \text{for } v_j - r > 0, \\ \mathcal{L}_j^{(v_j)}(x) = v_j! \\ \mathcal{L}_j^{(r)}(x) = 0 & \text{for } v_j - r < 0. \end{cases}$$

Then Wick's theorem yields:

$$(40) \quad \begin{aligned} \mathcal{L}_{j_1}(x_1) \dots \mathcal{L}_{j_q}(x_q) \\ &= \sum_{s_1, \dots, s_q} (\Omega, \mathcal{L}_{j_1}^{(s_1)}(x_1) \dots \mathcal{L}_{j_q}^{(s_q)}(x_q) \Omega) \frac{: A(x_1)^{s_1} \dots A(x_q)^{s_q} :}{s_1! \dots s_q!} \end{aligned}$$

and more generally,

$$(41) \quad \mathcal{L}_{j_1}^{(r_1)}(x_1) \dots \mathcal{L}_{j_q}^{(r_q)}(x_q) \\ = \sum_{s_1, \dots, s_q} (\Omega, \mathcal{L}_{j_1}^{(r_1+s_1)}(x_1) \dots \mathcal{L}_{j_q}^{(r_q+s_q)}(x_q) \Omega) \frac{: A(x_1)^{s_1} \dots A(x_q)^{s_q} :}{s_1! \dots s_q!}$$

Hence, formally, formula (35) leads to

$$(42) \quad T(\mathcal{L}(x_1) \dots \mathcal{L}(x_q)) \\ = \sum_r (\Omega, T(\mathcal{L}^{(r_1)}(x_1) \dots \mathcal{L}^{(r_q)}(x_q)) \Omega) \frac{: A(x_1)^{r_1} \dots A(x_q)^{r_q} :}{r_1! \dots r_q!}$$

From this we infer that it is useful to define chronological products not only for the \mathcal{L}_j themselves but also for their « derivatives » $\mathcal{L}_j^{(r_j)}$, and that it would be very convenient to have a formula such as (42) hold, since it would make the needed properties of translational invariance and mutual locality of the $T(J)$ very easy to impose. Actually it will be seen that, with trivial generalizations, this gives the most general solution.

We must now generalize equations (39)-(41) to the case when the \mathcal{L}_j contain derivatives of $A(x)$. Using Schwartz's notation, any partial derivative of $A(x)$ can be written $D^\alpha A(x)$, where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a quadri-index (i. e., $\alpha_\mu \geq 0$ is an integer for each μ); α can also be used as a multipower:

$$D^\alpha = \left(\frac{\partial}{\partial x} \right)^\alpha = \prod_{\mu=0}^3 \left(\frac{\partial}{\partial x_\mu} \right)^{\alpha_\mu} \\ x^\alpha = \prod_{\mu=0}^3 (x^\mu)^{\alpha_\mu}$$

As usual

$$|\alpha| = \sum_{\mu=0}^3 \alpha_\mu, \quad \alpha! = \prod_{\mu=0}^3 \alpha_\mu!$$

In order to denote formal partial derivatives of the \mathcal{L}_j with respect to the various $D^\alpha A(x)$, we introduce a new kind of multi-index; we call superquadri-index an integer-valued function $r: \alpha \rightarrow r(\alpha)$ defined on the set of all possible quadri-indices, such that $r(\alpha) \geq 0$ for all α and $r(\alpha) = 0$ for all sufficiently large α . The set of superquadri-indices will be denoted

$$(\mathbb{N})_0^{\mathbb{N}^4}$$

For any superquadri-index r , we denote

$$: A(x)^r : = : \prod_{\alpha \in \mathbb{N}^4} (D^\alpha A(x))^{r(\alpha)} :$$

Let X be a finite set of integers. A multi-superquadri-index indexed by X is a family $\{r_j\}_{j \in X}$ of superquadri-indices indexed by X . We denote in this case

$$\begin{aligned} |r| &= \sum_{j \in X} |r_j|, & |r_j| &= \sum_{\alpha} r_j(\alpha), \\ r! &= \prod_{j \in X} r_j!, & r_j! &= \prod_{\alpha} r_j(\alpha)!, \\ :A(X)^r: &= : \prod_{j \in X} A^{r_j}(x_j) : = : \prod_{j \in X} \prod_{\alpha} (D^{\alpha} A(x_j))^{r_j(\alpha)} : \end{aligned}$$

We now assume that the $\mathcal{L}_j(x)$ are defined as follows. For any

$$z \in \mathbb{R}^{N^4}$$

i. e., any family of real numbers z_{α} indexed by the quadri-indices, and for any superquadri-index r , we denote

$$z^r = \prod_{\alpha} z_{\alpha}^{r(\alpha)}.$$

We suppose given, for each $j = 1, \dots, p$, a polynomial

$$L_j(z) = \sum_{|r| \leq M_j} c_{jr} z^r$$

and that

$$\mathcal{L}_j(x) = :L_j(A(x)): = \sum_{|r| \leq M_j} c_{jr} :A(x)^r:$$

We define, for any superquadri-index r ,

$$\begin{aligned} \mathcal{L}_j^{(r)}(x) &= : \left(\frac{\partial}{\partial z} \right)^r L_j(z) \Big|_{z_{\alpha} = D^{\alpha} A(x)} : \\ \left(\frac{\partial}{\partial z} \right)^r L_j(z) &= \sum_{\substack{s \geq r \\ |s| \leq M_j}} \frac{s!}{(s-r)!} c_{js} z^{s-r} \\ \mathcal{L}_j^{(r)}(x) &= \sum_{\substack{s \geq r \\ |s| \leq M_j}} : \frac{s!}{(s-r)!} c_{js} \prod_{\alpha} (D^{\alpha} A(x))^{s(\alpha)-r(\alpha)} : \end{aligned}$$

With these notations, formula (38) remains valid.

We now supplement the induction hypothesis as follows:

(4) We extend the list of the $\mathcal{L}_j = \mathcal{L}_j^{(0)}$ by adding to it all the $\mathcal{L}_j^{(r)}$ with $r \neq 0$. All the assumptions made in Section 3 are now supposed to hold with the couples (j, r_j) now playing the role attributed to the indices j in Section 3. Thus we suppose that, for all $v \leq n - 1$ a family

$$T_{j_1 \dots j_v}^{r_1 \dots r_v}(x_1, \dots, x_v), \quad \bar{T}_{j_1 \dots j_v}^{r_1 \dots r_v}(x_1, \dots, x_v)$$

has been constructed with the same properties we assumed in Section 3 for the $T_{j_1 \dots j_v}$ and $\bar{T}_{j_1 \dots j_v}$. Here r_j runs through all possible superquadri-indices, but we assume that the

$$T_{j_1 \dots j_v}^{r_1 \dots r_v}(x_1, \dots, x_v)$$

have only a finite number of non-identically vanishing components. [Of course, the preceding requirements include, in the case $v = 1$,

$$T_j^r(x) = \mathcal{L}_j^r(x).]$$

(5) Moreover, we assume, for all $v \leq n - 1$ the validity (on D_0 , in the sense of distributions) of the formula

$$(43) \quad T_{j_1 \dots j_v}^{r_1 \dots r_v}(x_1, \dots, x_v) = \sum_{s_1, \dots, s_v} (\Omega, T_{j_1^{s_1} \dots j_v^{s_v}}^{r_1 + s_1 \dots r_v + s_v}(x_1, \dots, x_v) \Omega) \frac{:A(x_1)^{s_1} \dots A(x_v)^{s_v}:}{s_1! \dots s_v!}.$$

In the rest of this section, we shall use the abbreviated form $T(X)$ to denote:

$$T_{j_{u_1} \dots j_{u_q}}^{r_{u_1} \dots r_{u_1}}(x_{u_1}, \dots, x_{u_q})$$

where $\{u_1, \dots, u_q\} = X$. If it is necessary to mention explicitly the multi-superquadri-index $r = (r_{u_1}, \dots, r_{u_q})$ we write $T^r(X)$. Then (43) takes the abbreviated form

$$(44) \quad T^r(X) = \sum_s (\Omega, T^{r+s}(X) \Omega) \frac{:A(X)^s:}{s!}.$$

Since $(\Omega, T^r(X) \Omega)$ is a numerical distribution, and $:A(X)^s:$ an operator-valued distribution, it is not immediately obvious that (43) makes any sense. However, an easy calculation in momentum space yields:

THEOREM 0 (see [6] and Appendix 1). — *Let $F \in \mathcal{S}'(\mathbb{R}^{4n})$ be a tempered distribution such that $F(x_1, \dots, x_n) = F(x_1 + a, \dots, x_n + a)$ for all $a \in \mathbb{R}^4$. Then, for any multi-superquadri-index $r = (r_1, \dots, r_n)$ and any $f \in \mathcal{S}'(\mathbb{R}^{4n})$,*

$$\int F(x_1, \dots, x_n) :A(x_1)^{r_1} \dots A(x_n)^{r_n}: f(x_1, \dots, x_n) d^4x_1 \dots d^4x_n$$

is a well-defined operator on D_0 and maps D_0 into itself; it depends conti-

nuously on f in the sense that the vector obtained by applying it to any vector of D_0 depends continuously on f in the norm topology of \mathcal{F} .

This makes (43) and (44) meaningful. We have thus assumed that, for $|X| \leq n - 1$, $T(X)$ (when integrated with a test function in \mathcal{S}) maps D_0 into itself.

Now let E_1^r, \dots, E_M^r be operator-valued-tempered distributions mapping D_0 into itself, covariant under translations, and such that

$$E_k^r(X) = \sum_s (\Omega, E_k^{r+s}(X)\Omega) \frac{A(X)^s}{s!} :$$

for all r and all k . An easy calculation shows that:

$$\begin{aligned} E_1^r(X_1) \dots E_M^r(X_M) &= \sum_s (\Omega, E_1^{r+s_1}(X_1) \dots E_M^{r+s_M}(X_M)\Omega) \frac{A(X_1 \cup \dots \cup X_M)^s}{s_1! \dots s_M!} : \end{aligned}$$

As a consequence, we see that

$$(45) \quad \bar{T}^r(X) = \sum_s (\Omega, \bar{T}^{r+s}(X)\Omega) \frac{A(X)^s}{s!} :$$

and that the operators $D^r(x_1, \dots, x_{n-1}; x_n)$ computed as explained in the preceding section satisfy:

$$(46) \quad D^r(x_1, \dots, x_{n-1}; x_n) = \sum_s (\Omega, D^{r+s}(x_1, \dots, x_{n-1}; x_n)\Omega) \frac{A(X)^s}{s!} :$$

where $X = \{1, \dots, n\}$.

It is now clear that, in view of the discussion in the preceding section, a special solution of our problem will be found if we are able to find, for each r , and each set of indices j_1, \dots, j_n , a numerical tempered distribution in $4n$ variables, denoted

$$a_{j_1 \dots j_n}^{r,r}(x_1, \dots, x_{n-1}; x_n)$$

depending only on the differences $x_1 - x_n, \dots, x_{n-1} - x_n$, symmetrical in the variables (x_k, j_k, r_k) , $1 \leq k \leq n - 1$, with support in Γ^+ and coinciding with

$$(\Omega, D_{j_1 \dots j_n}^r(x_1, \dots, x_{n-1}; x_n)\Omega)$$

outside of Γ^- . Indeed we need only define:

$$(47) \quad t_{j_1 \dots j_n}^{r,r}(x_1, \dots, x_n) = a_{j_1 \dots j_n}^{r,r}(x_1, \dots, x_{n-1}; x_n) - (\Omega, A_{j_1 \dots j_n}^{r,r}(x_1, \dots, x_{n-1}; x_n)\Omega),$$

$$(48) \quad t_{j_1 \dots j_n}^r(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi} t_{j_{\pi 1} \dots j_{\pi n}}^{r, \dots, r}(x_{\pi 1}, \dots, x_{\pi n}),$$

$$(49) \quad T^r(X) = \sum_s t^{r+s}(X) \frac{A^s(X)}{s!}$$

for $X = (1, \dots, n)$.

It will be seen in the next section that it is an easy problem to find the a''^r . This will show, in conjunction with our discussion of the arbitrariness of the definition of T (Section 3) that *the most general solution* satisfies (49) and (43) (on D_0). Hence it maps D_0 into itself and there is no loss of generality in choosing $D_1 = D_0$.

5. GENERAL PRINCIPLES OF DISTRIBUTION SPLITTING

5.1. General facts.

Consider, in general, a tempered distribution $C \in \mathcal{S}'(\mathbb{R}^N)$ with support in $\Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = -\Gamma^-$ is a closed convex cone and $\Gamma^+ \cap \Gamma^- = \{0\}$. Then the dual cone $\tilde{\Gamma}^+$ of Γ^+ ,

$$\tilde{\Gamma}^+ = \{p : \forall x \neq 0 \text{ in } \Gamma^+, p \cdot x > 0\}$$

is a non-empty open convex one.

Splitting problem. — Is it possible to find a pair (T^+, T^-) of tempered distribution on \mathbb{R}^N , with supports in Γ^+ and Γ^- , respectively, such that

$$T^+ - T^- = C?$$

The answer is affirmative; indeed there exists a general theory of dissecting distributions into two parts with prescribed supports, provided these supports are « regularly separated » [20]. But the special problem set here is a very simple one and the general theory is not needed.

We first note that if there are two solutions (T_1^+, T_1^-) and (T_2^+, T_2^-) , the difference $T_1^+ - T_2^+ = T_1^- - T_2^-$ has support $\{0\}$ and must be of the form

$$\sum_{|\alpha|=0}^M c_\alpha D^\alpha \delta(x).$$

Here and in the following, we use Schwartz's notation for multi-indices: if a basis has been selected in \mathbb{R}^N and if α is a multi-index [i. e., a sequence of positive integers $(\alpha_1, \dots, \alpha_N)$, $\alpha_j \geq 0$], we denote

$$|\alpha| = \sum_{j=1}^N \alpha_j; \quad x^\alpha = \prod_{j=1}^N x_j^{\alpha_j}; \quad \alpha! = \prod_{j=1}^N (\alpha_j!);$$

$$D^\alpha f(x) = \left\{ \prod_{j=1}^N \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} \right\} f(x).$$

To show that a solution exists we use an auxiliary real function ω over \mathbb{R}^N with the following properties:

- (i) $0 \leq \omega(x) \leq 1$ and $\omega(0) = 0$;
- (ii) ω is infinitely differentiable at all $x \neq 0$;
- (iii) $\omega(x) = \omega(\rho x)$ for all $x \neq 0$ and all $\rho > 0$;
- (iv) outside of the origin ω takes the value 1 in a closed cone containing a neighbourhood of $\Gamma^+ - \{0\}$ and the value 0 in a closed cone containing a neighbourhood of $\Gamma^- - \{0\}$.

To construct such a function, it is sufficient to construct a \mathcal{C}^∞ function ω on the sphere $\{x : \|x\| = 1\}$ taking the value 1 (resp. 0) in a neighbourhood of the intersection of the sphere with Γ^+ (resp. Γ^-). This is possible since these intersections are disjoint compact subsets. (Here and in the following

we denote $\|x\|^2 = \sum_{j=1}^N |x_j|^2$, the co-ordinates being relative to the chosen basis.) Then we set $\omega(x) = \omega(x/\|x\|)$ for $x \neq 0$ and $\omega(0) = 0$. For $x \neq 0$ and $\rho > 0$ we have

$$(D^\alpha \omega)(\rho x) = \rho^{-|\alpha|} D_x^\alpha (\omega(\rho x)) = \rho^{-|\alpha|} D^\alpha \omega(x);$$

in particular

$$D^\alpha \omega(x) = \|x\|^{-|\alpha|} (D^\alpha \omega)(x/\|x\|)$$

so that, for every $|\alpha|$, there is a constant $B_{|\alpha|}$ such that

$$|D^\alpha \omega(x)| < B_{|\alpha|} \|x\|^{-|\alpha|} \quad \text{for all } x \neq 0.$$

The function ω may also be given (not in a unique way) by an integral representation of the form

$$\omega(x) = \int_0^\infty \frac{dr}{r} \psi(rx) \quad (x \neq 0)$$

where ψ is in \mathcal{D} or \mathcal{S} and vanishes in a neighbourhood of 0. For example ψ can be taken to be

$$\psi(x) = \alpha(\|x\|)\omega(x)$$

where α is an infinitely differentiable function of one real variable, with support in the interval $[1, 2]$ and satisfying

$$\int_0^\infty \alpha(r) \frac{dr}{r} = 1.$$

It is also convenient to adopt some classification of the possible singularities of tempered distributions at the origin. One such classification is provided by:

DÉFINITION 1. — *We say that a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^N)$ is sin-*

gular of order v at 0 if there exist integers $M \geq 0$, and $P \geq 0$, and, for each $\varepsilon > 0$, sufficiently small, a constant $K(\varepsilon) > 0$ such that, for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$| \langle T, \varphi \rangle | \leq K(\varepsilon) \sum_{|\alpha| \leq M} \sup_x (1 + \|x\|)^P \|x\|^{(-v+|\alpha|-\varepsilon)^+} | D^\alpha \varphi(x) |$$

Here $(-v + |\alpha| - \varepsilon)^+ = \max \{ 0, -v + |\alpha| - \varepsilon \}$.

Clearly, if a distribution is singular of order v at 0, it is *a fortiori* singular of order v' for any $v' > v$.

Let us now assume that the tempered distribution C (to be split) is singular at 0 of order ω , where ω is a (positive or negative) integer, i. e., that, for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and every $\varepsilon > 0$, sufficiently small,

$$| \langle C, \varphi \rangle | \leq K(\varepsilon) \sum_{|\alpha| \leq M} \sup_x (1 + \|x\|)^P \|x\|^{(-\omega+|\alpha|-\varepsilon)^+} | D^\alpha \varphi(x) |.$$

We shall distinguish two cases.

5.1.1. Case $\omega \leq -1$

$$| \langle C, \varphi \rangle | \leq K(\varepsilon) \sum_{|\alpha| \leq M} \sup_x (1 + \|x\|)^P \|x\|^{r+|\alpha|-\varepsilon} | D^\alpha \varphi(x) |, \quad r = -\omega \geq 1.$$

In this case C can be extended in a unique fashion into a continuous linear functional on the space of all functions φ , defined, continuous, and M times continuously differentiable in the complement of 0, and such that, for some $\varepsilon > 0$ (depending on φ),

$$| D^\alpha \varphi(x) | \leq \text{Const.} (1 + \|x\|)^{-P} \|x\|^{-r-|\alpha|+\varepsilon}, \quad x \neq 0, \quad 0 \leq |\alpha| \leq M.$$

Let $\omega\varphi$ be such a function. Then $\omega\varphi$ is continuous and M times continuously differentiable in the complement of 0, where it satisfies, for $0 \leq |\alpha| \leq M$:

$$\begin{aligned} & (1 + \|x\|)^P \|x\|^{r+|\alpha|-\varepsilon} | D^\alpha (\omega(x)\varphi(x)) | \\ & \leq \text{Const.} \sum_{\gamma \leq \alpha} (1 + \|x\|)^P \|x\|^{r+|\alpha|-\varepsilon} | D^{\alpha-\gamma} \omega(x) | | D^\gamma \varphi(x) | \\ & \leq \text{Const.} \sum_{\gamma \leq \alpha} (1 + \|x\|)^P \|x\|^{r+|\gamma|-\varepsilon} | D^\gamma \varphi(x) |. \end{aligned}$$

It follows that $\omega\varphi$ is also in the function space in which C acts and that the tempered distribution ωC defined by

$$\langle \omega C, \varphi \rangle = \langle C, \omega\varphi \rangle$$

is again of order ω at 0.

5.1.2. Case $\omega \geq 0$

In this case C can be uniquely extended into a continuous linear functional on the space of continuous functions φ , ω times continuously differentiable over the whole space \mathbb{R}^N , and M' times continuously differentiable in $\mathbb{R}^N - \{0\}$, $M' = \max \{ \omega + 1, M \}$, and such that, for at least one $\varepsilon > 0$ (depending on φ)

$$|D^\alpha \varphi(x)| < \text{Const.} (1 + \|x\|)^{-P}, \quad |\alpha| \leq \omega \quad (\text{all } x \in \mathbb{R}^N),$$

$$|D^\alpha \varphi(x)| < \text{Const.} (1 + \|x\|)^{-P} \|x\|^{-|\alpha| + \omega + \varepsilon}$$

for $\omega + 1 \leq |\alpha| \leq M'$, all $x \neq 0$.

Let $\psi \in \mathcal{S}$ and β be a multi-index with $|\beta| \geq \omega + 1$ and let φ be the function

$$\varphi(x) = x^\beta \sqcup(x) \psi(x).$$

Then φ belongs to the space just mentioned and

$$|D^\alpha \varphi(x)| \leq \text{Const.} \sum_{\substack{\gamma, \gamma', \gamma'' \\ \gamma + \gamma' + \gamma'' = \alpha \\ \gamma'' \leq \beta}} |D^{\gamma'} \sqcup(x) \cdot |x^{\beta - \gamma''}| \cdot |D^\gamma \psi(x)|$$

$$\leq \text{Const.} \sum_{\gamma \leq \alpha} \|x\|^{|\beta| - |\alpha| + |\gamma|} |D^\gamma \psi(x)|.$$

For $|\alpha| \leq \omega$,

$$(1 + \|x\|)^P |D^\alpha \varphi(x)| \leq \text{Const.} \sum_{\gamma \leq \alpha} (1 + \|x\|)^P \|x\|^{|\beta| + |\gamma| - |\alpha|} |D^\gamma \psi(x)|$$

$$\leq \text{Const.} \sum_{\gamma \leq \alpha} (1 + \|x\|)^{P + \omega + \varepsilon} \|x\|^{|\beta| - \omega + |\gamma| - \varepsilon} |D^\gamma \psi(x)|.$$

For $|\alpha| \geq \omega + 1$,

$$\|x\|^{|\alpha| - \omega - \varepsilon} (1 + \|x\|)^P |D^\alpha \varphi(x)|$$

$$\leq \text{Const.} \sum_{\gamma \leq \alpha} (1 + \|x\|)^P \|x\|^{|\beta| - \omega + |\gamma| - \varepsilon} |D^\gamma \psi(x)|$$

so that

$$(50) \quad \sum_{|\alpha| \leq M} \sup_s (1 + \|x\|)^P \|x\|^{(|\alpha| - \omega - \varepsilon)^+} |D^\alpha \varphi(x)|$$

$$\leq \text{Const.} \sum_{|\gamma| \leq M'} \sup_x (1 + \|x\|)^{P + \omega + 1} \|x\|^{|\beta| - \omega + |\gamma| - \varepsilon} |D^\gamma \psi(x)|.$$

Hence $\sqcup x^\beta C = F^{\beta+}$ can be defined (for $|\beta| \geq \omega + 1$) as a tempered distribution, singular of order $-|\beta| + \omega$ at 0, by the formula

$$\langle F^{\beta+}, \psi \rangle = \langle C, \sqcup x^\beta \psi \rangle.$$

We also define $F^{\beta-} = F^{\beta+} - x^\beta C$, which is also of order $-|\beta| + \omega$.

The quantity $\langle C, \omega x^\beta \psi \rangle$ can be obtained by an explicit limiting process. For example let χ be a C^∞ function on \mathbb{R}^N such that $0 \leq \chi \leq 1$, $\chi(x) = 0$ for $\|x\| < 1$, $\chi(x) = 1$ for $\|x\| > 2$. Let $\psi \in \mathcal{S}(\mathbb{R}^N)$ and $\psi_t(x) = \chi(tx)\psi(x)$ for $t > 0$. For $t > 1$, $\omega x^\beta \psi_t \in \mathcal{S}(\mathbb{R}^N)$ and

$$\left| \frac{d}{dt} \langle C, \omega x^\beta \psi_t \rangle \right| = \left| \left\langle C, \omega x^\beta \frac{d}{dt} \psi_t \right\rangle \right|$$

$$\leq \text{Const.} \sum_{|\alpha| \leq M'} \sup_x (1 + \|x\|)^{p+\omega+1} \|x\|^{|\beta|-\omega+|\alpha|-\varepsilon} \left| D^\alpha \sum_{j=1}^N x_j \chi_j(tx) \psi(x) \right|$$

where $\chi_j(x) = (\partial/\partial x_j)\chi(x)$. This is bounded by

$$\text{Const.} \sum_{|\alpha| \leq M'} \sup_x (1 + \|x\|)^{p+\omega+1} \|x\|^{|\beta|-\omega+|\alpha|-\varepsilon}$$

$$\sum_{\gamma \leq \alpha} \sum_{j=1}^N |D^{\alpha-\gamma}(x_j \chi_j(tx))| \cdot |D^\gamma \psi(x)|.$$

Since $\chi_j(y)$ vanishes for $\|y\| > 2$, only values of x such that $\|x\| \leq 2t^{-1}$ are relevant in the above expression. Furthermore

$$|D^{\alpha-\gamma}(x_j \chi_j(tx))| \leq \text{Const.} \|x\| t^{|\alpha-\gamma|} + \text{Const.} t^{|\alpha-\gamma|-1} \leq \text{Const.} t^{|\alpha-\gamma|-1}$$

so that

$$\left| \frac{d}{dt} \langle C, \omega x^\beta \psi_t \rangle \right|$$

$$\leq \text{Const.} t^{-(|\beta|+1-\omega-\varepsilon)} \sum_{|\gamma| \leq M'} \sup_x (1 + \|x\|)^{p+\omega+1} |D^\gamma \psi(x)|.$$

Since $|\beta| + 1 - \omega - \varepsilon \geq 2 - \varepsilon$, we have

$$\langle C, \omega x^\beta \psi \rangle = \lim_{t \rightarrow \infty} \langle C, \omega x^\beta \psi_t \rangle.$$

This shows, in particular, that $\langle C, \omega x^\beta \psi \rangle$ does not depend on a particular choice of ω . If ω' is some other function satisfying the same conditions (1), (2), (3), (4) as ω we have

$$\langle C, (\omega - \omega')x^\beta \psi \rangle = \lim_{t \rightarrow \infty} \langle C, (\omega - \omega')x^\beta \chi(tx) \psi \rangle.$$

The function $(\omega - \omega')x^\beta \chi(tx) \psi$ is in $\mathcal{S}(\mathbb{R}^N)$ for every finite $t > 0$ and its support does not intersect $\Gamma^+ \cup \Gamma^-$ so the result is zero.

