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**Time operators, position operators,
dilatation transformations and virtual particles
in relativistic
and nonrelativistic quantum mechanics**

by

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ABSTRACT. — We interpret the irreducible representations of the Weyl group (the group of inhomogeneous Lorentz transformations and dilatations in Minkowski space-time) as virtual (“ off-mass-shell ”) relativistic particles. There are two kinds of irreducible representations, corresponding to a particle with timelike or spacelike four-momentum. The irreducible representations are labelled by two invariant operators : the on-mass-shell mass, M , and $-W^2/P^2$, which, for a particle with timelike four-momentum is just $s(s+1)$ where s is the spin of the particle. The fact that the Weyl group Lie algebra contains one generator (the dilatation generator) more than that of the inhomogeneous Lorentz group allows us to construct a time operator as well as the usual four-momentum, angular momentum, and position operators. In fact we construct a hermitean four-position operator, R^μ , with the correct transformation properties under all the group transformations. For particles of non-zero spin, the different components of R^μ do not commute, and we explain why this must be so.

We also study the analogous situation in nonrelativistic quantum mechanics. Here the relevant group is the group of inhomogeneous Galilei transformations and dilatations $(t', \mathbf{x}') = (\lambda^2 t, \lambda \mathbf{x})$, which bear the same relation to the nonrelativistic Schrödinger equation as do the dilatations $x' = \lambda x$ to the Klein-Gordon equation. We interpret the irreducible representations of this group as describing virtual nonrelativistic

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tivistic particles. We find two kinds of irreducible representations, corresponding to Bargmann's ray representation of the Galilei group and Inönü and Wigner's Class II true representation of the Galilei group. The first kind is just a virtual massive nonrelativistic particle of mass m and spin s , and is the nonrelativistic limit of the timelike four-momentum irreducible representation of the Weyl group. The second kind is labelled by a helicity, λ , and is the nonrelativistic limit of a spacelike four-momentum irreducible representation of the Weyl group. For both these cases, we construct hermitean time and position operators which have all the properties we expect of them.

In both relativistic and nonrelativistic quantum mechanics, we discuss the system of two virtual particles. We also show the relation between the "dilatation change" in a scattering process and the Eisenbud-Bohm-Wigner time-delay.

RÉSUMÉ. — Nous interprétons les représentations irréductibles du groupe de Weyl (le groupe des transformations inhomogènes de Lorentz et dilatations dans l'espace-temps de Minkowski) comme particule relativistes virtuelles (« loin-la-couche-de-masse »). Deux genres des représentations irréductibles existent, qui correspondent pour une particule avec une quadri-quantité de mouvement de nature espace ou temps. Les représentations irréductibles sont définies par deux opérateurs invariants : la masse sur-la-couche-de-masse, M , et $-W^2/P^2$, qui est, pour une particule avec une quadri-quantité de mouvement de genre temps, $s(s+1)$, où s est le spin de la particule. Le fait que l'algèbre de Lie du groupe de Weyl contient un générateur de plus (le générateur des dilatations) que l'algèbre de Lie du groupe inhomogène de Lorentz nous permet de construire un opérateur de temps, ainsi que les opérateurs habituels de quadri-quantité de mouvement, spin, et position. En effet nous construisons un opérateur hermitien de quadri-position, R^μ , avec les propriétés correctes sous toutes les transformations du groupe. Pour les particules de spin non nul, les constituants différents ne commutent pas; et nous expliquons pourquoi c'est.

Nous étudions aussi la situation analogue dans la mécanique quantique nonrelativiste. Ici, le groupe qui intervient est le groupe des transformations inhomogènes de Galilée et les dilatations $(t, \mathbf{x}') = (\lambda^2 t, \lambda \mathbf{x})$, qui appartiennent à l'équation nonrelativiste de Schrödinger ainsi que les dilatations $x' = \lambda x$ pour l'équation de Klein-Gordon. Nous interprétons les représentations irréductibles de ce groupe comme particules nonrelativistes virtuelles. Nous trouvons deux types des représentations irréductibles, qui correspondent à la représentation rayon de Bargmann du groupe de Galilée et à la représentation vraie « classe II » de Inönü et Wigner du groupe de Galilée. Le premier genre est justement une par-

ticule nonrelativiste virtuelle massive de masse m et spin s , et c'est la limite nonrelativiste de la représentation irréductible avec une quadri-quantité de mouvement de genre temps du groupe de Weyl. Le deuxième type est repéré par une hélicité λ , et c'est la limite nonrelativiste d'une représentation irréductible avec une quadri-quantité de mouvement de genre espace du groupe de Weyl. Pour ces deux types, nous construisons des opérateurs hermitiens de temps et position, qui ont tous les propriétés désirées.

Dans la mécanique quantique relativiste et nonrelativiste, nous examinons le système des deux particules virtuelles. Nous montrons aussi le rapport entre le « changement de dilatation » dans un processus dispersif, et le délai de temps de Eisenbud-Bohm-Wigner.

I. INTRODUCTION

Ever since the work of Wigner [1], it has been generally accepted that the initial and final states in a scattering experiment consist of free particles (¹) which are described by unitary irreducible representations of the inhomogeneous Lorentz group. However, the situation is far from straightforward. Feynman ([3], Appendix B) has emphasised the point of view that all processes are, in fact, virtual, i. e. that the initial and final particles are off-mass-shell. Furthermore, Eden and Landshoff [4] have shown that a positive energy on-mass-shell particle (which is described by a unitary irreducible representation of the inhomogeneous Lorentz group) cannot be even approximately localised in time. This is somewhat disturbing. Also, even though the inhomogeneous Lorentz group allows us to construct a four-momentum, angular momentum, and position operator ([5]-[8], [2]), it does not allow us to construct a time operator. This difficulty is usually avoided by introducing the time co-ordinate t as a c -number parameter in the theory. However, this procedure is logically inconsistent; why should time be singled out in this way? The answer is, of course, that historically the Hamiltonian view of physics, which considers a system as continuously developing in time, was of immense importance in the early days of quantum mechanics, and that for Hamiltonian theories, parameter time is perfectly natural [even essential, as was shown by Pauli ([9], footnote, p. 60)]. However, since the work of Heisenberg [10], it has come to be realised that the S-matrix is the entity which is being studied, both experimentally and theoretically. In a scattering experiment, one makes measurements on the free particles which constitute the initial state a long time before the scattering, and on the free particles which constitute

(¹) By "particle", we really mean "elementary system" as defined by Newton and Wigner [2].

the final state a long time after the scattering. The energy of these free particles is on exactly the same footing as their momentum, spin, etc.; the Hamiltonian is just another observable. Clearly, in S-matrix theory, there is no justification for introducing time as a parameter; it should be introduced as an operator in the same way as other observables. Although the preceding discussion is for relativistic quantum mechanics, it is just as logically necessary to introduce a time operator for nonrelativistic quantum mechanics. In this paper, we shall do both.

In Section III of this paper, we interpret the unitary irreducible representations of the Weyl group, the group of inhomogeneous Lorentz transformations and dilatations in Minkowski space, as virtual ("off-mass-shell") relativistic particles. Previous authors, for example Kastrup [11], have rejected any straightforward interpretation of the dilatations in physical Hilbert space on the grounds that they lead to particles with continuous four-momentum squared. For virtual particles, this is precisely what we want. The irreducible representations fall naturally into two kinds: those with timelike four-momentum, $P^2 > 0$, and those with spacelike four-momentum, $P^2 < 0$. The irreducible representations with timelike four-momentum also have the sign of the energy as an invariant. Both kinds have $-W^2/P^2$, where $W^\mu = (1/2) \varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma$, as an invariant, and for the irreducible representations with timelike four-momentum, this invariant is just $s(s+1)$ where s is the spin of the particle. For all the irreducible representations, we construct a hermitean space-time position operator, R^μ , which has the correct transformation properties under homogeneous Lorentz transformations, space-time translations, and dilatations. Our eleven generators therefore give us eleven observables: the four-momentum, the four-position, and three angular momentum components of one kind or another. This is in accord with physical intuition. The on-mass-shell mass operator, M , is introduced in the same way as for the Galilei group i. e. because we are actually looking for true unitary representations of the central extension of the Weyl group by a one-dimensional abelian subgroup (see Appendix A), the main difference with the Galilei group being that the Weyl group has only (mathematically) trivial one-dimensional central extensions. The content of our work on the Weyl group is as follows: in Section III.1, we study the irreducible representations with timelike four-momentum and, in Section III.2, those with spacelike four-momentum. In Section III.3, we consider the system of two relativistic virtual particles and construct covariant centre-of-mass and relative operators. In Section III.4, we elucidate the connection between the "dilatation change" in a scattering process and the time-delay. In Section III.5, we suggest the existence of a

“supersuperselection rule” between states of timelike and spacelike four-momentum, this supersuperselection rule being the S-matrix-theoretic formulation of the microscopic causality of quantum field theory.

Before studying the Weyl group, however, we look at the analogous situation in nonrelativistic quantum mechanics, which is the content of Section II. Here the relevant group is the group of inhomogeneous Galilei transformations in space-time, together with the dilatation $(t', \mathbf{x}') = (\lambda^2 t, \lambda \mathbf{x})$, this dilatation bearing the same relation to the nonrelativistic Schrödinger equation ⁽²⁾, ⁽³⁾ :

$$(I.1) \quad \left(-\frac{1}{2m} \nabla^2 + \mathfrak{V} \right) \psi(\mathbf{x}, t) = i \frac{\partial \psi(\mathbf{x}, t)}{\partial t},$$

as does the relativistic dilatation, $x' = \lambda x$, to the Klein-Gordon equation

$$(I.2) \quad (\square + m^2 c^2) \varphi(x) = 0,$$

the “internal energy” term breaking the dilatation invariance in each case. We first of all derive the Lie algebra of our group from the matrices which generate the transformations in co-ordinate space-time. In Section II.1, we study the irreducible representations corresponding to Bargmann’s ([13], Section 6 *g*) ray representations of the Galilei group, and interpret these irreducible representations as virtual nonrelativistic particles of mass m and spin s . These irreducible representations which describe virtual particles, however, no longer have the internal energy, U , as an invariant operator, as do the corresponding representations of the Galilei group [12]. Our eleven parameter group gives the eleven observables (hermitean operators) : time, position, energy, momentum, and the three components of angular momentum. The mass is introduced exactly as for the Galilei group, and gives a superselection rule ([14], [12]) in the same way. In Section II.2, we study the irreducible representations corresponding to Inönü and Wigner’s [15] Class II true representations of the Galilei group, which have been interpreted by Lévy-Leblond [12] as zero-mass, infinite-speed particles. The corresponding irreducible representation of our group are labelled by one number, the helicity of the particle. We construct time and position operators for these particles but the different compo-

⁽²⁾ In this equation, \mathfrak{V} , the internal energy of the nonrelativistic particle, is an arbitrary real constant. The Schrödinger equation, (I.1), was studied by Lévy-Leblond [12] in his paper on the Galilei group. Physically, the arbitrariness of the internal energy of a nonrelativistic particle just means that we can choose freely the origin from which we count the energy.

⁽³⁾ Throughout the paper, we put \hbar equal to unity.

nents of the position operator do not commute. Section II.3 is devoted to a discussion of the system of two nonrelativistic virtual particles whilst, in Section II.4, we show the relationship between the "dilatation change" in a nonrelativistic scattering process and the time-delay.

The relevance of our work and possible directions for future research are discussed in the Conclusion, Section IV. Central extensions of Lie groups and Lie algebras, and their connection with ray representations, are reviewed in Appendix A, and the results of the general theory are then applied to the nonrelativistic dilatation group in Appendix A.I, and to the Weyl group in Appendix A.II. Appendix B is concerned with the evaluation of some integrals occurring in the relativistic Gartenhaus-Schwartz transformation, which is used to define the relative variables of the system of two relativistic virtual particles discussed in Section III.3.

II. THE GROUP OF INHOMOGENEOUS GALILEI TRANSFORMATIONS AND THE DILATATION $(t', \mathbf{x}') = (\lambda^2 t, \lambda \mathbf{x})$

The group with which we are concerned is an eleven parameter group consisting of the space rotations, pure Galilei transformations, displacements in space and time, and the nonrelativistic dilatations acting on co-ordinate space-time :

$$(II.1) \quad \begin{cases} \mathbf{x}' = \lambda R \mathbf{x} + \lambda^2 \mathbf{v} t + \mathbf{a}, \\ t' = \lambda^2 t + b \end{cases}$$

and a general element of the group will be denoted by

$$(II.2) \quad G = (\lambda, b, \mathbf{a}, \mathbf{v}, R),$$

where λ is a dilatation, b is a displacement in time, \mathbf{a} is a displacement in space, \mathbf{v} is a boost velocity, and R is a rotation matrix. The group multiplication law is

$$(II.3) \quad \begin{aligned} G' G &= (\lambda', b', \mathbf{a}', \mathbf{v}', R') (\lambda, b, \mathbf{a}, \mathbf{v}, R) \\ &= (\lambda' \lambda, \lambda'^2 b + b', \lambda' R' \mathbf{a} + \lambda'^2 \mathbf{v}' b \\ &\quad + \mathbf{a}', \lambda'^{-1} R' \mathbf{v} + \mathbf{v}', R' R) \end{aligned}$$

and the unit element, $\mathbf{1}$, is

$$(II.4) \quad \mathbf{1} = (1, 0, \mathbf{0}, \mathbf{0}, 1).$$

The inverse element, G^{-1} , is given by

$$(II.5) \quad G^{-1} = (\lambda^{-1}, -\lambda^{-2} b, -\lambda^{-1} R^{-1} (\mathbf{a} - b \mathbf{v}), -\lambda R^{-1} \mathbf{v}, R^{-1}).$$

To find the Lie algebra of the group, we define the five-dimensional co-ordinates y_i ($i = 0, 1, 2, 3, 4$) by ⁽⁴⁾ :

$$(II.6) \quad \begin{cases} y_0 = ty_4, \\ y_i = x_i y_4, \quad (i = 1, 2, 3), \end{cases}$$

where the co-ordinate y_4 "carries" the space-time translations in Equation (II.1) according to the formulae

$$(II.7) \quad \begin{cases} \mathbf{y}' = \lambda R \mathbf{y} + \lambda^2 \mathbf{v} y_0 + \mathbf{a} y_4, \\ y'_0 = \lambda^2 y_0 + b y_4, \\ y'_4 = y_4, \end{cases}$$

which we write as

$$(II.8) \quad \left\{ \begin{array}{l} y'_i = \sum_{j=0}^4 \mathcal{M}_{ij} y_j, \\ \mathcal{M} = e^{ib\mathcal{H}} e^{-ia\cdot\mathcal{X}} e^{-iv\cdot\mathcal{K}} e^{-i\varphi\cdot\mathcal{J}} e^{iz\mathcal{D}}, \end{array} \right.$$

where, $\lambda = e^{\alpha}$, $\varphi = \mathbf{n} \varphi$, where φ is the angle of rotation and \mathbf{n} a unit vector in the direction of the rotation, and where \mathcal{H} , \mathcal{X}_i , \mathcal{K}_i , \mathcal{J}_i , and \mathcal{D} are the 5×5 matrix generators of the "space-time representation" of the group. We evaluate these matrices in the usual way by restricting ourselves to the transformations which they generate and taking the derivative with respect to the group parameter at zero of that parameter, for example

$$(II.9) \quad \begin{aligned} \mathcal{D} &= -i \frac{d}{d\alpha} \left[\begin{array}{ccccc} e^{2\alpha} & 0 & 0 & 0 & 0 \\ 0 & e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & e^{\alpha} & 0 & 0 \\ 0 & 0 & 0 & e^{\alpha} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]_{\alpha=0} \\ &= \left[\begin{array}{ccccc} -2i & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right], \end{aligned}$$

⁽⁴⁾ Throughout Section II, we use the usual nonrelativistic convention

$$\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$$

for vectors.

also

$$(II.10) \quad \left\{ \begin{array}{l} \mathcal{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \mathcal{X}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{K}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \mathcal{J}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{array} \right.$$

etc., which are, of course, well known [13]. We evaluate the commutators and find

$$(II.11) \quad \left\{ \begin{array}{ll} [\mathcal{J}_i, \mathcal{J}_j] = i \varepsilon_{ijk} \mathcal{J}_k, & [\mathcal{J}_i, \mathcal{X}_j] = i \varepsilon_{ijk} \mathcal{X}_k, \\ [\mathcal{J}_i, \mathcal{K}_j] = i \varepsilon_{ijk} \mathcal{K}_k, & [\mathcal{K}_i, \mathcal{X}_j] = -i \mathcal{X}_i, \\ [\mathcal{K}_i, \mathcal{X}_i] = 0, & [\mathcal{K}_i, \mathcal{K}_j] = 0, \\ [\mathcal{J}_i, \mathcal{X}_j] = [\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{X}_i, \mathcal{X}_j] = 0, & \\ [\mathcal{O}, \mathcal{J}_i] = 0, & [\mathcal{O}, \mathcal{K}_i] = i \mathcal{K}_i, \\ [\mathcal{O}, \mathcal{X}_i] = -i \mathcal{X}_i, & [\mathcal{O}, \mathcal{X}_j] = -2i \mathcal{X}_j, \end{array} \right.$$

which is the group Lie algebra. Our task now is to find the unitary irreducible representations and physical interpretation of the group and this we shall do in Sections II.1 and II.2. We also note the parity and time-reversal matrices, I_s and I_t ,

$$(II.12) \quad I_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_t = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and their commutation and anticommutation relations with the generators :

$$(II.13) \quad \left\{ \begin{array}{ll} [I_s, \mathcal{X}_j] = 0, & \{I_t, \mathcal{X}_j\} = 0, \\ \{I_s, \mathcal{X}_i\} = 0, & [I_t, \mathcal{X}_i] = 0, \\ \{I_s, \mathcal{K}_i\} = 0, & \{I_t, \mathcal{K}_i\} = 0, \\ [I_s, \mathcal{J}_i] = 0, & [I_t, \mathcal{J}_i] = 0, \\ [I_s, \mathcal{O}] = 0, & [I_t, \mathcal{O}] = 0, \end{array} \right.$$

where curly brackets denote the anticommutator.

We also write down four operator identities involving commutators and anticommutators which will be used quite often in the rest of the paper :

$$(II.14 a) \quad \{ B, \{ C, A \} \} = \{ C, \{ B, A \} \} + [[B, C], A],$$

$$(II.14 b) \quad [A, \{ B, C \}] = \{ [A, B], C \} + \{ [A, C], B \},$$

$$(II.14 c) \quad [A, \{ B, C \}] = [\{ A, B \}, C] + [\{ A, C \}, B],$$

$$(II.14 d) \quad [[A, B], C] = [A, [B, C]] + [B, [C, A]],$$

the last one being, of course, the well-known Jacobi identity.

1. Irreducible representations corresponding to Bargmann's ray representations of the Galilei group

We now search for unitary ray representations of our group in Hilbert space i. e. unitary operators $U(G)$ which satisfy

$$(II.15) \quad U(G') U(G) = e^{i\xi(G', G)} U(G'G),$$

where $\xi(G', G)$ is a real function of G' and G , where G' , G , and $G'G$ are defined by Equation (II.3). Bargmann [13] showed that, for the rotation group and the homogeneous and inhomogeneous Lorentz groups, the phase factor in Equation (II.15) can be reduced to a sign factor of ± 1 and the unitary ray representations of these groups can be obtained from the true unitary representations of their universal covering groups, obtained by replacing the rotations by $SU(2)$ in each case. However, Bargmann also showed that, in the case of the inhomogeneous Galilei group ^(*), the phase factor in Equation (II-15) is non-trivial and the true unitary representations [15] of the universal covering group do not give all the unitary ray representations of the Galilei group. Furthermore, these nontrivial representations, which have been studied in detail by Lévy-Leblond [12], are the ones which describe physical non-relativistic particles. They can be found by looking at the true unitary representations of another group, the local group, which is a central extension of the universal covering group by a one-parameter group ([13], [12]) (see also Appendix A). In Appendix A.I, we show that the Lie algebra, Equations (II.11) has a non-trivial extension which is, in fact, the same as that which occurs for the Galilei group, and Section II.1.A is devoted to a discussion of this extended Lie algebra and its physical interpretation. In Section II.1.B, we calculate the exponent $\xi(G', G)$ which occurs in Equation (II.15) and, in Section II.1.C,

^(*) The Galilei group has recently been reviewed in detail by Lévy-Leblond [16].

we discuss the transformation properties of the physical single particle states under $U(G)$, i. e. we explicitly evaluate the unitary irreducible representations.