A second consequence is that if a function φ can be written in two different ways as

$$\varphi = \sum_{|\beta|=\tau} x^\beta \psi_\beta = \sum_{|\beta|=\tau} x^\beta \psi'_\beta$$

with $\psi_\beta \in \mathcal{S}$, $\psi'_\beta \in \mathcal{S}$, it makes sense to define

$$\langle C, \sqcup \varphi \rangle = \sum_{|\beta|=\tau} \langle C, \sqcup x^\beta \psi_\beta \rangle = \sum_{|\beta|=\tau} \langle C, \sqcup x^\beta \psi'_\beta \rangle$$

since the two last quantities are both equal to

$$\lim_{t \rightarrow \infty} \langle C, \sqcup \chi(tx) \varphi \rangle.$$

We are now in a position to define a solution (T^+, T^-) of the splitting problem. For this purpose we choose a fixed auxiliary function $w \in \mathcal{S}(\mathbb{R}^N)$ such that

$$w(0) = 1, \quad D^\alpha w(0) = 0 \quad \text{for} \quad 1 \leq |\alpha| \leq \omega.$$

For any φ in $\mathcal{S}(\mathbb{R}^N)$, we can define

$$(W\varphi)(x) = \varphi(x) - w(x) \sum_{|\alpha|=0}^{\omega} \frac{\alpha^\alpha}{\alpha!} D^\alpha \varphi(0).$$

The function $W\varphi$ can be written as

$$\begin{aligned} W\varphi &= \sum_{|\beta|=\omega+1} x^\beta \psi_\beta, \\ \psi_\beta(x) &= w(x) \varphi_\beta(x) - \varphi(x) w_\beta(x), \quad (|\beta| = \omega + 1), \\ \varphi_\beta(x) &= \frac{\omega + 1}{\beta!} \int_0^1 dt (1-t)^\omega (D^\beta \varphi)(tx), \\ w_\beta(x) &= \frac{\omega + 1}{\beta!} \int_0^1 dt (1-t)^\omega (D^\beta w)(tx). \end{aligned}$$

The functions ψ_β are in \mathcal{S} and we can define

$$\begin{aligned} \langle T^+, \varphi \rangle &= \langle C, \sqcup W\varphi \rangle \\ &= \sum_{|\beta|=\omega+1} \langle C, \sqcup x^\beta \psi_\beta \rangle, \end{aligned}$$

T^+ is a tempered distribution with support Γ^+ and coincides with C in the complement of Γ^- . If T^- is defined as $T^+ - C$, the pair (T^+, T^-) is a solution of the splitting problem. We shall now verify that T^+ (and therefore T^-) is singular of order ω at 0. We use the inequality (50) and must therefore estimate

$$\sum_{|\gamma| \leq M'} \sup_x (1 + \|x\|)^{P+\omega+1} \|x\|^{1+|\gamma|-\varepsilon} |D^\gamma \psi_\beta(x)|.$$

a. Contribution from $w(x)\varphi_\beta(x)$: it is bounded by

$$\begin{aligned} \text{Const.} \sum_{|\gamma| \leq M'} \sum_{\gamma' \leq \gamma} & \left\{ \sup_x \int_0^1 \frac{dt}{t^{1-\varepsilon}} \|tx\|^{1+|\gamma'|- \varepsilon} |(\mathbf{D}^{\beta+\gamma'}\varphi)(tx)| \right\} \\ & \times \left\{ \sup_y \|y\|^{|\gamma|-|\gamma'|} (1 + \|y\|)^{P+\omega+1} |\mathbf{D}^{\gamma-\gamma'}w(y)| \right\} \\ & \leq \text{Const.} \frac{1}{\varepsilon} \sum_{|\gamma| \leq M'+\omega+1} \sup_x \|x\|^{(-\omega+|\gamma|-\varepsilon)^+} |\mathbf{D}^\gamma\varphi(x)|. \end{aligned}$$

b. Contribution of $-\varphi(x)w_\beta(x)$: it is bounded by

$$\begin{aligned} \text{Const.} \sum_{|\gamma| \leq M'} \sum_{\gamma' \leq \gamma} & \sup_x (1 + \|x\|)^{P+\omega+1} \|x\|^{1+|\gamma|-\varepsilon} |\mathbf{D}^{\gamma'}\varphi(x)| \\ & \times \int_0^1 dt |(\mathbf{D}^{\gamma-\gamma'+\beta}w)(tx)| \\ & \leq \text{Const.} \sum_{|\gamma| \leq M'} \sup_x (1 + \|x\|)^{P+\omega+M'+2} \|x\|^{(-\omega+|\gamma|-\varepsilon)^+} |\mathbf{D}^\gamma\varphi(x)|. \end{aligned}$$

This completes the verification.

The most general solution (S^+, S^-) of the splitting problem which is singular of order ω at 0 is obtained from the special solution (T^+, T^-) just defined by

$$S^+(x) = T^+(x) + \sum_{|\alpha| \leq \omega} c_\alpha \mathbf{D}^\alpha \delta(x).$$

Thus the most general solution (of order ω) depends on an arbitrary polynomial of degree ω .

Note that, for any solution (S^+, S^-) of order ω , and for any multi-index β with $|\beta| \geq \omega + 1$,

$$(51) \quad x^\beta S^\pm = F^{\beta \pm}.$$

Conversely, if (S^+, S^-) is a solution of the splitting problem and satisfies (51) for all β with $|\beta| = \omega + 1$, we have

$$x^\beta (S^+ - T^+) = 0 \quad \text{for all } \beta \text{ with } |\beta| = \omega + 1.$$

Since $S^+ - T^+$ has support $\{0\}$, it must be of the form

$$S^+ - T^+ = \sum_{|\alpha| \leq \omega} c_\alpha \mathbf{D}^\alpha \delta$$

so that S^+ and S^- are also of order ω .

5.2. Further properties of the solutions.

In the applications of this general discussion we have in view, the distribution C will be $(\Omega, D(1, 2, \dots, n; n-1)\Omega)$. We know that the Fourier transform of this distribution vanishes in a large region of momentum space. To incorporate this fact into the general setting of this section, we add a new hypothesis to those made at the beginning of 5.1.

NEW HYPOTHESIS. — *The Fourier transform \tilde{C} of C vanishes in a connected open set \mathcal{R} of \mathbb{R}^N .*

Let \mathcal{R}_1 be an open subset of \mathcal{R} star shaped with respect to a point $a \in \mathcal{R}_1$. [In practice, when $C = (\Omega, D\Omega)$, in theories with non-zero masses, $0 \in \mathcal{R}$ and \mathcal{R} is star shaped with respect to 0. We shall then take $a = 0$, $\mathcal{R}_1 = \mathcal{R}$. In the case of zero masses a can be taken to be any real totally space-like point ($a_1^2 < 0$ for all 1) and \mathcal{R}_1 can be taken to be a ball centered at a .]

The tempered distribution $F^{\beta+}$ (resp. $F^{\beta-}$) has its support in Γ^+ (resp. Γ^-) and, by a classical theorem [21], its Fourier transform $\tilde{F}^{\beta+}$ (resp. $\tilde{F}^{\beta-}$) is the boundary value of a function $H^{\beta+}$ (resp. $H^{\beta-}$) analytic in the tube

$$\mathbb{R}^N + i\tilde{\Gamma} \text{ (resp. } \mathbb{R}^N - i\tilde{\Gamma}^+).$$

The functions $H^{\beta\pm}$ do not grow faster than a polynomial at infinity, nor than a negative power of the distance to the boundaries of the tubes where they are defined. Their boundary values $\tilde{F}^{\beta\pm}$ agree in \mathcal{R}_1 ; by the edge-of-the-wedge theorem, they are both restrictions (to $\mathbb{R}^N \pm i\tilde{\Gamma}^+$, respectively) of a single function H^β , holomorphic in a domain Δ_1 , the envelope of holomorphy of $(\mathbb{R}^N + i\tilde{\Gamma}^+) \cup (\mathbb{R}^N - i\tilde{\Gamma}^+) \cup \mathcal{R}_1$. The domain Δ_1 is really a domain in \mathbb{C}^N , i. e., it is one-sheeted and star shaped with respect to a .) In this domain:

$$D^\gamma H^\beta = i^{|\gamma|} H^{\beta+\gamma} \quad \text{for all } \gamma \text{ and } \beta \text{ with } |\beta| \geq \omega + 1.$$

These are necessary and sufficient conditions for the existence of a solution H , holomorphic in Δ_1 , of the « total » differential system

$$(52) \quad D^\beta H = i^{|\beta|} H^\beta \quad \text{for all } \beta \text{ with } |\beta| = \omega + 1.$$

For example, we can use the formula

$$(53) \quad H(k) = \sum_{|\alpha|=\omega+1} i^{\omega+1} \frac{(\omega+1)}{\alpha!} \int_0^1 dt (1-t)^\omega (k-a)^\alpha H^\alpha(a+t(k-a)).$$

This formula obviously defines a holomorphic function in Δ_1 , and, using the compatibility conditions $D^\gamma H^\beta = i^{|\gamma|} H^{\beta+\gamma}$ it is easy to verify that it is a solution of (22), indeed the unique solution verifying

$$D^\gamma H(a) = 0 \quad \text{for } |\gamma| \leq \omega.$$

Now let (S^+, S^-) be a solution of the splitting problem of order ω at 0 (the preceding subsection has shown their existence), i. e., satisfying (51).

The Fourier transform \tilde{S}^+ (resp. \tilde{S}^-) of S^+ (resp. S^-) is the boundary value from $\mathbb{R}^N + i\tilde{\Gamma}^+$ (resp. $\mathbb{R}^N - i\tilde{\Gamma}^+$) of a function G holomorphic in Δ_1 and satisfying

$$D^\beta G = i^{|\beta|} H^\beta \quad \text{whenever} \quad |\beta| \geq \omega + 1$$

Hence

$$(54) \quad G(k) = H(k) + \sum_{|\alpha| \leq \omega} \frac{(k-a)^\alpha}{\alpha!} D^\alpha G(a).$$

It follows, in particular, that the restriction of H to the tube $\mathbb{R}^N + i\tilde{\Gamma}^+$ (resp. $\mathbb{R}^N - i\tilde{\Gamma}^+$) is the Laplace transform of a tempered distribution F^+ (resp. F^-) with support in Γ^+ (resp. Γ^-). The Fourier transforms \tilde{F}^\pm of F^\pm are the boundary values of H from $\mathbb{R}^N \pm i\tilde{\Gamma}^+$ and satisfy

$$\begin{aligned} \tilde{F}^+ - \tilde{F}^- &= \tilde{C}, \\ D^\beta \tilde{F}^\pm &= i^{|\beta|} F^{\beta^\pm} \quad \text{for all } \beta \text{ with } |\beta| = \omega + 1. \end{aligned}$$

Hence (F^+, F^-) is also a solution of the splitting problem of order ω at 0.

[Actually this solution would correspond to the solution (T^+, T^-) constructed in the preceding subsection if the function w of the preceding section were formally taken to be e^{iax} . It is possible to show directly that such a choice does make sense; this is left as an exercise to the reader so inclined.]

It is easy to verify that

$$(55) \quad \tilde{C}(p) = \sum_{|\alpha| = \omega + 1} \int_0^1 dt \frac{\omega + 1}{\alpha!} (p-a)^\alpha D^\alpha \tilde{C}(a + t(p-a))(1-t)^\omega$$

$$(56) \quad \tilde{F}^\pm(p) = \sum_{|\alpha| = \omega + 1} i^{|\alpha|} \int_0^1 dt (1-t)^\omega \frac{(\omega + 1)}{\alpha!} (p-a)^\alpha \tilde{F}^{\alpha^\pm}(a + t(p-a)).$$

These formulae are to be understood in the sense of distributions, and they make sense because \tilde{C} and \tilde{F}^+ are \mathcal{C}^∞ in the neighbourhood \mathcal{R}_1 of a (indeed \tilde{C} vanishes in \mathcal{R}_1 and \tilde{F}^\pm are analytic there) (see Appendix 3).

6. MINIMAL SOLUTIONS OF THE INDUCTIVE PROBLEM : POWER COUNTING

6.1. Power counting in x space.

In this section we shall find estimates of the order of singularity at 0 (abbreviated as o. s.) of the distributions

$$(57) \quad \left\{ \begin{aligned} &(\Omega, T_{j_1, \dots, j_n}^{r_1, \dots, r_n}(x_1, \dots, x_n)\Omega), \\ &(\Omega, \tilde{T}_{j_1, \dots, j_n}^{r_1, \dots, r_n}(x_1, \dots, x_n)\Omega). \end{aligned} \right.$$

constructed in the preceding two sections. It will be seen that there are solutions for which these o. s. are as small as possible; these solutions will be called minimal.

To simplify the exposition we shall assume that the $\mathcal{L}_j(x)$ are Wick monomials, i. e.:

$$\mathcal{L}_j(x) = : A^{v_j}(x) :$$

where v_j is a superquadri-index. We shall prove inductively that the inductive construction can be carried out so that the o. s. of the distributions (57) is

$$(58) \quad \omega = 4 + \sum_{k=1}^n \left[\sum_{\alpha} (1 + |\alpha|)(v_{jk}(\alpha) - r_k(\alpha)) - 4 \right]$$

and the expression (57) vanishes unless $r_k \leq v_{jk}$ for all $k(1 \leq k \leq n)$. In this connection, (57) must be regarded as a distribution in $4(n - 1)$ variables. For example, if $n = 1$, $(\Omega, T'_j(x)\Omega) = (\Omega, \mathcal{L}_j^{(r)}(x)\Omega)$ is 0 if $r \neq v_j$; it is a constant (v_j) if $r = v_j$. This has o. s. 0 since it must be considered to depend on 0 variable.

Let us assume that $(\Omega, T^r(x)\Omega)$ and $(\Omega, \bar{T}^r(x)\Omega)$ have o. s. given by the formula (58) whenever $|X| \leq n - 1$. Since it is possible to split $(\Omega, D^r\Omega)$ without increasing its o. s., it is sufficient to show that $(\Omega, D^r(X)\Omega)$ has its o. s. given by (58) when $|X| = n$. For this purpose it is sufficient to investigate

$$(\Omega, T^r(X)\bar{T}^r(X')\Omega)$$

with

$$|X \cup X'| = n, \quad 1 \leq |X| \leq n - 1, \quad 1 \leq |X'| \leq n - 1.$$

This expression is a sum of terms of the form

$$(59) \quad \text{Const. } (\Omega, T_{j_1}^{r_1+a_1 \dots r_q+a_q}(x_1, \dots, x_q)\Omega) \times (\Omega, \bar{T}_{j'_1}^{r'_1+b_1 \dots r'_s+b_s}(x'_1, \dots, x'_s)\Omega) \\ \times \prod_{j=1}^l \frac{1}{i} D^{\alpha(j)} D^{\beta(j)} \Delta^+(x_{u(j)} - x'_{v(j)}; m).$$

In this formula, if $l \geq 1$, $u : j \rightarrow u(j)$ [resp. $v : j \rightarrow v(j)$] is a mapping of the set of integers $(1, \dots, l)$ into the set $(1, \dots, q)$ [resp. $(1, \dots, s)$] and for each $j(1 \leq j \leq l)$, $\alpha(j)$ and $\beta(j)$ are quadri-indices. a_k and b_k are superquadri-indices defined by:

$a_k(\gamma)$ is the number of times $u(j)$ takes the values k and $\alpha(j)$ takes at the same time the value γ .

$b_k(\gamma)$ is the number of times $v(j)$ takes the value k and $\beta(j)$ takes at the same time the value γ .

Hence

$$\begin{aligned} |a_k| &= \text{number of times that } u(j) \text{ takes the value } k, \\ |b_k| &= \quad \gg \quad \gg \quad \gg \quad \gg \quad v(j) \quad \gg \quad \gg \quad \gg \quad k, \end{aligned}$$

and

$$\sum_k |a_k| = \sum_k |b_k| = l.$$

Before we evaluate the o. s. of (59), it is useful to note some properties of the o. s. of a tempered distribution.

6.1.1. DIFFERENTIATION INCREASES THE O. S.

Let $F \in \mathcal{S}'(\mathbb{R}^N)$ be of o. s. ρ at 0. Then

$$\begin{aligned} |\langle D^\beta F, \varphi \rangle| &= |\langle F, D^\beta \varphi \rangle| \\ &\leq K(\varepsilon) \sum_{|\alpha| \leq M} \sup_x (1 + \|x\|)^P \|x\|^{(-\rho + |\alpha| - \varepsilon)^+} |D^{\alpha + \beta} \varphi(x)| \\ &\leq K(\varepsilon) \sum_{|\alpha| \leq M + |\beta|} \sup_x (1 + \|x\|)^P \|x\|^{(-\rho - |\beta| + |\alpha| - \varepsilon)^+} |D^\alpha \varphi(x)|. \end{aligned}$$

Hence $D^\beta F$ has o. s. $\rho + |\beta|$ at 0.

6.1.2. TENSOR PRODUCT RULE

Let $F \in \mathcal{S}'(\mathbb{R}^{N_1})$ and $G \in \mathcal{S}'(\mathbb{R}^{N_2})$ be singular of orders ρ_1 and ρ_2 (respectively) at 0. Then $F \otimes G$ is singular of order $\rho_1 + \rho_2$ at 0.

Proof. — For any $\varphi \in \mathcal{S}(\mathbb{R}^{N_1 + N_2})$,

$$\langle F \otimes G, \varphi \rangle = \langle F_x, \langle G_y, \varphi(x, y) \rangle \rangle.$$

Hence

$$\begin{aligned} |\langle F \otimes G, \varphi \rangle| &\leq K_1 \left(\frac{\varepsilon}{2}\right) \sum_{|\alpha| \leq M_1} \sup_{x \in \mathbb{R}^{N_1}} (1 + \|x\|)^{P_1} \|x\|^{[-\rho_1 + |\alpha| - (\varepsilon/2)]^+} |\langle G_y, D_x^\alpha \varphi(x, y) \rangle| \\ &\leq K_1 \left(\frac{\varepsilon}{2}\right) K_2 \left(\frac{\varepsilon}{2}\right) \sum_{|\alpha| \leq M_1} \sum_{|\beta| \leq M_2} \sup_{x \in \mathbb{R}^{N_1}} \sup_{y \in \mathbb{R}^{N_2}} (1 + \|x\|)^{P_1} (1 + \|y\|)^{P_2} \\ &\quad \times \| (x, y) \|^{[-\rho_1 + |\alpha| - (\varepsilon/2)]^+ + [-\rho_2 + |\beta| - (\varepsilon/2)]^+} |D_x^\alpha D_y^\beta \varphi(x, y)| \end{aligned}$$

$$\| (x, y) \|^2 = \|x\|^2 + \|y\|^2.$$

Since for any real numbers a, b , we have $a^+ + b^+ \geq (a + b)^+$, there exists an integer Q such that the above expression is majorized by

$$K_3(\varepsilon) \sum_{|\alpha| \leq M_1 + M_2} \sup_{x, y} (1 + \| (x, y) \|^Q) \| (x, y) \|^{(-\rho_1 - \rho_2 + |\alpha| - \varepsilon)^+} |D^\alpha \varphi(x, y)|.$$

6.1.3. IMPORTANT EXAMPLES

6.1.3.1. A non-zero constant, considered as an element of $\mathcal{S}'(\mathbb{R}^N)$ is singular at 0 of order $-N$. Indeed

$$|\langle 1, \varphi \rangle| = \left| \int \varphi(x) dx \right| \\ \leq [\sup_x \|x\|^{N-\varepsilon} (1 + \|x\|)^{\eta+\varepsilon} |\varphi(x)|] \int_y d^N \|y\|^{-N+\varepsilon} (1 + \|y\|)^{-\eta-\varepsilon}$$

for any $\eta > 0$ and any ε with $0 < \varepsilon < 1$.

6.1.3.2. In Minkowski space, the distributions

$$\Delta^+(x; m) = \frac{i}{(2\pi)^3} \int e^{-ipx} \theta(p^0) \delta(p^2 - m^2) d^4 p, \\ \Delta(x; m) = \frac{i}{(2\pi)^3} \int e^{-ipx} \varepsilon(p^0) \delta(p^2 - m^2) d^4 p, \\ \Delta_R(x; m) = \frac{1}{(2\pi)^4} \int e^{-ipx} \frac{d^4 p}{(p^0 + i0)^2 - \vec{p}^2 - m^2} = -\theta(x^0) \Delta(x; m), \\ \Delta_A(x; m) = \frac{1}{(2\pi)^4} \int e^{-ipx} \frac{d^4 p}{(p^0 - i0)^2 - \vec{p}^2 - m^2} = \theta(-x^0) \Delta(x; m), \\ \Delta_F(x; m) = \frac{1}{(2\pi)^4} \int e^{-ipx} \frac{d^4 p}{p^2 - m^2 - i0} = \theta(x^0) \Delta^+(x; m) + \theta(-x^0) \Delta^+(-x; m)$$

are singular of order -2 at 0.

Proof for $\Delta^+(x; m)$:

$$\Delta^+(x; m) = i \frac{\partial}{\partial x^0} \left(i \frac{\partial}{\partial x^0} + 1 \right)^2 F(x; m), \\ F(x; m) = \int \frac{e^{-i\omega x^0 + i\vec{p} \cdot \vec{x}} d^3 p}{2\omega^2(\omega + 1)^2}$$

where ω stands for $(\vec{p}^2 + m^2)^{1/2}$. $F(x; m)$ is a bounded continuous function on \mathbb{R}^4 . Moreover,

$$|F(x; m) - F(0; m)| = \left| \int \frac{(e^{-i\omega x^0 + i\vec{p} \cdot \vec{x}} - 1) d^3 p}{2\omega^2(\omega + 1)^2} \right| \\ \leq \int_{\omega \leq \mathbf{R}} \frac{|\omega x^0 - \vec{p} \cdot \vec{x}|}{2\omega^2(\omega + 1)^2} + \int_{\omega > \mathbf{R}} \frac{d^3 p}{\omega^2(\omega + 1)^2} \\ \leq 4\pi \|x\| \int_0^{\mathbf{R}} \frac{\omega d\omega}{(\omega + 1)^2} + 4\pi \int_{\mathbf{R}}^{\infty} \frac{d\omega}{(\omega + 1)^2} \\ \leq 4\pi \|x\| \log(\mathbf{R} + 1) + \frac{4\pi}{\mathbf{R} + 1}.$$

Choosing

$$R + 1 = \|x\|^{-1} \quad \text{for} \quad \|x\| < 1$$

and

$$R = 0 \quad \text{for} \quad \|x\| > 1$$

we get

$$|F(x; m) - F(0; m)| \leq 4\pi \|x\| (\|\log \|x\|\| + 1).$$

Hence

$$\begin{aligned} & \left| \int \Delta^+(x; m)\varphi(x)dx \right| \\ & \leq 4\pi \int \|x\| [\|\log \|x\|\| + 1] \left| -\frac{i\partial}{\partial x^0} \left(-\frac{i\partial}{\partial x^0} + 1 \right)^2 \varphi(x) \right| d^4x \\ & \leq \text{Const.} \sum_{|\alpha| \leq 3} \sup_x \|x\|^{5-\varepsilon} (1 + \|x\|)^{|\alpha|} |D^\alpha \varphi(x)| \\ & \leq \text{Const.} \sum_{|\alpha| \leq 3} \sup_x \|x\|^{2+|\alpha|-\varepsilon} (1 + \|x\|)^{|\alpha|} |D^\alpha \varphi(x)|. \end{aligned}$$

Thus $\Delta^+(x; m)$ is of order -2 . The same holds for $\Delta^+(-x; m)$, hence for $\Delta(x; m)$:

$$\Delta_R(x; m) = -\theta(x^0)\Delta(x; m)$$

is also singular of order -2 at 0. This can be seen from the fact that Δ_R can be defined as $-\square(x)\Delta(x)$ as in the preceding section. We leave this verification to the reader (a direct proof can also be easily found). The result then follows for Δ_A and Δ_F from the formulae

$$\begin{aligned} \Delta_A(x; m) &= \Delta_R(x; m) + \Delta(x; m), \\ \Delta_F(x; m) &= -\Delta_R(x; m) + \Delta^+(-x; m) \\ &= -\Delta_A(x; m) + \Delta^+(x; m). \end{aligned}$$

After these preliminaries, we come back to the expression (59) which we write in the form

$$(60) \quad F(\xi_1, \dots, \xi_{q-1})G(\xi'_1, \dots, \xi'_{s-1}) \prod_{j=1}^l D^{\alpha(j)+\beta(j)} \Delta^+(\xi_{u(j)} - \xi'_{v(j)} + \eta; m).$$

Here

$$\xi_k = x_k - x_q, \quad \xi'_k = x'_k - x'_s, \quad \eta = x_q - x'_s.$$

The above expression is indeed a tempered distribution in the variables

$$\xi_1, \dots, \xi_{q-1}, \quad \xi'_1, \dots, \xi'_{s-1}, \quad \eta$$

in view of the fact that

$$\prod_{j=1}^l D^{\gamma(j)} \Delta^+(t_j + \eta; m)$$

is a tempered distribution in η with values in the infinitely differentiable functions of t_1, \dots, t_j . Indeed it is the boundary value of

$$\prod_{j=1}^l D^{\gamma(j)} \Delta^+(t_j + \eta + i\eta'; m)$$

holomorphic when $\eta' \in V^-$. This will also appear in our calculation. We now assume that F (resp. G) is singular of order ω_1 (resp. ω_2) as a distribution in the variables ξ_1, \dots, ξ_{q-1} (resp. $\xi'_1, \dots, \xi'_{s-1}$). We show that (60) is singular at 0 of order

$$\omega = \omega_1 + \omega_2 + \sum_{j=1}^l [|\alpha(j)| + |\beta(j)| + 2] - 4,$$

when considered as a tempered distribution in $\xi_1, \dots, \xi_{q-1}, \xi'_1, \dots, \xi'_{s-1}, \eta$. In the case $l = 1$, it suffices to apply the tensor product rule since $D^\gamma \Delta^+$ is singular of order $|\gamma| - 2$. In the case $l = 0$, (60) reduces to $F \otimes G \otimes 1$ and since 1 is of order -4 we get $\omega_1 + \omega_2 - 4$. Thus we only have to treat the case $l \geq 2$. We define $\gamma(j) = \alpha(j) + \beta(j)$ and

$$\begin{aligned} R(t, \eta) &= \prod_{j=1}^l D^{\gamma(j)} \Delta^+(t_j + \eta; m) \\ &= \int d^4 P e^{-iP \cdot \eta - i \sum_{j=1}^l p_j t_j} \delta\left(P - \sum_{j=1}^l p_j\right) \prod_{j=1}^l \delta(p_j^2 - m^2) \theta(p_j^0) (-ip_j)^{\gamma(j)} d^4 p_j \\ &\quad [t = (t_1, \dots, t_l) \in \mathbb{R}^{4l}, \quad \eta \in \mathbb{R}^4]. \end{aligned}$$

For any multi-index α ,

$$D_i^\alpha R(t, \eta) = \left(i \frac{\partial}{\partial \eta^0} + 1\right) \left(i \frac{\partial}{\partial \eta^0}\right)^M B_\alpha(t, \eta)$$

with

$$M = |\alpha| + \sum_{j=1}^l (|\gamma(j)| + 2)$$

and

$$\begin{aligned} B_\alpha(t, \eta) &= \int \frac{d^4 P (-ip)^\alpha}{(P^0)^M (P^0 + 1)} e^{-iP \cdot \eta - i \sum_{j=1}^l p_j t_j} \delta\left(P - \sum_{j=1}^l p_j\right) \\ &\quad \times \prod_{j=1}^l \delta(p_j^2 - m^2) \theta(p_j^0) (-ip_j)^{\gamma(j)} d^4 p_j. \end{aligned}$$

Taking into account the inequality $|1 - e^{i\theta}| < |\theta|$ (valid for any real θ) and the fact that, in the domain of integration $\mathbf{P} \in \bar{\mathbf{V}}^+$,

$$|\bar{p}_j| \leq p_j^0 \leq P^0 \quad (1 \leq j \leq l),$$

we have, for all $A > 0$,

$$\begin{aligned} |B_\alpha(t, \eta) - B_\alpha(0, 0)| &\leq \text{Const.} \int_{\substack{\mathbf{P} \in \bar{\mathbf{V}}^+ \\ P^0 < A}} \frac{d^4 \mathbf{P} \|(t, \eta)\|}{(P^0 + 1)(P^0)^{M-|\alpha|-|\gamma|-1}} \rho_l(\mathbf{P}) \\ &+ \text{Const.} \int_{\substack{\mathbf{P} \in \bar{\mathbf{V}}^+ \\ P^0 \geq A}} \frac{d^4 \mathbf{P}}{(P^0 + 1)(P^0)^{M-|\alpha|-|\gamma|}} \rho_l(\mathbf{P}) \end{aligned}$$

where

$$\begin{aligned} \rho_l(\mathbf{P}) &= \int \delta\left(\mathbf{P} - \sum_{j=1}^l p_j\right) \prod_{j=1}^l \delta(p_j^2 - m^2) \theta(p_j^0) d^4 p_j, \\ \|(t, \eta)\|^2 &= \sum_{j=1}^l \|t_j\|^2 + \|\eta\|^2, \quad |\gamma| = \sum_{j=1}^l |\gamma(j)|. \end{aligned}$$

The constants depend only on $l, |\alpha|, |\gamma|$.