A. THE EXTENDED LIE ALGEBRA AND ITS PHYSICAL INTERPRETATION

In Appendix A.I, we show that the Lie algebra, Equations (II.11) has a non-trivial central extension of the form

$$(II.16) \quad \left\{ \begin{array}{ll} [J_i, J_j] = i \varepsilon_{ijk} J_k, & [J_i, P_j] = i \varepsilon_{ijk} P_k, \\ [J_i, K_j] = i \varepsilon_{ijk} K_k, & [K_i, H] = -i P_i, \\ [K_i, P_j] = -i M \delta_{ij}, & [K_i, K_j] = 0, \\ & [J_i, H] = [P_i, P_j] = [P_i, H] = 0, \\ [D, J_i] = 0, & [D, K_i] = i K_i, \\ [D, P_i] = -i P_i, & [D, H] = -2 i H, \\ [J_i, M] = [K_i, M] = [P_i, M] = [H, M] = [D, M] = 0. \end{array} \right.$$

In other words, the hermitean Hilbert space generators of the local group consist of the operators \mathbf{J} , \mathbf{K} , \mathbf{P} , H , and D which generate the transformations, Equations (II.1), in Hilbert space, together with the mass operator, M . This algebra has as its invariants the two operators

$$(II.17) \quad \left(\mathbf{J} + \frac{\mathbf{K} \times \mathbf{P}}{M} \right)^2 = \mathbf{S}^2, \quad M,$$

which means that a physical representation of our group is labelled by the intrinsic spin, $\mathbf{S}^2 = s(s+1)$, and the mass, M . In the case of the Galilei group, the extended Lie algebra is given by Equations (II.16) with the Lie brackets involving D removed ([17], [12]), and the internal energy $U = (H - \mathbf{P}^2/2M)$ is also an invariant operator. In our case, this is not so since

$$(II.18) \quad [D, U] = -2 i U.$$

Note, though, that sign (U) is still an invariant. However, since U is arbitrary, we shall neglect sign (U) in the labelling of the irreducible representations.

We now define the time operator, T , and position operator, \mathbf{R} , by the equations

$$(II.19) \quad \left\{ \begin{array}{l} T = \frac{1}{4} \left\{ \frac{1}{U}, D \right\} - \frac{\{K_i, P_i\}}{4 M U}, \\ R_i = \frac{T P_i - K_i}{M} = \frac{1}{4 M} \left\{ \frac{P_i}{U}, D \right\} - \frac{\{K_j, P_j P_i\}}{4 M^2 U} - \frac{K_i}{M} \end{array} \right.$$

and these hermitean operators satisfy the following commutation relations with the generators and with one another [as can be seen by using Equations (II.16) and (II.14)] :

$$(II.20) \quad \left\{ \begin{array}{ll} [T, H] = -i 1, & [R_i, H] = 0, \\ [T, P_i] = 0, & [R_i, P_j] = i \delta_{ij} 1, \\ [K_i, T] = 0, & [K_i, R_j] = -i \delta_{ij} T, \\ [J_i, T] = 0, & [J_i, R_j] = i \varepsilon_{ijk} R_k, \\ [D, T] = 2 i T, & [D, R_i] = i R_i, \\ [M, T] = 0, & [M, R_i] = 0, \\ [T, R_i] = 0, & [R_i, R_j] = 0, \end{array} \right.$$

which are precisely what we should expect of time and position operators in nonrelativistic quantum mechanics. Furthermore, if we have a unitary parity operator, \mathcal{P} , and an antiunitary time-reversal operator, \mathcal{T} , in Hilbert space, then Equations (II.13) become

$$(II.21) \quad \left\{ \begin{array}{ll} [\mathcal{P}, H] = 0, & [\mathcal{T}, H] = 0, \\ \{\mathcal{P}, \mathbf{P}\} = 0, & \{\mathcal{T}, \mathbf{P}\} = 0, \\ \{\mathcal{P}, \mathbf{K}\} = 0, & [\mathcal{T}, \mathbf{K}] = 0, \\ [\mathcal{P}, \mathbf{J}] = 0, & \{\mathcal{T}, \mathbf{J}\} = 0, \\ [\mathcal{P}, D] = 0, & \{\mathcal{T}, D\} = 0, \end{array} \right.$$

Furthermore, by applying \mathcal{P} and \mathcal{T} to the equation $[K_i, P_j] = -i M \delta_{ij}$, and using the fact that they are unitary and antiunitary respectively, we find

$$(II.22) \quad [\mathcal{P}, M] = 0, \quad [\mathcal{T}, M] = 0$$

and by using Equations (II.21) and (II.22) and the definitions of T and \mathbf{R} , Equations (II.19), we find :

$$(II.23) \quad \left\{ \begin{array}{ll} [\mathcal{P}, T] = 0, & \{\mathcal{T}, T\} = 0, \\ \{\mathcal{P}, \mathbf{R}\} = 0, & [\mathcal{T}, \mathbf{R}] = 0, \end{array} \right.$$

as we should expect.

The spin \mathbf{S} [the total angular momentum in the rest frame ($\mathbf{P} = 0$)] is defined by

$$(II.24) \quad \mathbf{S} = \mathbf{J} + \frac{\mathbf{K} \times \mathbf{P}}{M}.$$

It commutes with M , H , \mathbf{P} , T and \mathbf{R} , and satisfies $[S_i, S_j] = i \varepsilon_{ijk} S_k$.

We can express \mathbf{J} , \mathbf{K} , and D in terms of the twelve operators M , H , \mathbf{P} , T , \mathbf{R} , and \mathbf{S} by the formulae

$$(II.25) \quad \left\{ \begin{array}{l} \mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{S}, \\ \mathbf{K} = T \mathbf{P} - M \mathbf{R}, \\ D = \{ T, H \} - \frac{1}{2} \{ R_i, P_i \}. \end{array} \right.$$

We also note that if we restrict ourselves to the proper, orthochronous group, then the operators D , T , and \mathbf{R} are not unique, for the redefinitions

$$(II.26) \quad \left\{ \begin{array}{l} D \rightarrow D + 2a, \\ T \rightarrow T + \frac{a}{U}, \\ \mathbf{R} \rightarrow \mathbf{R} + \frac{a \mathbf{P}}{MU}, \end{array} \right.$$

where a is a constant, leave the commutation relations, Equations (II.20), unchanged. If however, we want the new operators to transform in the same way under parity and/or time-reversal, Equations (II.21) and (II.23), then this puts constraints on the constant a .

It is now clear that these irreducible representations describe virtual massive nonrelativistic particles; the internal energy is no longer an invariant, and the irreducible representations are labelled by the intrinsic spin and mass. Furthermore, the time and position operators allow us to construct states localised in space and time, (*see* Section II.1.C) which are clearly virtual particles.

One interesting consequence of Equations (II.20) is the equation

$$(II.27) \quad \left[T, \frac{\mathbf{P}^2}{2M} \right] = 0,$$

which tells us that the time and kinetic energy of a virtual nonrelativistic particle can be measured together with arbitrary precision. We also emphasise that our operators, T and \mathbf{R} , have no connection whatsoever with the space-time co-ordinates, t and \mathbf{x} , whose only function is to give us the space-time representation, Equations (II.9) and (II.10), from which we obtained the group Lie algebra, Equations (II.11).

The commutators of M , H , \mathbf{P} , T , and \mathbf{R} with one another, given in Equations (II.20), together with the properties of \mathbf{S} listed after Equations

tions (II.24), allow us to write the following canonical form for these operators :

$$(II.28) \quad \left\{ \begin{array}{l} \mathbf{M} \rightarrow m, \\ \mathbf{H} \rightarrow \mathbf{E} = \frac{\mathbf{P}^2}{2m} + \varpi, \\ \mathbf{P} \rightarrow \mathbf{p}, \\ \mathbf{T} \rightarrow -i \left(\frac{\partial}{\partial \mathbf{E}} \right)_{\mathbf{p}} = -i \left(\frac{\partial}{\partial \varpi} \right)_{\mathbf{p}}, \\ \mathbf{R} \rightarrow i \left(\frac{\partial}{\partial \mathbf{p}} \right)_{\mathbf{E}} = i \left(\frac{\partial}{\partial \mathbf{p}} \right)_{\varpi} - i \frac{\mathbf{p}}{m} \left(\frac{\partial}{\partial \mathbf{E}} \right)_{\mathbf{p}}, \\ \mathbf{S} \rightarrow \mathbf{s}, \end{array} \right.$$

where m is the particle mass, ϖ is the internal energy, and $\mathbf{s}^2 = s(s + 1)$ where s is the particle spin. The $(2s + 1)$ dimensional representation, \mathbf{s} , of the rotation group generators satisfies $[s_i, s_j] = i \varepsilon_{ijk} s_k$, and the derivatives in Equations (II.28) are taken at constant \mathbf{s} . By equations (II.25) and (II.28), the canonical forms for \mathbf{J} , \mathbf{K} , and \mathbf{D} are

$$(II.29) \quad \left\{ \begin{array}{l} \mathbf{J} \rightarrow -i \mathbf{p} \times \left(\frac{\partial}{\partial \mathbf{p}} \right)_{\mathbf{E}} + \mathbf{s}, \\ \mathbf{K} \rightarrow -i \mathbf{p} \left(\frac{\partial}{\partial \mathbf{E}} \right)_{\mathbf{p}} - im \left(\frac{\partial}{\partial \mathbf{p}} \right)_{\mathbf{E}} = -im \left(\frac{\partial}{\partial \mathbf{p}} \right)_{\varpi}, \\ \mathbf{D} \rightarrow -i \left(2\mathbf{E} \left(\frac{\partial}{\partial \mathbf{E}} \right)_{\mathbf{p}} + \mathbf{p}' \left(\frac{\partial}{\partial \mathbf{p}} \right)_{\mathbf{E}} + \frac{5}{2} \right) \end{array} \right.$$

and it is clear that these, together with the canonical forms for \mathbf{M} , \mathbf{H} , and \mathbf{P} given in Equations (II.28), satisfy the extended Lie algebra (II.16).

B. CALCULATION OF THE EXPONENT

To calculate the exponent, $\xi(G', G)$, occurring in Equation (II.15), we shall use the algebraic method of Lévy-Leblond ([16], Section III.C). We *define* the unitary operator $U(G)$, corresponding to the group element $G = (\lambda, b, \mathbf{a}, \mathbf{v}, \mathbf{R})$ by the equation

$$(II.30) \quad U(G) = e^{ib\mathbf{H}} e^{-i\mathbf{a}\cdot\mathbf{P}} e^{-i\mathbf{v}\cdot\mathbf{K}} e^{-i\boldsymbol{\varphi}\cdot\mathbf{J}} e^{i\alpha\mathbf{D}},$$

where \mathbf{H} , \mathbf{P} , \mathbf{K} , \mathbf{J} , and \mathbf{D} are the hermitean Hilbert space generators which satisfy the extended Lie algebra, Equations (II.16). We evaluate the product

$$(II.31) \quad U(G') U(G) = e^{ib'\mathbf{H}} e^{-i\mathbf{a}'\cdot\mathbf{P}} e^{-i\mathbf{v}'\cdot\mathbf{K}} e^{-i\boldsymbol{\varphi}'\cdot\mathbf{J}} e^{i\alpha'\mathbf{D}} e^{ib\mathbf{H}} \\ \times e^{-i\mathbf{a}\cdot\mathbf{P}} e^{-i\mathbf{v}\cdot\mathbf{K}} e^{-i\boldsymbol{\varphi}\cdot\mathbf{J}} e^{i\alpha\mathbf{D}},$$

by repeated use of the operator identity

$$(II.32) \quad e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \dots$$

and the commutators of the extended Lie algebra, Equations (II.16). For example, the first step in the evaluation is to take $e^{i\alpha'D}$ through and we find

$$(II.33) \quad U(G') U(G) = e^{ib'H} e^{-ia'P} e^{-iv'K} e^{-i\Phi'J} e^{i\lambda'^2 bH} \\ \times e^{-i\lambda'a'P} e^{-i\lambda'^{-1}v'K} e^{-i\Phi'J} e^{i(\alpha+\alpha')D}.$$

Proceeding in this way, we eventually obtain the result

$$(II.34) \quad U(G') U(G) = e^{iM(1/2)\lambda'^2 b\mathbf{v}'^2 + \lambda'\mathbf{v}'\cdot R'\mathbf{a}} e^{i(b' + \lambda'^2 b)H} \\ \times e^{-i(\mathbf{a}' + \lambda'^2 b\mathbf{v}' + \lambda'R'\mathbf{a})\cdot\mathbf{P}} e^{-i(\mathbf{v}' + \lambda'^{-1}R'\mathbf{v})\cdot\mathbf{K}} \\ \times e^{-i\Phi'J} e^{-i\Phi'J} e^{i(\alpha+\alpha')D},$$

which, on comparison with Equations (II.15) and (II.3), tells us that the exponent for a particle of mass m is given by

$$(II.35) \quad \xi_m(G', G) = m \left(\frac{1}{2} \lambda'^2 b \mathbf{v}'^2 + \lambda' \mathbf{v}' \cdot R' \mathbf{a} \right)$$

and on putting $\lambda' = 1$ in Equation (II.35), we retrieve the exponent of the Galilei group calculated by Lévy-Leblond ([16], Section III.C). If we subtract from $\xi_m(G', G)$ the trivial exponent $\xi'_m(G', G)$ defined by

$$(II.36) \quad \left\{ \begin{array}{l} \xi'_m(G', G) = \zeta_m(G'G) - \zeta_m(G') - \zeta_m(G), \\ \zeta_m(G) = \frac{1}{2} m \mathbf{a} \cdot \mathbf{v}, \end{array} \right.$$

we obtain another exponent, $\xi'_m(G', G)$, which is equivalent to $\xi_m(G', G)$:

$$(II.37) \quad \xi'_m(G', G) = \frac{m}{2} (\lambda' \mathbf{v}' \cdot R' \mathbf{a} - \lambda'^{-1} \mathbf{a}' \cdot R' \mathbf{v} - \lambda' b \mathbf{v}' \cdot R' \mathbf{v}),$$

which satisfies Equation (II.15) with operators

$$U'(G) = \exp\left(i \frac{1}{2} m \mathbf{a} \cdot \mathbf{v}\right) U(G).$$

On putting $\lambda' = 1$ in Equation (II.37) we retrieve the exponent of the Galilei group calculated by Bargmann ([13], Section 6f) using an analytic method and by Bernstein [18] using a global method [1].

Finally, we note that the Bargmann superselection rule ([14], [12]) for nonrelativistic mass is still valid. The unit transformation

$$(II.38) \quad (1, 0, \mathbf{0}, \mathbf{0}, 1) = (1, 0, \mathbf{0}, -\mathbf{v}, 1) (1, 0, -\mathbf{a}, \mathbf{0}, 1) \\ \times (1, 0, \mathbf{0}, \mathbf{v}, 1) (1, 0, \mathbf{a}, \mathbf{0}, 1),$$

is represented by the unitary operator

$$(II.39) \quad U(\mathbf{1}) = e^{i\mathbf{v}\cdot\mathbf{K}} e^{i\mathbf{a}\cdot\mathbf{P}} e^{-i\mathbf{v}\cdot\mathbf{K}} e^{-i\mathbf{a}\cdot\mathbf{P}} = e^{i\mathbf{M}\mathbf{a}\cdot\mathbf{v}},$$

as in the case of the Galilei group, and so the superposition principle cannot hold for states of different mass.

C. TRANSFORMATION OF STATES

We now consider the transformation properties of the physical single-particle states under the operator $U(G)$ which we have defined in Section II.1.B. From the work of Section II.1.A, the irreducible representations are labelled by the mass m and spin s (the eigenvalues of M and \mathbf{S}^2) and states within an irreducible representation can be labelled by the eigenvalues of the commuting operators H , \mathbf{P} , and S_3 . (States labelled by the eigenvalues of the commuting operators T , \mathbf{R} , and S_3 will be discussed later.) Our states will thus be denoted $|m, s; \mathbf{E}, \mathbf{p}, \sigma\rangle$ though we shall drop the (m, s) label for brevity. They are normalised according to

$$(II.40) \quad \langle E', p', \sigma' | \mathbf{E}, \mathbf{p}, \sigma \rangle = \delta_{\sigma'\sigma} \delta(E' - E) \delta^3(\mathbf{p}' - \mathbf{p}),$$

as is conventional.

The evaluation of $U(\lambda, b, \mathbf{a}, \mathbf{v}, R) | \mathbf{E}, \mathbf{p}, \sigma \rangle$ is simplified greatly by the fact that, from Equations (II.3), (II.15), and (II.35) [or directly from Equation (II.30)], $U(\lambda, b, \mathbf{a}, \mathbf{v}, R)$ can be written as

$$(II.41) \quad U(\lambda, b, \mathbf{a}, \mathbf{v}, R) = U(1, b, \mathbf{a}, \mathbf{v}, R) U(\lambda, 0, \mathbf{0}, \mathbf{0}, 1),$$

where $U(1, b, \mathbf{a}, \mathbf{v}, R)$ is a unitary operator of the Galilei group, whose effect on a physical state $| \mathbf{E}, \mathbf{p}, \sigma \rangle$ has been evaluated by Lévy-Leblond ([12], Equation (III.20)) ⁽⁶⁾ :

$$(II.42) \quad U(1, b, \mathbf{a}, \mathbf{v}, R) | \mathbf{E}, \mathbf{p}, \sigma \rangle = e^{i(bE' - \mathbf{a}\cdot\mathbf{p}')} \sum_{\sigma' = -s}^s | E', \mathbf{p}', \sigma' \rangle D^{s}_{\sigma'\sigma}(R),$$

where

$$D^{s}_{\sigma'\sigma}(R) = \langle s; \sigma' | \exp(-i \boldsymbol{\varphi} \cdot \mathbf{J}) | s; \sigma \rangle$$

is the well-known unitary irreducible representation of the rotation group ([19], 20) and

$$(II.43) \quad \begin{cases} \mathbf{p}' = R \mathbf{p} + m \mathbf{v}, \\ E' = E + \mathbf{v} \cdot R \mathbf{p} + \frac{1}{2} m \mathbf{v}^2. \end{cases}$$

⁽⁶⁾ He actually evaluated the equivalent representation denoted $U'(1, b, \mathbf{a}, \mathbf{v}, R)$ in Section II.1.B. He gets the factor $\exp(-i(1/2) m \mathbf{a} \cdot \mathbf{v})$ in his representation since his exponent $\xi(G', G)$ ([12], Equation (I.8)) is the negative of our $\xi'(G', G)$ [Equation (II.37) with $\lambda' = 1$].

We now define the action of $U(\lambda, 0, 0, 0, 1)$ on a state $|E, \mathbf{p}, \sigma\rangle$ by the equation

$$(II.44) \quad U(\lambda, 0, 0, 0, 1) |E, \mathbf{p}, \sigma\rangle = \lambda^{-5/2} |\lambda^{-2} E, \lambda^{-1} \mathbf{p}, \sigma\rangle,$$

the factor $\lambda^{-5/2}$ being required to preserve the normalisation, Equation (II.40), as is necessary for a unitary operator. We are now in a position to evaluate

$$U(\lambda, b, \mathbf{a}, \mathbf{v}, R) |E, \mathbf{p}, \sigma\rangle.$$

Using Equations (II.41) to (II.44), we find

$$(II.45) \quad U(\lambda, b, \mathbf{a}, \mathbf{v}, R) |E, \mathbf{p}, \sigma\rangle \\ = \lambda^{-5/2} e^{i(bE' - \mathbf{a} \cdot \mathbf{p}')} \sum_{\sigma' = -s}^s |E', \mathbf{p}', \sigma'\rangle D^s_{\sigma' \sigma}(R),$$

where

$$(II.46) \quad \begin{cases} \mathbf{p}' = \lambda^{-1} R \mathbf{p} + m \mathbf{v}, \\ E' = \lambda^{-2} E + \lambda^{-1} \mathbf{v} \cdot R \mathbf{p} + \frac{1}{2} m \mathbf{v}^2. \end{cases}$$

Equations (II.45) and (II.46) constitute a unitary ray representation of the group (II.1) in Hilbert space, with exponent given by Equation (II.35).

We shall now work out this representation using the basis states $|m, s; t, \mathbf{r}, \sigma\rangle$ which are eigenstates of the commuting operators T, \mathbf{R} , and S_3 . The states $|t, \mathbf{r}, \sigma\rangle$ are related to the set $|E, \mathbf{p}, \sigma\rangle$ by the formula

$$(II.47) \quad |t, \mathbf{r}, \sigma\rangle = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dE \int d^3 p e^{i(Et - \mathbf{p} \cdot \mathbf{r})} |E, \mathbf{p}, \sigma\rangle,$$

where we are at liberty to superpose states of positive and negative energy since the sign of the internal energy is not a measurable quantity [see discussion after Equation (II.18)]. The states $|t, \mathbf{r}, \sigma\rangle$ are normalised to

$$(II.48) \quad \langle t', \mathbf{r}', \sigma' | t, \mathbf{r}, \sigma\rangle = \delta_{\sigma' \sigma} \delta(t' - t) \delta^3(\mathbf{r}' - \mathbf{r}),$$

as can be seen from Equations (II.47) and (II.40). On applying $U(\lambda, b, \mathbf{a}, \mathbf{v}, R)$ to $|t, \mathbf{r}, \sigma\rangle$, given by Equation (II.47), and using Equation (II.45), we find

$$(II.49) \quad U(\lambda, b, \mathbf{a}, \mathbf{v}, R) |t, \mathbf{r}, \sigma\rangle = \frac{\lambda^{-5/2}}{(2\pi)^2} \int dE d^3 p e^{i(Et - \mathbf{p} \cdot \mathbf{r} + bE' - \mathbf{a} \cdot \mathbf{p}')} \\ \times \sum_{\sigma' = -s}^s |E', \mathbf{p}', \sigma'\rangle D^s_{\sigma' \sigma}(R),$$

where E' and \mathbf{p}' are given by Equations (II.46). Inverting Equations (II.46) gives :

$$(II.50) \quad \begin{cases} \mathbf{p} = \lambda R^{-1} (\mathbf{p}' - m \mathbf{v}), \\ E = \lambda^2 \left(E' - \mathbf{v} \cdot \mathbf{p}' + \frac{1}{2} m \mathbf{v}^2 \right) \end{cases}$$

and, on substituting Equations (II.50), together with

$$(II.51) \quad d^3 p dE = \lambda^5 d^3 p' dE',$$

into Equation (II.49), and using the definition of $|t, \mathbf{r}, \sigma\rangle$, Equation (II.47), we find

$$(II.52) \quad U(\lambda, b, \mathbf{a}, \mathbf{v}, R) |t, \mathbf{r}, \sigma\rangle = \lambda^{5/2} e^{i(-1/2)m\mathbf{v}^2(t-b) + m\mathbf{v} \cdot (\mathbf{r}-\mathbf{a})} \\ \times \sum_{\sigma'=-s}^s |t', \mathbf{r}', \sigma'\rangle D^s_{\sigma'\sigma}(R),$$

where

$$(II.53) \quad \begin{cases} t' = \lambda^2 t + b, \\ \mathbf{r}' = \lambda R \mathbf{r} + \lambda^2 \mathbf{v} t + \mathbf{a}. \end{cases}$$

The representation, defined by Equations (II.52) and (II.53) is, of course, equivalent to the one defined by Equations (II.45) and (II.46), the bases $|E, \mathbf{p}, \sigma\rangle$ and $|t, \mathbf{r}, \sigma\rangle$ being related to one another by the Fourier transform, Equation (II.47).

2. Irreducible representations corresponding to Inönü and Wigner's class II true representations of the Galilei group

In addition to the ray representations of the Galilei group studied by Bargmann ([13], Section 6 *g*), there are also true representations for which the phase factor in Equation (II.15) can be reduced to a sign factor, ± 1 , as for the rotation and Lorentz groups. These irreducible representations have been studied by Inönü and Wigner [15]. It is somewhat difficult to interpret these irreducible representations physically [17] but Ryder [21] has shown that they are the nonrelativistic limits of the representations of the inhomogeneous Lorentz group [1] with non-timelike four-momentum. In this section, we shall study the irreducible representations of our group, (II.1), which correspond to the Class II representations of Inönü and Wigner, i. e. we shall look for representations of the group Lie algebra, Equations (II.11), in Hilbert space (i. e. the matrices \mathcal{H} , \mathcal{X}_i , \mathcal{K}_i , \mathcal{J}_i and \mathcal{D} are replaced by hermitean operators H , P_i , K_i , J_i and D) when \mathbf{K} and \mathbf{P} are in the same direction. We shall construct a time operator and position operator but we shall find that the different components of the latter do not commute. We shall also explain why representations corresponding to Inönü and Wigner's Class I (i. e. \mathbf{K} and \mathbf{P} not parallel) do not occur.

We shall now look at the Lie algebra, (II.11), when \mathbf{K} and \mathbf{P} are parallel

$$(II.54) \quad \mathbf{K} = \mathbf{P} T,$$

where the operator $T = \mathbf{P} \cdot \mathbf{K} / \mathbf{P}^2$ satisfies, by Equations (II.11),

$$(II.55) \quad \begin{cases} [J_i, T] = 0, & [K_i, T] = 0, \\ [P_i, T] = 0, & [H, T] = i 1, \\ [D, T] = 2 i T \end{cases}$$

and can therefore be interpreted as a time operator. The invariant operator of the Lie algebra, (II.11), when \mathbf{K} and \mathbf{P} are parallel, (II.54), is

$$(II.56) \quad \frac{\mathbf{J} \cdot \mathbf{P}}{|\mathbf{P}|} = \Lambda,$$

which is the helicity of the particle. In the case of the Galilei group, the operator \mathbf{P}^2 is also an invariant ([15], [12]) but this is not so in our case since \mathbf{P}^2 does not commute with the dilatation generator

$$(II.57) \quad [D, \mathbf{P}^2] = -2 i \mathbf{P}^2.$$

We define the position operator, \mathbf{R} , by

$$(II.58) \quad R_i = -\frac{1}{2} \left\{ \frac{P_i}{\mathbf{P}^2}, D \right\} + \frac{\{H, K_i\}}{\mathbf{P}^2} - \frac{\varepsilon_{ijk} \{J_j, P_k\}}{2 \mathbf{P}^2}$$

and, on using the Lie algebra, Equations (II.11), and the operator identities, Equations (II.14), we find the commutation relations :

$$(II.59) \quad \begin{cases} [J_i, R_j] = i \varepsilon_{ijk} R_k, & [K_i, R_j] = -i \delta_{ij} T, \\ [R_i, H] = 0, & [R_i, P_j] = i \delta_{ij} 1, \\ [D, R_i] = i R_i, & [T, R_i] = 0, \\ [R_i, R_j] = -\frac{i}{|\mathbf{P}|^3} \varepsilon_{ijk} P_k \Lambda \end{cases}$$

which are all in agreement with physical intuition except perhaps for the last one which says that the different components of \mathbf{R} do not commute (except, of course, for a particle of zero spin). However, we shall see, when we study the Weyl group in Section III, that the different components of the four-position of a relativistic particle do not commute, and so the last of Equations (II.59) is perfectly natural (see particularly the discussion of the irreducible representations of the Weyl group with spacelike four-momentum given in Section III.2).

A unitary parity operator and an antiunitary time-reversal operator which act on the Hilbert space operators according to Equations (II.21) once again give Equations (II.23) when applied to the time operator and position operator defined in Equations (II.54) and (II.58) respectively.

As in Section II.1.A, if we restrict ourselves to the proper, orthochronous group, the operators are not unique, the redefinitions

$$(II.60) \quad \begin{cases} \mathbf{D} \rightarrow \mathbf{D} + \mathbf{a}, \\ \mathbf{R} \rightarrow \mathbf{R} - \frac{a \mathbf{P}}{\mathbf{P}^2}, \end{cases}$$

where a is a constant, leaving the Lie algebra, (II.11), and the commutation relations, (II.59), unchanged.

We have not been able to construct a canonical form for the operators. [Representations corresponding to Inönü and Wigner's Class I representation satisfy the Lie algebra, (II.11), with \mathbf{K} and \mathbf{P} not parallel. The invariant operator of this algebra is $(\mathbf{K} \times \mathbf{P})^2$, i. e. "continuous spin", and these particles are the nonrelativistic limit of virtual continuous spin relativistic particles with spacelike four-momentum (see Section III.2 of this paper, and also Ryder [21] for the corresponding statement in terms of "real" particles). We have not, however, been able to construct a position operator with reasonable properties for these Class I representations. We think that this is because the dilatation $(t', \mathbf{x}') = (\lambda^2 t, \lambda \mathbf{x})$ has been tailored to fit the Schrödinger equation, (I.1), which describes the physical representations of the Galilei group, and Lévy-Leblond [12] has shown that it is the Class II representations which are the zero-mass limit of the physical representations. To obtain virtual Class I particles, we feel that we should use the more general dilatation $(t', \mathbf{x}') = (\lambda_1 t, \lambda_2 \mathbf{x})$. Inönü and Wigner's Class III (i. e. $\mathbf{P} = \mathbf{0}$) and Class IV (i. e. $\mathbf{P} = \mathbf{0} = \mathbf{K}$) representations are not faithful representations of the Galilei group and we have not looked at the corresponding representations in our case (though Class IV representations certainly exist, giving \mathbf{J} , \mathbf{H} and \mathbf{T}). They are apparently the nonrelativistic limits of the relativistic null four-momentum representations [21].

3. The system of two nonrelativistic virtual particles

In this section we shall investigate the system of two nonrelativistic massive virtual particles ⁽¹⁾ of the type discussed in Section II.1. We shall find that the centre-of-mass variables form a reducible representation of the same type (the representation is not simply reducible except when the two particles have zero spin, the degeneracy parameter being the orbital and spin angular momenta), whilst the relative variables form a reducible (but undecomposable) representation of the type discussed in Section II.2 (i. e. Inönü-Wigner Class II), the reducibility

⁽¹⁾ See Lévy-Leblond [12], for the corresponding treatment for the system of two nonrelativistic massive real particles.

allowing us to construct a relative position operator whose different components commute.

We work with the momentum, energy, mass, time, position, and spin operators of the two particles which satisfy [*cf.* Equations (II.20) and remarks following Equation (II.24)] :

$$(II.61) \quad \left\{ \begin{array}{l} [T_i, H_i] = -i 1, \quad [R_{1i}, P_{1j}] = i \delta_{ij} 1, \\ [S_{1i}, S_{1j}] = i \varepsilon_{ijk} S_{1k}, \\ \text{all other commutators zero} \end{array} \right.$$

where $S_1^2 = s_1(s_1 + 1)$ with s_1 the spin of particle 1. There is a similar set of commutation relations for particle 2, and the operators of the different particles commute. In terms of these operators, the rotation, boost, and dilatation generators of particle 1 are defined by (*cf.* Equations (II.25)) :

$$(II.62) \quad \left\{ \begin{array}{l} \mathbf{J}_1 = \mathbf{R}_1 \times \mathbf{P}_1 + \mathbf{S}_1, \\ \mathbf{K}_1 = T_1 \mathbf{P}_1 - M_1 \mathbf{R}_1, \\ D_1 = \{ T_1, H_1 \} - \frac{1}{2} \{ R_{1i}, P_{1i} \} \end{array} \right.$$

and similiary for particle 2.

Our task now is to construct the centre-of-mass mass, energy, momentum, time, position and spin operators of the two-particle system. The mass, energy and momentum operators are defined by the equations

$$(II.63) \quad \left\{ \begin{array}{l} M = M_1 + M_2, \\ H = H_1 + H_2, \\ \mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 \end{array} \right.$$

and the centre-of-mass rotation and boost generators by

$$(II.64) \quad \left\{ \begin{array}{l} \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2, \\ \mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 \end{array} \right.$$

(note that we do not define the centre-of-mass dilatation generator in this way). We define the "preliminary" centre-of-mass time⁽⁸⁾ and position operators, \tilde{T} and $\tilde{\mathbf{R}}$, by

$$(II.65) \quad \left\{ \begin{array}{l} \tilde{T} = \frac{M_1 T_1 + M_2 T_2}{M_1 + M_2}, \\ \tilde{\mathbf{R}} = \frac{M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2}{M_1 + M_2} \end{array} \right.$$

(⁸) It should be noted that a formulation of quantum electrodynamics in which each electron has a different time was given forty years ago by Dirac [22]. The relation between Dirac's "many-time" formalism and the one using an electron field has been given by Tomonaga [23].

and these operators satisfy the following commutation relations with M , H , and \mathbf{P} :

$$(II.66) \quad \left\{ \begin{array}{l} [\tilde{T}, H] = -i 1, \quad [\tilde{\mathbf{R}}_i, P_j] = i \delta_{ij} 1, \\ \text{all other commutators zero.} \end{array} \right.$$

We also define the " preliminary " relative variables $(\tilde{h}, \tilde{\mathbf{q}}, \tilde{\tau}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ by

$$(II.67) \quad \left\{ \begin{array}{l} \tilde{h} = \frac{M_2 H_1 - M_1 H_2}{M_1 + M_2}, \\ \tilde{\mathbf{q}} = \frac{M_2 \mathbf{P}_1 - M_1 \mathbf{P}_2}{M_1 + M_2}, \\ \tilde{\tau} = T_1 - T_2, \\ \tilde{\mathbf{r}} = \mathbf{R}_1 - \mathbf{R}_2, \\ \tilde{\mathbf{s}} = \mathbf{S}_1 + \mathbf{S}_2, \end{array} \right.$$

which satisfy the following commutation relations among themselves :

$$(II.68) \quad \left\{ \begin{array}{l} [\tilde{\tau}, \tilde{h}] = -i 1, \quad [\tilde{r}_i, \tilde{q}_j] = i \delta_{ij} 1, \\ [\tilde{s}_i, \tilde{s}_j] = i \varepsilon_{ijk} \tilde{s}_k, \\ \text{all other commutators zero.} \end{array} \right.$$

The sets of operators $(M, H, \mathbf{P}, \tilde{T}, \tilde{\mathbf{R}})$ and $(\tilde{h}, \tilde{\mathbf{q}}, \tilde{\tau}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ commute with one another :

$$(II.69) \quad [(M, H, \mathbf{P}, \tilde{T}, \tilde{\mathbf{R}}), (\tilde{h}, \tilde{\mathbf{q}}, \tilde{\tau}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})] = 0.$$

Furthermore, the centre-of-mass energy can be written as the sum of kinetic energy and internal energy parts :

$$(II.70) \quad \left\{ \begin{array}{l} H = \frac{\mathbf{P}^2}{2M} + U, \\ U = \frac{\mathbf{q}^2}{2\mu} + U_1 + U_2 \end{array} \right.$$

[where the reduced mass $\mu = M_1 M_2 / (M_1 + M_2)$] as can be seen by expressing H_1 and H_2 as sums of kinetic energy and internal energy and using the expressions for M, H, \mathbf{P} and \mathbf{q} in Equations (II.63) and (II.67).

We next evaluate $(\mathbf{J}_1 + \mathbf{J}_2)$, $(\mathbf{K}_1 + \mathbf{K}_2)$, and $(D_1 + D_2)$ in terms of the operators $(M, H, \mathbf{P}, \tilde{T}, \tilde{\mathbf{R}})$ and $(\tilde{h}, \tilde{\mathbf{q}}, \tilde{\tau}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ using Equations (II.62), (II.63), (II.65) and (II.67) and find :

$$(II.71 a) \quad \mathbf{J}_1 + \mathbf{J}_2 = \tilde{\mathbf{R}} \times \mathbf{P} + \mathbf{r} \times \tilde{\mathbf{q}} + \tilde{\mathbf{s}},$$

$$(II.71 b) \quad \mathbf{K}_1 + \mathbf{K}_2 = \tilde{T} \mathbf{P} - M \tilde{\mathbf{R}} + \tilde{\mathbf{q}} \tilde{\tau},$$

$$(II.71 c) \quad D_1 + D_2 = \{H, \tilde{T}\} - \frac{1}{2} \{P_i, \tilde{R}_i\} + \{\tilde{h}, \tilde{\tau}\} - \frac{1}{2} \{\tilde{q}_i, \tilde{r}_i\}.$$

Now in terms of the operators $(M, H, \mathbf{P}, T, \mathbf{R}, \mathbf{S})$, the operators \mathbf{J} , \mathbf{K} and D are given by [cf. Equations (II.25)] :

$$(II.72 a) \quad \mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{S},$$

$$(II.72 b) \quad \mathbf{K} = T \mathbf{P} - M \mathbf{R},$$

$$(II.72 c) \quad D = \{ H, T \} - \frac{1}{2} \{ P_i, R_i \}$$

and so on using the formula for \mathbf{S} , Equations (II.24), together with Equations (II.64), (II.71 a) and (II.71 b), we find

$$(II.73) \quad \mathbf{S} = \left(\tilde{\mathbf{r}} - \frac{\mathbf{P} \tilde{\tau}}{M} \right) \times \tilde{\mathbf{q}} + \tilde{\mathbf{s}},$$

and, on comparing Equations (II.71 b) and (II.72 b), we find

$$(II.74) \quad \left\{ \begin{array}{l} T = \tilde{T} = \frac{M_1 T_1 + M_2 T_2}{M_1 + M_2}, \\ \mathbf{R} = \tilde{\mathbf{R}} - \frac{\tilde{\mathbf{q}} \tilde{\tau}}{M} = \frac{M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2}{M_1 + M_2} - \frac{\tilde{\mathbf{q}} \tilde{\tau}}{M}. \end{array} \right.$$

We see that the operators $(M, H, \mathbf{P}, \mathbf{J}, \mathbf{K}, D)$ defined by Equations (II.63) and (II.72) form a (reducible) representation of the type discussed in Section II.1, with invariant operators \mathbf{S}^2 and M .

We now wish to construct a set of relative operators $(h, \mathbf{q}, \tau, \mathbf{r}, \mathbf{s})$ which commute with the centre-of-mass operators $(M, H, \mathbf{P}, T, \mathbf{R})$. A technique for doing this has been given by Gartenhaus and Schwartz [24] and, in fact, the centre-of-mass dilatation generator, D , defined by Equation (II.72 c) is just the generator of the Gartenhaus-Schwartz transformation ⁽⁹⁾, now recognised as an infinite dilatation, which acts on the centre of mass operators according to :

$$(II.75) \quad \left\{ \begin{array}{l} \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} M e^{i\alpha D} = M, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} H e^{i\alpha D} = \lim_{\alpha \rightarrow \infty} H e^{-2\alpha} = 0, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \mathbf{P} e^{i\alpha D} = \lim_{\alpha \rightarrow \infty} \mathbf{P} e^{-\alpha} = 0, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} T e^{i\alpha D} = \lim_{\alpha \rightarrow \infty} T e^{2\alpha} = \infty, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \mathbf{R} e^{i\alpha D} = \lim_{\alpha \rightarrow \infty} \mathbf{R} e^{\alpha} = \infty. \end{array} \right.$$

If we denote by \tilde{O} the set of operators $(\tilde{h}, \tilde{\mathbf{q}}, \tilde{\tau}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ and by O the set of operators $(h, \mathbf{q}, \tau, \mathbf{r}, \mathbf{s})$, then O is constructed from \tilde{O} by the formula

$$(II.76) \quad O = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \tilde{O} e^{i\alpha D}.$$

⁽⁹⁾ Gartenhaus and Schwartz actually used the operator $D = \{ U, T \}$, since they worked with a theory which used c -number parameter time.