ρ_l is a continuous function with support $\{\mathbf{P}; P^0 \geq 0, P^2 \geq l^2 m^2\}$, bounded by $\text{const.} (P^0)^{2l-4} \theta(P^0 - |\bar{\mathbf{P}}|)$. Hence

$$\begin{aligned} |B_\alpha(t, \eta) - B_\alpha(0, 0)| &\leq \text{Const.} \int_0^A \frac{\|(t, \eta)\| dP^0}{P^0 + 1} \\ &+ \text{Const.} \int_A^\infty \frac{dP^0}{(P^0)^2} \leq \text{Const.} [\|(t, \eta)\| \log(A + 1) + A^{-1}]. \end{aligned}$$

Taking

$$A = \|(t, \eta)\|^{-1} \quad \text{for} \quad \|(t, \eta)\| \leq 1,$$

and

$$A = 1 \quad \text{for} \quad \|(t, \eta)\| > 1$$

we get

$$|B_\alpha(t, \eta) - B_\alpha(0, 0)| \leq \text{Const.} \|(t, \eta)\| [1 + |\log \|(t, \eta)\||].$$

For any $0 < \varepsilon < 1$, there exists a constant $C(\varepsilon)$ such that this is bounded by

$$C(\varepsilon) \|(t, \eta)\|^{1-\varepsilon} (1 + \|(t, \eta)\|).$$

Since $M > 1$,

$$D_i^\alpha \mathbf{R}(t, \eta) = \left(i \frac{\partial}{\partial \eta^0} + 1\right) \left(i \frac{\partial}{\partial \eta^0}\right)^M [B_\alpha(t, \eta) - B_\alpha(0, 0)].$$

Now, let $\varphi(\xi, \xi', \eta)$ be a function in $\mathcal{S}(\mathbb{R}^{4(q+s)-4})$ and

$$\psi(\xi, \xi') = \int \mathbf{R}(t, \eta) \varphi(\xi, \xi', \eta) d^4 \eta \Big|_{t_j = \xi_{u(j)} - \xi'_{v(j)}}$$

By the tensor product rule,

$$I = \left| \int F(\xi)G(\xi')R(t, \eta)\varphi(\xi, \xi', \eta)d\xi d\xi' d\eta \right|$$

$$\leq \text{Const.} \sum_{|\alpha| \leq K} \sup_{\xi, \xi'} (1 + \|(\xi, \xi')\|)^p \|(\xi, \xi')\|^{(-\omega_1 - \omega_2 + |\alpha| - \varepsilon)^+} \times |D_{\xi, \xi'}^\alpha \psi(\xi, \xi')|.$$

But

$$|D_{\xi, \xi'}^\alpha \psi(\xi, \xi')| \leq \sum_{|\beta| \leq |\alpha|} \text{Const.} \int \| (t, \eta) \|^{1-\varepsilon} (1 + \| (t, \eta) \|)$$

$$\times \left| D_{\xi, \xi'}^\beta \left(-i \frac{\partial}{\partial \eta^0} + 1 \right) \left(-i \frac{\partial}{\partial \eta^0} \right)^{|\alpha| - |\beta| + |\gamma| + 2l} \varphi(\xi, \xi', \eta) \right| d^4 \eta$$

$$\leq \sum_{|\beta| \leq |\alpha| + |\gamma| + 2l + 1} \sup_{\eta} (1 + \| (t, \eta) \|)^2 \| (t, \eta) \|^{1-\varepsilon} \|\eta\|^{4-\varepsilon} |D_{\xi, \xi'; \eta}^\beta \varphi(\xi, \xi', \eta)|.$$

Hence

$$I \leq \text{Const.} \sum_{|\beta| \leq K + |\gamma| + 2l + 1} \sup_{\xi, \xi', \eta} (1 + \|(\xi, \xi', \eta)\|)^{p'}$$

$$\times \|(\xi, \xi', \eta)\|^{(-\omega_1 - \omega_2 + |\beta| - |\gamma| - 2l - 1 - \varepsilon)^+ + 5 - 2\varepsilon} |D^\beta \varphi(\xi, \xi', \eta)|$$

$$\leq \text{Const.} \sum_{|\beta| \leq K'} \sup_{\xi, \xi', \eta} (1 + \|(\xi, \xi', \eta)\|)^q$$

$$\times \|(\xi, \xi', \eta)\|^{(-\omega_1 - \omega_2 + |\beta| - 2l + 4 - |\gamma| - 3\varepsilon)^+} |D^\beta \varphi(\xi, \xi', \eta)|.$$

This shows that

$$F(\xi)G(\xi') \prod_{j=1}^l D^{\nu(j)} \Delta^+(\xi_{u(j)} - \xi'_{v(j)} + \eta; m)$$

is a tempered distribution singular at 0 of order

$$\omega_1 + \omega_2 + |\gamma| + 2l - 4.$$

By the induction hypothesis, this is equal to

$$4 + \sum_{k=1}^q \left[\sum_{\alpha} (1 + |\alpha|)(v_{jk}(\alpha) - r_k(\alpha)) - 4 \right]$$

$$+ \sum_{k=1}^s \left[\sum_{\alpha} (1 + |\alpha|)(v_{jk}'(\alpha) - r_k'(\alpha)) - 4 \right]$$

$$- \sum_{k=1}^q \left[\sum_{\alpha} (1 + |\alpha|)a_k(\alpha) \right] + \sum_{j=1}^l (1 + \alpha(j))$$

$$- \sum_{k=1}^s \left[\sum_{\alpha} (1 + |\alpha|)b_k(\alpha) \right] + \sum_{j=1}^l (1 + \beta(j)).$$

The last terms cancel and the result is

$$4 + \sum_{k=1}^q \left[\sum_{\alpha} (1 + |\alpha|)(v_{jk}(\alpha) - r_k(\alpha)) - 4 \right] + \sum_{k=1}^s \left[\sum_{\alpha} (1 + |\alpha|)(v_{jk}(\alpha) - r_k(\alpha)) - 4 \right]$$

This is precisely (58). In particular, if there are no derivative couplings, i. e., if $v_j(\alpha) = 0$ for $\alpha \neq 0$, we find

$$\omega = \sum_{j=1}^n (v_j - r_j - 4) + 4 = \sum_{j=1}^n (v_j - 4) - \sum_{j=1}^n r_j + 4$$

[where v_j stands for $v_j(0)$, r_j for $r_j(0)$]. To reach an intuitive understanding of ω , we note that, owing to formula (43),

$$(\Omega, T_{j_1 \dots j_n}^{r_1 \dots r_n}(x_1, \dots, x_n)\Omega)$$

corresponds to a process with $|r_1| + \dots + |r_n|$ incoming or outgoing particles. In fact it coincides with what would be, in the usual presentation, the sum of all (renormalized) Feynman graphs with n vertices of type j_1, \dots, j_n to each of which are attached $|r_1|, \dots, |r_n|$ external lines (respectively). Among the $|r_k|$ external lines attached to the vertex j_k , $r_k(\alpha)$ have a D^α derivative. On the other hand, ω characterizes the inherent arbitrariness in the definition of

$$(\Omega, T_{j_1 \dots j_n}^{r_1 \dots r_n}\Omega).$$

Hence, the above estimate of ω allows the well-known classification of renormalizable, superrenormalizable, unrenormalizable theories [6], [22].

If

$$\left[\sum_{\alpha} (1 + |\alpha|)v_j(\alpha) \right] \leq 4 \quad \text{for all } j,$$

the theory is renormalizable (i. e., processes with a large number of external lines are not « superficially » ambiguous). If

$$\left[\sum_{\alpha} (1 + |\alpha|)v_j(\alpha) \right] > 4 \quad \text{for some } j,$$

the theory is unrenormalizable (i. e., for any given number of external lines the amplitudes are « superficially » ambiguous above a certain order in g).

6.2. Remark for the case $m > 0$, $\omega > 0$.

It has been noted in Section 5 that, among the solutions of the splitting problem of $(\Omega, D'\Omega)$ which are singular of order ω at 0, there is one and only one characterized by the following property: the Fourier transform of $a'''(x_1, \dots, x_{n-1}; x_n)$ vanishes $\omega + 1$ times at 0, i. e.,

$$(61) \quad D^\alpha \tilde{a}'''(0) = 0 \quad \text{for all } |\alpha| \leq \omega.$$

As explained previously, the true retarded functions $a'(x_1, \dots, x_{n-1}; x_n)$ are obtained by a symmetrization procedure [see Equation (48) and (35)-(37)]. From the remark at the end of Section 6, we know that a' is a solution of the splitting problem for $(\Omega, D'\Omega)$. Hence its Fourier transform \tilde{a}' is the boundary value of a function H_r , holomorphic in the same domain as \tilde{a}''' (from which it may differ only by a polynomial of degree ω). However, this function is analytic in the full domain of the n point Green function, since \tilde{a}' coincides in real regions with other generalized retarded functions (see Section 8). Due to the symmetry of T and the definition of the various advanced functions, we see that $H_r(k) = H_{\pi r}(\pi k)$ for all permutations π of $(1, \dots, n)$. The function H_r' of which a''' is the boundary value is the unique solution of

$$\begin{aligned} D^\beta H_r'' &= D^\beta H_r, & \text{for all } |\beta| \geq \omega + 1, \\ D^\gamma H_r''(0) &= 0 & \text{for all } |\gamma| \leq \omega, \end{aligned}$$

hence it also satisfies

$$H_{\pi r}''(\pi k) = H_r''(k).$$

Hence [due to (37), (47)-(49)]:

$$T'(1, \dots, n) - T(1, \dots, n)$$

is actually symmetric in $1, \dots, n$ and is therefore 0. Thus, the symmetrization is not necessary in this case.

Similarly, if the $T(X)$ for $|X| \leq n - 1$ have been constructed so as to transform correctly under Lorentz transformations, the same will be true for the $T(1, \dots, n)$ just mentioned. Indeed, since $(\Omega, D'\Omega)$ is Lorentz covariant, and is uniquely characterized by the Lorentz covariant condition (61), it is Lorentz covariant. (This solution of the splitting problem will be called the « central solution » for future reference.)

6.3. Remark for the case $m = 0$.

The inductive construction we have described and the theory of power counting are valid for the case $m = 0$. It is slightly more complicated to show the existence of Lorentz covariant minimal solutions. We shall only indicate the procedure for scalar theories. If we assume Lorentz

covariance for all $T(X)$ with $|X| \leq n - 1$, $(\Omega, D^r\Omega)$ is Lorentz covariant. Let $H(k_1, \dots, k_n)$ be the analytic function associated with a (non-necessarily covariant) solution of the splitting problem. Since the H^β for $|\beta| \geq \omega + 1$ do not depend on the choice of ω , they are Lorentz covariant. H and the H^β are analytic in a domain which is invariant under the whole complex Lorentz group $L_+(C)$. Hence we can define

$$G(k) = \int_{O(4)} d\Lambda H(\Lambda k)$$

where $O(4)$ is a maximal compact subgroup of $L_+(C)$ and $d\Lambda$ is the invariant measure on $O(4)$ such that $\int d\Lambda = 1$. Clearly, because of the covariance of the H^β

$$D^\beta G(k) = i^{|\beta|} H(k) \quad \text{for} \quad |\beta| \geq \omega + 1$$

so that $G(k)$ also provides a minimal solution of the splitting problem which is Lorentz invariant.

6.4. Unitarity.

The unitarity of $S(g)$ as a formal power series is expressed by

$$S(g) = (S^{-1}(g))^*$$

or

$$T(X) = \bar{T}(X)^* \quad \text{on} \quad D_1,$$

or

$$(62) \quad (\Phi, T(X)\bar{\Psi}) = (\bar{T}(X)\Phi, \Psi)$$

for all $\Phi, \Psi \in D_1$. Owing to (43), (45), this equivalent to

$$(\Omega, T_{j_1 \dots j_n}^{r_1 \dots r_n}(X)\Omega) = (\Omega, \bar{T}_{j_1 \dots j_n}^{r_1 \dots r_n}(X)\Omega)^*$$

for all $X = (1, \dots, n), r, j$. Denoting $X = (1, \dots, n), Y = (1, \dots, n - 1)$, we note that (17) yields

$$A(Y; n) = \sum_{\substack{I \cup J = X \\ I \cap J = \emptyset \\ n \in I}} (-1)^{|I|} T(J)\bar{T}(I) = - \sum_{\substack{I \cup J = X \\ I \cap J = \emptyset \\ n \in I}} (-1)^{|I|} T(J)\bar{T}(I)$$

so that, on D_1 , (62) implies

$$(63) \quad \begin{cases} A(Y; n)^* = - \sum_{\substack{I \cup J = X \\ I \cap J = \emptyset \\ n \in I}} (-1)^{|X|} (-1)^{|J|} T(I)\bar{T}(J), \\ A(Y; n)^* = (-1)^{|X|+1} A(Y; n) \end{cases}$$

and similarly

$$(64) \quad R(Y; n)^* = (-1)^{|X|+1} R(Y; n).$$

Conversely, (63) and (64) imply (62).

If we assume that the inductive construction has been carried out so as to satisfy (62) for $|X| \leq n-1$, it is easy to verify that $D(1, \dots, n-1; n)$ satisfies

$$D(Y; n) = (-1)^{n+1} D(Y; n)^* \quad \text{on} \quad D_1.$$

Thus, if $A_1(Y; n)$ and $R_1(Y; n)$ are a solution of the splitting problem for $D(Y; n)$, setting

$$A(Y; n) = \frac{1}{2} \{ A_1(Y; n) + (-1)^{n+1} A_1(Y; n)^* \},$$

$$R(Y; n) = \frac{1}{2} \{ R_1(Y; n) + (-1)^{n+1} R_1(Y; n)^* \}$$

(on D_1), we satisfy (63), (64) and (62) for $|X| = n$.

Moreover, it is easy to check that, in the case $m > 0$, the « central solution » (or any solution of the splitting problem defined by requiring the Fourier transforms of the retarded functions to vanish $\omega + 1$ times at a real point of analyticity) needs no such symmetrization.

6.5. Power counting in momentum space.

It is possible to base the power counting theory on a characterization of the growth at infinity of the Fourier transforms of retarded, advanced, chronological, etc., functions. This is explained in Appendix 3. A by-product of this theory is the possibility to replace the function ω by an ordinary step function, for example $\theta(e \cdot x)$ where $e \in \tilde{\Gamma}^+$. Specifically, if

$$\Gamma^+ = \{ x: x_j - x_n \in \tilde{V}^+, \quad 1 \leq j \leq n-1 \}$$

considered as a subset of $\mathbb{R}^{4(n-1)}$ we can use

$$\theta \left(\sum_{j=1}^{n-1} e_j \cdot (x_j - x_n) \right) = \theta \left(\sum_{j=1}^n e_j \cdot x_j \right)$$

where the e_j satisfy

$$\sum_{j=1}^n e_j = 0, \quad e_j \in V^+ \quad \text{for} \quad 1 \leq j \leq n-1.$$

Starting from this remark it is possible to prove that (for $m > 0$) the

central solution of the splitting problem has the following integral representation

$$H(\lambda e) = \frac{\lambda^{\omega+1}}{2\pi i} \int \frac{\tilde{C}(\lambda' e)}{\lambda'^{\omega+1}(\lambda' - \lambda)} d\lambda'.$$

Here \tilde{C} is the Fourier transform of $(\Omega, D(1, \dots, n-1; n)\Omega)$; e is a real vector in $\mathbb{R}^{4(n-1)}$ with $e_j \in V^+$ for $1 \leq j \leq n$; λ may vary in the cut plane

$$\{ \lambda \in \mathbb{C}, \quad \lambda^2 \notin \mathbb{R}^+ + \kappa^2 \}$$

$\kappa^2 = \inf (m^2/e_1^2)$. (I runs through all subsets of $\{1, \dots, n-1\}$ and

$$e_1 = \sum_{j \in I} e_j$$

Other such representations are valid in the various tubes of analyticity of H.

7. THE POINT OF VIEW OF INFINITE COUNTERTERMS. CONNECTION WITH OTHER FORMALISMS

While the formal solution provided by (38) makes no sense, it becomes meaningful if the $\mathcal{L}_j^{(n)}(x)$ are replaced by smoother objects by using one of the well-known regularizing procedures.

A general class of regularizations consists of replacing the free field $A(x)$ by a generalized free field B_ρ whose two-point function is chosen to be

$$(65) \quad (\Omega, B_\rho(x)B_\rho(y)\Omega) = \frac{1}{(2\pi)^3} \int \rho(p)e^{-ip \cdot (x-y)} d^4p$$

where ρ is a Lorentz invariant tempered distribution with support in \bar{V}^+ . (Of course, the Fock space \mathcal{F}_ρ where B_ρ operates is no longer a Hilbert space; its vectors are sequences of smooth « wave functions », etc.). If ρ is such that it decreases sufficiently fast at infinity when regularized with any element of $\mathcal{S}(\mathbb{R}^4)$, the two-point function (65) is continuous and has a finite number of continuous derivatives. Then products of operators of the form $: D^{\alpha_1}B_\rho(x) \dots D^{\alpha_n}B_\rho(x) :$ can be freely multiplied by step functions provided the $|\alpha_1|, \dots, |\alpha_n|, \dots$, are not too large. This device has the virtue of preserving the locality of the fields. The method of analytic regularization [10] and the Pauli-Villars regularization are in this class.

We shall restrict our attention to the case when ρ is a *measure* of the form

$$\rho(p) = \mu(p^2)$$

where μ is a measure on \mathbb{R} with support in \mathbb{R}^+ . In this case, we can define

$$|\rho|(p) = |\mu|(p^2)$$

(and \mathcal{F}_ρ can be identified with $\mathcal{F}_{|\rho|}$ and given a Hilbert space structure).

We shall, moreover, assume that $|\mu|$ has moments of arbitrarily high order (it has compact support in the usually chosen Pauli-Villars regularizations):

$$\int |\mu| (\kappa^2)^n d\kappa^2 = K_n < \infty$$

and we shall denote $K_0 = \|\mu\|$, and

$$\int \frac{|\mu| (\kappa^2) d\kappa^2}{1 + \kappa^2} = \|\mu\|.$$

The two-point function of B_ρ (denoted B_μ from now on),

$$\begin{aligned} (\Omega, B_\mu(x)B_\mu(y)\Omega) &= \frac{1}{i} \int \mu(\kappa^2)\Delta^+(x - y; \kappa) d\kappa^2 \\ &= \frac{1}{(2\pi)^3} \int e^{-ip(x-y)} \delta(p^2 - \kappa^2) \theta(p^0) \mu(\kappa^2) d\kappa^2 d^4p, \end{aligned}$$

depends continuously on μ , when $\|\mu\|$ stays bounded, in the sense of the norm $\|\mu\|$, when considered as an element of the space of tempered distributions singular of order -2 at 0 .

To see this, we write

$$\frac{1}{(2\pi)^3} \int e^{-ipx} \delta(p^2 - \kappa^2) \theta(p^0) \mu(\kappa^2) d\kappa^2 d^4p = \left(i \frac{\partial}{\partial x^0} + 1\right) \left(i \frac{\partial}{\partial x^0}\right)^2 G(x),$$

$$G(x) = \frac{1}{(2\pi)^3} \int \frac{(e^{-ipx} - 1)}{(p^0)^2(p^0 + 1)} \mu(\kappa^2) \delta(p^2 - \kappa^2) \theta(p^0) d^4p d\kappa^2,$$

$$\begin{aligned} |G(x)| &\leq \frac{1}{(2\pi)^3} \int_{p^0 \leq A} \frac{2\|x\|}{p^0(p^0 + 1)} h(p^0) dp^0 \\ &\quad + \frac{1}{(2\pi)^3} \int_{p^0 > A} \frac{2}{(p^0)^2(p^0 + 1)} h(p^0) dp^0 \end{aligned}$$

where

$$h(p^0) = 4\pi\theta(p^0) \int r^2 dr |\mu(\kappa^2)| d\kappa^2 \delta((p^0)^2 - r^2 - \kappa^2)$$

$$= 2\pi\theta(p^0) \int_{p^0 \geq \kappa} \sqrt{(p^0)^2 - \kappa^2} |\mu(\kappa^2)| d\kappa^2 \leq \begin{cases} 2\pi\theta(p^0)p^0 \|\mu\|, \\ 2\pi\theta(p^0)p^0(1 + p^0)^2 \|\mu\|. \end{cases}$$

Hence

$$h(p^0) \leq 2\pi\theta(p^0)p^0(1 + p^0)^{2\epsilon} \|\mu\|^{1-\epsilon} \|\mu\|^\epsilon$$

and

$$\begin{aligned} |G(x)| &\leq \frac{1}{(2\pi)^2} \int_0^A \frac{2\|x\|}{(\omega + 1)^{1-2\epsilon}} \|\mu\|^{1-\epsilon} \|\mu\|^\epsilon d\omega \\ &\quad + \frac{1}{(2\pi)^2} \int_A^\infty \frac{2}{\omega(\omega + 1)^{1-2\epsilon}} \|\mu\|^{1-\epsilon} \|\mu\|^\epsilon d\omega, \end{aligned}$$

$$\begin{aligned} |G(x)| &\leq \frac{2}{(2\pi)^2} \frac{1}{2\epsilon} \|\mu\|^{1-\epsilon} \|\mu\|^\epsilon \|x\| (A + 1)^{2\epsilon} \\ &\quad + \frac{2}{(2\pi)^2} \frac{1}{1 - 2\epsilon} \|\mu\|^{1-\epsilon} \|\mu\|^\epsilon A^{-1+2\epsilon} \end{aligned}$$

Taking

$$A = \|x\|^{-1} \quad \text{for} \quad \|x\| \leq 1$$

and

$$A = 1 \quad \text{for} \quad \|x\| \geq 1,$$

we get

$$|G(x)| \leq \frac{4}{(2\pi)^2} \varepsilon^{-1} \|\mu\|^{1-\varepsilon} \|\mu\|^\varepsilon \|x\|^{1-2\varepsilon}$$

for any positive $\varepsilon < 1/4$.

Thus

$$\begin{aligned} & \left| \int (\Omega, B_\mu(x) B_\mu(0) \Omega) \varphi(x) d^4x \right| \\ & \leq \frac{4}{(2\pi)^2} \varepsilon^{-1} \|\mu\|^{1-\varepsilon} \|\mu\|^\varepsilon \int \|x\|^{1-2\varepsilon} \left| \left(-i \frac{\partial}{\partial x^0} + 1 \right) \left(-i \frac{\partial}{\partial x^0} \right)^2 \varphi(x) \right| dx \\ & \leq 4\varepsilon^{-2} \|\mu\|^{1-\varepsilon} \|\mu\|^\varepsilon \sum_{2 \leq |\alpha| \leq 3} \sup_x (1 + \|x\|) \|x\|^{5-3\varepsilon} |D^\alpha \varphi(x)| \\ & \leq 4\varepsilon^{-2} \|\mu\|^{1-\varepsilon} \|\mu\|^\varepsilon \sum_{|\alpha| \leq 3} \sup_x (1 + \|x\|)^2 \|x\|^{2+|\alpha|-3\varepsilon} |D^\alpha \varphi(x)|. \end{aligned}$$

Furthermore, the quantity

$$\rho_{l,\mu}(s) = \int \delta \left(P - \sum_{j=1}^l p_j \right) \prod_{j=1}^l \left\{ \mu(\kappa_j^2) \delta(p_j^2 - \kappa_j^2) \theta(p_j^0) d^4 p_j d\kappa_j^2 \right\}$$

[where $P = (\sqrt{s}, 0)$, $s > 0$] also depends continuously on μ . To see this let us denote

$F(s; \mu_1, \dots, \mu_n)$

$$= \int \delta \left(P - \sum_{j=1}^l p_j \right) \prod_{j=1}^l \left\{ \mu_j(\kappa_j^2) \delta(p_j^2 - \kappa_j^2) \theta(p_j^0) d^4 p_j d\kappa_j^2 \right\}$$

where μ_1, \dots, μ_n are measures on \mathbb{R}^+ with $\|\mu_j\| < \infty$ for all j . We wish to show by induction on n that (for $n \geq 2$) F is continuous in s and

$$(66) \quad |F(s; \mu_1, \dots, \mu_n)| \leq \theta(s) s^{n-2} \left(\frac{\pi}{2} \right)^{n-1} \frac{(n-1)}{[(n-1)!]^2} \prod_{j=1}^l \|\mu_j\|.$$

The reader will easily verify that this is true for $n = 2$. Assuming it to be true for some $n \geq 2$, we have

$F(s; \mu_1, \dots, \mu_n, \mu_{n+1})$

$$= \int \mu_{n+1}(\kappa^2) d\kappa^2 \delta(p^2 - \kappa^2) \theta(p^0) F((P - p)^2; \mu_1, \dots, \mu_n) d^4 p$$

[where $P = (\sqrt{s}, 0)(s > 0)$]

$$\begin{aligned} &= \int \mu_{n+1}(\kappa^2) \delta(p^2 - \kappa^2) \theta(p^0) F(s + \kappa^2 - 2\sqrt{sp^0}; \mu_1, \dots, \mu_n) \\ &\quad \theta(\sqrt{s} - p^0) \theta(s + \kappa^2 - 2\sqrt{sp^0}) d^4 p d\kappa^2 \\ &= 2\pi \int \mu_{n+1}(\kappa^2) d\kappa^2 \int_{\kappa}^{\sqrt{s}/2 + \kappa^2/2\sqrt{s}} d\omega \sqrt{\omega^2 - \kappa^2} F(s + \kappa^2 - 2\sqrt{s\omega}; \mu_1, \dots, \mu_n). \end{aligned}$$

The integrand of the second integral vanishes unless $s \geq \kappa^2$. Changing to the variable $\omega' = \omega - \kappa^2/2\sqrt{s}$, we get

$$(67) \quad F(s; \mu_1, \dots, \mu_{n+1}) = 2\pi \int_0^\infty \mu_{n+1}(\kappa^2) d\kappa^2 \int_{\kappa - \kappa^2/2\sqrt{s}}^{\sqrt{s}/2} d\omega' \sqrt{\left(\omega' + \frac{\kappa^2}{2\sqrt{s}}\right)^2 - \kappa^2} F(s - 2\sqrt{s\omega'}; \mu_1, \dots, \mu_n).$$

In the range of integration,

$$\left(\omega' + \frac{\kappa^2}{2\sqrt{s}}\right)^2 - \kappa^2 \leq \omega'^2 + \frac{\kappa^2}{2} + \frac{\kappa^4}{4s} - \kappa^2 \leq \omega'^2 - \frac{\kappa^2}{4} \leq \omega'^2.$$

Hence

$$\begin{aligned} |F(s; \mu_1, \dots, \mu_{n+1})| &\leq 2\pi \|\mu_{n+1}\| \left\| \int_0^{\sqrt{s}/2} d\omega' \omega' |F(s - 2\sqrt{s\omega'}; \mu_1, \dots, \mu_n)| \right\| \\ &\leq 2\pi \left(\prod_{j=1}^{n+1} \|\mu_j\| \right) \left(\frac{\pi}{2}\right)^{n-1} \frac{n-1}{[(n-1)!]^2} \theta(s) s^{n-2} \int_0^{\sqrt{s}/2} \omega' \left(1 - \frac{2\omega'}{\sqrt{s}}\right)^{n-2} d\omega' \\ &= \left(\frac{\pi}{2}\right)^n \left(\prod_{j=1}^{n+1} \|\mu_j\| \right) \frac{1}{(n-1)!(n-2)!} \theta(s) s^{n-1} \int_0^1 t(1-t)^{n-2} dt. \end{aligned}$$

The last integral equals $1/n(n-1)$, which proves our contention.

As a consequence,

$$\begin{aligned} |\rho_{l,\mu}(s) - \rho_{l,\mu'}(s)| &= |F(s; \mu, \dots, \mu) - F(s; \mu, \dots, \mu, \mu') \\ &\quad + F(s; \mu, \dots, \mu, \mu') - F(s; \mu, \dots, \mu, \mu', \mu') + \dots| \\ &= \left| \sum F(s; \mu, \dots, \mu, \mu - \mu', \mu', \dots, \mu') \right| \\ &\leq l \|\mu - \mu'\| [\max\{\|\mu\|, \|\mu'\|\}]^{l-1} \left(\frac{\pi}{2}\right)^{l-1} \frac{l-1}{[(l-1)!]^2} \theta(s) s^{l-2} \end{aligned}$$

On the other hand, since, in the range of integration of (67), we have $\sqrt{s} > \kappa$, we can write

$$\begin{aligned}
 & |F(s; \mu_1, \dots, \mu_{n+1})| \\
 & \leq 2\pi \int_0^\infty \frac{\mu_{n+1}(\kappa^2)d\kappa^2}{1 + \kappa^2} \int_0^{\sqrt{s/2}} (1 + s)\omega' |F(s - 2\sqrt{s}\omega'; \mu_1, \dots, \mu_n)| d\omega' \\
 & \leq \left(\frac{\pi}{2}\right)^n \{\mu_{n+1}\} \left(\prod_{j=1}^n \|\mu_j\|\right) \frac{n}{(n!)^2} \theta(s)s^{n-1}(1 + s).
 \end{aligned}$$

Taking an average of the two bounds,

$$\begin{aligned}
 & |F(s; \mu_1, \dots, \mu_{n+1})| \\
 & \leq \left(\frac{\pi}{2}\right)^n \left(\prod_{j=1}^n \|\mu_j\|\right) \frac{n!}{(n!)^2} \|\mu_{n+1}\|^{1-\varepsilon} \{\mu_{n+1}\}^\varepsilon \times \theta(s)s^{n-1}(1 + s)^\varepsilon.
 \end{aligned}$$

As a consequence,

$$\begin{aligned}
 |\rho_{l,\mu}(s) - \rho_{l,\mu'}(s)| & \leq \frac{l(l-1)}{[(l-1)!]^2} \left(\frac{\pi}{2}\right)^{l-1} [\max\{\|\mu\|, \|\mu'\|\}]^{l-1} \\
 & \quad \times \|\mu - \mu'\|^{1-\varepsilon} \{\mu - \mu'\}^\varepsilon \theta(s)s^{l-2}(1 + s)^\varepsilon.
 \end{aligned}$$

Using these remarks, it is easy to see that, if $A(x)$ is replaced by B_μ in $\mathcal{L}_j^{(r)}(x)$, the inductive construction of the $T(X)$ goes through without change, including the power counting theory. Moreover, denoting (for $X = \{1, \dots, n\}$):

$$\begin{aligned}
 & \|(\Omega, T_j^r(X)\Omega)\|_{\omega, M, P, \varepsilon} \\
 & = \sup_{\varphi \in \mathcal{S}^r} \left| \int (\Omega, T_{j_1 \dots j_n}^{r_1 \dots r_n}(y_1, \dots, y_n)\Omega)\varphi(y_1 - y_n, \dots, y_{n-1} - y_n) \right. \\
 & \quad \left. d(y_1 - y_n) \dots d(y_{n-1} - y_n) \right| \\
 & \quad \times \left[\sum_{|\alpha| \leq M} \sup_x (1 + \|x\|)^P \|x\|^{(-\omega + |\alpha| - \varepsilon)^+} |D^\alpha \varphi(x)| \right]^{-1}
 \end{aligned}$$

with ω given by (58), we see that (for the minimal solutions) there are integers M and P such that, for all sufficiently small $\varepsilon > 0$, and if $\|\mu\|$ and $\|\mu'\|$ remain bounded,

$$\|(\Omega, T_j^r(X)\Omega)_\mu - (\Omega, T_j^r(X)\Omega)_{\mu'}\|_{\omega, M, P, \varepsilon} \leq \text{Const.} \{\mu - \mu'\}^{\varepsilon/R},$$

where the constant depends on $n, r, j, \|\mu\|, \|\mu'\|$. Here $(\Omega, T_j^r(X)\Omega)_\mu$ and $(\Omega, T_j^r(X)\Omega)_{\mu'}$ denote solutions obtained (in the case of B_μ and $B_{\mu'}$, respectively) by applying an identical minimal prescription: for example, using the same auxiliary function w (see Section 5) to perform the splittings in

both cases, or requiring, in both cases, the Fourier transforms of the retarded functions to vanish $\omega + 1$ times at the same point of analyticity (e. g. 0; in the case $m > 0$ we shall restrict our attention to measures μ vanishing in a neighbourhood of 0). In particular if $\|\mu\|$ remains bounded and μ tends to $\delta(\kappa^2 - m^2)$ in the sense of the norm $\{\mu\}$, we recover the solutions corresponding to $A(x)$.

We now assume that the measure μ has been selected so that, when A is replaced by B_μ in the various $\mathcal{L}_j^{(r)}$, the Wightman functions of the latter possess a certain number of continuous derivatives. We denote

$$T_{j_1 \dots j_n}^{r_1 \dots r_n}(x_1, \dots, x_n)$$

a minimal solution of the inductive problem, obtained as described in the preceding sections. [For example, we can consider, in the case $m > 0$, the solution obtained by requiring the Fourier transforms of the retarded functions

$$(\Omega, R_{j_1 \dots j_n}^{r_1 \dots r_n}(x_1, \dots, x_{n-1}; x_n)\Omega)$$

to vanish $\omega + 1$ times at 0, with ω given by (58). This solution will be called the « central solution » for brevity.]

Now, given local fields (in a space with indefinite metric) having continuous Wightman functions, say $\Phi_1(x), \dots, \Phi_l(x), \dots$, we shall denote

$$\Theta(\Phi_{j_1}(x_{j_1}) \dots \Phi_{j_q}(x_{j_q})) = \Theta(j_1, \dots, j_q) = \Theta(X) \quad \text{for} \quad X = \{j_1, \dots, j_q\},$$

the operator given by

$$\sum_{\pi} \theta(x_{\pi j_1}^0 - x_{\pi j_2}^0) \dots \theta(x_{\pi j_{q-1}}^0 - x_{\pi j_q}^0) \Phi_{\pi j_1}(x_{\pi j_1}) \dots \Phi_{\pi j_q}(x_{\pi j_q}).$$

This definition is supposed, at first, to refer to a fixed Lorentz frame (it will be seen shortly that it does not depend on it).

Let us suppose that, for $|X| \leq n - 1$, it has been proved that $\Theta(X)$ has the property of causal factorization, i. e., that if $X = P \cup Q, P \cap Q = \emptyset$, in the region $\{x\}_P \succeq \{x\}_Q$, $\Theta(X)$ coincides with $\Theta(P)\Theta(Q)$. Let now $X = \{1, \dots, n\}$ and $X = P \cup Q, P \cap Q = \emptyset, P \neq \emptyset, Q \neq \emptyset$ and let x satisfy $\{x\}_P \succeq \{x\}_Q$. We can distinguish two cases.

First case. — There is an index $j \in X$ such that $x_j^0 > x_k^0$ for all $k \neq j$. Then it is clear that, near the point x ,

$$\Theta(X) = \Phi_j(x_j)\Theta(X \setminus j) = \Phi_j(x_j)\Theta((X \setminus j) \cap P)\Theta((X \setminus j) \cap Q).$$

In case $j \in P$, this is equal to

$$\Theta_j(x_j)\Theta(P \setminus j)\Theta(Q)$$

and (since $Q \neq \emptyset$), to

$$\Theta(P)\Theta(Q).$$

If $j \in Q$ we have $\{x_j\} \sim \{x\}_p$ so the expression coincides with

$$\Theta(P)\Phi_f(x_j)\Theta(Q \setminus j) = \Theta(P)\Theta(Q).$$

Second case. — $x_j^0 = x_k^0$ for some j and k in X . This case is of measure zero.

Therefore $\Theta(1, \dots, n)$ is the only locally bounded solution of the splitting problem if the $\Theta(X)$ are given for $|X| \leq n - 1$. It follows, by induction, that the $\Theta(X)$ are independent (in the sense of distributions) of the Lorentz frame chosen to define them. With these notations, we wish to prove that the central solution $T(X)$ can be expressed by the following identity between formal power series:

$$\begin{aligned} S(\underline{g}) &\equiv \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int dx_1 \dots dx_n T(x_1, \dots, x_n) \underline{g}(x_1) \dots \underline{g}(x_n) \\ &= \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int dx_1 \dots dx_n \underline{g}(x_1) \dots \underline{g}(x_n) \Theta(\mathcal{R}(x_1; \underline{g}) \dots \mathcal{R}(x_n; \underline{g})) \end{aligned}$$

where $\mathcal{R}(x; \underline{g})$ is a formal series of Wick powers:

$$\mathcal{R}(x; \underline{g}) = \sum_{n=1}^{\infty} \mathcal{H}_n(x; \underline{g}),$$

$$\mathcal{H}_1(x; \underline{g}) = \mathcal{L}(x) \quad \text{and, for } q \geq 2,$$

$$\int \mathcal{H}_q(x; \underline{g}) \underline{g}(x) dx = \sum_{r,s,j} \int g_{j_1}^{r_1}(x_1) \dots g_{j_q}^{r_q}(x_q) \rho_{j_1^{r_1} \dots j_q^{r_q}}^{r_1 + s_1 \dots r_q + s_q}(x_1, \dots, x_q) \times (s!)^{-1} : A(x_1)^{s_1} \dots A(x_q)^{s_q} : dx_1 \dots dx_q,$$

$$(68) \quad \rho_{j_1^{r_1} \dots j_q^{r_q}}^{r_1 + s_1 \dots r_q + s_q}(x_1, \dots, x_q) = \sum_{|\alpha| \leq \omega} \rho_{j; \alpha}^r D^\alpha \delta(x_1 - x_2) \dots \delta(x_{q-1} - x_q),$$

with ω given by formula (58) (for $n = q$). Assuming this to be true for $q \leq n - 1$, it is clear from the preceding remarks that the n th order part of

$$\Theta(\mathcal{R}(x_1; \underline{g}) \dots \mathcal{R}(x_n; \underline{g}))$$

(because it has the property of causal factorization) yields one minimal solution of the n th order splitting problem. Hence, it differs from the central solution $T(1, \dots, n)$ by a term of the form

$$\sum_s \rho_{j_1^{r_1} \dots j_n^{r_n}}^{r_1 + s_1 \dots r_n + s_n}(x_1, \dots, x_n) (s!)^{-1} : A(X)^s : \quad (X = \{1, \dots, n\}),$$

where ρ is of the form (68). We use these ρ to define the n th order terms of $\mathcal{R}(x; \underline{g})$.

The n th order terms ($n > 1$) of $\mathcal{R}(x; g)$ have no limit when the regulators are removed, i. e., when $\mu(\kappa^2) \rightarrow \delta(\kappa^2 - m^2)$. They become « infinite counterterms ». Since the list of these counterterms is the same as in the Bogoliubov-Parasiuk-Hepp ([5], [7]) formalism, and since (in the presence of regulators) they are uniquely determined by requiring the Fourier transforms of the retarded functions to vanish $\omega + 1$ times at some point, the present formalism coincides with that of Bogoliubov-Parasiuk-Hepp.

8. PROPERTIES OF THE SOLUTIONS. PROBLEM OF THE ADIABATIC LIMIT

8.1. General facts.

In the preceding sections, a formal power series $S(g)$ has been constructed. It is easy to verify that all the properties required in Section 1 are satisfied. The only non-trivial point is to verify that, defining

$$(69) \quad V(\underline{g}, \underline{h}) = S(\underline{g})^{-1}S(\underline{g} + \underline{h}) \quad W(\underline{g}, \underline{h}) = S(\underline{g} + \underline{h})S(\underline{g})^{-1}$$

we have

$$(70) \quad V(\underline{g} + \underline{h}_2, \underline{h}_1) = V(\underline{g}, \underline{h}_1) \quad \text{if } \sup \underline{h}_2 \gtrsim \sup \underline{h}_1.$$

Indeed, this means

$$S(\underline{g} + \underline{h}_2)^{-1}S(\underline{g} + \underline{h}_2 + \underline{h}_1) = S(\underline{g})^{-1}S(\underline{g} + \underline{h}_1)$$

which [by multiplying on the left by $S(\underline{g})^{-1}S(\underline{g} + \underline{h}_2)$] is equivalent to

$$\text{i. e.,} \quad S(\underline{g})^{-1}S(\underline{g} + \underline{h}_2 + \underline{h}_1) = S(\underline{g})^{-1}S(\underline{g} + \underline{h}_2)S(\underline{g})^{-1}S(\underline{g} + \underline{h}_1)$$

$$(71) \quad V(\underline{g}, \underline{h}_1 + \underline{h}_2) = V(\underline{g}, \underline{h}_2)V(\underline{g}, \underline{h}_1).$$

This is, in turn, equivalent to the same condition for

$$W(\underline{g}, \underline{h}) = S(\underline{g})V(\underline{g}, \underline{h})S^{-1}(\underline{g}).$$

To perform the verification, let us denote

$$(72) \quad V(\underline{g}, \underline{h}) = \sum_{n,m=0}^{\infty} \frac{(i)^{n+m}}{n!m!} \int \mathbf{R}(y_1, \dots, y_n; x_1, \dots, x_m) \\ \times \underline{g}(y_1) \dots \underline{g}(y_n)\underline{h}(x_1) \dots \underline{h}(x_m)dy_1 \dots dy_ndx_1 \dots dx_m.$$

The condition (70) then means that

$$(73) \quad \text{Sup. } \mathbf{R}(Y; X) \subset \{ (y, x) : \{ y \}_Y \subset \{ x \}_X + \bar{V}^- \},$$

i. e., for every $j \in Y$ there must be a $k \in X$ such that $y_j - x_k \in \bar{V}^-$. From (69) and (70), we deduce

$$\mathbf{R}(Y; X) = \sum_{I \subset Y} (-1)^{|I|} \bar{T}(I)T(Y \setminus I, X).$$

If a point (y, x) is not in the set (73), there are subsets P and Q of Y such that

$$P \neq \emptyset, \quad P \cup Q = Y, \quad P \cap Q = \emptyset$$

and

$$\{y\}_P \cap \{x\}_X + \bar{V}^- = \emptyset, \quad \{y\}_Q \subset \{x\}_X + \bar{V}^-.$$

As a consequence, in a neighbourhood of this point,

$$\begin{aligned} R(Y; X) &= \sum_{\substack{J \subset P \\ K \subset Q}} (-1)^{|J|+|K|} \bar{T}(J \cup K) T(P \setminus J, Q \setminus K, X) \\ &= \sum_{\substack{J \subset P \\ K \subset Q}} (-1)^{|J|+|K|} \bar{T}(K) \bar{T}(J) T(P \setminus J) T(Q \setminus K, Y) \end{aligned}$$

which vanishes since $P \neq \emptyset$.

We can now define the interacting fields $\hat{A}(x; g)$ and, more generally,

$$\hat{\mathcal{L}}(x; g) = \{ \mathcal{L}_j^{(r)}(x; g) \}_{j,r}$$

by the formula

$$(74) \quad \hat{\mathcal{L}}(x; g) = S(g)^{-1} \frac{\delta}{i \delta g(x)} S(g).$$

Similarly

$$(75) \quad \hat{T}(x_1, \dots, x_n; g) = S(g)^{-1} \frac{\delta^n}{i^n \delta g(x_1) \dots \delta g(x_n)} S(g),$$

$$(76) \quad \hat{T}(x_1, \dots, x_n; g) = \sum_{r=0}^{\infty} \frac{i^r}{r!} \int R(y_1, \dots, y_r; x_1, \dots, x_n) g(y_1) \dots g(y_r) dy_1 \dots dy_r,$$

$$(77) \quad V(g, h) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \hat{T}(x_1, \dots, x_n; g) h(x_1) \dots h(x_n) dx_1 \dots dx_n.$$

The factorization property (71) implies that, if $X = P \cup Q$, $P \cap Q = \emptyset$, in the region $\{x\}_P \supseteq \{x\}_Q$,

$$(78) \quad \hat{T}(X; g) = \hat{T}(P; g) \hat{T}(Q; g).$$

Thus the $\hat{T}(X; g)$ are a set of « chronological products » for the fields $\hat{\mathcal{L}}(x; g)$ (in the sense of formal power series).

A set of chronological products [such as the $T'(X)$] for a set of local fields [such as the $\mathcal{L}^{(r)}(x)$] automatically defines a complete set of *generalized retarded products* (GRP) for these fields, special cases of which are the Steinmann GRP (SGRP) or Steinmann monomials. Hence these are well defined for the $\mathcal{L}^{(r)}(x)$: as an example, we recall the appearance, in the inductive construction, of the totally advanced or retarded products

$$A(Y; n), \quad R(Y; n).$$

Actually, the whole inductive construction can be written in terms of the SGRP (instead of T products) (this method has been summarized in [16]). We recall that a GRP of n fields is labelled by a « cell » S in $(n - 1)$ dimensional real space: consider in \mathbb{R}^n the set Ξ_n of points (s_1, \dots, s_n) such that $s_1 + \dots + s_n = 0$; the complement of the union of all the hyperplanes of the form $s_I = 0 \left(s_I \equiv \sum_{j \in I} s_j; I \text{ is a proper subset of } \{ 1, \dots, n \} \right)$

is composed of disjoint open polyhedral cones called cells. Thus a cell S can be described by specifying for all I the sign ε_I^S taken by s_I in S , or, equivalently, the list \mathcal{S} of proper subsets I of $\{ 1, \dots, n \}$ such that $\varepsilon_I^S = 1$. With this notation, the GRP corresponding to this cell is denoted $R_{\mathcal{S}}$ and given by

$$(79) \quad R_{\mathcal{S}}(x_1, \dots, x_n) = T(X) + \sum_{v=2}^n (-1)^{v-1} \sum_{\substack{J_1, \dots, J_v \\ J_1 \cup \dots \cup J_v = X, \\ J_j \cap J_k = \emptyset \text{ for } j \neq k, \\ J_k \neq \emptyset \text{ for all } k, \\ J_v \in \mathcal{S}, \dots, J_r \cup \dots \cup J_v \in \mathcal{S} \\ \text{for all } r \geq 2}} T(J_1)T(J_2) \dots T(J_v)$$

The properties of these GRP are given in ([23], [24]). A presentation adapted to the purposes of this paper is given in [25]. While the SGRP can be studied as special cases of the above formula, they can also be introduced as follows.

Let $\mathcal{O}(\underline{g})$ be a formal power series in \underline{g} (with operator values). We denote

$$(80) \quad \hat{\mathcal{L}}(x; \underline{g}) \downarrow \mathcal{O}(\underline{g}) = \frac{1}{i} \frac{\delta}{\delta \underline{g}(x)} \mathcal{O}(\underline{g}),$$

$$(81) \quad \hat{\mathcal{L}}(x; \underline{g}) \uparrow \mathcal{O}(\underline{g}) = S^{-1}(\underline{g})[\hat{\mathcal{L}}(x; \underline{g}) \downarrow S(\underline{g})\mathcal{O}(\underline{g})S(\underline{g})^{-1}]S(\underline{g}).$$

The Steinmann monomials for the interacting fields are the operators of the form

$$\hat{\mathcal{L}}(x_1; \underline{g}) \downarrow \hat{\mathcal{L}}(x_2; \underline{g}) \downarrow \dots \hat{\mathcal{L}}(x_{n-1}; \underline{g}) \downarrow \hat{\mathcal{L}}(x_n; \underline{g}).$$

The Steinmann monomials for the fields $\mathcal{L}(x)$ (Wick polynomials of free fields) are obtained by setting $\underline{g} = 0$ in the preceding expression. The rules of the arrow calculus are:

(i) Arrows of the same direction commute:

$$(82) \quad \begin{cases} \hat{\mathcal{L}}(x; \underline{g}) \uparrow \hat{\mathcal{L}}(y; \underline{g}) \uparrow \mathcal{O}(\underline{g}) = \hat{\mathcal{L}}(y; \underline{g}) \uparrow \hat{\mathcal{L}}(x; \underline{g}) \uparrow \mathcal{O}(\underline{g}), \\ \hat{\mathcal{L}}(x; \underline{g}) \downarrow \hat{\mathcal{L}}(y; \underline{g}) \downarrow \mathcal{O}(\underline{g}) = \hat{\mathcal{L}}(y; \underline{g}) \downarrow \hat{\mathcal{L}}(x; \underline{g}) \downarrow \mathcal{O}(\underline{g}). \end{cases}$$

(ii)

$$(83) \quad \hat{\mathcal{L}}(x; \underline{g}) \uparrow \mathcal{O}(\underline{g}) - \hat{\mathcal{L}}(x; \underline{g}) \downarrow \mathcal{O}(\underline{g}) = [\hat{\mathcal{L}}(x; \underline{g}), \mathcal{O}(\underline{g})].$$

(iii)

$$(84) \quad \hat{\mathcal{L}}(x; g) \uparrow \hat{\mathcal{L}}(y; g) = \hat{\mathcal{L}}(y; g) \downarrow \hat{\mathcal{L}}(x; g).$$

We shall use the following abbreviated notation for Steinmann monomials:

$$\begin{aligned} \hat{j}_1 \uparrow \hat{j}_2 \downarrow \dots \uparrow \hat{j}_n & \text{ stands for } \hat{\mathcal{L}}(x_{j_1}; \underline{g}) \downarrow \dots \downarrow \hat{\mathcal{L}}(x_{j_n}; \underline{g}), \\ j_1 \downarrow j_2 \downarrow \dots \downarrow j_n & \text{ stands for } \mathcal{L}(x_{j_1}) \downarrow \dots \downarrow \mathcal{L}(x_{j_n}). \end{aligned}$$

Moreover, if $I_1, \dots, I_n, \{j\}$ are disjoint sets of integers,

$$\hat{I}_1 \downarrow \hat{I}_2 \uparrow \dots \uparrow \hat{I}_n \uparrow \hat{j}$$

stands for

$$\left[\prod_{k \in I_1} (\hat{\mathcal{L}}(x_k; \underline{g}) \downarrow) \right] \left[\prod_{k \in I_2} (\hat{\mathcal{L}}(x_k; \underline{g}) \uparrow) \right] \dots \left[\prod_{k \in I_n} \hat{\mathcal{L}}(x_k; \underline{g}) \uparrow \right] \hat{\mathcal{L}}(x_j; \underline{g})$$

and similarly $I_1 \downarrow I_2 \uparrow \dots \uparrow I_n \uparrow j$ stands for the same expression with the $\hat{\mathcal{L}}(x; \underline{g})$ replaced by $\mathcal{L}(x)$.

Starting from the definitions, it is possible to find explicit formulae giving $I_1 \downarrow I_2 \uparrow \dots \uparrow I_n \uparrow j$ in terms of T operators and to show that they possess all the properties described in [24]. We also denote

$$(85) \quad Y \downarrow T(X_1)T(X_2) \dots T(X_p) = \hat{Y} \downarrow \hat{T}(X_1; \underline{g}) \dots \hat{T}(X_p; \underline{g})|_{\underline{g}=0}.$$

And similarly for

$$(86) \quad Y \uparrow T(X_1) \dots T(X_p).$$

It is easy to see that

$$\begin{aligned} Y \downarrow T(X_1) \dots T(X_p) &= \sum_{\substack{Y_1 \cup \dots \cup Y_p = Y \\ Y_j \cap Y_k = \emptyset \text{ if } j \neq k}} (Y_1 \downarrow T(X_1))(Y_2 \downarrow T(X_2)) \dots (Y_p \downarrow T(X_p)) \end{aligned}$$

(similarly for $Y \uparrow$).

Moreover, the support of (85) is contained in

$$\{ \{y\}_Y, \{x\}_{X_1 \cup \dots \cup X_p} : \{y\}_Y \subset \{x\}_{X_1 \cup \dots \cup X_p} + \bar{V}^- \}$$

and the support of (86) is contained in

$$(87) \quad \{ \{y\}_Y, \{x\}_{X_1 \cup \dots \cup X_p} : \{y\}_Y \subset \{x\}_{X_1 \cup \dots \cup X_p} + \bar{V}^+ \}$$

This follows easily from the causality property (70), (71).