The right hand side of Equation (II.76) is evaluated first for finite α , either by using Equation (II.32) or by the formulae :

$$(II.77) \quad \left\{ \begin{array}{l} \tilde{O}(\alpha) = e^{-i\alpha D} \tilde{O} e^{i\alpha D}, \\ \frac{d\tilde{O}(\alpha)}{d\alpha} = -i e^{-i\alpha D} [D, \tilde{O}] e^{i\alpha D}, \\ 0 = \tilde{O}(\infty) = \tilde{O} + \int_0^\infty \frac{d\tilde{O}(\alpha)}{d\alpha} d\alpha. \end{array} \right.$$

The evaluation of \mathbf{q} , τ , and \mathbf{s} is trivial since D commutes with $\tilde{\mathbf{q}}$, $\tilde{\tau}$ and $\tilde{\mathbf{s}}$ and the evaluation of h and \mathbf{r} is straightforward, using Equations (II.68) and (II.69) :

$$(II.78 a) \quad h = \tilde{h} - \frac{\mathbf{P} \cdot \tilde{\mathbf{q}}}{M} = \frac{M_2 H_1 - M_1 H_2}{M_1 + M_2} - \frac{\mathbf{P} \cdot \mathbf{q}}{M},$$

$$(II.78 b) \quad \mathbf{q} = \tilde{\mathbf{q}} = \frac{M_2 \mathbf{P}_1 - M_1 \mathbf{P}_2}{M_1 + M_2},$$

$$(II.78 c) \quad \tau = \tilde{\tau} = (T_1 - T_2),$$

$$(II.78 d) \quad \mathbf{r} = \tilde{\mathbf{r}} - \frac{\mathbf{P} \tilde{\tau}}{M} = (\mathbf{R}_1 - \mathbf{R}_2) - \frac{\mathbf{P} \tau}{M},$$

$$(II.78 e) \quad \mathbf{s} = \tilde{\mathbf{s}} = (\mathbf{S}_1 + \mathbf{S}_2)$$

and it is easily seen that these operators commute with the centre of mass operators (M , H , \mathbf{P} , T , \mathbf{R}), and satisfy the same commutation relations with one another as do the "preliminary" relative variables [Equations (II.68)]. On defining the relative rotation, boost, and dilatation generators by

$$(II.79 a) \quad \mathbf{j} = \mathbf{r} \times \mathbf{q} + \mathbf{s},$$

$$(II.79 b) \quad \mathbf{x} = \mathbf{q} \tau,$$

$$(II.79 c) \quad d = \{h, \tau\} - \frac{1}{2} \{q_i, r_i\},$$

we see that the operators (h , \mathbf{q} , \mathbf{j} , \mathbf{x} , d), given by Equations (II.78 a), (II.78 b) and (II.79 c), form a (reducible) representation of the type discussed in Section II.2, with invariant operator $^{(10)} \mathbf{j} \cdot \hat{\mathbf{q}}$ which is just equal to the difference of the helicities of the two particles ($\mathbf{S}_1 \cdot \hat{\mathbf{P}}_1 - \mathbf{S}_2 \cdot \hat{\mathbf{P}}_2$) in the centre-of-mass frame. This reducible representation also has the operators $(\mathbf{S}_1 + \mathbf{S}_2)^2$, \mathbf{S}_1^2 , \mathbf{S}_2^2 , $\mathbf{S}_1 \cdot \hat{\mathbf{q}}$ and $\mathbf{S}_2 \cdot \hat{\mathbf{q}}$ as invariants, but only the sets $(\mathbf{S}_1^2, \mathbf{S}_2^2, (\mathbf{S}_1 + \mathbf{S}_2)^2, \mathbf{j} \cdot \hat{\mathbf{q}})$ or $(\mathbf{S}_1^2, \mathbf{S}_2^2, \mathbf{S}_1 \cdot \hat{\mathbf{q}}, \mathbf{S}_2 \cdot \hat{\mathbf{q}})$ can be diagonalised simultaneously, and we shall use the former set.

Looking at Equations (II.71 c), (II.72 c), (II.74), (II.78) and (II.79 c), we see that

$$(II.80) \quad D_1 + D_2 = D + d,$$

i. e. the sum of single-particle dilatation generators splits up into a sum of the centre-of-mass and relative dilatation generators. This pheno-

⁽¹⁰⁾ The symbol $\hat{\mathbf{q}}$ denotes a unit vector in the \mathbf{q} direction. Similarly for $\hat{\mathbf{P}}_1$ and $\hat{\mathbf{P}}_2$.

menon [which also occurs in the relativistic case, see Equation (III.104)] seems to be connected with the noncompactness of the dilatation transformation.

We note that the centre-of-mass spin and position operators, given by Equations (II.73) and (II.74), can be written in terms of the relative operators \mathbf{j} and \mathbf{x} as

$$(II.81) \quad \begin{cases} \mathbf{S} = \mathbf{j} = \mathbf{r} \times \mathbf{q} + \mathbf{s}, \\ \mathbf{R} = \frac{M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2}{M_1 + M_2} - \frac{\mathbf{x}}{M}. \end{cases}$$

We have mentioned previously that different components of the relative position operator, \mathbf{r} , commute even though the relative variables form a representation of the type discussed in Section II.2. This is because of the reducibility of the representation [i. e. basically because of the extra degree of freedom afforded to us by the operator $s = (\mathbf{S}_1 + \mathbf{S}_2)$], and the expression for \mathbf{r} in terms of h , \mathbf{q} , \mathbf{j} , \mathbf{x} , d and \mathbf{s} is

$$(II.82) \quad r_i = -\frac{1}{2} \left\{ \frac{q_i}{\mathbf{q}^2}, d \right\} + \frac{\{h, x_i\}}{\mathbf{q}^2} - \frac{\varepsilon_{ijk} \{ (j_j - s_j), q_k \}}{2 \mathbf{q}^2},$$

which should be compared with Equation (II.58).

We also note that, in analogy with Equations (II.26) and (II.60), our entire theory is invariant under the redefinitions :

$$(II.83) \quad \left\{ \begin{array}{l} D_1 \rightarrow D_1 + \frac{2aU_1}{U}, \\ D_2 \rightarrow D_2 + \frac{2aU_2}{U}, \\ D \rightarrow D + 2a, \\ d \rightarrow d - \frac{2a}{U} \left(\frac{\mathbf{q}^2}{2\mu} \right), \\ T_1 \rightarrow T_1 + \frac{a}{U}, \\ T_2 \rightarrow T_2 + \frac{a}{U}, \\ T \rightarrow T + \frac{a}{U}, \\ \mathbf{R}_1 \rightarrow \mathbf{R}_1 + \frac{a\mathbf{P}_1}{M_1U}, \\ \mathbf{R}_2 \rightarrow \mathbf{R}_2 + \frac{a\mathbf{P}_2}{M_2U}, \\ \mathbf{R} \rightarrow \mathbf{R} + \frac{a\mathbf{P}}{MU}, \\ \mathbf{r} \rightarrow \mathbf{r} + \frac{a\mathbf{q}}{\mu U}, \end{array} \right.$$

where a is a constant. The invariance operation (II.83) seems to be connected with the existence of a set of relative operators which commute with the centre-of-mass operators, as we have been unable to find a corresponding redefinition invariance in the relativistic case (see Section III.3).

Finally, if we denote an irreducible representation of the type discussed in Section II.1, of mass m and spin s , by $[m | s]$, and a reducible representation of the type defined by Equations (II.78 a), (II.78 b) and (II.79) by $(O | s_1 s_2 s \lambda)$, then the reduction carried out in this section may be written as

$$(II.84) \quad [m_1 | s_1] \otimes [m_2 | s_2] = \left(\bigoplus_{l=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{j=|l-s|}^{l+s} [m_1 + m_2 | j] \right)_{c.m.} \\ \times \left(\bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{\lambda=-s}^s (O | s_1 s_2 s \lambda) \right)_{rel.}$$

It might be thought that the “relative” part could be written

$$(II.85) \quad \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{\lambda=-s}^s \{ O | \lambda \},$$

where $\{ O | \lambda \}$ is an irreducible representation of mass zero and helicity λ of the type discussed in Section II.2. However the reducible representation defined by Equations (II.78 a), (II.78 b) and (II.79) contains $\{ O | s \}$ to $\{ O | -s \}$ in an undecomposable way and (II.84) seems a better notation. The centre-of-mass and relative parts of $[m_1 | s_1] \otimes [m_2 | s_2]$ are simply reducible and irreducible respectively if and only if $s_1 = s_2 = 0$.

4. Scattering theory

In this section we shall elucidate the connection between the dilatation change in a scattering process and the Eisenbud [25], Bohm ([26], p. 290 et seq.), Wigner [27] time-delay.

According to Equations (II.70), (II.72 b) and (II.72 c), we can write the centre-of-mass dilatation generator as

$$(II.86) \quad D = \{ U, T \} + \frac{\{ P_i, K_i \}}{2M},$$

where $U = (H - \mathbf{P}^2/2M)$ is the centre-of-mass internal energy and T, M, \mathbf{P} and \mathbf{K} are the centre-of-mass time operator, mass operator, momentum operator, and boost generator respectively. Taking the commutator of Equation (II.86) with a Galilei-invariant scattering operator, S , gives

$$(II.87) \quad [D, S] = \{ U, [T, S] \},$$

since S commutes with U , \mathbf{P} , \mathbf{K} and M . However, U and $[T, S]$ commute since, by the Jacobi identity (II.14 *d*) :

$$(II.88) \quad [[T, S], U] = [T, [S, U]] + [S, [U, T]] = 0.$$

So we find, multiplying Equation (II.87) by S^\dagger from the left

$$(II.89) \quad S^\dagger [D, S] = 2 U S^\dagger [T, S].$$

Now $S^\dagger [T, S]$ is just the time-delay [28], as can easily be seen by writing

$$U \rightarrow \varpi, \quad S \rightarrow \exp(2i\delta(\varpi)), \quad T \rightarrow -i\frac{\partial}{\partial\varpi},$$

which gives

$$(II.90) \quad S^\dagger [T, S] = 2\frac{d\delta}{d\varpi},$$

which is the usual expression for the time-delay ([25], [26], [27]). So Equation (II.89) becomes

$$(II.91) \quad S^\dagger [D, S] = 4\varpi\frac{d\delta}{d\varpi},$$

which is the connection between the dilatation change and the time-delay in a scattering process. We therefore see that a scattering process is dilatation-invariant when the time-delay is zero or when, the centre-of-mass energy is zero. However, since ϖ is arbitrary up to an additive constant ($\varpi_1 + \varpi_2$), see Equation (II.70), we are not sure whether the latter condition has any physical meaning.

III. THE WEYL GROUP

The Weyl group is an eleven parameter group consisting of the homogeneous Lorentz transformations, displacements, and dilatations acting on Minkowski space-time $x^\mu = (x^0, \mathbf{x}) = (ct, \mathbf{x})$, according to

$$(III.1) \quad x'^\mu = \lambda L^\mu_\nu x^\nu + a^\mu,$$

where λ , the dilatation, is a real positive constant, a^μ is a constant four-vector displacement, and L^μ_ν , the homogeneous Lorentz transformation, is a matrix satisfying

$$(III.2) \quad L^\mu_\nu g_{\mu\rho} L^\rho_\sigma = g_{\nu\sigma},$$

where the Minkowski metric tensor, $g_{\mu\nu}$, is given by

$$(III.3) \quad g_{00} = +1, \quad g_{ij} = -\delta_{ij}, \quad g_{0i} = 0 = g_{i0}.$$

The significance of Equation (III.1) is that it is the most general transformation which preserves the timelike, spacelike, or lightlike character of the distance $(x - y)^2$ between two points, x and y , in Minkowski space [29]. A general element of the group will be denoted by

$$(III.4) \quad G = (\lambda, L, a)$$

and the group multiplication law is

$$(III.5) \quad G' G = (\lambda', L', a') (\lambda, L, a) = (\lambda' \lambda, L' L, \lambda' L' a + a'),$$

whilst the unit element, $\mathbf{1}$, is

$$(III.6) \quad \mathbf{1} = (1, 1, 0)$$

and the inverse element G^{-1} is

$$(III.7) \quad G^{-1} = (\lambda^{-1}, L^{-1}, -\lambda^{-1} L^{-1} a).$$

The Lie algebra of the Weyl group is found from the space-time representation in a way similar to that which gave Equations (II.11) in Section II. However, since the Weyl group Lie algebra is well-known, we omit the details and write

$$(III.8) \quad \begin{cases} [\mathfrak{M}^{\mu\nu}, \mathfrak{M}^{\rho\sigma}] = i (\mathfrak{M}^{\mu\rho} g^{\nu\sigma} - \mathfrak{M}^{\mu\sigma} g^{\nu\rho} + \mathfrak{M}^{\nu\sigma} g^{\mu\rho} - \mathfrak{M}^{\nu\rho} g^{\mu\sigma}), \\ [\mathfrak{M}^{\mu\nu}, \mathfrak{P}^\sigma] = -i (\mathfrak{P}^\mu g^{\nu\sigma} - \mathfrak{P}^\nu g^{\mu\sigma}), \\ [\mathfrak{P}^\mu, \mathfrak{P}^\nu] = 0, \quad [\mathfrak{D}, \mathfrak{M}^{\mu\nu}] = 0, \quad [\mathfrak{D}, \mathfrak{P}^\mu] = -i \mathfrak{P}^\mu, \end{cases}$$

where the 5×5 matrices $\mathfrak{M}^{\mu\nu}$, \mathfrak{P}^μ and \mathfrak{D} are the generator of homogeneous Lorentz transformations, displacements and dilatations respectively. Their commutation and anticommutation relations with the parity and time-reversal matrices, I_s and I_t [given in Equations (II.12)] are

$$(III.9) \quad \begin{cases} I_s \mathfrak{M}^{\mu\nu} I_s^{-1} = \eta(\mu, \nu) \mathfrak{M}^{\mu\nu}, & I_t \mathfrak{M}^{\mu\nu} I_t^{-1} = \eta(\mu, \nu) \mathfrak{M}^{\mu\nu}, \\ I_s \mathfrak{P}^\mu I_s^{-1} = \eta(\mu) \mathfrak{P}^\mu, & I_t \mathfrak{P}^\mu I_t^{-1} = -\eta(\mu) \mathfrak{P}^\mu, \\ I_s \mathfrak{D} I_s^{-1} = \mathfrak{D}, & I_t \mathfrak{D} I_t^{-1} = \mathfrak{D}, \end{cases}$$

where

$$\eta(0) = +1, \quad \eta(i) = -1,$$

and

$$\eta(\mu, \nu) = \eta(\mu) \eta(\nu).$$

If we define \mathfrak{H} , \mathfrak{J}^i , and \mathfrak{K}^i by ⁽¹⁾ :

$$(III.10) \quad \begin{cases} \mathfrak{H} = \mathfrak{P}^0, \\ \mathfrak{J}^i = -\frac{1}{2} \varepsilon^{ijk} \mathfrak{M}^{jk}, \\ \mathfrak{K}^i = -\mathfrak{M}^{0i}, \end{cases}$$

⁽¹⁾ The antisymmetric tensor ε^{ijk} is defined such that $\varepsilon^{123} = +1$.

then Equations (III.9) becomes *formally* the same as Equations (II.13). Note, however, that \mathcal{K}^i and \mathcal{O} are *not* the same as the corresponding matrices defined in Section II (although \mathcal{K}^i , \mathcal{X}^i and \mathcal{Y}^i are the same).

1. Irreducible representations with timelike four-momentum

We now search for unitary ray representations of the Weyl group in Hilbert space, i. e. unitary operators $U(G)$ which satisfy Equation (II.15) with G' , G , and $G'G$ given by Equation (III.5). It is well-known that the inhomogeneous Lorentz group has only trivial central extensions, i. e. that the phase factor in Equation (II.15) can be reduced to a sign factor ± 1 , and that the unitary ray representations of the inhomogeneous Lorentz group can be found by searching for the true unitary representations of its universal covering group ([1], [13]) which has the same Lie algebra. In Appendix A.II, we show that the Weyl group Lie algebra (III.8) has only (mathematically) trivial central extensions, i. e. isomorphic to a direct sum of the Weyl group Lie algebra and a one-dimensional abelian subalgebra. However, the central extension is *physically* non-trivial, its hermitean Hilbert space generator giving the on-mass-shell mass of the virtual relativistic particle which the irreducible representation describes. In Section III.1.A, we discuss the extended Lie algebra and its physical interpretation whilst, in Section III.1.B, we show that, in the nonrelativistic limit, $c \rightarrow \infty$, the extended Lie algebra goes over into the nonrelativistic extended Lie algebra, Equation (II.16). Finally, in Section III.1.C, we discuss the transformation properties of the physical single particle states under $U(G)$, i. e. we explicitly evaluate the unitary irreducible representations.

A. THE EXTENDED LIE ALGEBRA AND ITS PHYSICAL INTERPRETATION

In Appendix A.II, we show that the Weyl group Lie algebra, (III.8), has only mathematically trivial central extensions. The local group is therefore the direct product of the Weyl group (or rather its universal covering group) with the phase group, $U(1)$, and we shall therefore be concerned with the Lie algebra

$$(III.11) \quad \begin{cases} [M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\rho}g^{\nu\sigma} - M^{\mu\sigma}g^{\nu\rho} + M^{\nu\sigma}g^{\mu\rho} - M^{\nu\rho}g^{\mu\sigma}), \\ [M^{\mu\nu}, P^\sigma] = -i(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma}), \\ [P^\mu, P^\nu] = 0, \quad [D, M^{\mu\nu}] = 0, \quad [D, P^\mu] = -iP^\mu, \\ [M^{\mu\nu}, M] = 0, \quad [P^\mu, M] = 0, \quad [D, M] = 0, \end{cases}$$

where the hermitean operators $M^{\mu\nu}$, P^μ and D generate the transformations (III.1) in Hilbert space and where M is the on-mass-shell-mass operator. The introduction of the mass operator via the trivial central extension may seem a little *ad hoc* but it is not really so.

Hermann [30] has shown that it is just such an extension of the Lie algebra of the inhomogeneous Lorentz group for timelike four-momentum which goes over into extended Galilei Lie algebra ([17], [12]) as $c \rightarrow \infty$. However, in the case of the inhomogeneous Lorentz group, the trivial extension gives nothing new *physically* since the mass operator, M , is related to P^μ by $P^2 = M^2 c^2$, which is not true for the operators in Equations (III.11) ⁽¹²⁾. It is, of course, physically obvious that the Weyl group must reproduce the on-mass-shell mass if it is to describe virtual relativistic particles, as was emphasised by Taylor [33].

We now look for the invariant operators of the extended Lie algebra (III.11). We first define the Pauli-Lubanski spin pseudo-vector ⁽¹³⁾, $W^\mu = (1/2) \varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma$, which has the commutation relations

$$\begin{aligned} \text{(III.12 a)} \quad & [M^{\mu\nu}, W^\sigma] = -i (W^\mu g^{\nu\sigma} - W^\nu g^{\mu\sigma}), \\ \text{(III.12 b)} \quad & [P^\mu, W^\nu] = 0, \\ \text{(III.12 c)} \quad & [D, W^\mu] = -i W^\mu, \\ \text{(III.12 d)} \quad & [M, W^\mu] \neq 0, \\ \text{(III.12 e)} \quad & [W^\mu, W^\nu] = -i \varepsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma \end{aligned}$$

and also satisfies $P \cdot W = 0$. We shall also need the antisymmetric second-rank spin tensor ([34], [35]), $W^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma / P^2$ which satisfies

$$\text{(III.13)} \quad \left\{ \begin{aligned} [M^{\mu\nu}, W^{\rho\sigma}] &= i (W^{\mu\rho} g^{\nu\sigma} - W^{\mu\sigma} g^{\nu\rho} \\ &\quad + W^{\nu\sigma} g^{\mu\rho} - W^{\nu\rho} g^{\mu\sigma}), \\ [P^\mu, W^{\rho\sigma}] &= 0, \\ [D, W^{\mu\nu}] &= 0, \\ [M, W^{\mu\nu}] &= 0, \\ [W^{\mu\nu}, W^\sigma] &= -i \left(W^\mu \left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2} \right) \right. \\ &\quad \left. - W^\nu \left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2} \right) \right), \\ [W^{\mu\nu}, W^{\rho\sigma}] &= i \left(W^{\mu\rho} \left(g^{\nu\sigma} - \frac{P^\nu P^\sigma}{P^2} \right) - W^{\mu\sigma} \left(g^{\nu\rho} - \frac{P^\nu P^\rho}{P^2} \right) \right. \\ &\quad \left. + W^{\nu\sigma} \left(g^{\mu\rho} - \frac{P^\mu P^\rho}{P^2} \right) \right. \\ &\quad \left. - W^{\nu\rho} \left(g^{\mu\sigma} - \frac{P^\mu P^\sigma}{P^2} \right) \right) \end{aligned} \right.$$

⁽¹²⁾ The fact that the mass operator squared, M^2 , is not given by P^2/c^2 means that we can combine the Weyl group with an internal symmetry group in such a way that the space-time and internal transformations commute, and yet still get mass-splitting [31]. This, of course, cannot be done for the inhomogeneous Lorentz group (Mc Glinn's theorem) [32].

⁽¹³⁾ The antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$ is defined such that $\varepsilon^{0123} = +1$.

and in terms of which W^μ can be written as

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} W_{\nu\rho} P_\sigma.$$

The invariants of the Lie algebra (III.11) are

$$(III.14) \quad -\frac{W^2}{P^2} = \frac{1}{2} W^{\mu\nu} W_{\mu\nu}, \quad M, \quad \text{sign}(P^2),$$

which means that a representation of the Weyl group is labelled by the spin, $-W^2/P^2$, the mass, M , and the sign of the four-momentum squared. For timelike four-momentum, i. e. $\text{sign}(P^2) = +1$, the sign of the energy, $\text{sign}(P^0)$, is also an invariant and, as we are going to interpret these representations as virtual particles we shall need $\text{sign}(P^0) = +1$, the case $\text{sign}(P^0) = -1$ being connected with the existence of antiparticles. For $\text{sign}(P^2) = +1$, the operator $-W^2/P^2$ is just equal to $s(s+1)$ where s is the spin of the particle. In the case of the inhomogeneous Lorentz group, P^2 is also an invariant and is equal to $M^2 c^2$, the mass-squared. In the case of the Weyl group, this is not so since

$$(III.15) \quad [D, P^2] = -2iP^2.$$

We now define the hermitean space-time position operator,

$$(III.16) \quad R^\mu = (R^0, R^i) = (cT, \mathbf{R}),$$

$$R^\mu = \frac{1}{2} \left\{ \frac{P^\mu}{P^2}, D \right\} - \frac{\{P_\nu, M^{\mu\nu}\}}{2P^2}$$

[cf. Equations (II.19) and (II.58)]. Its commutation relations with the various operators are

$$(III.17 a) \quad [M^{\mu\nu}, R^\sigma] = -i(R^\mu g^{\nu\sigma} - R^\nu g^{\mu\sigma}),$$

$$(III.17 b) \quad [R^\mu, P^\nu] = -i g^{\mu\nu},$$

$$(III.17 c) \quad [D, R^\mu] = iR^\mu,$$

$$(III.17 d) \quad [M, R^\mu] = 0,$$

$$(III.17 e) \quad [R^\mu, R^\nu] = -\frac{i \varepsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma}{P^2},$$

$$(III.17 f) \quad [R^\mu, W^\nu] = -i \frac{P^\mu W^\nu - P^\nu W^\mu}{P^2},$$

$$(III.17 g) \quad [R^\mu, W^{\rho\sigma}] = -i \frac{P^\rho W^{\sigma\mu} - P^\sigma W^{\rho\mu}}{P^2},$$

as can be verified by using Equations (II.14) and the extended Lie algebra (III.11). The first three equations give the behaviour of R^μ under Lorentz transformations, translations and dilatations, and are as

we should expect. The fact that the different components of R^μ do not commute, (III.17 e), for a particle of non-zero spin seems strange at first sight but has been noted by previous authors ([5]-[7], ^(13a)). It is easily seen using Equation (III.28) [which expresses W^μ in terms of the spin operator $\mathbf{S}(P, W)$], that, for timelike four-momentum, the commutator $[R^\mu, R^\nu]$ goes to zero as $c \rightarrow \infty$. (See also Section III.1.B).

If we have a unitary parity operator, \mathcal{P} , and an antiunitary time-reversal operator, \mathcal{T} , in Hilbert space, then Equations (III.9) become

$$(III.18) \quad \left\{ \begin{array}{ll} \mathcal{P} M^{\mu\nu} \mathcal{P}^{-1} = \eta(\mu, \nu) M^{\mu\nu}, & \mathcal{T} M^{\mu\nu} \mathcal{T}^{-1} = -\eta(\mu, \nu) M^{\mu\nu}, \\ \mathcal{P} P^\mu \mathcal{P}^{-1} = \eta(\mu) P^\mu, & \mathcal{T} P^\mu \mathcal{T}^{-1} = \eta(\mu) P^\mu, \\ \mathcal{P} D \mathcal{P}^{-1} = D, & \mathcal{T} D \mathcal{T}^{-1} = -D, \end{array} \right.$$

where

$$\eta(0) = +1, \quad \eta(i) = -1,$$

and

$$\eta(\mu, \nu) = \eta(\mu) \eta(\nu).$$

Hence we find

$$(III.19) \quad \mathcal{P} W^\mu \mathcal{P}^{-1} = -\eta(\mu) W^\mu, \quad \mathcal{T} W^\mu \mathcal{T}^{-1} = \eta(\mu) W^\mu$$

and, on using the definition of R^μ , Equation (III.16), and Equations (III.18), we obtain

$$(III.20) \quad \mathcal{P} R^\mu \mathcal{P}^{-1} = \eta(\mu) R^\mu, \quad \mathcal{T} R^\mu \mathcal{T}^{-1} = -\eta(\mu) R^\mu,$$

as expected. The behaviour of M under \mathcal{P} and \mathcal{T} does not follow from Equations (III.9) but the only reasonable assumption is that it transforms in the same way as P^0 :

$$(III.21) \quad \mathcal{P} M \mathcal{P}^{-1} = M, \quad \mathcal{T} M \mathcal{T}^{-1} = M.$$

We can express $M^{\mu\nu}$ and D in terms of the operators R^μ , P^μ , and $W^{\mu\nu}$ by the equations

$$(III.22) \quad \left\{ \begin{array}{l} M^{\mu\nu} = \frac{1}{2} (\{ P^\mu, R^\nu \} - \{ P^\nu, R^\mu \}) + W^{\mu\nu}, \\ D = \frac{1}{2} \{ R^\mu, P_\mu \}. \end{array} \right.$$

^(13a) It is elementary exercise to show that, even classically, a relativistic spinning particle cannot be localised at a point. Let the particle have rest mass m and radius r . Then in the centre-of-mass frame the magnitude of the angular momentum \mathbf{s} is given by $s \sim mrv$ where \mathbf{v} is the velocity of some point on the particle's surface. Now we must have $v < c$, and therefore $s \lesssim mrc$. Hence for a particle of angular momentum \mathbf{s} , we must have $r \gtrsim s/mc$, which is the content of the quantum-mechanical commutator (III.17 e). A rather more sophisticated argument has been given by Møller ([7], [8], p. 179). An interesting corollary of Equation (III.17 e) is that a high spin particle (i. e. $s > 1$) cannot be localised even to a length \approx its Compton wavelength (i. e. $1/mc$) and so virtual pair creation for such particles will be inhibited.

As in the nonrelativistic case, the operators D and R^μ are not unique, since we can make the redefinitions

$$(III.23) \quad \begin{cases} D \rightarrow D + a, \\ R^\mu \rightarrow R^\mu + \frac{a P^\mu}{P^2}, \end{cases}$$

where a is a constant, which leave the commutation relations (III.17) unchanged. Once again, though, if we want the new operators to have the correct transformation properties under parity and/or time-reversal, then this puts constraints on the constant a .

At this stage, we note parenthetically that Kastrup [36] and Castell [37] attempted to introduce a relativistic position operator by working with the conformal group and interpreting the generator of special conformal transformations as the required operator. We felt that this was incorrect because one can introduce a space position operator within the context of the inhomogeneous Lorentz group ([5]-[8]) (it is essentially the boost generator, M^{0i}) so only one more group generator should be required to give a time operator, too. The dilatation generator, D , is that group generator.

We now continue with our analysis. To define the spin operator of our virtual particle we go to the frame where it has zero three-momentum [38]. We define the spin operator, $\mathbf{S}(P, W)$, by the equation

$$(III.24) \quad (P^2)^{1/2} S^i(P, W) = L^{-1}(P)^i_\nu W^\nu = W^i - \frac{P^i W^0}{(P^2)^{1/2} + P^0},$$

where $L^{-1}(P)^\mu_\nu$ is the operator matrix which takes P^ν into $((P^2)^{1/2}, \mathbf{0})$:

$$(III.25) \quad L^{-1}(P)^\mu_\nu = \begin{pmatrix} \frac{P^0}{(P^2)^{1/2}} & \frac{P_j}{(P^2)^{1/2}} \\ -\frac{P^i}{(P^2)^{1/2}} & \delta^i_j - \frac{P^i P_j}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)} \end{pmatrix}.$$

Our Equation (III.24) agrees with Macfarlane's ([39], Equation (3.10)) definition of $\mathbf{S}(P, W)$ if one allows for the fact that he is discussing on-mass-shell particles (i. e. the inhomogeneous Lorentz group) ⁽¹⁴⁾. Using Equations (III.24) and (III.12 e), we find that $\mathbf{S}(P, W)$ satisfies

$$(III.26) \quad [S^i(P, W), S^j(P, W)] = i \varepsilon^{ijk} S^k(P, W)$$

and therefore

$$(III.27) \quad -\frac{W^2}{P^2} = \mathbf{S}(P, W)^2 = s(s+1),$$

⁽¹⁴⁾ Note however that Macfarlane denotes as $L(P)$ the matrix we call $L^{-1}(P)$, ([39], Equation (2.8)).

where s is the spin of the particle. The operator W^μ can be written in terms of $\mathbf{S}(P, W)$ and P^μ as :

$$(III. 28) \quad W^\mu = \left(\mathbf{P} \cdot \mathbf{S}, (P^2)^{1/2} \mathbf{S} + \frac{\mathbf{P} \mathbf{P} \cdot \mathbf{S}}{(P^2)^{1/2} + P^0} \right),$$

where, for convenience, we have written $\mathbf{S} = \mathbf{S}(P, W)$. Similarly, the operator

$$W^{\mu\nu} = \frac{\varepsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma}{P^2}$$

is given by

$$(III. 29 a) \quad W^{ij} = -\varepsilon^{ijk} \left(S^k + \frac{(\mathbf{P} \times (\mathbf{S} \times \mathbf{P}))^k}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)} \right).$$

$$(III. 29 b) \quad W^{0k} = \frac{(\mathbf{P} \times \mathbf{S})^k}{(P^2)^{1/2}}.$$

The operator $\mathbf{S}(P, W)$ commutes with P^μ :

$$(III. 30) \quad [P^\mu, S^i] = 0$$

and its commutation relations with the space-time position operator, R^μ , defined by Equation (III.16), are found by using Equations (III.24), (II.14 b), (III.17 f) and the fact that the commutator of R^μ with any function of P^ν alone is given by

$$(III. 31) \quad [R^\mu, f(P^\nu)] = -i \frac{\partial f(P^\nu)}{\partial P_\mu}.$$

We find

$$(III. 32 a) \quad [R^0, S^i] = 0,$$

$$(III. 32 b) \quad [R^i, S^j] = \frac{i (P^i S^j - \mathbf{P} \cdot \mathbf{S} \delta^{ij})}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)}.$$

Hence, we find, using Equations (III.26) and (III.30) that

$$(III. 33) \quad \left[R^i + \frac{(\mathbf{P} \times \mathbf{S})^i}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)}, S_i \right] = 0$$

and Equations (III.32 a) and (III.33), together with the equations

$$(III. 34 a) \quad \left[R^0, R^i + \frac{(\mathbf{P} \times \mathbf{S})^i}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)} \right] = 0,$$

$$(III. 34 b) \quad \left[R^k + \frac{(\mathbf{P} \times \mathbf{S})^k}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)}, R^i + \frac{(\mathbf{P} \times \mathbf{S})^i}{(P^2)^{1/2} ((P^2)^{1/2} + P^0)} \right] = 0,$$

which have been evaluated using Equations (III.17 e), (III.29 b), (III.17 g), (III.31) and (III.26), allow us to construct a canonical form for R^μ if we can find a canonical form for $\mathbf{S}(P, W)$. This is done by defining the spin operator, $\mathbf{S}(p)$, of a state of four-momentum p^μ by [39], Equations (3.5) and (3.6) [cf. Equation (III.24)] :

$$(III. 35) \quad (p^2)^{1/2} S^i(p) = U^{-1} (L^{-1}(p)) W^i U (L^{-1}(p)) = L^{-1}(p)^i_\nu W^\nu.$$

The operator $\mathbf{S}(p)$ satisfies equations analogous to (III.26) and (III.27) for $\mathbf{S}(P, W)$:

$$(III.36) \quad [S^i(p), S^j(p)] = i \varepsilon^{ijk} S^k(p),$$

$$(III.37) \quad -\frac{W^2}{p^2} = \mathbf{S}(p)^2 = s(s+1).$$

It is easily seen that, when operating on a state of four-momentum p^μ , $\mathbf{S}(p)$ is given by the usual $(2s+1)$ -dimensional representation of the rotation group generators ([39, Equation (3.4)]. Thus, we can now write the canonical forms :

$$(III.38) \quad \left\{ \begin{array}{l} P^\mu \rightarrow p^\mu, \\ \mathbf{S}(P, W) \rightarrow \mathbf{S}(p), \\ R^0 \rightarrow -i \left(\frac{\partial}{\partial p^0} \right)_{\mathbf{p}}, \\ \mathbf{R} \rightarrow i \left(\frac{\partial}{\partial \mathbf{p}} \right)_{p^0} - \frac{\mathbf{p} \times \mathbf{S}(p)}{(p^2)^{1/2} ((p^2)^{1/2} + p^0)} \\ \quad = i \left(\frac{\partial}{\partial \mathbf{p}} \right)_{p^2} - \frac{i \mathbf{p}}{p^0} \left(\frac{\partial}{\partial p^0} \right)_{\mathbf{p}} - \frac{\mathbf{p} \times \mathbf{S}(p)}{(p^2)^{1/2} ((p^2)^{1/2} + p^0)}, \end{array} \right.$$

where the derivatives are taken at constant $\mathbf{S}(p)$. The canonical forms (III.38) satisfy Equations (III.26), (III.30), (III.32), (III.33) and (III.34). We can now write a canonical form for the generators $M^{\mu\nu}$ and D using Equations (III.22), (III.29) and (III.38). Writing $J^i = -(1/2) \varepsilon^{ijk} M^{jk}$ and $N^i = -M^{0i}$, we find

$$(III.39) \quad \left\{ \begin{array}{l} \mathbf{J} \rightarrow -i \mathbf{p} \times \left(\frac{\partial}{\partial \mathbf{p}} \right)_{p^0} + \mathbf{S}(p), \\ \mathbf{N} \rightarrow -i \mathbf{p} \left(\frac{\partial}{\partial p^0} \right)_{\mathbf{p}} - i p^0 \left(\frac{\partial}{\partial \mathbf{p}} \right)_{p^0} - \frac{\mathbf{p} \times \mathbf{S}(p)}{(p^2)^{1/2} + p^0} \\ \quad = -i p^0 \left(\frac{\partial}{\partial \mathbf{p}} \right)_{p^2} - \frac{\mathbf{p} \times \mathbf{S}(p)}{(p^2)^{1/2} + p^0}, \\ D \rightarrow -i \left(p^0 \left(\frac{\partial}{\partial p^0} \right)_{\mathbf{p}} + \mathbf{p} \left(\frac{\partial}{\partial \mathbf{p}} \right)_{p^0} + 2 \right) \end{array} \right.$$

and it is clear that Equations (III.39), together with the canonical form for P^μ given in Equations (III.38) and the trivial form for the mass operator $M \rightarrow m$, satisfy the extended Lie algebra (III.11). Note that the canonical forms for \mathbf{J} and \mathbf{N} are the same as in the case of the inhomogeneous Lorentz group ([39], Equations (3.18) and (3.19)).

Although the canonical forms (III.38) and (III.39) are the most obvious and useful, there exists another canonical form which can be constructed using non-hermitean operators and which is valid for space-

like, as well as timelike, four-momentum. We first define the Gürsey-Radicati ([40], [41]) spin tensor, $X^{\mu\nu}$, by

$$(III.40) \quad X^{\mu\nu} = W^{\mu\nu} + \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} W_{\rho\sigma} = \frac{\varepsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma - i(P^\mu W^\nu - P^\nu W^\mu)}{P^2}.$$

This is a self-dual antisymmetric second-rank tensor

$$(III.41) \quad X^{\mu\nu} = \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} X_{\rho\sigma}$$

and is non-hermitean. It has the important property that it satisfies the same commutation relations as $M^{\mu\nu}$:

$$(III.42) \quad [X^{\mu\nu}, X^{\rho\sigma}] = i(X^{\mu\rho} g^{\nu\sigma} - X^{\mu\sigma} g^{\nu\rho} + X^{\nu\sigma} g^{\mu\rho} - X^{\nu\rho} g^{\mu\sigma})$$

and has the following commutation relations with the generators :

$$(III.43) \quad \begin{cases} [M^{\mu\nu}, X^{\rho\sigma}] = i(X^{\mu\rho} g^{\nu\sigma} - X^{\mu\sigma} g^{\nu\rho} + X^{\nu\sigma} g^{\mu\rho} - X^{\nu\rho} g^{\mu\sigma}), \\ [P^\mu, X^{\rho\sigma}] = [D, X^{\rho\sigma}] = [M, X^{\rho\sigma}] = 0. \end{cases}$$

We next define the corresponding orbital angular momentum tensor, $\hat{M}^{\mu\nu}$, by

$$(III.44) \quad \hat{M}^{\mu\nu} = M^{\mu\nu} - X^{\mu\nu}.$$

This also satisfies angular momentum commutation relations

$$(III.45) \quad [\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = i(\hat{M}^{\mu\rho} g^{\nu\sigma} - \hat{M}^{\mu\sigma} g^{\nu\rho} + \hat{M}^{\nu\sigma} g^{\mu\rho} - \hat{M}^{\nu\rho} g^{\mu\sigma})$$

and has the following commutators with the generators :

$$(III.46) \quad \begin{cases} [\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = i(\hat{M}^{\mu\rho} g^{\nu\sigma} - \hat{M}^{\mu\sigma} g^{\nu\rho} + \hat{M}^{\nu\sigma} g^{\mu\rho} - \hat{M}^{\nu\rho} g^{\mu\sigma}), \\ [\hat{M}^{\mu\nu}, P^\sigma] = -i(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma}), \\ [D, \hat{M}^{\mu\nu}] = [M, \hat{M}^{\mu\nu}] = 0. \end{cases}$$

Furthermore, $\hat{M}^{\mu\nu}$ and $X^{\rho\sigma}$ commute :

$$(III.47) \quad [\hat{M}^{\mu\nu}, X^{\rho\sigma}] = 0$$

(cf. the corresponding formulae in nonrelativistic quantum mechanics with $M^{\mu\nu} \rightarrow \mathbf{J}$, $\hat{M}^{\mu\nu} \rightarrow \mathbf{L}$, $X^{\mu\nu} \rightarrow \mathbf{S}$). We next define the modified position operator, \hat{R}^μ , by

$$(III.48) \quad \hat{R}^\mu = R^\mu + \frac{i W^\mu}{P^2},$$

whose different components commute :

$$(III.49) \quad [\hat{R}^\mu, \hat{R}^\nu] = 0$$

and which has the following commutation relations with the generators :

$$(III.50) \quad \begin{cases} [M^{\mu\nu}, \hat{R}^\sigma] = -i(\hat{R}^\mu g^{\nu\sigma} - \hat{R}^\nu g^{\mu\sigma}), \\ [\hat{R}^\mu, P^\nu] = -i g^{\mu\nu}, \quad [D, \hat{R}^\mu] = i \hat{R}^\mu, \quad [M, \hat{R}^\mu] = 0 \end{cases}$$

and with $\hat{M}^{\mu\nu}$ and $X^{\mu\nu}$:

$$(III.51 a) \quad [\hat{M}^{\mu\nu}, \hat{R}^\sigma] = -i(\hat{R}^\mu g^{\nu\sigma} - \hat{R}^\nu g^{\mu\sigma}),$$

$$(III.51 b) \quad [X^{\mu\nu}, \hat{R}^\sigma] = 0.$$

We can express $M^{\mu\nu}$ and D in terms of \hat{R}^μ , P^μ and $X^{\mu\nu}$ by the equations

$$(III.52) \quad \left\{ \begin{array}{l} M^{\mu\nu} = \hat{M}^{\mu\nu} + X^{\mu\nu} = \frac{1}{2}(\{P^\mu, \hat{R}^\nu\} - \{P^\nu, \hat{R}^\mu\}) + X^{\mu\nu}, \\ D = \frac{1}{2}\{\hat{R}^\mu, P_\mu\}, \end{array} \right.$$

which follow from Equations (III.22), (III.40) and (III.48). Now Equations (III.49), (III.50) and (III.51 a) allow us to write the following canonical form for P^μ , \hat{R}^μ , $M^{\mu\nu}$ and D :

$$(III.53) \quad \left\{ \begin{array}{l} P^\mu \rightarrow p^\mu, \\ \hat{R}^\mu \rightarrow -i \frac{\partial}{\partial p_\mu} \equiv \hat{r}^\mu, \\ \hat{M}^{\mu\nu} \rightarrow \frac{1}{2}(\{p^\mu, \hat{r}^\nu\} - \{p^\nu, \hat{r}^\mu\}), \\ D \rightarrow \frac{1}{2}\{\hat{r}^\mu, p_\mu\}, \end{array} \right.$$

so, if we can find a canonical form for $X^{\mu\nu}$, we have solved the problem. For timelike four-momentum, this can be done by replacing P^μ and $\mathbf{S}(P, W)$ in Equations (III.28) and (III.40) by p^μ and $\mathbf{S}(p)$ respectively :

$$(III.54) \quad \left\{ \begin{array}{l} W^\mu \rightarrow w^\mu(p, \mathbf{S}(p)) = \left(\mathbf{p} \cdot \mathbf{S}(p), (p^2)^{1/2} \mathbf{S}(p) + \frac{\mathbf{p}\mathbf{p} \cdot \mathbf{S}(p)}{(p^2)^{1/2} + p^0} \right), \\ X^{\mu\nu} \rightarrow x^{\mu\nu}(p, \mathbf{S}(p)) = \frac{\varepsilon^{\mu\nu\rho\sigma} p_\rho w_\sigma - i(p^\mu w^\nu - p^\nu w^\mu)}{p^2}. \end{array} \right.$$

Because of Equation (III.51 b), the partial derivative, $\partial/\partial p_\mu$, in Equations (III.53) should be taken at constant $x^{\rho\sigma}$. Then, by using the definition of \hat{R}^μ , Equation (III.48) and $\hat{M}^{\mu\nu}$, Equation (III.44), we find the following canonical form for our operators :

$$(III.55) \quad \left\{ \begin{array}{l} P^\mu \rightarrow p^\mu, \\ R^\mu \rightarrow -i \left(\frac{\partial}{\partial p_\mu} \right)_{x^{\rho\sigma}} - i \frac{w^\mu}{p^2} \equiv \hat{r}^\mu - i \frac{w^\mu}{p^2}, \\ M^{\mu\nu} \rightarrow \frac{1}{2}(\{p^\mu, \hat{r}^\nu\} - \{p^\nu, \hat{r}^\mu\}) + x^{\mu\nu}, \\ D \rightarrow \frac{1}{2}\{\hat{r}^\mu, p_\mu\}, \end{array} \right.$$

which together with the trivial form for the mass operator, $M \rightarrow m$, satisfy the extended Lie algebra (III.11). We also note the existence of the hermitean conjugate set of operators $(P^u, \hat{R}^{u\dagger}, \hat{M}^{u,v\dagger}, X^{u,v\dagger})$ which give another canonical form which is in some way equivalent to (III.55), and it has occurred to us that the non-hermitean operators with which we have been working should be interpreted using an indefinite metric as was done by Kastrup [42] for the conformal group. We have not, however, had any success with this idea, and a more likely interpretation is that \hat{R}^u is the non-normal ⁽¹⁵⁾ position operator of Kálnay and Toledo [43], the hermitean part, R^u , being the position of the centre-of-mass, and the antihermitean part, W^u/P^2 , being the radius of the particle.