8.2. Existence of the weak adiabatic limit.

We have shown that when the coupling function $g(x)$ is in \mathcal{S} the various field operators $T(\hat{\mathcal{L}}_{i_1}(x_1)\hat{\mathcal{L}}_{i_2}(x_2) \dots \hat{\mathcal{L}}_{i_p}(x_p), g)$ exist as tempered operator valued distributions on the domain \mathcal{D}_0 and possess all the required pro-

erties in the sense of formal power series. The n th order expansion coefficient of such an operator is of the form

$$(88) \quad \hat{T}_n(\hat{X}; g) = \frac{i^n}{n!} \int \mathcal{L}(y_1) \downarrow \dots \downarrow \mathcal{L}(y_n) \downarrow T(\mathcal{L}_{i_1}(x_1) \dots \mathcal{L}_{i_p}(x_p)) \\ \times g(y_1) \dots g(y_n) dy_1 \dots dy_n \equiv \frac{i^n}{n!} \int Y \downarrow T(X)g(Y)dY.$$

We want to show that the « adiabatic limit » $g(y) \rightarrow \lambda = \text{const.}$ of the vacuum expectation value of (88) always exists in the case $m > 0$ in the sense of tempered distributions in the variables X . This limit possesses all the expected *linear* properties of a Green's function such as causality, Lorentz covariance, the spectral condition. In order to show this we must study also the adiabatic limit of the vacuum expectation values of the n th order expansion coefficient of a product of different T products:

$$(89) \quad [\hat{T}(X_1)\hat{T}(X_2) \dots \hat{T}(X_p)]_n = \frac{i^n}{n!} \int Y \downarrow \{ T(X_1) \dots T(X_p) \} g(Y)dY.$$

This will yield us in particular the existence of the Wightman functions in the case when all the X_i consist of single points, and the vacuum expectation values of the \hat{T} products and various retarded functions, which are all linear combinations of the monomials (89).

Let $O(X), |X| = v$, be a shorthand notation for such a product. The starting point will be the support and spectral properties of the pair of distributions

$$(90) \quad F^-(X, Y) = (\Omega, Y \downarrow O(X)\Omega) \equiv \langle Y \downarrow O(X) \rangle$$

and

$$F^+(X, Y) = \langle Y \uparrow O(X) \rangle.$$

They have, as we know from (87), the « retarded » resp. « advanced » support

$$(91) \quad \text{supp. } F^\pm \subset C_\pm = \{ (x, y) \in \mathbb{R}_{4(n+v)} : \{ y \}_Y \subset \{ x \}_X + \bar{V}_\pm \}.$$

More precisely:

$$(92) \quad C_\pm = \{ (x_1, \dots, x_v, y_1, \dots, y_n) \in \mathbb{R}_{4(n+v)} : y_i - x_{u(i)} \in \bar{V}_\pm, \\ i = 1, \dots, n \text{ for at least a mapping } u: \{ 1, \dots, n \} \rightarrow \{ 1, \dots, v \} \}.$$

The last definition shows that $C_+ = -C_-$ are two opposed closed cones pointed at the origin (generally non convex!).

On the other hand we have

$$(93) \quad F^+ - F^- = \langle Y \uparrow \{ T(X_1) \dots T(X_p) \} \rangle - \langle Y \downarrow \{ T(X_1) \dots T(X_p) \} \rangle \\ = - \sum_{\mathcal{Q}_{2p}(Y)} \langle [\check{T}(Y_1), T(Y_2, X_1)\check{T}(Y_3)T(Y_4, X_2) \dots \check{T}(Y_{2p-1})T(Y_{2p}, X_p)] \rangle.$$

Here the sum runs over all subdivisions of the set Y into $2p$ mutually disjoint subsets Y_1, \dots, Y_{2p} , and $\check{T}(I) = (-)^{|I|}T(I)$. In deriving this formula we have used Leibnitz's rule for differentiating a product of several factors and

$$(94) \quad \begin{aligned} Y_r \downarrow T(X_s) &= \sum_{I \subset Y_r} \check{T}(I)T(Y_r \setminus I, X_s), \\ Y_r \uparrow T(X_s) &= \sum_{I \subset Y_r} T(Y_r \setminus I, X_s)\check{T}(I). \end{aligned}$$

By « inserting intermediary states » between the factors of the commutators in (93) one gets readily the following support property of the Fourier transform of (93)

$$(95) \quad \left. \begin{aligned} \check{F}^+(p, q) - \check{F}^-(p, q) &= 0 \quad \text{in} \quad \mathcal{R} = \{ (p, q) \in \mathbb{R}_{4(n+v)} : \\ q^2_{\mathbf{1}} \equiv \left(\sum_{j \in \mathbf{1}} q_j \right)^2 &< 4m^2 \text{ and } \neq m^2 \quad \text{for all} \quad \mathbf{I} \subset \{ 1, \dots, n \} \} \end{aligned} \right\}.$$

Here the p_i are the conjugate momenta of x and q those of y :

$$(96) \quad \check{F}^\pm(p, q) = \int e^{i \sum_i p_i x_i + i \sum_i q_i y_i} F^\pm(X, Y) dX dY.$$

Note that the coincidence region \mathcal{R} is independent of p . The main result of this section will be:

THEOREM. — *If a pair of tempered distributions $F^\pm \in \mathcal{S}'(\mathbb{R}_{4(n+v)})$ have the support (91) and their Fourier transforms coincide in the region \mathcal{R} (95), then their Fourier transforms $\check{F}^\pm(p, q)$ are tempered distributions $\in \mathcal{S}'(\mathbb{R}_{4v})$ in p and infinitely differentiable in q for all $q \in \mathcal{R}$.*

The proof is an easy consequence of the following lemma first systematically used by K. Hepp:

LEMMA (Hepp [26]). — *If a tempered distribution $T \in \mathcal{S}'(\mathbb{R}_N)$ vanishes in an open cone C then for any test function $\varphi \in \mathcal{S}(\mathbb{R}_N)$ the convolution $(T * \varphi)(x)$ is a C_∞ function vanishing at infinity faster than any inverse power of $|x|$ in any closed cone $\Gamma \subset C$.*

The lemma follows from the fact that any distribution is of finite order and from the very definition of the support of a distribution: there is an integer $P \geq 0$ and a constant $M_\varepsilon < 0$ such that

$$|\langle T, \varphi \rangle| \leq M_\varepsilon \sum_{|\alpha| \leq P} \sup_{x \in C + K_\varepsilon} (1 + |x|)^P |D^\alpha \varphi(x)|$$

for all $\varphi \in \mathcal{S}(\mathbb{R}_N)$. Here \tilde{C} is the complement of C in \mathbb{R}_N and K_ε the sphere $|x| \leq \varepsilon$ with arbitrarily small radius $\varepsilon > 0$ (M depends in general on ε). If we insert the definition of a convolution $(T * \varphi)(x) = \langle T_y, \varphi(x - y) \rangle$ into the above inequality, we get from $|D^\alpha \varphi(x - y)| < C_R(1 + |x - y|)^{-R}$ for all $|\alpha| \leq P$ and R arbitrarily large

$$(97) \quad |(T * \varphi)(x)| < C'_R M_\varepsilon (1 + |x|d^{-1}(x, \tilde{C}_\varepsilon))^P (1 + d(x, \tilde{C}_\varepsilon))^{P-R}.$$

Here d is the Euclidean distance from the point x to $\tilde{C}_\varepsilon = \tilde{C} + K_\varepsilon$. When x is in $\Gamma = \bar{\Gamma} \subset C$ the first factor in (97) remains bounded by a constant C_Γ while the second tends to zero at infinity as $|x|^{P-R}$ for $R > P$. The same conclusion holds for any $D^\beta(T * \varphi)(x)$, i. e., for φ replaced by $D^\beta \varphi$ in the above argument. This proves Hepp's lemma.

In order to prove the theorem we multiply (96) with the test function

$$\tilde{\varphi}(p, q) = \tilde{\psi}(-p)\tilde{\alpha}(q)$$

where $\tilde{\psi}$ is any test function in $\mathcal{S}(\mathbb{R}_{4v})$ while $\tilde{\alpha} \in \mathcal{S}(\mathbb{R}_{4n})$ has its support in the coincidence region \mathcal{R} . For such a $\tilde{\varphi}$ we have then identically in $\mathbb{R}_{4(n+v)}$:

$$(98) \quad \tilde{F}^+ \tilde{\varphi} = \tilde{F}^- \tilde{\varphi},$$

or after the Fourier transformation

$$(99) \quad (F^+ * \varphi)(X, Y) = (F^- * \varphi)(X, Y) \equiv \Phi(X, Y).$$

From the general theory of convolution one knows that Φ is infinitely differentiable and of polynomial growth at infinity. What we assert is that $\Phi(0, y) \in \mathcal{S}(\mathbb{R}_{4n})$. Now by Hepp's lemma $\Phi(x, y)$ is of fast decrease at infinity in Γ_+ by the first part of (99); the same is also true in Γ_- by the second part of that formula, where Γ_\pm are any two closed cones contained in the open cones $\tilde{C}_\pm = \bigcup C_\pm$. What we have to show is that Γ_+ and Γ_- can be chosen so that $(\Gamma_+ \cup \Gamma_-) \cap L = L \setminus \{0\}$, where L is the linear subspace $\{x = 0\}$ of $\mathbb{R}_{4(n+v)}$ and $\{0\}$ is the origin. Now we evidently have

$$(\tilde{C}_+ \cup \tilde{C}_-) \cap L = L \setminus \{0\}$$

since

$$C_+ \cap C_- \cap L = \{x = 0, y_i \in \bar{V}_+, i = 1, \dots, n\} \cap \{x = 0, y_i \in V_-, i = 1, \dots, n\} = \{0\}.$$

Consider the unit sphere $S = \{|(x, y)| = 1\}$, where $|(x, y)|$ is the Euclidean distance in $\mathbb{R}_{4(n+v)}$. By what has just been said $\tilde{C}_+ \cap L \cap S, \tilde{C}_- \cap L \cap S$ is an open covering of the (closed) sphere $S \cap L$ in the relative topology of $S \cap L$. From it we can extract a closed covering E_+, E_- :

$$S \cap L = E_+ \cup E_-, E_\pm \subset \tilde{C}_\pm \cap L.$$

Since E_\pm are also closed considered as subsets of S in the induced topology on S , the closed cones $\Gamma_\pm = \{(x, y) \in \mathbb{R}_{4(n+v)} : (x, y) = rE_\pm, 0 \leq r < \infty\} \subset \tilde{C}_\pm$

will have the required properties. Therefore $\Phi(0, y) \in \mathcal{S}$. By taking the Fourier transform with respect to y we get upon insertion of the explicit form of φ :

$$\int \tilde{F}^+(p, q)\tilde{\psi}(p)dp\tilde{\alpha}(q) = \int \tilde{F}^-(p, q)\tilde{\psi}(p)dp\tilde{\alpha}(q) \in \mathcal{S}(\mathbb{R}_{4n}).$$

By taking $\tilde{\alpha}(q) = 1$ in any relatively compact open set $\omega \subset \mathcal{R}$ such that also $\bar{\omega} \subset \mathcal{R}$, we arrive at the assertion of the theorem:

$$(100) \quad \int \tilde{F}^+(p, q)\tilde{\psi}(p)dp = \int \tilde{F}^-(p, q)\tilde{\psi}(p)dp \in C_\infty(\mathcal{R}) \quad \text{for all } \tilde{\psi} \in \mathcal{S}(\mathbb{R}_{4v})$$

or equivalently

$$(101) \quad \int F_+(X, Y)\psi(X)e^{iqY}dXdY \\ = \int F^-(X, Y)\psi(X)e^{iqY}dXdY \in C_\infty(\mathcal{R}) \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}_{4v}).$$

REMARK 1. — In the above theorem we have not explicitly taken into account the translational invariance of F_\pm . This is easily remedied by writing

$$(102) \quad F^\pm(X, Y) = f_\pm(x_1 - x_v, \dots, x_{v-1} - x_v, y_1 - x_v, \dots, y_n - x_v)$$

and noticing that the cones C_\pm are translationally invariant. For the Fourier transform of f_\pm , i. e.

$$(103) \quad \tilde{F}_\pm(p, q) = \delta\left(\sum_1^v p_i + \sum_1^n q_i\right)\tilde{f}_\pm(p_1, \dots, p_{v-1}, q_1, \dots, q_n)$$

where the p_r are the conjugate momenta of $x_r - x_v$ ($r = 1, \dots, v - 1$) and q_s those of $y_s - x_v$ ($s = 1, \dots, n$), our theorem holds unchanged :

$$\tilde{f}_\pm \text{ are } C_\infty \quad \text{in } q \in \mathcal{R}$$

when smeared in the p 's with test functions in $\mathcal{S}(\mathbb{R}_{4(v-1)})$.

REMARK 2. — When tested with a test-function ψ of compact support in x space, the functions

$$\tilde{F}_\psi^\pm(q) = \int \tilde{F}_\pm(p, q)\tilde{\psi}(p)dp$$

are even analytic in a complex neighbourhood of the real points \mathcal{R} . This can be seen either by noticing that the $F_\psi^\pm(y)$ have then their support in asymptotically cone-like regions, which upon the Fourier transformation yield analyticity in the two tubes $\text{Im } k_i \in V_\pm$ ($i = 1, \dots, n$), $k_i = p_i + iq_i$, or by studying the expressions $\tilde{\alpha}_\beta(q)D^\beta \tilde{F}^\pm(q)$ in y space. By choosing conve-

niently the $\tilde{\alpha}_\beta(q)$ under the condition $\tilde{\alpha}_\beta(q_0) = 1$, $\text{supp. } \tilde{\alpha}_\beta \subset \mathcal{R}$, one can show that the derivatives $D^\beta F_\psi(q_0)$ satisfy Cauchy inequalities guaranteeing the convergence of the corresponding Taylor series. This method yields an alternative proof of the edge-of-the wedge theorem. The statement about analyticity is, however, in general *not* true when ψ does not have a compact support: a counter-example is easily produced by exploiting the presence of the δ function in the expression (103).

The existence of the adiabatic limit for the Green's functions follows now readily from the above theorem when $m > 0$. In that case the region \mathcal{R} contains a neighbourhood of the origin $q_i = 0$ ($i = 1, \dots, n$). Take any fixed $\psi(X) \in \mathcal{S}(\mathbb{R}_{4,v})$ and take for $g(Y)$ any sequence of test functions

$$g_m(Y) \in \mathcal{S}(\mathbb{R}_{4n})$$

such that

$$\text{supp. } \tilde{g}_m(q_1, \dots, q_n) \subset \mathcal{R} \quad \text{and} \quad \lim_{m \rightarrow \infty} \tilde{g}_m = \lambda^n \delta_4(q_1) \dots \delta_4(q_n)$$

in the sense of distributions. Then the limit

$$(104) \quad \lim_{m \rightarrow \infty} \int F^-(X, Y) \psi(X) g_m(Y) dX dY = \lambda^n \int \tilde{F}^+(p, 0) \tilde{\psi}(p) dp \equiv \lambda^n \langle \Phi_m, \psi \rangle,$$

exists and is independent of the choice of the sequence g_m . Moreover, by the weak completeness of tempered distributions, the limit is again a tempered distribution: $\Phi \in \mathcal{S}'(\mathbb{R}_{4,v})$, as suggested by the last formula ⁽³⁾. What remains to be checked is that Φ satisfies the following list of *linear* properties (as expected):

$$(Trinv) \quad \Phi_n(x + a) = \Phi_n(x)$$

for $a \in \mathbb{R}_4$.

This follows immediately from Remark 1 upon testing formula (103) with a $\tilde{\psi}(p) \in \mathcal{S}$ and taking $q = 0$;

$$(Linv) \quad \Phi_n(\Lambda^{-1}X) = \Phi_n(X)$$

for all $\Lambda \in \mathcal{L}_+^\dagger$.

By the previous construction we have

$$\tilde{F}_+(\Lambda^{-1}p, \Lambda^{-1}q) - \tilde{F}_+(p, q) = 0.$$

Testing this identity with $\tilde{\psi} \in \mathcal{S}$ and letting $q \rightarrow 0$ we get the above assertion since $F_\psi^-(q)$ is continuous at the origin.

In order to check *causality* we introduce the following notation [compare with (89)]:

⁽³⁾ It would be interesting to see what become our renormalisation norms on F- in the adiabatic limit. This, however, will not be attempted in the present paper.

$$(105) \quad \lambda^n \hat{t}_n(X_1, X_2, \dots, X_p) = \lim_{m \rightarrow \infty} \int \langle Y \downarrow \{T(X_1) \dots T(X_p)\} \rangle g_m(Y) dY \frac{i^n}{n!}, \quad |Y| = n$$

where $\bigcup_1^p X_r = X$ is a subdivision of X into any p disjoint sets. Now, we have

$$(106) \quad \langle Y \downarrow T(X) \rangle = \langle Y \downarrow \{T(X_1)T(X_2) \dots T(X_p)\} \rangle$$

if $\{x\}_{X_r} \succeq \{x\}_{X_s}$

for all $r < s$ and all y [compare with (78)]. Testing this formula with g_m and any $\psi(x)$ having its support in $\{x\}_{X_r} \succeq \{x\}_{X_s}$, $r > s$, and passing to the limit $m \rightarrow \infty$, we obtain

$$(Caus) \quad \hat{t}_n(X) = \hat{t}_n(X_1, X_2, \dots, X_p) \quad \text{if} \quad \{x\}_r \succeq \{x\}_s$$

for all $r < s$. In particular this yields the usual coincidence formula for the various permuted Wightman functions if the subsets X_r consist of single points.

In order to check the *spectral condition* we study the Fourier transform of

$$(107) \quad \langle Y \downarrow \{T(X_1) \dots T(X_p)\} \rangle = \sum_{\mathcal{D}_p(Y)} \langle Y_1 \downarrow T(X_1) \dots Y_p \downarrow T(X_p) \rangle$$

where the sum runs over all possible subdivisions of Y into p disjoint subsets [compare with (87)]. By « putting intermediary states » between the various terms $Y_r \downarrow T(X_r)$, we find that the Fourier transform

$$\tilde{F}_{Y_1, \dots, Y_p}(p, q) = \delta_4 \left(\sum_1^v p_i + \sum_1^n q_i \right) \tilde{f}_{Y_1, \dots, Y_p}(p, q)$$

of a generic term of the sum (107) has its support contained in

$$(108) \quad S_{Y_1, \dots, Y_p} = \left\{ (p, q) \in \mathbb{R}_{4(v+n)} : \sum_1^v p_i + \sum_1^n q_i = 0, \right. \\ \left. \sum_{i=1}^q (p_{X_i} + q_{Y_i}) \in \bar{V}_+(0, m, 2m), \quad q = 1, 2, \dots, p-1 \right\}.$$

Here

$$p_i = \sum_{i \in I} p_i, \quad q_i = \sum_{i \in I} q_i$$

and $\bar{V}_+(0, m, 2m)$ denotes the set in \mathbb{R}_4 :

$$\bar{V}_+(0, m, 2m) = \{p = 0\} \cup \{p_0 = +\sqrt{\bar{p}^2 + m^2}\} \cup \{p_0 \geq \sqrt{\bar{p}^2 + 4m^2}\}.$$

Therefore the Fourier transform $\tilde{F}(p, q)$ of the left-hand side of (107) has its support in

$$(109) \quad S = \bigcup_{\mathcal{D}_p(Y)} S_{Y_1, \dots, Y_p}.$$

It follows that the Fourier transform of $\hat{f}_n(X_1, \dots, X_p)$ has the support:

$$(Sp. I) \quad \text{supp. } \delta_4 \left(\sum_1^v p_i \right) \tilde{f}_{X_1, \dots, X_p}(p) = \left\{ p \in \mathbb{R}_{4v} : \sum_1^v p_i = 0, \right. \\ \left. \sum_1^q p_{X_i} \in \bar{V}_+(0, m, 2m), \quad q = 1, \dots, p - 1 \right\} \equiv \Sigma.$$

Indeed, take any $\tilde{\psi}(p) \in \mathcal{D}(\mathbb{R}_{4v})$ with its (compact!) support outside Σ . Then by compactness arguments we have $\tilde{F}_\psi^+(q) = 0$ for all q in a sufficiently small neighbourhood of $q = 0$. Hence $\langle \hat{f}(X_1, \dots, X_p), \psi \rangle = 0$ also.

REMARK 3. — Our proof shows also that the above adiabatic limit exists in the following traditional sense: take

$$g_\varepsilon(x) = \prod_{i=1}^n g(\varepsilon x_i),$$

where $g(x_i)$ is any function in $\mathcal{S}(\mathbb{R}_4)$ with $g(0) = \lambda$ and $\varepsilon > 0$. Then the limit $\varepsilon \rightarrow +0$ of the expression (104) with g_m replaced by g_ε exists and is independent of the particular choice of $g(x)$. In order to see this, observe that the Fourier transform of g_ε is of the form

$$\tilde{g}_\varepsilon(q) = \varepsilon^{-4n} \prod_1^n \tilde{g}(q_i/\varepsilon),$$

where $\tilde{g}(q_i)$ is the Fourier transform of $g(x_i)$, choose $\tilde{\alpha}(q) \in \mathcal{D}(\mathbb{R}_{4n})$ with its support in the coincidence region \mathcal{R} such that $\tilde{\alpha}(0) = 1$, and write

$$\tilde{g}_\varepsilon = \tilde{\alpha} \tilde{g}_\varepsilon + (1 - \tilde{\alpha}) \tilde{g}_\varepsilon.$$

Now

$$\lim_{\varepsilon \rightarrow +0} \tilde{\alpha} \tilde{g}_\varepsilon = \lambda^n \delta_{4n}(q)$$

in the sense of distributions in $\mathcal{D}'(\mathcal{R})$, while, as it is easy to check,

$$\lim_{\varepsilon \rightarrow +0} (1 - \tilde{\alpha}) \tilde{g}_\varepsilon = 0$$

in the sense of test functions in $\mathcal{S}(\mathbb{R}_{4n})$. This proves our assertion.

REMARK 4. — We warn the reader that the adiabatic limit does in general *not* exist separately for the individual terms in the decomposition (107).

For this to be true, it is necessary to perform first the mass and field renormalizations. Such questions we reserve, however, for a subsequent publication.

Thus we have shown that \hat{t}_n has the support properties of a vacuum expectation value of a product of p local operators operating on the Fock space. More precisely, \hat{t}_n behaves as if the equality

$$(110) \quad \hat{t}_n(X_1, \dots, X_p) = \sum_{n_1 + \dots + n_p = n} \langle \hat{T}_{n_1}(X_1) \dots \hat{T}_{n_p}(X_p) \rangle$$

were true, where $\hat{T}_m(X)$ is the m th order expansion coefficient of the time-ordered product of $|X|$ local fields after the adiabatic limit has been performed

$$\hat{T}(X) = \sum_{m=0}^{\infty} \lambda^m \hat{T}_m(X), \quad \hat{T}_m(X) = \frac{i^m}{m!} \int Y \downarrow T(X) dY, \quad |Y| = m.$$

In order to make complete the list of the linear properties of the Green's functions, the vacuum and the one-particle structure of the spectrum remain to be studied. Closely related to the vacuum structure is the question of the « vacuum renormalization », which arises when the adiabatic limit

$$(111) \quad \langle S\hat{T}(X) \rangle_n = \frac{i^n}{n!} \lim_{\epsilon \rightarrow +0} \int \langle T(X, Y) \rangle g_\epsilon(Y) dY, \quad |Y| = n$$

of « the off-mass shell matrix elements » of the S matrix is calculated. We discuss this problem first.

By Fourier-transforming the formula

$$(112) \quad \langle Y \downarrow T(X) \rangle = \sum_{I \subset Y} \langle \check{T}(I) T(Y \setminus I, X) \rangle$$

we find

$$(113) \quad \mathcal{F}(\langle Y \downarrow T(X) \rangle)(p, q) = \sum_{I \subset Y} \mathcal{F}(\langle \check{T}(I) \rangle) \langle T(Y \setminus I, X) \rangle(p, q)$$

if $q_1^2 < m^2$ for all $I \subset Y$. Here again p are the conjugate momenta of X and q those of Y . Indeed, by inserting intermediary states between the factors T and T in (112), it easily follows that only the vacuum state contributes for $q_1^2 < m^2$. But for the presence of the vacuum amplitudes

$$\langle \check{T}(I) \rangle, \quad I \subset Y,$$

the adiabatic limit of $\langle T(X, Y) \rangle$ would therefore be the same as that for

$$\langle Y \downarrow T(X) \rangle,$$

also the configuration space support properties. Thus the analog of (91) and (106), namely

$$(91') \text{ supp. } \langle Y \downarrow \{ T(X_1) \dots T(X_p) \} \rangle_c \subset \{ \{ y \}_Y \subset \{ x \}_X + \bar{V}_- \},$$

$$(106') \langle Y \downarrow T(X) \rangle_c = \langle Y \downarrow \{ T(X_1) \dots T(X_p) \} \rangle_c$$

if $\{ x \}_{X_r} \supseteq \{ x \}_{X_s}$ for all $r < s$,

remain valid, as it immediately follows from the corresponding properties of the respective generating functionals v . The usefulness of the truncated functions comes, however, from their modified support properties in momentum space: the truncation procedure, as it is well known, eliminates the contribution of the vacuum state as an intermediary state. For example, in contradistinction with (108), the Fourier transform of

$$\langle Y_1 \downarrow T(X_1) \dots Y_p \downarrow T(X_p) \rangle_c$$

has its support contained in

$$(108') \ S_{Y_1, \dots, Y_p}^c = \left\{ (p, q) \in \mathbb{R}_{4(v+n)} : \sum_{i=1}^v p_i + \sum_{i=1}^n q_i = 0, \right. \\ \left. \sum_{i=1}^2 (p_{X_i} + q_{Y_i}) \in V_+(m, 2m), \quad q = 1, \dots, p-1 \right\}$$

where $\bar{V}_+(m, 2m)$ is the set

$$\{ p_0 = +\sqrt{\bar{p}^2 + m^2} \} \cup \{ p_0 \geq \sqrt{\bar{p}^2 + 4m^2} \}$$

in \mathbb{R}_4 . The contribution of the vacuum state is missing. We also have the following well-known identity:

$$(119) \ \mathcal{F} \{ \langle Y_1 \downarrow T(X_1) \dots Y_p \downarrow T(X_p) \rangle_c \} (p, q) \\ = \mathcal{F} \{ \langle Y_1 \downarrow T(X_1) \dots Y_p \downarrow T(X_p) \rangle \} (p, q)$$

if $p_i + q_j \neq 0$ for all $I \subset X$ and $J \subset Y$ with $I \cup J \neq X \cup Y$ and \emptyset .