B. NONRELATIVISTIC LIMIT

We shall now show that the extended Lie algebra (III.11) goes over into the nonrelativistic extended Lie algebra (II.16) as $c \rightarrow \infty$. We merely define the operators

$$(III.56) \quad \left\{ \begin{array}{l} M' = M, \\ H' = P^0 c - M c^2, \\ P'^i = P^i, \\ J'^i = -\frac{1}{2} \varepsilon^{ijk} M'^k, \\ K'^i = -\frac{M^{0i}}{c}, \\ D' = \frac{1}{2} \left\{ \frac{P^2 - M^2 c^2}{P^2}, D \right\} - \frac{\{P^0 P^i, M^{0i}\}}{2 P^2} \end{array} \right.$$

and it is easily seen, using the commutators of the extended Lie algebra (III.11), that, as $c \rightarrow \infty$, the primed operators go over into the corresponding nonrelativistic operators defined in Section II.1.A, and generate the Lie algebra (II.16). It is also easily seen, using Equations (III.56) and the fact that $(P^2 - M^2 c^2) \rightarrow 2 MU$ as $c \rightarrow \infty$, that R^0/c and \mathbf{R} , defined by Equation (III.16), go over into the nonrelativistic time and position operator, given by Equations (II.19), as $c \rightarrow \infty$.

Note the rather complicated definition of D' in Equations (III.56). (In fact, the definition is not unique, we could replace either P^2 in the denominators of the two terms in the expression for D' by $M^2 c^2$.) The limit $c \rightarrow \infty$ is not a simple "contraction" ([44], [45]) as is that which gives the Lie algebra of the extended Galilei group from the Lie

⁽¹⁵⁾ A non-normal operator is one which does not commute with its hermitean conjugate.

algebra of the direct product of the inhomogeneous Lorentz group and the phase group [30]. Another indication of the complicated nature of the relationship between the relativistic and nonrelativistic dilatations is given by considering the trivial exponent, $\xi_m^{\lambda'}(G', G)$, generated by M :

$$(III.57) \quad \begin{cases} \xi_m^{\lambda'}(G', G) = \xi_m(G'G) - \xi_m(G') - \xi_m(G), \\ \xi_m(G) = mca^0, \end{cases}$$

where G, G', and G'G are given by Equation (III.5). Saletan [46] showed that, in the case of the contraction of the inhomogeneous Lorentz group into the Galilei group, the trivial exponent, $\xi_m^{\lambda'}(G', G)$, given by (III.57), goes over into the *non*-trivial exponent of the Galilei group [Equation (II.35) with $\lambda' = 1$]. Now, in our case, using the formulae :

$$(III.58) \quad \begin{cases} \xi_m^{\lambda'}(G', G) = mc(\lambda' L'^0_\nu a^\nu - a^0), \\ L'^{\mu\nu}_\nu = \begin{pmatrix} \gamma' & -\frac{v'_k \gamma'}{c} \\ \frac{v'^i \gamma'}{c} & \delta^i_k - \frac{v'^i v'_k (\gamma' - 1)}{v'^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R'^k_j \end{pmatrix} \end{cases}$$

where we have written $\gamma' = (1 - \mathbf{v}'^2/c^2)^{-1/2}$ for convenience, we find

$$(III.59) \quad \xi_m^{\lambda'}(G', G) = mca^0 \left(\frac{\lambda'}{(1 - \mathbf{v}'^2/c^2)^{1/2}} - 1 \right) + \frac{m \lambda' \mathbf{v}' \cdot \mathbf{R}' \mathbf{a}}{(1 - \mathbf{v}'^2/c^2)^{1/2}}$$

and the first term is not defined as $c \rightarrow \infty$ (except, of course, when $\lambda' = 1$, when we reproduce the result of Saletan [46]). Thus, we do not retrieve the non-trivial exponent, (II.35), as $c \rightarrow \infty$, and the representations considered in Sections II.1.C and III.1.C are not simply related to one another as are the corresponding representations of the Galilei group and inhomogeneous Lorentz group.

C. TRANSFORMATION OF STATES

We now consider the transformation properties of the single-particle states under the operator U(G) which satisfies Equation (II.15), with G', G and G'G given by Equation (III.5) and with $\xi(G', G) = 0$, as we are at liberty to do from the work of Appendix A.II⁽¹⁶⁾. From the

(¹⁶) Actually the equation $U(G')U(G) = U(G'G)$ is true only for the universal covering group, not for the Weyl group itself. However, the transition from the covering group to the Weyl group introduces into the equation only a possible sign factor, which we have ignored. (The sign factor also occurs for the nonrelativistic group of Section II.) The transition from the covering group to the group itself is by no means trivial though, e. g. the covering group of the two-dimensional Galilei group has a central extension which is not an extension of the group itself ([16], Section III.B).

work of Section III.1.A, the irreducible representations are labelled by the mass m and spin s [the eigenvalues of M and $\mathbf{S}(P, W)^2$], and states within an irreducible representation can be labelled by the eigen values of the commuting operators P^μ and $S^3(P, W)$. Our states will thus be denoted $|m, s; p, \sigma\rangle$, though we shall drop the (m, s) label for brevity. They are normalised according to

$$(III.60) \quad \langle p', \sigma' | p, \sigma \rangle = \delta_{\sigma', \sigma} \delta^4(p' - p).$$

As in the nonrelativistic case discussed in Section II.1.C, the evaluation of $U(\lambda, L, a) |p, \sigma\rangle$ is simplified by the fact that, from Equations (III.5), (II.15) and the work of Appendix A.II, we can write

$$(III.61) \quad U(\lambda, L, a) = U(1, L, a) U(\lambda, 1, 0),$$

where $U(1, L, a)$ is a unitary operator of the inhomogeneous Lorentz group, whose effect on a physical state $|p, \sigma\rangle$ has been given by Wigner [1] :

$$(III.62) \quad U(1, L, a) |p, \sigma\rangle = e^{ip'a} \sum_{\sigma'=-s}^s |p', \sigma'\rangle D^s_{\sigma'\sigma}(L^{-1}(p')LL(p)),$$

where the matrix $(^{17}) L^{-1}(p')LL(p)$ is a pure rotation, and where p'^μ is given by $p' = Lp$. We now define the action of $U(\lambda, 1, 0)$ on a state $|p, \sigma\rangle$ by the equation

$$(III.63) \quad U(\lambda, 1, 0) |p, \sigma\rangle = \lambda^{-2} |\lambda^{-1} p, \sigma\rangle,$$

where the factor λ^{-2} is required to preserve the normalisation (III.60), as is necessary for $U(\lambda, 1, 0)$ to be unitary, as in the nonrelativistic case. We now evaluate $U(\lambda, L, a) |p, \sigma\rangle$, using Equations (III.62) and (III.63), and find :

$$(III.64) \quad U(\lambda, L, a) |p, \sigma\rangle = \lambda^{-2} e^{ip'a} \sum_{\sigma'=-s}^s |p', \sigma'\rangle D^s_{\sigma'\sigma}(L^{-1}(p')LL(p)),$$

where $p' = \lambda^{-1} Lp$. In writing Equation (III.64), we have used the fact that $L(\lambda^{-1} p) = L(p)$, since $L(p)$ should more correctly be written $L(\mathbf{p}/p^0)$. Equation (III.64) constitutes a unitary up-to-a sign representation of the Weyl group (III.1) in Hilbert space.

We shall now discuss the transformation properties of single particle states localised in space and time. The discussion here is more involved than the nonrelativistic case discussed in Section II.1.C. There are two separate factors which complicate the issue. Firstly, the different

(¹⁷) The matrix $L^{-1}(p)$ is defined in Equation (III.25).

components of the space-time position operator, R^μ , do not commute for particles of non-zero spin, *see* Equation (III.17 e), so we cannot construct a state which is an eigenstate of the four-components of R^μ . Secondly, we construct states localised in space and time by superposing states of different four-momentum, and we are limited in the integration over the momentum, p , to positive p^2 and positive p^0 [*cf.* the discussion following Equation (III.14)]. (*See* Section III.5 for a discussion of this and other possible “supersuperselection rules”.) Hence the most localised state which we can construct for a relativistic virtual particle is

$$(III.65) \quad |m, s; r, \sigma\rangle = \frac{1}{(2\pi)^2} \int d^4 p \theta(p^2) \theta(p^0) e^{ip \cdot r} |m, s; p, \sigma\rangle,$$

where we use the notation $|r, \sigma\rangle$, rather than $|r, \sigma\rangle$, to denote that $|r, \sigma\rangle$ is *not* an eigenstate of R^μ . Using Equation (III.60), it is easily seen that the state $|r, \sigma\rangle$, defined in Equation (III.65), is normalised to

$$(III.66) \quad \langle r', \sigma' | r, \sigma\rangle = \delta_{\sigma'\sigma} \frac{1}{2\pi^3} \left(P \frac{1}{(s^2)^2} + \pi i \varepsilon(s^0) \delta^{(1)}(s^2) \right),$$

where we have written $s = (r' - r)$, and P denotes “principal part”, ε denotes “sign”, and $\delta^{(1)}$ denotes the first derivative of the delta-function. On applying $U(\lambda, L, a)$ to $|r, \sigma\rangle$, given by Equation (III.65) and using Equation (III.64), we find

$$(III.67) \quad U(\lambda, L, a) |r, \sigma\rangle = \frac{\lambda^{-2}}{(2\pi)^2} \int d^4 p \theta(p^2) \theta(p^0) e^{i(p \cdot r + p' \cdot a)} \\ \times \sum_{\sigma'=-s}^s |p', \sigma'\rangle D^s_{\sigma'\sigma}(L^{-1}(p')LL(p)),$$

where $p' = \lambda^{-1} L p$. On using the equation

$$(III.68) \quad \begin{cases} p \cdot r = p' \cdot (\lambda L r), \\ \theta(p^2) \theta(p^0) = \theta(p'^2) \theta(p'^0), \\ d^4 p = \lambda^4 d^4 p', \end{cases}$$

in Equation (III.67), we find

$$(III.69) \quad U(\lambda, L, a) |r, \sigma\rangle = \frac{\lambda^2}{(2\pi)^2} \int d^4 p' \theta(p'^2) \theta(p'^0) e^{i(p' \cdot r' + p' \cdot a)} \\ \times \sum_{\sigma'=-s}^s |p', \sigma'\rangle D^s_{\sigma'\sigma}(L^{-1}(p')LL(\lambda L^{-1} p')),$$

where $r' = \lambda L r + a$. We have not been able to express the right hand side of Equation (III.69) in a simpler form due to the presence

of the p' -dependent Wigner rotation matrix in the integrand (though it can, of course, be written as the convolution of a localised state, $| \)$, and the Fourier transform of the Wigner rotation matrix), but it is clearly localised around $r' = \lambda L r + a$. For a spinless particle, Equation (III.69) becomes :

$$(III.70) \quad U(\lambda, L, a) | r \rangle = \lambda^3 | r' \rangle.$$

an extremely simple transformation law.

2. Irreducible representations with spacelike four-momentum

We now look at the extended Lie algebra, Equations (III.11), when $P^2 < 0$, and interpret it as describing a virtual particle with spacelike four-momentum. The first thing that we note about the irreducible representations with $P^2 < 0$ is that all the work of Section III.1.A, up to and including Equations (III.23), remains valid whether P^2 is positive or negative, the difference between the two cases being in the definition of the "spin", as in the case of the inhomogeneous Lorentz group ([1], [21]). The point is that we can no longer choose a Lorentz transformation which takes the four-momentum, P^ν , into $((P^2)^{1/2}, \mathbf{0})$, [cf. Equation (III.25)]. We can, however, choose a Lorentz transformation which takes P^ν into $(0, (-P^2)^{1/2}, 0, 0)$, and we call this Lorentz transformation $\Lambda(P)^{\mu_\nu}$. Then in analogy with Equation (III.24) for $\mathbf{S}(P, W)$, we define

$$(III.71) \quad (-P^2)^{1/2} \Sigma^\mu(P, W) = \Lambda(P)^{\mu_\nu} W^\nu$$

and the $\Sigma^\mu(P, W)$ are given in terms of the transformed generators

$$M'^{\mu\nu} = \Lambda(P)^{\mu_\rho} \Lambda(P)^{\nu_\sigma} M^{\rho\sigma}$$

by the equations

$$(III.72) \quad \begin{cases} \Sigma^0(P, W) = -M'^{23} = J'^1, \\ \Sigma^1(P, W) = 0, \\ \Sigma^2(P, W) = -M'^{03} = N'^3, \\ \Sigma^3(P, W) = M'^{02} = -N'^2. \end{cases}$$

From the commutation relations of the W^μ , (III.12 e) and Equation (III.71), we find the Lie algebra of our "little group" [analogous to Equations (III.26)] :

$$(III.73) \quad \begin{cases} [\Sigma^0(P, W), \Sigma^2(P, W)] = i \Sigma^3(P, W), \\ [\Sigma^0(P, W), \Sigma^3(P, W)] = -i \Sigma^2(P, W), \\ [\Sigma^2(P, W), \Sigma^3(P, W)] = -i \Sigma^0(P, W), \end{cases}$$

which is the Lie algebra of $SU(1, 1)$, as in the case of the inhomogeneous Lorentz group, for spacelike four-momentum [47]. The uni-

tary irreducible representations of the group $SU(1, 1)$, have been studied in detail by Bargmann [48] [for a review of $SU(1, 1)$ using the "algebraic" approach, see Biedenharn ([49], Section III)], but we shall merely make a few comments on the extended Weyl group Lie algebra (III.11) and its nonrelativistic limit. For a discussion of the nonrelativistic limit of the irreducible representations of the inhomogeneous Lorentz group from the "little group" viewpoint, see Ryder [21].

The invariant operators of the Lie algebra (III.11) are given by Equation (III.14), and states within an irreducible representation can be labelled by the eigenvalues of P^μ and $\Sigma^0(P, W)$. To investigate the nonrelativistic limit of Equations (III.11), we define the primed operators

$$(III.74) \quad \left\{ \begin{array}{l} M' = M, \\ H' = \frac{P^0}{c}, \\ P'^i = P^i, \\ J'^i = -\frac{1}{2} \varepsilon^{ijk} M^{jk}, \\ K'^i = -\frac{M^{0i}}{c}, \\ D' = D + \frac{\{P^0 P^i, M^{0i}\}}{2P^2} \end{array} \right.$$

and it is easily seen, using the commutators of the extended Lie algebra (III.11), that the primed operators go over into the corresponding nonrelativistic operators defined in Section II.2 (i. e. the operators describing Inönü-Wigner [15] Class II virtual particles) as $c \rightarrow \infty$, provided that $(P' \times K')$ and M' both go to zero in this limit. It is also easily seen using Equations (III.74) that the operators R^0/c and \mathbf{R} , defined by Equation (III.16), go over into the nonrelativistic time and position operators, given by Equations (II.54) and (II.58) respectively, as $c \rightarrow \infty$.

Particles for which $(P' \times K')$ and M' do *not* go to zero as $c \rightarrow \infty$ go over into Inönü-Wigner Class II virtual particles, which we conjectured in Section II.2 to be described by irreducible representations of the group of inhomogeneous Galilei transformations and dilatations $(t', \mathbf{x}') = (\lambda_1 t, \lambda_2 \mathbf{x})$.

3. The system of two relativistic virtual particles

In this section we shall investigate the system of two relativistic massive virtual particles, each of which has timelike four-momentum ⁽¹⁸⁾.

⁽¹⁸⁾ See Macfarlane [39] for a detailed discussion of the system of two relativistic on-mass-shell particles, and for a complete bibliography of previous work on the subject.

We shall find that the centre-of-mass variables form a (not simply) reducible representation of the extended Weyl group with timelike four-momentum, whilst the relative variables, which are found, as in the nonrelativistic case, by applying the Gartenhaus-Schwartz ([24], [50], [51]) technique, form a reducible (but undecomposable) representation of the Weyl group with spacelike on-mass-shell four-momentum. Both the centre-of-mass and relative variables are completely covariant. However, unlike the nonrelativistic case, the centre-of-mass and relative operators do not commute with one another.

We work with on-mass-shell mass, four-momentum, four-position, and spin tensor operators of the two particles which satisfy [*cf.* Equations (III.11), (III.13) and (III.17)] :

$$(III.75) \left\{ \begin{aligned} & [P_1^\mu, P_1^\nu] = 0, \quad [R_1^\mu, R_1^\nu] = -\frac{i W_1^{\mu\nu}}{P_1^2}, \\ & [W_1^{\mu\nu}, W_1^{\rho\sigma}] \\ & = i \left(W_1^{\mu\rho} \left(g^{\nu\sigma} - \frac{P_1^\nu P_1^\sigma}{P_1^2} \right) - W_1^{\mu\sigma} \left(g^{\nu\rho} - \frac{P_1^\nu P_1^\rho}{P_1^2} \right) \right. \\ & \quad \left. + W_1^{\nu\sigma} \left(g^{\mu\rho} - \frac{P_1^\mu P_1^\rho}{P_1^2} \right) - W_1^{\nu\rho} \left(g^{\mu\sigma} - \frac{P_1^\mu P_1^\sigma}{P_1^2} \right) \right), \\ & [R_1^\mu, P_1^\nu] = -i g^{\mu\nu}, \quad [P_1^\mu, W_1^{\rho\sigma}] = 0, \\ & [R_1^\mu, W_1^{\rho\sigma}] = -i \frac{P_1^\rho W_1^{\sigma\mu} - P_1^\sigma W_1^{\rho\mu}}{P_1^2}, \\ & [M_1, P_1^\mu] = [M_1, R_1^\mu] = [M_1, W_1^{\mu\nu}] = 0, \end{aligned} \right.$$

where $(1/2) W_1^{\mu\nu} W_{1\mu\nu} = s_1 (s_1 + 1)$ with s_1 the spin of particle 1. There is a similar set of equations for particle 2 and the operators of the different particles commute. In terms of these operators, the generators of homogeneous Lorentz transformations and dilatations for particle 1 are defined by [*cf.* Equations (III.22)] :

$$(III.76) \left\{ \begin{aligned} M_1^{\mu\nu} &= \frac{1}{2} (\{ P_1^\mu, R_1^\nu \} - \{ P_1^\nu, R_1^\mu \}) + W_1^{\mu\nu}, \\ D &= \frac{1}{2} \{ R_1^\mu, P_{1\mu} \} \end{aligned} \right.$$

and similarly for particle 2.

Our task now is to construct the centre-of-mass on mass-shell mass four-momentum, four-position, and spin-tensor operators of the two particle system. We proceed as we did in the nonrelativistic case

(Section II.3). The centre-of-mass four-momentum operator is defined by

$$(III.77) \quad P^\mu = P_1^\mu + P_2^\mu$$

and the mass operator, M , is given by

$$(III.78) \quad M^2 = c^{-2} P^2 |_{\text{on shell}} = c^{-2} (P_1 + P_2)^2 |_{\text{on shell}}$$

and this can be evaluated in the centre-of-mass frame to give

$$(III.79) \quad M^2 = M_1^2 + M_2^2 + 2 \frac{\mathbf{e}^2}{c^2} + 2 \left(M_1^2 + \frac{\mathbf{e}^2}{c^2} \right)^{1/2} \left(M_2^2 + \frac{\mathbf{e}^2}{c^2} \right)^{1/2},$$

where

$$e^i = L^{-1} (P)^{i\nu} P_{1\nu} = -L^{-1} (P)^{i\nu} P_{2\nu},$$

and $L^{-1} (P)^{i\nu}$ is defined in Equation (III.25). We therefore see that the operator M has eigenvalues ranging between $(m_1 + m_2)$ and ∞ , as is well known [39]. The centre-of-mass generator of homogeneous Lorentz transformations, $M^{\mu\nu}$, is given by

$$(III.80) \quad M^{\mu\nu} = M_1^{\mu\nu} + M_2^{\mu\nu}.$$

The centre-of-mass dilatation generator will be discussed later. [It turns out, as in the nonrelativistic case, *see* Equation (II.80), that $(D_1 + D_2)$ is equal to the sum of the centre-of-mass and relative dilatation generators.] We define the "preliminary" centre-of-mass position operator, \tilde{R}^μ , by the equation

$$(III.81) \quad \tilde{R}^\mu = \frac{M_1 R_1^\mu + M_2 R_2^\mu}{M_1 + M_2},$$

which satisfies the following commutators with P^ν [defined by Equation (III.77)] and \tilde{R}^ν :

$$(III.82 a) \quad [\tilde{R}^\mu, P^\nu] = -ig^{\mu\nu},$$

$$(III.82 b) \quad [\tilde{R}^\mu, \tilde{R}^\nu] = -\frac{i}{(M_1 + M_2)^2} \left(\frac{M_1^2 W_1^{\mu\nu}}{P_1^2} + \frac{M_2^2 W_2^{\mu\nu}}{P_2^2} \right).$$

We also define the "preliminary" variables $(\tilde{q}^\mu, \tilde{r}^\mu, w^{\mu\nu})$ by

$$(III.83) \quad \left\{ \begin{array}{l} \tilde{q}^\mu = \frac{M_2 P_1^\mu - M_1 P_2^\mu}{M_1 + M_2}, \\ \tilde{r}^\mu = R_1^\mu - R_2^\mu, \\ \tilde{w}^{\mu\nu} = W_1^{\mu\nu} + W_2^{\mu\nu} \end{array} \right.$$

and these satisfy the following commutation relations among themselves

$$(III. 84) \left\{ \begin{aligned} [\tilde{q}^\mu, \tilde{q}^\nu] &= 0, & [\tilde{r}^\mu, \tilde{r}^\nu] &= -i \left(\frac{W_1^{\mu\nu}}{P_1^2} + \frac{W_2^{\mu\nu}}{P_2^2} \right), \\ [\tilde{w}^{\mu\nu}, \tilde{w}^{\rho\sigma}] &= [W_1^{\mu\nu}, W_1^{\rho\sigma}] + [W_2^{\mu\nu}, W_2^{\rho\sigma}], \\ [\tilde{r}^\mu, \tilde{q}^\nu] &= -ig^{\mu\nu}, & [\tilde{q}^\mu, \tilde{w}^{\rho\sigma}] &= 0, \\ [\tilde{r}^\mu, \tilde{w}^{\rho\sigma}] &= -i \frac{P_1^\rho W_1^{\sigma\mu} - P_1^\sigma W_1^{\rho\mu}}{P_1^2} \\ &\quad + i \frac{P_2^\rho W_2^{\sigma\mu} - P_2^\sigma W_2^{\rho\mu}}{P_2^2}, \end{aligned} \right.$$

where $[W_1^{\mu\nu}, W_1^{\rho\sigma}]$ and $[W_2^{\mu\nu}, W_2^{\rho\sigma}]$ are given in Equations (III.15). The set of operators $(\tilde{q}^\mu, \tilde{r}^\mu, \tilde{w}^{\mu\nu})$ has the following commutation relations with P^μ and \tilde{R}^μ :

$$(III. 85 a) \quad [P^\mu, \tilde{q}^\nu] = [P^\mu, \tilde{r}^\nu] = [P^\mu, \tilde{w}^{\rho\sigma}] = 0,$$

$$(III. 85 b) \left\{ \begin{aligned} [\tilde{R}^\mu, \tilde{q}^\nu] &= 0, \\ [\tilde{R}^\mu, \tilde{r}^\nu] &= -i \left(\frac{(M_1 W_1^{\mu\nu}/P_1^2) - (M_2 W_2^{\mu\nu}/P_2^2)}{M_1 + M_2} \right), \\ [\tilde{R}^\mu, \tilde{w}^{\rho\sigma}] &= -i \left(\frac{\{ M_1 [(P_1^\rho W_1^{\sigma\mu} - P_1^\sigma W_1^{\rho\mu})/P_1^2] \right.}{M_1 + M_2} \left. + M_2 [(P_2^\rho W_2^{\sigma\mu} - P_2^\sigma W_2^{\rho\mu})/P_2^2] \right). \end{aligned} \right.$$

We now evaluate $(M_1^{\mu\nu} + M_2^{\mu\nu})$ and $(D_1 + D_2)$ in terms of the operators $P^\mu, \tilde{R}^\mu, \tilde{q}^\mu, \tilde{r}^\mu$ and $\tilde{w}^{\mu\nu}$, using Equations (III. 76), (III. 77), (III. 81) and (III. 83) :

$$(III. 86 a) \quad M_1^{\mu\nu} + M_2^{\mu\nu} = \frac{1}{2} (\{ P^\mu, \tilde{R}^\nu \} - \{ P^\nu, \tilde{R}^\mu \}) \\ + \frac{1}{2} (\{ \tilde{q}^\mu, \tilde{r}^\nu \} - \{ \tilde{q}^\nu, \tilde{r}^\mu \}) + \tilde{w}^{\mu\nu},$$

$$(III. 86 b) \quad D_1 + D_2 = \frac{1}{2} \{ P^\mu, \tilde{R}_\mu \} + \frac{1}{2} \{ \tilde{q}^\mu, \tilde{r}_\mu \}.$$

But, in terms of the centre-of-mass four-momentum, four-position, and spin tensor operators $(P^\mu, R^\mu, W^{\mu\nu})$, the operators $M^{\mu\nu}$ and D are given by [cf. Equations (III. 22)] :

$$(III. 87 a) \quad M^{\mu\nu} = \frac{1}{2} (\{ P^\mu, R^\nu \} - \{ P^\nu, R^\mu \}) + W^{\mu\nu},$$

$$(III. 87 b) \quad D = \{ P^\mu, R_\mu \}.$$

So, on using Equation (III.80) to equate of the right-hand sides Equations (III.86 a) and (III.87 a), we find

$$(III.88) \quad W^{\mu\nu} = \frac{1}{2} (\{P^\mu, \tilde{R}^\nu - R^\nu\} - \{P^\nu, \tilde{R}^\mu - R^\mu\}) \\ + \frac{1}{2} (\{\tilde{q}^\mu, \tilde{r}^\nu\} - \{\tilde{q}^\nu, \tilde{r}^\mu\}) + \tilde{w}^{\mu\nu}.$$

On contracting Equation (III.88) with $(1/2) \varepsilon_{\lambda\mu\nu} P^\lambda$, and using the relation $W_x = (1/2) \varepsilon_{x\lambda\mu\nu} P^\lambda W^{\mu\nu}$, together with the facts that P^λ commutes with both $(\tilde{R}^\mu - R^\mu)$ and the relative operators $(\tilde{q}^\mu, \tilde{r}^\mu, \tilde{w}^{\mu\nu})$ [Equations (III.85)], we find

$$(III.89) \quad W^x = \frac{1}{2} \varepsilon^{x\lambda\mu\nu} P_\lambda \left(\tilde{w}_{\mu\nu} + \frac{1}{2} (\{\tilde{q}_\mu, \tilde{r}_\nu\} - \{\tilde{q}_\nu, \tilde{r}_\mu\}) \right),$$

which is the expression for the centre-of-mass spin pseudovector in terms of the single particle operators [via Equations (III.77) and (III.83)]. The expression for the spin tensor, $W^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma / P^2$, is easily found from Equation (III.89) :

$$(III.90) \quad W^{\mu\nu} = \left(\frac{g^{\mu\rho} - P^\mu P^\rho}{P^2} \right) \left(\frac{g^{\nu\sigma} - P^\nu P^\sigma}{P^2} \right) \\ \times \left(\tilde{w}_{\rho\sigma} + \frac{1}{2} (\{\tilde{q}_\rho, \tilde{r}_\sigma\} - \{\tilde{q}_\sigma, \tilde{r}_\rho\}) \right)$$

and on substituting this expression for $W^{\mu\nu}$ back into Equation (III.88), we can solve for R^μ :

$$(III.91) \quad R^\mu = \tilde{R}^\mu - P_\nu \frac{\tilde{w}^{\mu\nu} + (1/2) (\{\tilde{q}^\mu, \tilde{r}^\nu\} - \{\tilde{q}^\nu, \tilde{r}^\mu\})}{P^2},$$

where we have put a possible additive term $a P^\mu / P^2$ equal to zero, as we are at liberty to do by Equations (III.23). The expression (III.91) for R^μ gives for the centre-of-mass dilatation generator, D , by Equation (III.87 b) :

$$(III.92) \quad D = \frac{1}{2} \{P^\mu, \tilde{R}_\mu\}.$$

It is straightforward but tedious to verify, using Equations (II.14), (III.75), (III.84) and (III.85), that the operators M , P^μ , R^μ and $W^{\mu\nu}$, defined by Equations (III.78), (III.77), (III.91) and (III.90) respectively, satisfy commutation relations of the same form as those for particle 1, Equations (III.75). The only evaluation which is a little tricky is $[M, R^\mu]$. By Equation (III.79), we have to show that $[e^2, R^\mu] = 0$

where $e^i = L^{-1}(P)^i, \tilde{q}^\nu$. Evaluating e^2 and using Equations (III.2) and (III.25) to give

$$(III.93) \quad L^{-1}(P)^i, L^{-1}(P)^i{}_\sigma \\ = L^{-1}(P)^0, L^{-1}(P)^0{}_\sigma - g_{\nu\sigma} = - \left(g_{\nu\sigma} - \frac{P_\nu P_\sigma}{P^2} \right),$$

we find ⁽¹⁹⁾ :

$$(III.94) \quad e^2 = -\tilde{q}^2 + \frac{(P \cdot \tilde{q})^2}{P^2}$$

and it is easily seen that R^μ commutes with both \tilde{q}^2 and $(P \cdot \tilde{q})^2/P^2$, [see Equations (III.109) and (III.111 c)]. Thus, by Equation (III.79) :

$$(III.95) \quad [M, R^\mu] = 0,$$

as required. We therefore see that the operators $(M, P^\mu, M^{\mu\nu}, D)$ defined by Equations (III.77), (III.78) and (III.87) form a (reducible) representation of the extended Weyl group with timelike four-momentum, with invariant operators $(1/2) W^{\mu\nu} W_{\mu\nu} = \mathbf{S}(P, W)^2$ and M . A discussion of the actual way in which the spin operator $\mathbf{S}(P, W)$ is constructed, from the spin operators of the individual particles and their relative angular momentum, will be delayed until we have found the set of covariant relative operators of the two-particle system, which is the next topic on the agenda.

We shall now construct a set of covariant relative operators $(q^\mu, r^\mu, w^{\mu\nu})$ which satisfy commutation relations analogous to Equations (III.75) for particle 1 (except for $[M_1, R_1^\mu] = 0$, see below). We do this by the same method as in the nonrelativistic case viz. we apply the Gartenhaus-Schwartz transformation ([24], [50], [51]) (infinite dilatation) to the preliminary relative variables $(\tilde{q}^\mu, \tilde{r}^\mu, \tilde{w}^{\mu\nu})$. This transformation, whose generator is the centre-of-mass dilatation generator, D , acts on the centre-of-mass operators M, P^μ, R^μ and $W^{\mu\nu}$ according to

$$(III.96) \quad \left\{ \begin{array}{l} \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} M e^{i\alpha D} = M, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} P^\mu e^{i\alpha D} = \lim_{\alpha \rightarrow \infty} e^{-\alpha} P^\mu = 0, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} R^\mu e^{i\alpha D} = \lim_{\alpha \rightarrow \infty} e^\alpha R^\mu = \infty, \\ \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} W^{\mu\nu} e^{i\alpha D} = W^{\mu\nu}, \end{array} \right.$$

as can be seen by applying Equations (III.11), (III.13) and (III.17 c) to the centre-of-mass operators. If we denote by \tilde{O} the set of opera-

⁽¹⁹⁾ All that Equation (III.94) says is that if we define the four-vector $e^\lambda = L^{-1}(P)^\lambda, \tilde{q}^\nu$, then $e^0 = (P \cdot \tilde{q})/(P^2)^{1/2}$ as is obvious from Equation (III.25).

tors $(\tilde{q}^\mu, \tilde{r}^\mu, \tilde{w}^{\mu\nu})$ and by O the set of operators $(q^\mu, r^\mu, w^{\mu\nu})$, then O is constructed from \tilde{O} by the formula

$$(III.97) \quad O = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \tilde{O} e^{i\alpha D},$$

as in the nonrelativistic case [Equation (II.76)]. The right-hand side of Equation (III.97) is then evaluated by using Equations (II.77). The details are given in Appendix B; here we merely reproduce the results :

$$(III.98 a) \quad q^\mu = \tilde{q}^\mu,$$

$$(III.98 b) \quad r^\mu = \tilde{r}^\mu + \frac{(M_1 + M_2)}{((P \cdot q)^2 - P^2 q^2)} \\ \times \left(\frac{(P_1^2)^{1/2}}{M_1} W_1^{\mu\rho} q_\rho P \cdot Q_1 + \frac{(P_2^2)^{1/2}}{M_2} W_2^{\mu\rho} q_\rho P \cdot Q_2 \right),$$

$$(III.98 c) \quad W_1^{\mu\nu}(\infty) = W_1^{\mu\nu} + \frac{(P_1^2)^{1/2} (M_1 + M_2)}{M_1 ((P \cdot q)^2 - P^2 q^2)} \\ \times ((P^\mu q \cdot Q_1 - q^\mu P \cdot Q_1) W_1^{\nu\rho} q_\rho - \mu \leftrightarrow \nu),$$

$$(III.98 d) \quad W_2^{\mu\nu}(\infty) = W_2^{\mu\nu} + \frac{(P_2^2)^{1/2} (M_1 + M_2)}{M_2 ((P \cdot q)^2 - P^2 q^2)} \\ \times ((P^\mu q \cdot Q_2 - q^\mu P \cdot Q_2) W_2^{\nu\rho} q_\rho - \mu \leftrightarrow \nu),$$

$$(III.98 e) \quad w^{\mu\nu} = W_1^{\mu\nu}(\infty) + W_2^{\mu\nu}(\infty),$$

where we have written

$$(III.99) \quad W_1^{\mu\nu}(\infty) = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} W_1^{\mu\nu} e^{i\alpha D}$$

[similarly for $W_2^{\mu\nu}(\infty)$], and

$$(III.100) \quad \begin{cases} Q_1^\mu = \frac{P_1^\mu}{(P_1^2)^{1/2}} - \frac{q^\mu}{(q^2)^{1/2}}, \\ Q_2^\mu = \frac{P_2^\mu}{(P_2^2)^{1/2}} + \frac{q^\mu}{(q^2)^{1/2}}. \end{cases}$$

It is easily seen, by applying the transformation (III.97) to Equations (III.84), that

$$(III.101) \quad \left\{ \begin{array}{l} [q^\mu, q^\nu] = 0, \quad [r^\mu, r^\nu] = -\frac{iw^{\mu\nu}}{q^2}, \\ [w^{\mu\nu}, w^{\rho\sigma}] = i \left(w^{\mu\rho} \left(g^{\nu\sigma} - \frac{q^\nu q^\sigma}{q^2} \right) - w^{\mu\sigma} \left(g^{\nu\rho} - \frac{q^\nu q^\rho}{q^2} \right) \right. \\ \quad \left. + w^{\nu\sigma} \left(g^{\mu\rho} - \frac{q^\mu q^\rho}{q^2} \right) - w^{\nu\rho} \left(g^{\mu\sigma} - \frac{q^\mu q^\sigma}{q^2} \right) \right), \\ [r^\mu, q^\nu] = -ig^{\mu\nu}, \quad [q^\mu, w^{\rho\sigma}] = 0, \\ [r^\mu, w^{\rho\sigma}] = -i \frac{q^\rho w^{\sigma\mu} - q^\sigma w^{\rho\mu}}{q^2}, \end{array} \right.$$

as anticipated. Note that the on-mass-shell four-momentum squared, given by

$$(III.102) \quad m^2 = c^{-2} q^2 |_{\text{on shell}} = M_1 M_2 \left(1 - \frac{M^2}{(M_1 + M_2)^2} \right),$$

does *not* commute with r^μ since M^2 is a function of e^z where $e^i = L^{-1}(P)^i, q^\nu$, [see Equation (III.79)], and so the relative operators do not form a representation of the *extended* Weyl group. Note also that, since M^2 has eigenvalues ranging from $(m_1 + m_2)^2$ to ∞ , m^2 has eigenvalues ranging from 0 to $-\infty$. We define the relative generators of homogeneous Lorentz transformations and dilatations by

$$(III.103 a) \quad m^{\mu\nu} = \frac{1}{2} (\{ q^\mu, r^\nu \} - \{ q^\nu, r^\mu \}) + w^{\mu\nu},$$

$$(III.103 b) \quad d = \frac{1}{2} \{ q^\mu, r_\mu \} = \frac{1}{2} \{ q^\mu, \tilde{r}_\mu \}$$

and it is easily seen that the operators $(q^\mu, m^{\mu\nu}, d)$ defined by Equations (III.98 a) and (III.103) generate a representation of the Weyl group Lie algebra with invariant operators

$$\frac{1}{2} w^{\mu\nu} w_{\mu\nu}, \quad \frac{1}{2} W_1^{\mu\nu}(\infty) W_{1\mu\nu}(\infty),$$

and

$$\frac{1}{2} W_2^{\mu\nu}(\infty) W_{2\mu\nu}(\infty),$$

whose physical meaning will be discussed later.

Looking at Equations (III.86 b), (III.92) and (III.103 b), we see that the sum of the single-particle dilatation generators splits up into a sum of the centre-of-mass and relative dilatation generators :

$$(III.104) \quad D_1 + D_2 = D + d,$$

exactly as in the nonrelativistic case [cf. Equation (II.80)]. Note also that D and d commute.