For the sake of completeness we give a proof of the last two statements. Let more generally

$$(120) \ F_r(f_r) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int F_r(X) f_r(x_1) \dots f_r(x_n) dX, \quad r = 1, \dots, p, \quad |X| = n,$$

be p operator-valued formal power series in the test functions $f_r \in \mathcal{S}$ with the coefficients $F_r(X)$ having the same linear properties as those of the S matrix (domain, L invariance, etc.) including $F_r(\emptyset) = 1$. Defining, as before,

$$(121) \ \begin{cases} v(f) = \langle F_1(f_1) \dots F_p(f_p) \rangle \equiv 1 + \rho(f) & \text{with } \rho(0) = 0, \\ \eta(f) = \ln v(f) = \ln(1 + \rho) = \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{q} \rho^q \end{cases}$$

we have

$$(122) \quad \langle F_1(X_1) \dots F_p(X_p) \rangle = X_1(f_1) \downarrow \dots \downarrow X_p(f_p) \downarrow v(f) |_{f=0}$$

and by definition

$$(123) \quad \langle F_1(X_1) \dots F_p(X_p) \rangle_c = X_1(f_1) \downarrow \dots \downarrow X_p(f_p) \downarrow \eta(f) |_{f=0} \\ = \sum_{q=1}^{|\mathbf{X}|} \frac{(-)^{q-1}}{q} \sum_{\mathcal{D}'_q(\mathbf{X}/\mathbf{l})} \prod_{s=1}^q \langle F_1(X_1 \cap I_s) \dots F_v(X_v \cap I_s) \dots F_p(X_p \cap I_s) \rangle.$$

The last formula is obtained by differentiating termwise the formal power series (121) with due regard of $\rho(0) = 0$. In (123) $X = X_1 \cup \dots \cup X_p$ and the sum \mathcal{D}'_q runs over all subdivision of X into q disjoint non-empty

sets $I_s : X = \bigcup_1^q I_s, I_r \cap I_s = \emptyset$ for $r \neq s, I_s \neq \emptyset, s = 1, \dots, q$. If p are the conjugate momenta of x we have the

LEMMA . — *The Fourier transform of (123) has its support contained in the set*

$$S' = \left\{ p \in \mathbb{R}(4 | \mathbf{X}) : \sum_{r=1}^v p_{X_r} \in \bar{V}_+(m, 2m), v = 1, \dots, p-1; \sum_1^p p_{X_r} = 0 \right\}$$

and we have

$$(124) \quad \mathcal{F} \{ \langle \dots \rangle_c \} (p) = \mathcal{F} \{ \langle \dots \rangle \} (p) \\ \text{if } \sum_{1 \leq r \leq v} p_{X_r} \neq 0 \quad \text{for } v = 1, \dots, p-1.$$

The proof follows directly from the Fourier analysis of the right-hand side of (123): by « inserting intermediary states » between all the pairs of operators in the product (123), one sees that the Fourier transform of a generic term $\Pi \langle \dots \rangle$ has its support contained in the set:

$$(125) \quad S_{\Pi} = \left\{ p \in \mathbb{R}_{4|\mathbf{X}|} : \sum_{s=1}^v p_{X_s \cap I_r} \in \bar{V}_+(0, m, 2m), v = 1, \dots, p-1, \right. \\ \left. r = 1, \dots, q; \sum_{s=1}^p p_{X_s \cap I_r} = 0, r = 1, \dots, q \right\}.$$

Since

$$(126) \quad \sum_{s=1}^v p_{X_s} = \sum_{r=1}^q \sum_{s=1}^v p_{X_s \cap I_r}$$

it follows that $S_{\Pi} \subset S$ [compare with (108)]. Now for a fixed $v < p$ the four-vector on the left-hand side of (126) can vanish only if each individual term of the r sum on the right-hand side vanishes, all those vectors being in \bar{V}_+ according to condition (125). Therefore a contribution to the Fourier transform of (123) corresponding to the value zero of the four-vector (126) with a fixed $v < p$ can come only from the terms

$$(127) \quad \mathcal{F} \left\{ \sum \frac{(-)^{q-1}}{q} \sum_{s=1}^q \prod_{s=1}^q \langle F_1(X_1 \cap I_s) \dots F_v(X_v \cap I_s) \rangle \right. \\ \left. \langle F_{v+1}(X_{v+1} \cap I_s) \dots F_p(X_p \cap I_s) \rangle \right\}$$

which is the Fourier transform of the sum (123) with the vacuum state inserted every-where between the v th and $(v + 1)$ first operator. But the sum (127) is the appropriate expansion coefficient of the formal power series

$$\ln \{ \langle F_1(f_1) \dots F_v(f_v) \rangle \langle F_{v+1}(f_{v+1}) \dots F_p(f_p) \rangle \}$$

and the coefficient in question vanishes because of the additivity of the logarithm. This proves the first assertion of our lemma. The second is an immediate consequence of the support property (125).

By choosing

$$F_r(f_r) = S^{-1}(g_r)S(g_r + h_r),$$

$f_r = (g_r, h_r)$ we get the statements (108') and (119).

From what has just been said about truncation, it is clear that the adiabatic limits

$$(105') \quad \lambda^n \hat{t}_n^c(X_1, \dots, X_p) = \frac{i^n}{n!} \lim_{m \rightarrow \infty} \int \langle Y \downarrow \{ T(X_1) \dots T(X_p) \} \rangle_c g_m(Y) dY$$

exist also and share all the properties of the corresponding untruncated distributions \hat{t}_n , except the momentum space support:

$$(Sp. Ia) \quad \text{sup. } \mathcal{F} \{ \hat{t}_n^c(X_1, \dots, X_p) \} (p)$$

$$= \left\{ p \in \mathbb{R}_{4v} : \sum_{r=1}^q p_{X_r} \in \bar{V}_+(m, 2m), q = 1, \dots, p-1, \sum_{i=1}^v p_i = 0 \right\}.$$

We are now ready to analyze the adiabatic limit (111) of the « off-mass shell » S matrix. We replace first in formula (112) everywhere the vacuum expectation values by the truncated ones and find

$$(113') \quad \mathcal{F} \langle Y \downarrow T(X) \rangle_c(p, q) \\ = \mathcal{F} \langle T(X, Y) \rangle_c(p, q) \quad \text{if } q_1^2 < m^2 \quad \text{for all } I \subset Y \text{ and } X \neq \emptyset.$$

This ensures that the « truncated part » of the limit (112) exists and is equal to

$$(128) \quad \langle S\hat{T}(X) \rangle_n = \tilde{t}_n^c(X) \quad \text{if} \quad X \neq \emptyset.$$

In order to show the untruncated analogue of (128) we express $\langle T(X, Y) \rangle$ in terms of truncated amplitudes. This can be achieved by remarking that

$$(129) \quad \langle T(Y, X) \rangle = Y(g) \downarrow X(h) \downarrow v(g+h) |_{g=h=0} \quad \text{with} \quad v(g+h) = \langle S(g+h) \rangle$$

and writing

$$v = e^\eta = \sum_{q=0}^{\infty} \frac{1}{q!} \eta^q \quad \text{with} \quad \eta(0) = 0.$$

Upon inserting the last power series into (129) we obtain

$$(130) \quad \langle T(Y, X) \rangle = \sum_{q=1}^{|Z|} \frac{1}{q!} \sum_{\mathcal{D}_q(Z/I)} \langle T(Z \cap I_1) \rangle_c \dots \langle T(Z \cap I_q) \rangle_c.$$

Here $Z = X \cup Y$ and the inner sum extends over all subdivisions of Z into q disjoint non-empty subsets I_1, \dots, I_q . But according to (128) every factor in this sum has an adiabatic limit provided $X \cap I \neq \emptyset$. Therefore $\langle T(Y, X) \rangle$ will have an adiabatic limit provided all $\langle T(I) \rangle_c$ with $I \subset Y$, $I \neq \emptyset$, have an adiabatic limit. Usually one requires even more:

$$(131) \quad \lim_{\epsilon \rightarrow 0} \int \langle T(Y) \rangle_{c, g_\epsilon} dY = 0 \quad \text{for all} \quad |Y| > 0.$$

Now

$$(132) \quad \mathcal{F}(\langle T(Y) \rangle_c)(q) = \mathcal{F}(\langle Y_1 \downarrow \mathcal{L}(y_n) \rangle)(q) \equiv \delta_4 \left(\sum_1^n q_i \right) \tilde{t}_n^c(q_1, \dots, q_{n-1})$$

if $q_i^2 < m^2$ for all $\emptyset \neq I \subset Y_1$ with $Y_1 = Y \setminus \{n\}$ and $|Y| = n > 0$

according to (113'). By Remark 1, accompanying formula (103) \tilde{t}_c is a C^∞ (and even an analytic) function of its arguments in a neighbourhood of the origin. Therefore the presence of the δ function in (132) forces \tilde{t}_n^c to have a zero of at least fifth order at $q = 0$ if we want $\langle T(Y) \rangle_c$ to satisfy the adiabatic condition (131), as it can be seen by an elementary calculation. By our argument this can always be achieved with four subtractions at the origin in each order of perturbation theory. Under this condition the adiabatic limit (111) will exist and, as the reader may verify, be equal to

$$(128') \quad \langle S\hat{T}(X) \rangle_n = \tilde{t}_n^c(X).$$

The existence of the limit (105') allows also to establish the correct vacuum structure — i. e., the correct « clustering properties » — of the Green's functions \hat{t}_n . By expressing $\langle Y \downarrow \{ T(X_1) \dots T(X_p) \} \rangle$ in terms

of the corresponding connected vacuum expectation values in analogy to formula (130), we easily obtain

$$(133) \quad \hat{t}_n(X_1, \dots, X_p) = \left[\sum_{q=1}^{|\mathcal{X}|} \frac{1}{q!} \sum_{\mathcal{O}'(X/I)} \prod_{r=1}^q \hat{t}^c(X_1 \cap I_r, \dots, X_p \cap I_r) \right]_n.$$

Here t^c is the formal power series in λ :

$$\hat{t}^c(J_1, \dots, J_p) = \sum_{n=1}^{\infty} \lambda^n \hat{t}_n^c(J_1, \dots, J_p)$$

and $[\]_n$ is the n th coefficient of the formal power series within the brackets. Since the \hat{t}^c have all their support in $\bar{V}_+(m, 2m)$ according to (Sp. I a), Equation (133) simply means that \hat{t}_n behaves indeed as if it were the vacuum expectation value (110) of a product of p operators as far the vacuum state is concerned (« non-degeneracy of the vacuum state »).

What remains to be studied is the correct one-particle structure of \hat{t}_n . If \hat{t}_n were indeed the vacuum expectation value (110):

$$(Sp. II) \quad \sup. \mathcal{F} \{ \hat{t}_n(X_1, \dots, X_p) \} (p)(p_{1q}^2 - m^2) \\ \subset \{ p \in \mathbb{R}_{4|\mathcal{X}|} : p_{1q} \in \bar{V}_+(0, 2m) \} \text{ for } q=1, \dots, p-1, \text{ with } p_{1q} = \sum_{r=1}^q p_{X_r}$$

should be valid. This simply means that the Klein-Gordon operator kills the contribution of the one-particle state as an intermediary state. $\bar{V}_+(0, 2m)$ is here obviously the set

$$\{ p = 0 \} \cup \{ p_0 \geq \sqrt{\vec{p}^2 + 4m^2} \}$$

in \mathbb{R}_4 . What one can conclude, however, from a Fourier analysis of the decomposition (107) is that the relation (Sp. II) is in general valid only if $(p_1^2 - m^2)$ is replaced in it by $(p_1^2 - m^2)^n$. This comes roughly speaking from the fact that n one-particle poles arising from different y 's will contribute to give a n th order pole $(p^2 - m^2)^{-n}$ in the limit $q = 0$. For the relation (Sp. II) to be true a necessary condition is the correct mass normalisation of the two point function: by writing

$$(134) \quad \mathcal{F} \{ \langle Y \downarrow A(x_1) \downarrow A(x_2) \rangle \} (p_1, p_2, q_1, \dots, q_n) \\ = \delta_4 \left(p_1 + p_2 + \sum_1^n q_i \right) \tilde{r}_n(p, q)$$

where $n = |Y|$ and \tilde{r}_n is defined only on the linear manifold

$$p_1 + p_2 + q_1 + \dots + q_n = 0,$$

the mass renormalization condition for the two-point function reads:

$$(135) \quad (p_1^2 - m^2)(p_2^2 - m^2)\tilde{r}_n(p_1, p_2, q_1, \dots, q_n) = 0$$

for $q = 0, p_1^2 = p_2^2 = m^2$ and for all $n \geq 1$.

By defining similarly \tilde{a}_n as the Fourier transform of $\langle Y \uparrow A(X_1) \uparrow A(X_2) \rangle$ we see that (135) is just a condition for fixing partially the ambiguity in our « splitting procedure » for $\tilde{d}_n = \tilde{a}_n - \tilde{r}_n$. Since

$$\tilde{d}_n = 0 \quad \text{for} \quad (p_1, q) \in \mathcal{R} = \{ (p_1, q) \in \mathbb{R}_{4(1+n)} : (p_1 + q)^2 < 4m^2$$

and $\neq m^2, q_1^2 < 4m^2$ and $\neq m^2$, for all $I \subset \{1, \dots, n\}$;

\tilde{r}_n and \tilde{a}_n coincide and are analytic in a complex neighbourhood of the real points \mathcal{R} according to the edge-of-the-wedge theorem. Therefore

$$\tilde{r}_n(p_1) = \tilde{r}_n(p, q)|_{q=0}$$

and similarly $\tilde{a}_n(p_1)$ are different boundary values of a function analytic in the cut plane $p_1^2 \neq 4m^2 + \rho, \rho \geq 0, p_1^2 \neq m^2$, which is the correct analyticity domain of the two-point function. Since \hat{r}_n is in addition Lorentz invariant (135) can be achieved by fixing a single subtraction constant in each order of perturbation theory as it can be shown by induction on n . Let us mention that in condition (135) \tilde{r}_n may be replaced by \tilde{r}_n^c , where \tilde{r}_n^c is the Fourier transform of

$$r_n^c = \langle Y \downarrow T(A(x_1)A(x_2)) \rangle_c$$

defined as in (134). This follows from the relation

$$\tilde{r}_n^c(p_1) \equiv \tilde{r}_n^c(p_1, q)|_{q=0} = \begin{cases} \tilde{r}_n(p_1) & \text{for } p_1 \notin \bar{V}_+(m, 2m), \\ \tilde{a}_n(p_1) & \text{for } p_1 \notin \bar{V}_-(m, 2m). \end{cases}$$

which is obtained by Fourier transformation of the identity

$$\langle Y \downarrow A(x_1) \downarrow A(x_2) \rangle = \langle Y \downarrow A(x_1) \downarrow A(x_2) \rangle_c = \langle Y \downarrow T(A(x_1)A(x_2)) \rangle_c$$

$$+ \sum_{Y_1 \cup Y_2 = Y} \langle \check{T}(Y_1, A(x_1))T(Y_2, A(x_2)) \rangle_c \quad \text{with} \quad \check{T}(I) = (-)^{|I|}\bar{T}(I)$$

and of a similar identity for $\langle Y \uparrow A(x_1) \uparrow A(x_2) \rangle$.

In order to prove that the two-point function mass renormalization entails the correct one-particle structure (Sp. II), it is necessary to introduce « one-particle irreducible » amplitudes $\langle Y \downarrow \{ T(X_1) \dots T(X_p) \} \rangle_i$ which have the same configuration-space causal structure as the corresponding truncated or untruncated vacuum expectation values but in whose momentum space support the contributions of the vacuum and the one-particle intermediary states have been suppressed. These new amplitudes can be best defined with the help of generating functionals which are a generaliza-

tion of the functional $\Gamma(\mathcal{A})$ discussed by Symanzik [27]. This procedure is similar in spirit to the truncation procedure just described but much more involved. We shall therefore omit its exposition here, especially in view of the fact that in a subsequent paper we intend to prove by a quite different method a much stronger result, namely the existence of the adiabatic limit in the strong operator sense on a suitably defined dense domain \mathcal{D}_1 of the Fock space. In order to achieve this, it will be necessary to perform also the wave function renormalization, which more precisely consists in the requirement that also all the first derivatives of the expression (135) vanish at $q = 0$ and $p_1^2 = p_2^2 = m^2$.

Let us remark here that the existence of the S matrix in the weak sense follows from a weaker requirement than (Sp. II), namely (Sp. II) only when $|I_q| = 1$. In that case the amplitudes

$$K_{x_1} \dots K_{x_v} \tilde{t}_n^c(x_1, \dots, x_p) \equiv \tau_n^c(x_1, \dots, x_p)$$

with $K_x = \square_x - m^2$ meet all the configuration and momentum space requirements necessary for the application of the Hepp theorems [26]: $\tilde{\tau}_n^c(p)$ defined by

$$\mathcal{F} \{ \tau_n^c(X) \} (p) = \delta_4 \left(\sum_1^v p_i \right) \tilde{\tau}_n^c(p), \quad v = |X|,$$

can be restricted to the mass shell $p_i^2 = m^2, i = 1, \dots, v$. More precisely:

$$\sigma_n(p_1, \dots, p_\mu; p_{\mu+1}, \dots, p_v) = \tilde{\tau}_n^c(p) |_{p_i = m^2, p_i \in \bar{V}_+(m) \text{ for } i = 1, \dots, \mu, p_i \in \bar{V}_-(m) \text{ for } i = \mu + 1, \dots, v}$$

is a tempered distribution on the mass shell manifold provided it is applied to test functions vanishing in a neighbourhood of any two parallel momenta. If the wave function renormalization has been also performed, $\sigma_n \delta_4(\sum p_i)$ is expected to be equal to the connected part of the S matrix element

$$\langle p_1, \dots, p_\mu | S_n | p_{\mu+1}, \dots, p_v \rangle.$$

However, the unitarity in the sense of perturbation theory of the S matrix defined in this manner can be guaranteed only if the existence of the adiabatic limit in the strong operator sense has been shown to exist ⁽⁴⁾.

We end this paper with the remark that the weaker form of the spectral condition (Sp. II) follows in a simple manner from the condition (135) once the analyticity properties of the function $t_n^c(X)$ in momentum space have been established. These are based on the support properties of the set of G. R. P. introduced in 8. 1. A proof along these lines has been given in [1] and will not be repeated here.

(4) A proof of this fact has been given in [28].

APPENDIX 1

PROOF OF THEOREM O

To prove this theorem (stated in Section 4) it is sufficient to prove the following statement.

Let \tilde{F} be a tempered distribution in $\mathcal{S}'(\mathbb{R}^{4(r-1)})$. Let I_1, \dots, I_r be a partition of $\{1, \dots, l'\}$ and let J_1, \dots, J_r be a partition of $\{1, \dots, s\}$. Let $\Phi \in \mathcal{S}'(\mathbb{R}^{4(l+l')})$ and $f \in \mathcal{S}'(\mathbb{R}^{4r})$. Then, for any sequence $\{\alpha_j\}_{1 \leq j \leq l'}$ of quadri-indices

$$\begin{aligned} \Psi(p_1, \dots, p_{s+l}) &= \int F(q_1 - p'_{l_1} + p_{j_1}, \dots, q_{r-1} - p'_{l_{r-1}} + p_{j_{r-1}}) \\ &\quad \times \delta\left(\sum_{j=1}^r q_j - \sum_{j=1}^{l'} p'_j + \sum_{j=1}^s p_j\right) f(q_1, \dots, q_r) d^4 q_1 \dots d^4 q_r \\ &\quad \times \Phi(p'_{l_1}, \dots, p'_{l_r}, p_{s+1}, \dots, p_{s+l}) \prod_{j=1}^{l'} \delta(p_j'^2 - m^2) \theta(p_j'^0) p_j'^{\alpha_j} d^4 p'_j \end{aligned}$$

is a \mathcal{C}^∞ function of p_1, \dots, p_{s+l} , which, when restricted to $\{p : p_j^2 = m^2, p_j^0 > 0, 1 \leq j \leq s+l\}$, is of fast decrease (obviously the same will then follow for all derivatives of Ψ). First, changing

from the variables q_1, \dots, q_r to the variables $q_1, \dots, q_{r-1}, Q = \sum_{k=1}^r q_k$ and denoting

$$\hat{f}(q_1, \dots, q_{r-1}, Q) = f\left(q_1, \dots, q_{r-1}, Q - \sum_{k=1}^{r-1} q_k\right),$$

we can rewrite this in the form

$$\Psi(p_1, \dots, p_{s+l}) = \int F(q_1, \dots, q_{r-1}) h(q; p) dq_1 \dots dq_r,$$

$$\begin{aligned} h(q_1, \dots, q_{r-1}; p_1, \dots, p_{s+l}) &= \int \hat{f}(q_1 - p_{j_1} + p'_{l_1}, \dots, q_{r-1} - p_{j_{r-1}} + p'_{l_{r-1}}, P' - P) \\ &\quad \times \Phi(p'_{l_1}, \dots, p'_{l_r}, p_{s+1}, \dots, p_{s+l}) \prod_{j=1}^{l'} \delta(p_j'^2 - m^2) \theta(p_j'^0) p_j'^{\alpha_j} d^4 p'_j \end{aligned}$$

where P stands for $\sum_{j=1}^s p_j$, $P' = \sum_{j=1}^{l'} p'_j$ (and, as usual, $p'_l = \sum_{j \in I_l} p'_j$, $p_j = \sum_{j \in J_l} p_j$).

Since $F \in \mathcal{S}'(\mathbb{R}^{4(r-1)})$ there is a constant $\{F\}$ and integers U and $V \geq 0$ such that

$$|\langle F, \psi \rangle| < \{F\} \sum_{|\beta| \leq U} \sup_q \left(1 + \sum_{j=1}^{r-1} \|q_j\|\right)^V |D^\beta \psi(q)|.$$

Let us assume that, for every β such that $|\beta| \leq U$, and every integer $M \geq 0$ there are constants $C_M \geq 0$ and $K_M \geq 0$ such that

$$|\Phi(p'_{l_1}, \dots, p'_{l_r}, p_{s+1}, \dots, p_{s+l})| \leq K_M \left(1 + \sum_{j=1}^{l'} \|p'_j\|\right)^{-M} \left(1 + \sum_{j=s+1}^{s+l} \|p_j\|\right)^{-M}$$

and

$$|D^{\beta} \hat{f}(q_1, \dots, q_{r-1}; Q)| \leq C_M \left(1 + \sum_{k=1}^{r-1} \|q_j\|\right)^{-V} (1 + \|Q\|)^{-M}.$$

Then

$$\begin{aligned} \sum_{|\beta| \leq U} \sup_q \left(1 + \sum_{j=1}^{r-1} \|q_j\|\right)^V |D_q^{\beta} h(p; q)| \\ \leq \text{Const. } C_N K_M \int \left(\prod_{k=1}^{l'} \frac{d\vec{p}'_k}{2\omega'_k}\right) \sup_q \frac{\left(1 + \sum_{j=1}^{r-1} \|q_j\|\right)^V}{\left(1 + \sum_{j=1}^{r-1} \|q_j - p_{j_j} + p'_{j_j}\|\right)^V} \\ \times (1 + \|P' - P\|)^{-N} \left(1 + \sum_{j=1}^{l'} \|p'_j\|\right)^{|\alpha| - M} \left(1 + \sum_{j=s+1}^{s+l} \|p_j\|\right)^{-M} \end{aligned}$$

where

$$\omega'_k = \sqrt{\vec{p}'_k{}^2 + m^2}, \quad |\alpha| = \sum_j |\alpha_j|.$$

Since

$$\begin{aligned} \left(1 + \sum_{j=1}^{r-1} \|q_j\|\right) \leq \left(1 + \sum_{j=1}^{r-1} \|q_j - p_{j_j} + p'_{j_j}\| + \sum_{k=1}^{l'} \|p'_k\| + \sum_{k=1}^s \|p_k\|\right) \\ \leq \left(1 + \sum_{j=1}^{r-1} \|q_j - p_{j_j} + p'_{j_j}\|\right) \left(1 + \sum_{k=1}^{l'} \|p'_k\|\right) \left(1 + \sum_{k=1}^s \|p_k\|\right), \end{aligned}$$

we have

$$\begin{aligned} |\Psi(p)| \leq \text{Const. } \{F\} C_N K_M \int \left(\prod_{k=1}^{l'} \frac{d\vec{p}'_k}{2\omega'_k}\right) \left(1 + \sum_{k=1}^{l'} \|p'_k\|\right)^{V + |\alpha| - M} \\ \left(1 + \sum_{k=1}^s \|p_k\|\right)^V (1 - \|P' - P\|)^{-N} \left(1 + \sum_{j=s+1}^{s+l} \|p_j\|\right)^{-M}. \end{aligned}$$

Now

$$(1 + \|P' - P\|)^N \left(1 + \sum_{j=1}^{l'} \|p'_j\|\right)^N \geq \left(1 + \sum_{j=1}^{l'} \|p'_j\| + \|P - P'\|\right)^N \geq (1 + \|P\|)^N.$$

Let us choose $M = N + |\alpha| + V + 2l' + 1$. Then

$$|\Psi(p)| \leq \text{Const. } C_N K_M \{F\} (1 + \|P\|)^{-N} \left(1 + \sum_{j=s+1}^{s+l} \|p_j\|\right)^{-M} \left(1 + \sum_{k=1}^s \|p_k\|\right)^V.$$

Hence, if the p_j are restricted to satisfy $p_j^2 = m^2, p_j^0 > 0$, we have

$$\|P\| \geq \sum_{j=1}^s p_j^0 \geq \frac{1}{2} \sum_{j=1}^s \|p_j\|$$

and our assertion follows. This estimate also shows the continuous dependence of Ψ on F, f and Φ .

APPENDIX 2

PROOF OF AMBIGUITY CHARACTERIZATION

Let $B(x_1, \dots, x_n)$ be an operator-valued tempered distribution with the following properties,

1. For every $f \in \mathcal{S}(\mathbb{R}^{4n})$, the quantity

$$B(f) = \int B(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1 \dots dx_n$$

is an operator defined on a dense subspace D_1 of \mathcal{F} and $B(f)D_1 \subset D_1$. Moreover, for every pair (Φ, Ψ) of vectors of D_1 ,

$$f \rightarrow (\Phi, B(f)\Psi)$$

is a tempered distribution. We assume that, for every $g \in \mathcal{S}(\mathbb{R}^{4p})$,

$$\int A(x_1) \dots A(x_p) g(x_1, \dots, x_p) dx_1 \dots dx_p$$

is defined on D_1 and maps D_1 into itself. D_1 is supposed to be Poincaré invariant and to contain the vacuum. It follows that D_1 contains D_0 .

2. B has the natural transformation law under translations

$$U(a, 1)B(x_1, \dots, x_n)U(a, 1)^{-1} = B(x_1 + a, \dots, x_n + a) \quad (\text{on } D_1).$$

3. B is local with respect to the free field:

$$[B(x_1, \dots, x_n), A(y)] = 0 \quad (\text{on } D_1)$$

(in the sense of tempered distributions) when $(x_j - y)^2 < 0, \forall j = 1, \dots, n$.

4. The support of $B(x_1, \dots, x_n)$ is concentrated on $\{x : x_j - x_n = 0 \text{ for all } j = 1, \dots, n-1\}$. This means: for every $\Phi \in D_1, B(f)\Phi = 0$ if f vanishes in a neighbourhood of the above set.

NOTATION :

$$A(x) = \int e^{ipx} \tilde{A}(p) d^4p,$$

$$\tilde{A}(p) = \delta(p^2 - m^2) \{ a^*(\vec{p})\theta(p^0) + a(-\vec{p})\theta(-p^0) \} \sqrt{2|p^0|},$$

$$[a(\vec{p}), a^*(\vec{p}')] = \delta_3(\vec{p} - \vec{p}'),$$

$$a^\#(p) = a(p) \quad \text{or} \quad a^*(p).$$

For every $f \in \mathcal{S}(\mathbb{R}^{3n}), \int a^\#(p_1) \dots a^\#(p_l) a^\#(p_{l+1}) \dots a^\#(p_n) f(p_1, \dots, p_n) dp_1 \dots dp_n$ is an operator defined on D_1 which maps D_1 into itself (it maps also D_0 into itself).

We now define inductively (on the integer l) two sequences B_l and C_l of operator-valued tempered distribution having the same properties as those postulated for B except that they are defined on D_0 and map D_0 into D_1 , such that:

$$\begin{aligned}
 & C_{l+1} = C_l - B_l, \\
 & C_0 = B, \quad B_0 = (\Omega, B\Omega), \\
 (A.2.1) \quad & \left\{ \begin{array}{l} (\Omega; A(y_1) \dots A(y_r); C_l(x_1, \dots, x_n); A(y_{r+1}) \dots A(y_p); \Omega) = 0 \\ \text{if } p < l, \end{array} \right. \\
 (A.2.2) \quad & \left\{ \begin{array}{l} (\Omega; A(y_1) \dots A(y_r); B_l(x_1, \dots, x_n); A(y_{r+1}) \dots A(y_p); \Omega) = 0 \\ \text{if } p < l; \text{ if } p = l, \text{ it is equal to:} \\ (\Omega; A(y_1) \dots A(y_r); C_l(x_1, \dots, x_n); A(y_{r+1}) \dots A(y_l); \Omega), \\ \text{and if } p > l, \text{ it is equal to:} \\ \sum_{\substack{I \subset \{1, \dots, n\} \\ J \subset \{r+1, \dots, p\} \\ |I| + |J| = l \\ r - |I| = p - r - |J| \geq 0}} \left(\Omega; \prod_{i \in I} A(y_i); C_l(x_1, \dots, x_n); \prod_{j \in J} A(y_j); \Omega \right) \\ \times \sum_{\substack{\tau = \text{mapping}(1 \text{ to } 1) \\ \tau: \{1, \dots, r\} \rightarrow \{r+1, \dots, p\} - J}} \prod_{\substack{1 \leq j \leq r \\ j \neq I}} (i)^{-1} \Delta^+(y_j - y_{\tau(j)}; m). \end{array} \right.
 \end{aligned}$$

Suppose C_0, \dots, C_l and B_0, \dots, B_{l-1} have already been constructed. Then we can denote for $r + s = l$,

$$\begin{aligned}
 C_l^{(r,s)}(p'_1, \dots, p'_r; x_1, \dots, x_n; p_1, \dots, p_s) &= a^*(p'_1) \dots a^*(p'_r) a(p_1) \dots a(p_s) \frac{1}{r! s!} \\
 &\times (\Omega, a(p'_1) \dots a(p'_r) C_l(x_1, \dots, x_n) a^*(p_1) \dots a^*(p_s) \Omega).
 \end{aligned}$$

The matrix element

$$(\Omega, a(p'_1) \dots a(p'_r) C_l(x_1, \dots, x_n) a^*(p_1) \dots a^*(p_s) \Omega)$$

is, by virtue of our assumptions and the nuclear theorem, a tempered distribution in $\mathcal{S}'(\mathbb{R}^{3r+3s+4n})$. Its Fourier transform with respect to the variables x_1, \dots, x_n (with conjugate variables q_1, \dots, q_n) is of the form

$$\delta(q_1 + \dots + q_n + p_1 + \dots + p_s - p'_1 - \dots - p'_r) F(\vec{p}_1, \dots, \vec{p}_s, \vec{p}'_1, \dots, \vec{p}'_r, q_1, \dots, q_{n-1})$$

where F is a tempered distribution in $\mathcal{S}'(\mathbb{R}^{4(n-1)+3l})$. From this it follows that

$$\int C_l^{(r,s)}(p'_1, \dots, p'_r; x_1, \dots, x_n; p_1, \dots, p_s) d\vec{p}'_1 \dots d\vec{p}'_r d\vec{p}_1 \dots d\vec{p}_s = C_l^{(r,s)}(x_1, \dots, x_n)$$

is a well-defined operator-valued tempered distribution over D_0 and maps D_0 into itself. The proof is the same as that of Theorem 0. We can now denote

$$B_l(x_1, \dots, x_n) = \sum_{\substack{r,s \\ r+s=l}} C_l^{(r,s)}(x_1, \dots, x_n).$$

B_l is an operator-valued tempered distribution defined on D_0 and mapping D_0 into itself with the property (A.2.2) and, as a consequence $C_{l+1} = C_l - B_l$ has the property (A.2.1). The only point we must verify is that, on D_0 ,

$$[B_l(x_1, \dots, x_n), A(y)] = 0$$

if $(x_i - y)^2 < 0$ for all $1 \leq i \leq n$. But this immediately follows from the induction hypothesis (i. e., that C_l has this locality property) and formulae (2).

We now denote

$$\begin{aligned} G_{l,r}(y_1, \dots, y_l, x_1, \dots, x_n) &= (\Omega, : A(y_1) \dots A(y_r) : B_l(x_1, \dots, x_n) : A(y_{r+1}) \dots A(y_l) : \Omega) \\ &= (\Omega, A(y_1) \dots A(y_r) B_l(x_1, \dots, x_n) A(y_{r+1}) \dots A(y_l) \Omega) \\ &= \int e^{i \sum_{j=1}^l p_j y_j + i \sum_{k=1}^n q_k x_k} \delta \left(\sum_j p_j + \sum_k q_k \right) \tilde{G}_{l,r}(\bar{p}_1, \dots, \bar{p}_l, q_1, \dots, q_{n-1}) \\ &\quad \times \prod_{j=1}^l \{ \theta(-p_j^0) \delta(p_j^2 - m^2) d^4 p_j \} \prod_{i=r+1}^l \{ \theta(p_i^0) \delta(p_i^2 - m^2) d^4 p_i \} d^4 q_1 \dots d^4 q_n. \end{aligned}$$

$\tilde{G}_{l,r}$ is a tempered distribution. We can also denote

$$K_l(y_1, \dots, y_l; x_1, \dots, x_n) = \sum_{r=0}^l (-1)^r \sum_{\substack{1 \leq i_1 < \dots < i_r \leq l \\ 1 \leq j_1 < \dots < j_{l-r} \leq l \\ i_k \neq j_t}} G_{l,r}(y_{i_1}, \dots, y_{i_r}, y_{j_1}, \dots, y_{j_{l-r}}, x_1, \dots, x_n).$$

If we take as independent variables $\eta_j = y_j - x_n$ ($1 \leq j \leq l$) and $\xi_k = x_k - x_n$ ($1 \leq k \leq n-1$), this tempered distribution satisfies $(\square_{\eta_j} + m^2) K_l = 0$ for every j ($1 \leq j \leq l$) and hence can be restricted to the manifold $\{ \eta, \xi : \eta_j^0 = 0 \text{ for all } j \}$ where its support is the origin. The same is true for all the time derivatives of K_l . Let us denote

$$\begin{aligned} K_l(y_1, \dots, y_l; x_1, \dots, x_n) &= \int e^{i \sum_{j=1}^l p_j (y_j - x_n) + i \sum_{k=1}^{n-1} q_k (x_k - x_n)} \\ &\quad \times \prod_{j=1}^l \delta(p_j^2 - m^2) d p_j \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_l) \\ \alpha_j = \pm 1}} \prod_{j=1}^l \varepsilon_j p_j^{\alpha_j} \theta(\varepsilon_j p_j^0) F_\alpha(\bar{p}_1, \dots, \bar{p}_l, q_1, \dots, q_{n-1}) d q_1 \dots d q_{n-1}. \end{aligned}$$

We find that: for every $\alpha = (\alpha_1, \dots, \alpha_l)$, $\alpha_j = 0$ or 1 ,

$$\sum_{\varepsilon} \prod_{j=1}^l (\varepsilon_j \omega(\bar{p}_j))^{\alpha_j} F_\alpha(\bar{p}, q)$$

must be a polynomial $P_\alpha(\bar{p}, q)$ in the variables $\bar{p}_1, \dots, \bar{p}_l, q_1, \dots, q_{n-1}$. It follows that on D_0 , $B_l(x_1, \dots, x_n)$ is of the form

$$\begin{aligned} B_l(x_1, \dots, x_n) &= \int d y_1 \dots d y_l P_l \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_l}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right) \\ &\quad \times : A(y_1) \dots A(y_l) : \delta(y_1 - x_n) \dots \delta(y_l - x_n) \delta(x_1 - x_n) \dots \delta(x_{n-1} - x_n). \end{aligned}$$

Moreover, for every pair Ψ, Φ of vectors of D_0 , and every f in $\mathcal{S}(\mathbb{R}^{4n})$,

$$(\Psi, B(f)\Phi) = \sum_{l=0}^{\infty} (\Psi, B_l(f)\Phi).$$

In particular,

$$B(f)\Omega = \sum_{l=0}^{\infty} B_l(f)\Omega$$

in the sense of the strong convergence, since $B_l(f)\Omega$ is precisely the l particle projection of $B(f)\Omega$: for if we denote

$$\Phi_N(f) = \sum_{l=0}^N B_l(f)$$

we have

$$B(f)\Omega = \Phi_N(f)\Omega + C_{N+1}(f)\Omega$$

and

$$(\Phi_N(f)\Omega, C_{N+1}(f)\Omega) = 0.$$

Hence,

$$\|B(f)\Omega\|^2 = \sum_{l=0}^{\infty} \|B_l(f)\Omega\|^2.$$

The map $f \rightarrow B_l(f)\Omega$ is continuous in the strong topology. For the map

$$(f, g) \rightarrow (B_l(f)\Omega, B_l(g)\Omega)$$

is separately continuous in f and g , hence simultaneously continuous by the nuclear theorem. It follows that $\|B_l(f)\Omega\|^2$ is continuous in f and since $B_l(f)\Omega$ is weakly continuous in f , the assertion follows.

From the argument given in [19], it follows that for every $f \in \mathcal{S}(\mathbb{R}^{4n})$ there is an $L(f)$ such that $l > L(f) \rightarrow B_l(f) = 0$ (on D_0).

Let F_N be the set of functions

$$F_N = \left\{ f \in \mathcal{S}(\mathbb{R}^{4n}) : (\Phi, B_l(f)\Psi) = 0 \quad \text{for all } l \geq N \right. \\ \left. \text{and all pairs } (\Phi, \Psi) \in D_0 \times D_0 \right\}.$$

F_N is a closed subspace of $\mathcal{S}(\mathbb{R}^{4n})$ (as an intersection of closed subspaces). Moreover, the sequence $\{F_N\}$ is an increasing one, whose union is the whole of $\mathcal{S}(\mathbb{R}^{4n})$. The latter is a Frechet space and has the Baire property. Hence, there is a certain N such that F_N has an interior point; therefore it is the whole space.

APPENDIX 3

THEORY OF POWER COUNTING
IN MOMENTUM SPACE

The splitting of $(\Omega, D(1, \dots, n; n + 1)\Omega)$ discussed in Sections 5 and 6 will now be studied in momentum space. The reason for this supplementary investigation is that it yields more detailed information about the behaviour of the solution.

NOTATION. — In this Appendix, if φ is a continuous function on \mathbb{R}^N , we denote, for any real μ ,

$$\|\varphi\|_\mu = \sup_{p \in \mathbb{R}^N} (1 + \|p\|)^\mu |\varphi(p)|.$$

A. 3.1. Definition of a degree of growth for tempered distributions.

DEFINITION 2. — Let T be a tempered distribution, $T \in \mathcal{S}'(\mathbb{R}^N)$. We say that T is of degree ω if there is an integer $l > 0$, and, for each $\varepsilon > 0$ a constant $K(\varepsilon)$ such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$|\langle T, \varphi \rangle| \leq K(\varepsilon) \sum_{\substack{\alpha \\ |\alpha| \leq l}} \|D^\alpha \varphi\|_{N+\omega+|\alpha|+\varepsilon}.$$

PROPERTIES OF THIS DEGREE

It may be positive or negative. A polynomial of degree ω in the usual sense is also of degree ω according to Definition 2. If T is of degree $\omega \geq -N$ and if $\varphi \in \mathcal{S}(\mathbb{R}^N)$, and if φ_r denotes

$$\varphi_r(p) = r^{-N} \varphi(r^{-1}p),$$

then, for $r \geq 1$,

$$|\langle T, \varphi_r \rangle| \leq K(\varepsilon) r^{\omega+\varepsilon} \sum_{\substack{\alpha \\ |\alpha| \leq l}} \|D^\alpha \varphi\|_{N+\omega+|\alpha|+\varepsilon}.$$

The two most important properties of the degree are:

(deg 1): Differentiation decreases the degree

For, if T is of degree ω , and $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned} |\langle D^\alpha T, \varphi \rangle| &= |\langle T, D^\alpha \varphi \rangle| \leq K(\varepsilon) \sum_{\substack{\beta \\ |\beta| \leq m}} \|D^{\alpha+\beta} \varphi\|_{N+\omega+\varepsilon+|\beta|} \\ &\leq K(\varepsilon) \sum_{\substack{\beta \\ |\beta| \leq m+|\alpha|}} \|D^\beta \varphi\|_{N+(\omega-|\alpha|)+|\beta|+\varepsilon}. \end{aligned}$$

(deg 2): Tensor product rule

Let $T \in \mathcal{S}'(\mathbb{R}^{N_1})$ and $S \in \mathcal{S}'(\mathbb{R}^{N_2})$ be of degrees ω_1 and ω_2 respectively, and suppose

$$\omega_1 + N_1 \geq 0 \quad \text{and} \quad \omega_2 + N_2 \geq 0.$$

Then $T \otimes S$ is of degree $\omega_1 + \omega_2$.

Proof:

$$\langle T \otimes S, \varphi \rangle = \langle T_p, \langle S_q, \varphi(p, q) \rangle \rangle,$$

hence

$$|\langle T \otimes S, \varphi \rangle| \leq K_1(\varepsilon) \sum_{|\alpha| \leq m_1} \sup_p (1 + \|p\|)^{N_1 + \omega_1 + |\alpha| + \varepsilon} \times |D_p^\alpha \langle S_q, \varphi(p, q) \rangle|$$

and

$$|D_p^\alpha \langle S_q, \varphi(p, q) \rangle| = |\langle S_q, D_p^\alpha \varphi(p, q) \rangle| \leq \sum_{|\beta| \leq m_2} K_2(\varepsilon) \sup_q (1 + \|q\|)^{N_2 + \omega_2 + |\beta| + \varepsilon} |D_q^\beta D_p^\alpha \varphi(p, q)|.$$

So that

$$|\langle T \otimes S, \varphi \rangle| \leq \sum_{|\alpha| \leq m_1 + m_2} K_1(\varepsilon) K_2(\varepsilon) \|D^\alpha \varphi\|_{N_1 + N_2 + \omega_1 + \omega_2 + |\alpha| + 2\varepsilon}$$

where we have used

$$(1 + \|p\|)^v \leq (1 + \|(p, q)\|)^v \quad \text{if } v \geq 0.$$

Another obvious property of the degree is that: if T is of degree ω , it is also of degree ω' for all $\omega' \geq \omega$.

EXAMPLES :

$$\delta_+(p^2 - m^2) = \delta(p^2 - m^2)\theta(p^0) \quad \text{and} \quad (p^2 - m^2)_{\mathbb{R}, A, F}^{-1}$$

are of degree -2 in Minkowski four-dimensional space-time.

A.3.2. Inductive estimate of the degree of Green distributions.

This estimate will be performed in two steps closely paralleling those of Sections 5 and 6. The first step is the momentum space version of subsection 6.1; it consists of showing that if the degrees of the Fourier transforms of the distributions $(\Omega, T_j(X)\Omega)$ are given by the formula (58) (Section 6), for $|X| \leq n - 1$, the same holds for the Fourier transform of $(\Omega, D(1, \dots, n - 1; n)\Omega)$. Let \tilde{F} and \tilde{G} be two tempered distributions in $\mathcal{S}'(\mathbb{R}^{4(q-1)})$, and $\mathcal{S}'(\mathbb{R}^{4(s-1)})$, respectively, having degrees ω_1 and ω_2 with $\omega_1 \geq -4(q-1)$, $\omega_2 \geq -4(s-1)$. Let

$$S(q_1, \dots, q_{q-1}, q'_1, \dots, q'_{s-1}, Q) = \int \tilde{F}(q_1 + p_1, \dots, q_{q-1} + p_{q-1}) \tilde{G}(q'_1 - p_1, \dots, q'_{s-1} - p_{s-1}) \times \delta\left(Q - \sum_{j=1}^l p_j\right) \prod_{j=1}^l p_j^{\gamma(j)} \delta(p_j^2 - m^2) \theta(p_j^0) d^4 p_j$$

where I_1, \dots, I_q are disjoint subsets of $\{1, \dots, l\}$ with $I_1 \cup \dots \cup I_q = \{1, \dots, l\}$, and similarly for J_1, \dots, J_s ; for each j , $\gamma(j)$ is a quadri-index; p_K denotes $\sum_{j \in K} p_j$. This expression is meant to be the Fourier transform of (60) of Section 6.1, and we wish to show that S [considered as an element of $\mathcal{S}'(\mathbb{R}^{4(q+s-1)})$] is of degree

$$\omega = \omega_1 + \omega_2 - 4 + \sum_{j=1}^l (|\gamma(j)| + 2)$$

[and therefore satisfies $\omega \geq -4(q + s - 1)$].

The cases $l = 0$ and $l = 1$ follow immediately from the tensor product rule (since the δ function in \mathbb{R}^N is of degree $-N$). In the case $l \geq 2$ we can write:

$$\int S(q, q', Q)\varphi(q, q', Q)dq_1 \dots dq_{q-1}dq'_1 \dots dq'_{s-1}dQ = \int \tilde{F}(q)\tilde{G}(q')\psi(q, q')dq_1 \dots dq'_{s-1}$$

where

$$\begin{aligned} \psi(q, q') &= \int \varphi(q_1 - p_1, \dots, q_{q-1} - p_{1_{q-1}}, q'_1 + p_{j_1}, \dots, q'_{s-1} + p_{j_{s-1}}, Q) \\ &\quad \times \delta\left(Q - \sum_{j=1}^l p_j\right) dQ \prod_{j=1}^l \{ \delta(p_j^2 - m^2)\theta(p_j^0)p_j^{(j)}d^4 p_j \}. \end{aligned}$$

Denoting

$$N_1 = 4(q - 1), \quad N_2 = 4(s - 1), \quad |\gamma| = \sum_{j=1}^l |\gamma(j)|,$$

and applying the proof of the tensor product rule, we find

$$\begin{aligned} | \langle S, \varphi \rangle | &= | \langle \tilde{F} \otimes \tilde{G}, \psi \rangle | \\ &\leq \text{Const.} \sum_{\substack{\alpha, \alpha' \\ |\alpha| + |\alpha'| \leq M}} \sup_{q, q'} (1 + \|q\|)^{N_1 + \omega_1 + |\alpha| + \varepsilon} (1 + \|q'\|)^{N_2 + \omega_2 + |\alpha'| + \varepsilon} \\ &\quad \times | D_q^\alpha D_{q'}^{\alpha'} \psi(q, q') | \\ &\leq \text{Const.} \sum_{\substack{\alpha \\ |\alpha| \leq M}} \sup_{q, q'} \int [1 + \|\hat{q}\| + \|\hat{q}'\| + \|Q\|]^{N_1 + N_2 + |\alpha| + \omega_1 + \omega_2 + 2l + |\gamma| + 3\varepsilon} \\ &\quad \times | D^\alpha \varphi(\hat{q}, \hat{q}', Q) | \frac{dQ}{[1 + \|Q\|]^{2l + |\gamma| + \varepsilon}} \rho_l(Q) \end{aligned}$$

(where $\hat{q} = \{q_1 - p_1, \dots, q_{q-1} - p_{1_{q-1}}\}$,
 $\hat{q}' = \{q'_1 + p_{j_1}, \dots, q'_{s-1} + p_{j_{s-1}}\}$)

$$\leq C(\varepsilon) \sum_{\substack{\alpha \\ |\alpha| \leq M}} \sup_{q, q', Q} [1 + \|q\| + \|q'\| + \|Q\|]^{(N_1 + N_2 + 4) + (\omega_1 + \omega_2 + 2l + |\gamma| - 4) + |\alpha| + 3\varepsilon} \times | D^\alpha \varphi(q, q', Q) |.$$

This is just what is needed since $N_1 + N_2 + 4 = 4(q + s - 1)$.

The second step is to show that the splitting operation can be performed so as to preserve the degree. We denote $C(x) = (\Omega, D(x_1, \dots, x_{n-1}; x_n)\Omega)$ (considered as a distribution in the variables $x_j - x_n$ $1 \leq j \leq n - 1$, \tilde{C} its Fourier transform [$\tilde{C} \in \mathcal{S}'(\mathbb{R}^{4(n-1)})$]). The splitting process will consist of first multiplying C by $x^\beta \omega$ then integrating $|\beta|$ times in momentum space. Let ω be the degree of \tilde{C} , supposed to be an integer satisfying $\omega \geq -N = -4(n - 1)$. We shall see that:

1. if $\omega \leq -1$, $\tilde{\omega} * \tilde{C}$ can be defined as a distribution of degree ω ; if $\omega \geq 0$, for $|\alpha| = \omega + 1$, it is possible to define $D^\alpha \tilde{\omega} * \tilde{C}$ as a tempered distribution of degree -1 ;
2. taking $|\alpha| = \omega + 1$ and integrating $\omega + 1$ times by means of fomula (56) yields a distribution of degree ω .

The first assertion follows from:

LEMMA A. 3.1. — Let $\tilde{\omega}$ be a distribution defined over \mathbb{R}^N by

$$\tilde{\omega}(p) = \int_0^\infty \frac{dr}{r^{N+1}} \psi\left(\frac{p}{r}\right)$$

with $\psi \in \mathcal{S}(\mathbb{R}^N)$; in other words

$$\langle \tilde{\omega}, \varphi \rangle = \int_0^\infty \frac{dr}{r^{N+1}} \int \psi\left(\frac{p}{r}\right) \varphi(p) dp = \int_0^\infty \frac{dr}{r} \int \psi(p) \varphi(rp) dp.$$

Here ψ is assumed to be of the form

$$\psi(p) = \sum_{\mu=1}^N \frac{\partial}{\partial p_{\mu}} \chi_{\mu}(p), \quad \chi_{\mu} \in \mathcal{S}(\mathbb{R}^N) \quad \text{for } 1 \leq \mu \leq N.$$

This condition is equivalent to $\tilde{\psi}(0) = 0$, $\tilde{\psi}$ being the Fourier transform of ψ .

Then, for every v with $0 \leq v \leq N - 1$, for every integer $|\alpha| \leq 0$, and every $\varepsilon > 0$ sufficiently small, there exists a constant $K(\varepsilon, |\alpha|, v)$ such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$(1 + \|p\|)^{v+|\alpha|+\varepsilon} |D^{\alpha} \tilde{\omega} * \varphi(p)| \leq \sum_{\substack{\beta \\ |\beta| \leq |\alpha|+1}} K(\varepsilon, |\alpha|, v) \sup_p (1 + \|p'\|)^{v+|\beta|+3\varepsilon} |D^{\beta} \varphi(p')|.$$

The proof of Lemma A.3.1 is given in Appendix 4.

If \tilde{C} is of degree $\omega \leq -1$ (with $\omega \geq -N$), applying Lemma 1 with $v = \omega + N$, we find that $\tilde{\omega} * \tilde{C}$ is well defined and of degree ω . If \tilde{C} is of degree $\omega \geq 0$, for $|\alpha| = \omega + 1$,

$$\begin{aligned} |\langle \tilde{C}, D^{\alpha} \tilde{\omega} * \varphi \rangle| &\leq \text{Const.} \sum_{\substack{\alpha' \\ |\alpha'| \leq M}} \sup_p (1 + \|p\|)^{N+\omega+|\alpha'|+\varepsilon} |D^{\alpha'+\alpha} \tilde{\omega} * \varphi(p)| \\ &\leq \text{Const.} \sum_{\substack{\beta \\ |\beta| \leq M+\omega+1}} \sup_p (1 + \|p\|)^{N-1+|\beta|+3\varepsilon} |D^{\beta} \varphi(p)|. \end{aligned}$$

Hence $D^{\alpha} \tilde{\omega} * \tilde{C}$ is well defined and of degree -1 .

The next step is to integrate $\omega + 1$ times, using the formula (56), i. e.,

$$\tilde{F}^+(p) = \sum_{|\alpha|=\omega+1} i^{|\alpha|} \int_0^1 dt (1-t)^{\omega} \frac{\omega+1}{\alpha!} (p-a)^{\alpha} D^{\alpha} \tilde{\omega} * \tilde{C}(a+t(p-a))$$

understood in the sense of distributions, a being a point where $D^{\alpha} \tilde{\omega} * \tilde{C}$ is \mathcal{C}^{∞} (actually analytic). We shall, in the sequel take a as the origin, and consider first the case of a family T^{β} of tempered distributions of degree -1 which vanish in $\{p : \|p\| < 3\}$. Here β runs through the set of multi-indices with $|\beta| = \omega + 1$. We define a tempered distribution T by

$$(A.3.1) \quad T(p) = \sum_{|\alpha|=\omega+1} i^{|\alpha|} \int_0^1 dt (1-t)^{\omega} \frac{\omega+1}{\alpha!} p^{\alpha} T^{\alpha}(tp)$$

which must be understood as

$$(A.3.2) \quad \langle T, \varphi \rangle = \sum_{|\alpha|=\omega+1} i^{|\alpha|} \int_0^1 dt (1-t)^{\omega} t^{-N} \int T^{\alpha}(p) \left(\frac{p}{t}\right)^{\alpha} \varphi\left(\frac{p}{t}\right) dp.$$

Let u be a \mathcal{C}^{∞} function on \mathbb{R}^N such that $0 \leq u \leq 1$, $u(p) = 0$ for $\|p\| \leq 1$, $u(p) = 1$ for $\|p\| \geq 2$ and let $\psi \in \mathcal{S}(\mathbb{R}^N)$. We have

$$(A.3.3) \quad D_p^{\beta} \int_0^1 dt (1-t)^{\omega} t^{-N} u(p) \psi\left(\frac{p}{t}\right) = \sum_{\substack{\gamma, \lambda \\ \gamma+\lambda=\beta}} \int_0^1 dt (1-t)^{\omega} t^{-N-|\gamma|} \frac{\beta!}{\gamma! \lambda!} D^{\lambda} u(p) (D^{\gamma} \psi)\left(\frac{p}{t}\right).$$

a. The term in (A.3.3) corresponding to $\lambda = 0$ is 0 for $\|p\| \leq 1$. For $\|p\| > 1$, if $\nu \leq N - 1$ we find

$$\begin{aligned} (1 + \|p\|)^{\nu+|\beta|} \left| \int_0^1 dt (1-t)^\omega t^{-N-|\beta|} u(p) (D^\beta \psi) \left(\frac{p}{t} \right) \right| \\ \leq \text{Const.} \int_0^1 \frac{dt}{t^{1-\varepsilon}} \left\| \frac{p}{t} \right\|^{N-1+\varepsilon+|\beta|} \left| (D^\beta \psi) \left(\frac{p}{t} \right) \right| \\ \leq \text{Const.} \varepsilon^{-1} \sup_p (1 + \|p\|)^{N-1+|\beta|+\varepsilon} |D^\beta \psi(p)|. \end{aligned}$$

b. If $\lambda > 0$, $\|p\| > 1$, $\nu \leq N - 1$, using the fact that $D^\lambda u(p)$ vanishes for $\|p\| > 2$, we find

$$\begin{aligned} (1 + \|p\|)^{\nu+|\beta|} \left| \int_0^1 dt (1-t)^\omega t^{-N-|\nu|} \frac{\beta!}{\gamma! \lambda!} (D^\gamma \psi) \left(\frac{p}{t} \right) D^\lambda u(p) \right| \\ \leq \text{Const.} \int_0^1 dt \cdot t^{-1+\varepsilon} \left\| \frac{p}{t} \right\|^{N-1+|\gamma|+\varepsilon} \left| (D^\gamma \psi) \left(\frac{p}{t} \right) \right| \\ \leq \text{Const.} \varepsilon^{-1} \sup_p (1 + \|p\|)^{N-1+|\nu|+\varepsilon} |D^\nu \psi(p)|. \end{aligned}$$

Hence, if $\nu \leq N - 1$, we have

$$\begin{aligned} (1 + \|p\|)^{\nu+|\beta|} \left| D_p^\beta \int_0^1 dt (1-t)^\omega t^{-N} u(p) \psi \left(\frac{p}{t} \right) \right| \\ \leq \text{Const.} \varepsilon^{-1} \sum_{|\gamma| \leq |\beta|} \sup_p (1 + \|p\|)^{N-1+|\nu|+\varepsilon} |D^\nu \psi(p)|. \end{aligned}$$

[For $\nu > N - 1$, it is easy to see that the same quantity is bounded by

$$\text{Const.} \sum_{\gamma \leq \beta} \sup_p (1 + \|p\|)^{\nu+|\gamma|} |D^\gamma \psi(p)|.]$$

Applying this to the case $\psi(p) = p^\alpha \varphi(p)$ we find that if the T^α are of degree -1 , the distribution T defined by (A.3.1) has degree ω .

This result remains true if, instead of vanishing in the ball $\{p : \|p\| < 3\}$ the T^β are only supposed to be \mathcal{C}^∞ there. To see this, we note that, replacing $T^\alpha(p)$ by $(1 - u(p))T^\alpha(p)$ in formula (A.3.2) yields a bounded \mathcal{C}^∞ function which (considered as a tempered distribution) is of degree 0.

To apply this result to the case $T^\alpha = D^\alpha \tilde{\omega} * \tilde{C}$, we need only prove that these distributions are \mathcal{C}^∞ where \tilde{C} vanishes. In the case of interest, these points are points of analyticity of the T^α by the edge-of-the-wedge theorem. But it is easy to obtain a direct proof by splitting $D^\alpha \tilde{\omega}$ into a \mathcal{C}^∞ function decreasing at infinity like $\|p\|^{-N-|\alpha|}$ (which, when convoluted with \tilde{C} , yields a \mathcal{C}^∞ function) plus a distribution of arbitrarily small compact support around the origin. The latter, when convoluted with \tilde{C} has a support arbitrarily close to the support of \tilde{C} .

A.3.4. Remarks.

Let C satisfy the assumptions of the beginning of Section 6 and suppose that its Fourier transform \tilde{C} is of degree $\omega \geq -N$ (in the sense of definition 2), i. e., that for each $\varepsilon > 0$ ($\varepsilon < 1$) and $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$|\langle \tilde{C}, \varphi \rangle| \leq K(\varepsilon) \sum_{|\alpha| \leq M} \sup_p (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} |D^\alpha \varphi(p)|.$$

Let $\rho \in \mathcal{D}(\mathbb{R}^N)$ have support in $\{x: \|x\| \leq 1\}$ and satisfy

$$\int \rho(x) dx = 1.$$

For $0 < \tau \leq 1$ we denote $\rho_\tau(x) = \tau^{-N} \rho(\tau^{-1}x)$. The Fourier transform $\tilde{\rho}_\tau$ of ρ_τ is

$$\tilde{\rho}_\tau(p) = \tilde{\rho}(\tau p), \quad \tilde{\rho} = \text{Fourier transform of } \rho.$$

For $\varphi \in \mathcal{S}$ we have

$$\begin{aligned} (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} |D_p^\alpha \{ (1 - \tilde{\rho}(\tau p))\varphi(p) \}| \\ = (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} \left| D_p^\alpha \int_0^\tau d\tau' \frac{\partial}{\partial \tau'} \tilde{\rho}(\tau' p) \varphi(p) \right| \\ = \left| \int_0^\tau d\tau' (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} \frac{\partial}{\partial \tau'} D_p^\alpha \{ \tilde{\rho}(\tau' p) \varphi(p) \} \right|. \end{aligned}$$

The integrand is bounded by

$$\text{Const.} \sum_{\gamma \leq \alpha} (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} \left| \frac{\partial}{\partial \tau'} \tau'^{|\alpha-\gamma|} [(D^{\alpha-\gamma} \tilde{\rho})(\tau' p)] D^\gamma \varphi(p) \right|.$$

The contribution of the term with $\gamma = \alpha$ is bounded by

$$\begin{aligned} \text{Const.} (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} |D^\alpha \varphi(p)| \sum_{j=1}^N |p_j (\partial_j \tilde{\rho})(\tau' p)| \\ \leq \text{Const.} (1 + \|p\|)^{\omega+N+|\alpha|+2\varepsilon} |D^\alpha \varphi(p)| \tau'^{\varepsilon-1} \sup_q \sum_{|\beta|=1} (1 + \|q\|)^{1-\varepsilon} |D^\beta \tilde{\rho}(q)| \\ \leq \text{Const.} \tau'^{\varepsilon-1} (1 + \|p\|)^{\omega+N+|\alpha|+2\varepsilon} |D^\alpha \varphi(p)|. \end{aligned}$$

The contribution of the terms with $|\alpha - \gamma| \geq 1$ is bounded by

$$\begin{aligned} \text{Const.} (1 + \|p\|)^{\omega+N+|\alpha|+\varepsilon} |D^\gamma \varphi(p)| \left[|\alpha - \gamma| \tau'^{|\alpha-\gamma|-1} |(D^{\alpha-\gamma} \tilde{\rho})(\tau' p)| \right. \\ \left. + \sum_{j=1}^N \tau'^{|\alpha-\gamma|} |(D^{\alpha-\gamma+\delta_j} \tilde{\rho})(\tau' p)| \cdot |p_j| \right]. \end{aligned}$$

Since $|\alpha - \gamma| - \varepsilon$ is positive, this is bounded by

$$\begin{aligned} \text{Const.} (1 + \|p\|)^{\omega+N+|\gamma|+2\varepsilon} |D^\gamma \varphi(p)| \\ \times \left[|\alpha - \gamma| \tau'^{|\alpha-\gamma|-1} (1 + \|p\|)^{|\alpha-\gamma|-\varepsilon} |(D^{\alpha-\gamma} \tilde{\rho})(\tau' p)| \right. \\ \left. + \tau'^{|\alpha-\gamma|} (1 + \|p\|)^{|\alpha-\gamma|+1-\varepsilon} \sum_{j=1}^N |(D^{\alpha-\gamma+\delta_j} \tilde{\rho})(\tau' p)| \right] \\ \leq \text{Const.} \tau'^{\varepsilon-1} (1 + \|p\|)^{\omega+N+|\gamma|+2\varepsilon} |D^\gamma \varphi(p)| \\ \times \sup_q \left[|\alpha - \gamma| (1 + \|q\|)^{|\alpha-\gamma|-\varepsilon} |D^{\alpha-\gamma} \tilde{\rho}(q)| + (1 + \|q\|)^{|\alpha-\gamma|+1-\varepsilon} \sum_{j=1}^N |D^{\alpha-\gamma+\delta_j} \tilde{\rho}(q)| \right] \end{aligned}$$

so that the integrand is bounded by

$$\text{Const.} \tau'^{\varepsilon-1} \sum_{\gamma \leq \alpha} (1 + \|p\|)^{\omega+N+|\gamma|+2\varepsilon} |D^\gamma \varphi(p)|.$$

Therefore

$$\begin{aligned}
 |\langle \tilde{C}\tilde{\rho}_\tau, \varphi \rangle - \langle \tilde{C}, \varphi \rangle| &= |\langle \tilde{C}, \tilde{\rho}_\tau\varphi - \varphi \rangle| \\
 &\leq \text{Const. } \tau^\epsilon \sum_{|\alpha| \leq M} \sup_p (1 + \|p\|)^{\omega + N + |\alpha| + 2\epsilon} |D^\alpha \varphi(p)|
 \end{aligned}$$

which shows that $\tilde{C}\tilde{\rho}_\tau$ tends to \tilde{C} in the sense of the above norm. It follows that, if ω verifies conditions (i) to (iv) of Section 5,

$$\langle \tilde{C}\tilde{\rho}_\tau, D^\beta \tilde{\omega} * \varphi \rangle - \langle \tilde{C}, D^\beta \tilde{\omega} * \varphi \rangle$$

tends to zero when $\tau \rightarrow 0$ for $|\beta| \geq \omega + 1$ and for any φ in \mathcal{S} . In x space, $\langle C * \rho_\tau, x^\beta \omega \varphi \rangle$ tends to $\langle C, x^\beta \omega \varphi \rangle$, as was announced in Section 6.

Moreover, we have

$$|C * \rho_\tau(x)| \leq K'(\eta)(1 + \|x\|)^M \tau^{-(N + \omega + \eta)}$$

for all $\tau > 0$, $\tau \leq 1$ and all η with $0 < \eta \leq 1$, as can be verified easily. Let χ be the characteristic function of a closed convex cone containing a neighbourhood of $\Gamma^+ - \{0\}$ and having only the point 0 in common with Γ^- . For any multi-index β with $|\beta| \geq \omega + 1$ and any $\varphi \in \mathcal{S}(\mathbb{R}^N)$, the function

$$x^\beta(\omega(x) - \chi(x))\varphi(x)C * \rho_\tau(x)$$

has its support in a compact set $B_\tau = \tau B_1$. Hence

$$\begin{aligned}
 \left| \int x^\beta(\omega(x) - \chi(x))\varphi(x)C * \rho_\tau(x) dx \right| &\leq K'(\eta) \int_{B_\tau} \|x\|^{|\beta|} \tau^{-(N + \omega + \eta)} (1 + \|x\|)^M |\varphi(x)| dx \\
 &\leq K''(\eta) \sup_y (1 + \|y\|)^M |\varphi(y)| \tau^{|\beta| - \omega - \eta} \int_{B_1} \|x\|^{|\beta|} dx
 \end{aligned}$$

which tends to zero as $\tau \rightarrow 0$. Hence

$$F^{\beta+} = \lim_{\tau \rightarrow 0} (C * \rho_\tau)\chi$$

in the sense of tempered distributions. In particular, χ may be chosen as the characteristic function of a closed half-space containing Γ^+ and having only 0 in common with Γ^- ; that means

$$\chi = \theta(n \cdot x), \quad n \in \tilde{\Gamma}^+.$$

One can also use a finite product of such functions. The above calculations show again that $F^{\beta+}$ does not depend on ω .

APPENDIX 4

PROOF OF LEMMA A.3.1

In this Appendix, $\tilde{\omega}$ denotes a distribution over \mathbb{R}^N given (formally) by

$$(A.4.1) \quad \tilde{\omega}(p) = \int_0^\infty dr \int r^{-(N+1)} \psi(r^{-1}p)$$

where

$$\psi(p) = \sum_{\mu=1}^N \frac{\partial}{\partial p_\mu} \chi_\mu(p)$$

and $\chi_\mu \in \mathcal{S}(\mathbb{R}^N)$ for $1 \leq \mu \leq N$. The precise meaning of (A.4.1) is

$$(A.4.2) \quad \langle \tilde{\omega}, \varphi \rangle = \int_0^\infty dr \int r^{-(N+1)} \psi(r^{-1}p) \varphi(p) dp.$$

φ being any function in $\mathcal{S}(\mathbb{R}^N)$. The existence of $\tilde{\omega}$ follows from the considerations of Section 6 and will be demonstrated again in the course of proving Lemma 1. In this Appendix, we denote, for every $f \in \mathcal{S}(\mathbb{R}^N)$ and any real number v :

$$\|f\|_v = \sup_{p \in \mathbb{R}^N} (1 + \|p\|)^v |f(p)|, \quad \left(\|p\|^2 = \sum_{\mu=1}^N |p_\mu|^2 \right).$$

We now propose to estimate, for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $0 \leq v \leq N - 1$, the quantity

$$(1 + \|p\|)^{v+|\alpha|+\varepsilon} |D^\alpha \tilde{\omega} * \varphi(p)|,$$

$$D^\alpha \tilde{\omega} * \varphi(p) = \int_0^\infty dr \int r^{-(N+1)} \psi(r^{-1}p') D^\alpha \varphi(p - p') dp'.$$

It will be shown that the integral over r is absolutely convergent. For this purpose we split the range of integration $[0, \infty]$ into the intervals $[1, \infty]$ and $[0, 1]$.

A.4.1. Contribution from $1 < r < \infty$.

In order to majorize

$$(1 + \|p\|)^{v+|\alpha|+\varepsilon} \int_1^\infty dr \cdot r^{-(N+1)} \left| \int \psi(r^{-1}p') D^\alpha \varphi(p - p') dp' \right|$$

$$= (1 + \|p\|)^{v+|\alpha|+\varepsilon} \int_1^\infty dr \cdot r^{-(N+1+|\alpha|)} \left| \int (D^\alpha \psi) \left(\frac{p - p'}{r} \right) \varphi(p') dp' \right|,$$

we first assume that $\|p\| \geq 1$ and write φ as a sum

$$\varphi(p') = u \left(\frac{p'}{\|p\|} \right) \varphi(p') + v \left(\frac{p'}{\|p\|} \right) \varphi(p').$$

Here $u \in \mathcal{D}(\mathbb{R}^N)$ is a function such that

$$0 \leq u \leq 1; \quad u(q) = 0 \quad \text{for} \quad \|q\| \geq \frac{1}{2}; \quad u(q) = 1 \quad \text{for} \quad \|q\| \leq \frac{1}{4};$$

$$v = 1 - u.$$

A.4.1 a. CONTRIBUTION OF THE TERM CONTAINING u

It is majorized by

$$\| 2p \|^{v+|a|+\varepsilon} \int_1^\infty dr \cdot r^{-(N+1+|a|)} \int_{2\|p'\| \leq \|p\|} \left| (D^\alpha \psi) \left(\frac{p-p'}{r} \right) \right| \cdot |\varphi(p')| dp'.$$

Taking into account

$$\begin{aligned} 2\|p'\| < \|p\| &\Rightarrow 2\|p\| \leq 2\|p-p'\| + 2\|p'\| \leq 2\|p-p'\| + \|p\| \\ &\Rightarrow \|p\| \leq 2\|p-p'\| \end{aligned}$$

it is majorized by

$$\begin{aligned} 4^{v+|a|+\varepsilon} \int_1^\infty dr \cdot r^{-(N+1+|a|)} \int_{2\|p'\| \leq \|p\|} dp' \|p-p'\|^{v+|a|+\varepsilon} \left| (D^\alpha \psi) \left(\frac{p-p'}{r} \right) \right| \cdot |\varphi(p')| \\ \leq 4^{v+|a|+\varepsilon} \int_1^\infty dr \cdot r^{-(1+\varepsilon)} \\ \times \int_{2\|p'\| \leq \|p\|} \|D^\alpha \psi\|_{N+|a|-\varepsilon} \|\varphi\|_{v+3\varepsilon} \|p-p'\|^{-(N-v-2\varepsilon)} \|p'\|^{-(v+3\varepsilon)} dp'. \end{aligned}$$

If $0 < \varepsilon < 1/6$, the integral

$$\int_{2\|p'\| \leq \|p\|} \|p-p'\|^{-(N-v-2\varepsilon)} \|p'\|^{-(v+3\varepsilon)} dp'$$

is convergent; it is a homogeneous function of p' of degree $-\varepsilon$ bounded by

$$\left\| \frac{p}{2} \right\|^{-(N-v-2\varepsilon)} \int_0^{\|p\|/2} N 2^N \rho^{N-1-v-3\varepsilon} d\rho \leq N 2^N (N-v-3\varepsilon)^{-1} \left\| \frac{p}{2} \right\|^{-\varepsilon}.$$

Hence the contribution of the term we consider is bounded by

$$2^{3N+2|a|} \varepsilon^{-1} \|D^\alpha \psi\|_{N+|a|-\varepsilon} \|\varphi\|_{v+3\varepsilon}.$$

A.4.1 b. CONTRIBUTION OF THE TERM CONTAINING v

By partial integrations, using

$$(D^\alpha \psi) \left(\frac{p-p'}{r} \right) = (-1)^{|a|} r^{|a|} D_p^\alpha \left(\psi \left(\frac{p-p'}{r} \right) \right)$$

we can shift the derivatives D^α to the product $v\varphi$. The term we consider is bounded by

$$\begin{aligned} \alpha! \| 2p \|^{v+|a|+\varepsilon} \sum_{\gamma \leq \alpha} \int_1^\infty dr \cdot r^{-(N+1)} \left| \int \psi \left(\frac{p-p'}{r} \right) \|p\|^{-|\alpha-\gamma|} (D^{\alpha-\gamma} v) \left(\frac{p'}{\|p\|} \right) D^\gamma \varphi(p') dp' \right| \\ \leq \alpha! 8^{v+|a|+\varepsilon} \sum_{\gamma \leq \alpha} \|D^{\alpha-\gamma} v\|_0 \int_1^\infty dr \cdot r^{-(N+1)} \\ \times \int_{\|p'\| \geq \|p\|/4} \left| \psi \left(\frac{p-p'}{r} \right) \right| \cdot \|p'\|^{v+|\gamma|+\varepsilon} |D^\gamma \varphi(p')| dp' \\ \leq \alpha! 8^{v+|a|+\varepsilon} \sum_{\gamma \leq \alpha} \|D^{\alpha-\gamma} v\|_0 \|\psi\|_{N-\varepsilon} \|D^\gamma \varphi\|_{v+|\gamma|+3\varepsilon} \\ \times \int_1^\infty dr \cdot r^{-(1+\varepsilon)} \int_{\|p'\| \geq \|p\|/4} \|p-p'\|^{-(N-\varepsilon)} \|p'\|^{-2\varepsilon} dp'. \end{aligned}$$

The last integral is again convergent and bounded by $\text{const.} \|p\|^{-\varepsilon}$.

To see this more precisely we may split the integration range into the two subsets:

$$\left\{ p' : \|p - p'\| \geq \left\| \frac{p'}{2} \right\| \geq \left\| \frac{p}{8} \right\| \right\} \quad \text{and} \quad \left\{ p' : \|p - p'\| < \left\| \frac{p'}{2} \right\| \geq \left\| \frac{p}{8} \right\| \right\}.$$

We have

$$\begin{aligned} \int_{2\|p-p'\| \geq \|p'\| \geq \|p/4\|} \|p - p'\|^{-(N-\varepsilon)} \|p'\|^{-2\varepsilon} dp' &\leq 2^{N-\varepsilon} \int_{\|p'\| \geq \|p/4\|} \|p'\|^{-(N+\varepsilon)} dp' \\ &\leq N 2^{2N-\varepsilon} \int_{\|p/4\|}^{\infty} \rho^{-(1+\varepsilon)} d\rho = N 2^{2N+2\varepsilon} \varepsilon^{-1} \|p\|^{-\varepsilon}. \end{aligned}$$

In the second subset,

$$\begin{aligned} \|p'\| &\leq \|p - p'\| + \|p\| \leq \left\| \frac{p'}{2} \right\| + \|p\|, \\ \|p'\| &\leq 2\|p\|, \quad \|p - p'\| \leq \|p\| \end{aligned}$$

so that the corresponding integral is bounded by

$$\left\| \frac{p}{4} \right\|^{-2\varepsilon} \int_{\|p-p'\| < \|p\|} \|p - p'\|^{-(N-\varepsilon)} dp' \leq \left\| \frac{p}{4} \right\|^{-2\varepsilon} N 2^N \int_0^{\|p\|} \rho^{-(1-\varepsilon)} d\rho = N 2^{N+4\varepsilon} \|p\|^{-\varepsilon} \varepsilon^{-1}.$$

Finally, the contribution from the term containing v is bounded by

$$\alpha! 2^{3\nu+3|a|+2N+1+6\varepsilon} N \varepsilon^{-2} \sum_{\gamma \leq \alpha} \|D^{\alpha-\gamma} v\|_0 \|\psi\|_{N-\varepsilon} \|D^\gamma \varphi\|_{\nu+|\gamma|+3\varepsilon}.$$

Let us now suppose that $\|p\| < 1$. We have

$$\begin{aligned} \int_1^\infty dr \cdot r^{-(N+1+|a|)} \int_{\|p'\| < 4} \left| (D^\alpha \psi) \left(\frac{p-p'}{r} \right) \right| \cdot |\varphi(p')| dp' &\leq N 8^N (N + |\alpha|)^{-1} \|D^\alpha \psi\|_0 \|\varphi\|_0; \\ \int_1^\infty dr \cdot r^{-(N+1+|a|)} \int_{\|p'\| > 4} \left| (D^\alpha \psi) \left(\frac{p-p'}{r} \right) \right| \cdot |\varphi(p')| dp' &\leq \int_1^\infty dr \cdot r^{-(1+\varepsilon+|a|)} \int_{\|p'\| > 4} \|D^\alpha \psi\|_{N-\varepsilon} \|\varphi\|_{2\varepsilon} \|p - p'\|^{-N+\varepsilon} \|p'\|^{-2\varepsilon} dp' \\ &\leq N 2^{2N+1+2\varepsilon} (\varepsilon + |\alpha|)^{-1} \varepsilon^{-1} \|D^\alpha \psi\|_{N-\varepsilon} \|\varphi\|_{2\varepsilon}. \end{aligned}$$

In conclusion, the term coming from the integration (in r) on the interval $[1, \infty]$ is bounded (for sufficiently small $\varepsilon > 0$) by

$$C(N, \alpha, \psi) \varepsilon^{-2} \sum_{\gamma \leq \alpha} \|D^\gamma \varphi\|_{\nu+|\gamma|+3\varepsilon}.$$

A.4.2. Contribution from $[0, 1]$.

[Note : if $a \geq b > 0$ and if $\rho > 0$ we have $(a + b)^\rho \leq (2a)^\rho \leq 2^\rho (a^\rho + b^\rho)$.]

We seek a bound for

$$\begin{aligned} \|p\|^{\nu+\varepsilon+|a|} \int_0^1 dr \cdot r^{-(N+1)} \left| \int \psi \left(\frac{p-p'}{r} \right) D^\alpha \varphi(p') dp' \right| &\leq \int_0^1 dr \cdot r^{-N} \int 2^{\nu+\varepsilon+|a|} [\|p - p'\|^{\nu+\varepsilon+|a|} + \|p'\|^{\nu+\varepsilon+|a|}] \\ &\quad \times \sum_{|\beta|=1} \left| \chi_\beta \left(\frac{p-p'}{r} \right) \right| \cdot \|D^{\alpha+\beta} \varphi(p')\| dp'. \end{aligned}$$

Performing the change of integration variable $p' = rp''$, we find that this expression is bounded by

$$\begin{aligned} & \sum_{|\beta|=1} 2^{\nu+\varepsilon+|\alpha|} \int_0^1 dr \cdot r^{\nu+\varepsilon+|\alpha|} \int \left\| \frac{p}{r} - p'' \right\|^{\nu+\varepsilon+|\alpha|} \left| \chi_\beta \left(\frac{p}{r} - p'' \right) \right| \times |(\mathbf{D}^{\alpha+\beta} \varphi)(rp'')| dp'' \\ & + \sum_{|\beta|=1} 2^{\nu+\varepsilon+|\alpha|} \int_0^1 dr \int \left| \chi_\beta \left(\frac{p}{r} - p'' \right) \right| \cdot \|rp''\|^{\nu+\varepsilon+|\alpha|} |(\mathbf{D}^{\alpha+\beta} \varphi)(rp'')| dp'' \\ & \leq \text{Const.} \sum_{|\beta|=1} \|\mathbf{D}^{\alpha+\beta} \varphi\|_{\nu+\varepsilon+|\alpha|}. \end{aligned}$$

A similar bound is easily obtained if $\|p\|^{\nu+\varepsilon+|\alpha|}$ is replaced by $(1 + \|p\|)^{\nu+\varepsilon+|\alpha|}$. Assembling the various bounds obtained above, we find that, for sufficiently small ε (e. g., $0 < \varepsilon < 1/6$):

$$\begin{aligned} & (1 + \|p\|)^{\nu+|\alpha|+\varepsilon} |\mathbf{D}^\alpha \tilde{\omega} * \varphi(p)| \\ & \leq C(N, \psi, |\alpha|) \varepsilon^{-2} \sum_{0 \leq |\gamma| \leq |\alpha|+1} \sup_{p' \in \mathbb{R}^N} (1 + \|p'\|)^{\nu+|\gamma|+3\varepsilon} |\mathbf{D}^\gamma \varphi(p')|. \end{aligned}$$

This proves Lemma 1.

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(Manuscrit reçu le 17 mai 1973).