We can express the centre-of-mass spin and position operators, $W^{\mu\nu}$ and R^μ , in terms of the relative operator $m^{\mu\nu}$ by means of Equations (III.96) and

$$(III.105) \quad R^\mu = \lim_{\alpha \rightarrow \infty} e^{-\alpha} e^{-i\alpha D} R^\mu e^{i\alpha D}$$

(which is derived from them) applied to Equations (III.90) and (III.91). Using Equations (III.97) and (III.103 a), we find :

$$(III.106 a) \quad W^{\mu\nu} = \left(g^{\mu\rho} - \frac{p^\mu p^\rho}{P^2} \right) \left(g^{\nu\sigma} - \frac{p^\nu p^\sigma}{P^2} \right) m_{\rho\sigma},$$

$$(III.106 b) \quad R^\mu = \bar{R}^\mu - \frac{P_\nu m^{\mu\nu}}{P^2},$$

where

$$(III.107) \quad \bar{R}^\mu = \lim_{\alpha \rightarrow \infty} e^{-\alpha} e^{-i\alpha D} \tilde{R}^\mu e^{i\alpha D},$$

which can be evaluated using Equations (II.77) and (III.82 *b*) to give (see Appendix B) :

$$(III.108) \quad \bar{R}^\mu = \tilde{R}^\mu - \frac{(M_1 + M_2)}{((P \cdot q)^2 - P^2 q^2)} \\ \times \left(\frac{(P_1^2)^{1/2}}{M_1} W_1^{\mu\nu} q_\nu q \cdot Q_1 + \frac{(P_2^2)^{1/2}}{M_2} W_2^{\mu\nu} q_\nu q \cdot Q_2 \right),$$

which agrees with the result obtained by comparing Equations (III.91) and (III.106 *b*) for R^μ , using Equations (III.98) and (III.103 *a*). By applying a finite centre-of-mass dilatation to Equation (III.82 *b*), multiplying by $e^{-2\alpha}$, and letting $\alpha \rightarrow \infty$, we find that

$$(III.109) \quad [\bar{R}^\mu, \bar{R}^\nu] = 0,$$

since the right-hand side of Equation (III.82 *b*) gives a finite expression under the Gartenhaus-Schwartz transformation. Equations (III.106) are the relativistic analogues of the nonrelativistic Equations (II.81). If we can find the commutation relations of $(q^\mu, r^\mu, w^{\mu\nu})$ with \bar{R}^μ , then we can find their commutators with R^μ from Equation (III.106 *b*). To obtain the commutation relations with \bar{R}^μ we use the same trick on Equations (III.85 *b*) as gave Equation (III.109) from Equation (III.82 *b*) viz. we apply a finite centre-of-mass dilatation, $e^{-i\alpha D} () e^{i\alpha D}$, to Equations (III.85 *b*), multiply by $e^{-\alpha}$, and let $\alpha \rightarrow \infty$, to obtain ⁽²⁰⁾ :

$$(III.110) \quad [\bar{R}^\mu, q^\nu] = [\bar{R}^\mu, r^\nu] = [\bar{R}^\mu, w^{\sigma\tau}] = 0,$$

since the right-hand sides of Equations (III.85 *b*) give finite expressions under application of the Gartenhaus-Schwartz transformation. The commutators of the sets of operators (M, P^μ, R^μ) and $(q^\mu, r^\mu, w^{\mu\nu})$ can now be evaluated using Equations (III.77), (III.79), (III.101), (III.106 *b*) and (III.110) :

$$(III.111 a) \quad \left\{ \begin{array}{l} [M, q^\rho] = [M, w^{\rho\sigma}] = 0, \\ [M, r^\rho] = -\frac{4i}{c^2} \left(\frac{M^2}{M^2 - M_1^2 - M_2^2 - 2\mathbf{e}^2/c^2} \right) \left(q^\rho - \frac{P \cdot q P^\rho}{P^2} \right), \end{array} \right.$$

⁽²⁰⁾ Note that Equations (III.109) and (III.110), together with $[\bar{R}^\mu, P^\nu] = -ig^{\mu\nu}$, allow us to write the canonical form $\bar{R}^\mu \rightarrow -i(\partial/\partial P_\mu)_{m^{\rho\sigma}}$. The verification of commutation relations of M, P^μ, R^μ , and $W^{\mu\nu}$ is much easier using Equations (III.106) than by the method mentioned after Equation (III.92).

III. 111 b) $[P^\mu, q^\rho] = [P^\mu, r^\rho] = [P^\mu, w^{\rho\sigma}] = 0,$

(III. 111 c) $\left\{ \begin{aligned} [R^\mu, q^\rho] &= i P_\nu \frac{q^\mu g^{\nu\rho} - q^\nu g^{\mu\rho}}{P^2}, \\ [R^\mu, r^\rho] &= i P_\nu \frac{r^\mu g^{\nu\rho} - r^\nu g^{\mu\rho}}{P^2}, \\ [R^\mu, w^{\rho\sigma}] &= -i P_\nu \frac{w^{\mu\rho} g^{\nu\sigma} - w^{\mu\sigma} g^{\nu\rho} + w^{\nu\sigma} g^{\mu\rho} - w^{\nu\rho} g^{\mu\sigma}}{P^2}, \end{aligned} \right.$

together with

(III. 112) $[R^\mu, q^2] = [R^\mu, r^2] = \left[R^\mu, \frac{1}{2} w^{\rho\sigma} w_{\rho\sigma} \right] = 0.$

The situation regarding commutativity of the centre-of-mass and relative operators can be improved by defining the “deboosted” set of relative operators $(e^\mu, \rho^\mu, s^{\mu\nu})$ by the equations

(III. 113) $\left\{ \begin{aligned} e^\mu &= L^{-1} (P)^{\mu}_\nu q^\nu, \\ \rho^\mu &= L^{-1} (P)^{\mu}_\nu r^\nu, \\ s^{\mu\nu} &= L^{-1} (P)^{\mu}_\rho L^{-1} (P)^\nu_\sigma w^{\rho\sigma}, \end{aligned} \right.$

where P^μ is the centre-of-mass four-momentum, (III. 77), and $L^{-1} (P)^\mu_\nu$ is given by Equation (III. 25). The set $(e^\mu, \rho^\mu, s^{\mu\nu})$ satisfy the same commutation relations among themselves as do the set $(q^\mu, r^\mu, w^{\mu\nu})$, i. e. Equation (III. 101). Using Equations (III. 25), (III. 31), (II. 14 b) and (III. 111), we find :

(III. 114 a) $[M, e^\rho] = [M, s^{\rho\sigma}] = [M, \rho^0] = 0, \quad [M, \rho^k] \neq 0,$

(III. 114 b) $[P^\mu, e^\rho] = [P^\mu, \rho^\rho] = [P^\mu, s^{\rho\sigma}] = 0,$

(III. 114 c) $\left\{ \begin{aligned} [R^\mu, e^0] &= [R^\mu, \rho^0] = 0, \\ [R^\mu, e^2] &= [R^\mu, \rho^2] = 0, \\ [R^0, e^k] &= [R^0, \rho^k] = [R^0, s^{\rho\sigma}] = 0, \\ [R^i, e^k] &\neq 0, \quad [R^i, \rho^k] \neq 0, \quad [R^i, s^{\rho\sigma}] \neq 0. \end{aligned} \right.$

The actual expressions for the commutators which are non-zero are complicated and uninspiring, and so have not been reproduced here.

We shall now discuss the centre-of-mass spin operator $\mathbf{S} (P, W)$ and the relative invariant operator $(1/2) w^{\mu\nu} w_{\mu\nu}$, and their relation to the single-particle spin operators. The centre-of-mass spin operator is defined in terms of $W^{\mu\nu}$ by

(III. 115) $S^k (P, W) = -\frac{1}{2} \varepsilon^{kij} L^{-1} (P)^i_\rho L^{-1} (P)^\nu_\sigma W^{\rho\sigma},$

where $L^{-1} (P)^\mu_\nu$ is given by Equation (III. 25). On using Equation (III. 106 a), we therefore find :

(III. 116) $S^k (P, W) = -(\mathbf{e} \times \boldsymbol{\rho})^k - \frac{1}{2} \varepsilon^{kij} s^{ij},$

where \mathbf{e} , ρ , and s^{ij} are given by Equations (III.113). Since the invariant of the relative operators, $(1/2) w^{\mu\nu} w_{\mu\nu}$, can be written as $(1/2) s^{\mu\nu} s_{\mu\nu}$, it is clear that we need to evaluate $s^{\mu\nu}$, which is equal to

$$L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma (W_1^{\rho\sigma}(\infty) + W_2^{\rho\sigma}(\infty)),$$

where $W_1^{\mu\nu}(\infty)$ and $W_2^{\mu\nu}(\infty)$ are given by Equations (III.98 c) and (III.98 d). The operator $L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma W_1^{\rho\sigma}(\infty)$ is easily evaluated using Equations (III.29) [with $P \rightarrow L^{-1}(P)P_1$, $\mathbf{S} \rightarrow \mathbf{S}_1$, $(L^{-1}(P)P_1, L^{-1}(P)W_1)$] together with Equations (III.100) and (III.113) in Equation (III.98 c), and we find

$$(III.117 a) \quad L^{-1}(P)^i_\rho L^{-1}(P)^j_\sigma W_1^{\rho\sigma}(\infty) = -\varepsilon^{ijk} \left(S_1^k + \frac{(\mathbf{e} \times (\mathbf{S}'_1 \times \mathbf{e}))^k}{(q^2)^{1/2} ((q^2)^{1/2} + e^0)} \right),$$

$$(III.117 b) \quad L^{-1}(P)^0_\rho L^{-1}(P)^k_\sigma W_1^{\rho\sigma}(\infty) = \frac{(\mathbf{e} \times \mathbf{S}'_1)^k}{(q^2)^{1/2}},$$

where we have written $\mathbf{S}'_1 = \mathbf{S}_1(L^{-1}(P)P_1, L^{-1}(P)W_1)$. The operator $L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma W_2^{\rho\sigma}(\infty)$ is given by Equations (III.117) with $e^\mu \rightarrow -e^\mu$, $\mathbf{S}'_1 \rightarrow \mathbf{S}'_2 = \mathbf{S}_2(L^{-1}(P)P_2, L^{-1}(P)W_2)$. Incidentally, note that if we "evaluate" $L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma W_1^{\rho\sigma}(\infty)$ by using

$$(III.118) \quad L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma W_1^{\rho\sigma}(\infty) = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma W_1^{\rho\sigma} e^{i\alpha D}$$

and Equations (III.29) [with $P \rightarrow L^{-1}(P)P_1$, $\mathbf{S} \rightarrow \mathbf{S}_1(L^{-1}(P)P_1, L^{-1}(P)W_1)$] we just obtain Equations (III.117) with \mathbf{S}'_1 replaced by $\lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \mathbf{S}'_1 e^{i\alpha D}$. Hence \mathbf{S}'_1 is invariant under the Gartenhaus-Schwartz transformation (similarly for \mathbf{S}'_2):

$$(III.119) \quad \left\{ \begin{array}{l} \mathbf{S}_1(L^{-1}(P)P_1, L^{-1}(P)W_1) \\ = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \mathbf{S}_1(L^{-1}(P)P_1, L^{-1}(P)W_1) e^{i\alpha D}, \\ \mathbf{S}_2(L^{-1}(P)P_2, L^{-1}(P)W_2) \\ = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \mathbf{S}_2(L^{-1}(P)P_2, L^{-1}(P)W_2) e^{i\alpha D}. \end{array} \right.$$

We also note that the spin operator $\mathbf{S}'_1 = \mathbf{S}_1(L^{-1}(P)P_1, L^{-1}(P)W_1)$ is related to the spin operator $\mathbf{S}_1 = \mathbf{S}_1(P_1, W_1)$ by the equation

$$(III.120) \quad S_1^i = (R(P_1, L^{-1}(P))^{-1})^{ij} S_1^j,$$

where the Wigner rotation matrix operator is given by ⁽²¹⁾:

$$(III.121) \quad (R(P_1, L^{-1}(P))^{-1})^{ij} = L^{-1}(L^{-1}(P)P_1)^\mu_\rho L^{-1}(P)^\nu_\sigma L(P_1)^\rho_j,$$

⁽²¹⁾ Our definition of a Wigner rotation, $R(P, L)$, agrees with that of Macfarlane [39], Equation (2.7) if one allows for the difference in the definition of $L(P)$ mentioned in footnote ⁽¹⁴⁾.

as can be seen by using Equation (III.24) (similarly for particle 2). We now return to the calculation of $\mathbf{S}(P, W)$. Substituting Equations (III.117) and the corresponding expressions for

$$L^{-1}(P)^\mu_\rho L^{-1}(P)^\nu_\sigma W_2^{\rho\sigma}(\infty)$$

into Equation (III.116) we find :

$$(III.122) \quad \mathbf{S}(P, W) = \mathbf{S}'_1 + \mathbf{S}'_2 - \mathbf{e} \\ \times \left(\rho - \frac{(\mathbf{S}'_1 \times \mathbf{e})}{(q^2)^{1/2} ((q^2)^{1/2} + e^0)} - \frac{(\mathbf{S}'_2 \times \mathbf{e})}{(q^2)^{1/2} ((q^2)^{1/2} - e^0)} \right)$$

Now it can easily be seen that

$$(III.123) \quad \left\{ \begin{aligned} & \left[\rho^i - \frac{(\mathbf{S}'_1 \times \mathbf{e})^i}{(q^2)^{1/2} ((q^2)^{1/2} + e^0)} - \frac{(\mathbf{S}'_2 \times \mathbf{e})^i}{(q^2)^{1/2} ((q^2)^{1/2} - e^0)}, \right. \\ & \qquad \qquad \qquad \left. (\mathbf{S}'_1 + \mathbf{S}'_2)^j \right] = 0, \\ & \left[\rho^j - \frac{(\mathbf{S}'_1 \times \mathbf{e})^j}{(q^2)^{1/2} ((q^2)^{1/2} + e^0)} - \frac{(\mathbf{S}'_2 \times \mathbf{e})^j}{(q^2)^{1/2} ((q^2)^{1/2} - e^0)}, \right. \\ & \qquad \qquad \qquad \left. \rho^j - \frac{(\mathbf{S}'_1 \times \mathbf{e})^j}{(q^2)^{1/2} ((q^2)^{1/2} + e^0)} - \frac{(\mathbf{S}'_2 \times \mathbf{e})^j}{(q^2)^{1/2} ((q^2)^{1/2} - e^0)} \right] = 0, \end{aligned} \right.$$

by applying the Gartenhaus-Schwartz transformation to the equations

$$(III.124) \quad \left\{ \begin{aligned} & \left[L^{-1}(P)^i_\rho \tilde{\rho}^\rho - \frac{(\mathbf{S}'_1 \times \mathbf{P}'_1)^i}{(P_1^2)^{1/2} ((P_1^2)^{1/2} + P_1^0)} \right. \\ & \qquad \qquad \qquad \left. + \frac{(\mathbf{S}'_2 \times \mathbf{P}'_2)^i}{(P_2^2)^{1/2} ((P_2^2)^{1/2} + P_2^0)}, (\mathbf{S}'_1 + \mathbf{S}'_2)^j \right] = 0, \\ & \left[L^{-1}(P)^i_\rho \tilde{\rho}^\rho \right. \\ & \qquad \qquad \qquad \left. - \frac{(\mathbf{S}'_1 \times \mathbf{P}'_1)^i}{(P_1^2)^{1/2} ((P_1^2)^{1/2} + P_1^0)} + \frac{(\mathbf{S}'_2 \times \mathbf{P}'_2)^i}{(P_2^2)^{1/2} ((P_2^2)^{1/2} + P_2^0)}, \right. \\ & \left. L^{-1}(P)^j_\rho \tilde{\rho}^\rho \right. \\ & \qquad \qquad \qquad \left. - \frac{(\mathbf{S}'_1 \times \mathbf{P}'_1)^j}{(P_1^2)^{1/2} ((P_1^2)^{1/2} + P_1^0)} + \frac{(\mathbf{S}'_2 \times \mathbf{P}'_2)^j}{(P_2^2)^{1/2} ((P_2^2)^{1/2} + P_2^0)} \right] = 0 \end{aligned} \right.$$

[where $P'_1 = L^{-1}(P) P_1$, $P'_2 = L^{-1}(P) P_2$], which are themselves consequences of Equations (III.33) and (III.34 b). Equations (III.123), together with $[\rho^i, e^j] = i \delta^{ij}$ allow us to write the following canonical forms for ρ , \mathbf{S}'_1 and \mathbf{S}'_2 :

$$(III.125) \quad \left\{ \begin{aligned} & \rho^i \rightarrow i \left(\frac{\partial}{\partial e^i} \right)_{(\mathbf{S}'_1, (\rho_1)^\nu + \mathbf{S}'_2 (\rho_2)^\nu)}, \\ & S_1^i \rightarrow S_1^i(p_1)' = (R(p_1, L^{-1}(p)))^{-1ij} S_j(p_1), \\ & S_2^i \rightarrow S_2^i(p_2)' = (R(p_2, L^{-1}(p)))^{-1ij} S_j(p_2), \end{aligned} \right.$$

and therefore $\mathbf{S}(P, W)$:

$$(III.126) \quad \mathbf{S}(P, W) \rightarrow \mathbf{S}(p) = \mathbf{S}_1(p_1)' + \mathbf{S}_2(p_2)' - i \mathbf{e} \\ \times \left(\frac{\partial}{\partial \mathbf{e}} \right)_{(\mathbf{S}_1(p_1)' + \mathbf{S}_2(p_2)')},$$

where \mathbf{e} in Equations (III.125) and (III.126) is a c -number, not an operator. Our expression for the canonical form of the spin operator, $\mathbf{S}(p)$, agrees with that of Macfarlane ([39], Equations (5.18) and (5.20)). (Note that although Macfarlane's definition of q^μ , and therefore e^μ , is not the same as ours ([39], Equation (4.11)), his \mathbf{e} is in the same direction as ours, which is all that is required for agreement with his result.)

The three invariants of the relative operators,

$$\frac{1}{2} W_1^{\mu\nu}(\infty) W_{1\mu\nu}(\infty), \quad \frac{1}{2} W_2^{\mu\nu}(\infty) W_{2\mu\nu}(\infty),$$

and

$$\frac{1}{2} w^{\mu\nu} w_{\mu\nu}$$

are evaluated in the centre-of-mass frame, using Equations (III.117). We find

$$(III.127 a) \quad \frac{1}{2} W_1^{\mu\nu}(\infty) W_{1\mu\nu}(\infty) = \mathbf{S}'_1{}^2,$$

$$(III.127 b) \quad \frac{1}{2} W_2^{\mu\nu}(\infty) W_{2\mu\nu}(\infty) = \mathbf{S}'_2{}^2,$$

$$(III.127 c) \quad \frac{1}{2} w^{\mu\nu} w_{\mu\nu} = (\mathbf{S}'_1 + \mathbf{S}'_2)^2 - 2((\mathbf{S}'_1 + \mathbf{S}'_2)^2 - \mathbf{S}'_1{}^2 - \mathbf{S}'_2{}^2 \\ - \Lambda^2 + \Lambda_1{}^2 + \Lambda_2{}^2),$$

where ⁽²²⁾ : $\Lambda_1 = \hat{\mathbf{e}} \cdot \mathbf{S}'_1$, $\Lambda_2 = -\hat{\mathbf{e}} \cdot \mathbf{S}'_2$ are the helicities of the particles in the centre-of-mass frame, and where $\Lambda = \Lambda_1 - \Lambda_2$. So we see that the operators $(\mathbf{S}'_1 + \mathbf{S}'_2)^2$, Λ , Λ_1 and Λ_2 which are separately invariants in the nonrelativistic case [see the discussion following Equations (II.79)] are now combined into the one invariant given by Equation (III.127 c).

Finally, if we denote an irreducible representation of the type discussed in Section III.1, of mass m and spin s , by $[m, s]$, and a reducible representation of the type defined by Equations (III.98 a) and (III.103) by $(s_1 s_2 \sigma (s_1 s_2; s \lambda_1 \lambda_2))$ where

$$(III.128) \quad \sigma(s_1, s_2; s \lambda_1 \lambda_2) = -s(s+1) \\ + 2s_1(s_1+1) + 2s_2(s_2+1) - 4\lambda_1\lambda_2,$$

⁽²²⁾ The symbol $\hat{\mathbf{e}}$ denotes a unit vector in the \mathbf{e} direction.

then the reduction carried out in this section may be written as

$$(III.129) \quad [m_1, s_1] \otimes [m_2, s_2] \\ = \left(\int_{m_1+m_2}^{\infty} \bigoplus_{\substack{\ell=0 \\ s=|s_1-s_2|}}^{\infty} dm \bigoplus_{\substack{s_1+s_2 \\ s=|s_1-s_2|}}^{\infty} \bigoplus_{\substack{j=|\ell-s| \\ \lambda_1=-s_1 \\ \lambda_2=-s_2}}^{s_1+s_2} [m, j] \right)_{\text{c. m.}} \\ \times \left(\bigoplus_{\substack{s_1+s_2 \\ s=|s_1-s_2|}}^{\infty} \bigoplus_{\substack{s_1 \\ \lambda_1=-s_1}}^{\infty} \bigoplus_{\substack{s_2 \\ \lambda_2=-s_2}}^{\infty} (s_1 \ s_2 \ \sigma(s_1 \ s_2; s \ \lambda_1 \ \lambda_2)) \right)_{\text{rel}} .$$

The centre-of-mass and relative parts of $[m_1, s_1] \otimes [m_2, s_2]$ are simply reducible and irreducible respectively if and only if $s_1 = s_2 = 0$.

We shall conclude this section by delving a little more deeply into why the centre-of-mass and relative operators do not commute [Equations (III.111) and (III.114)]. We have the centre-of-mass momentum and position operators, P^μ and R^μ , defined by Equations (III.77) and (III.91), together with the set of “preliminary” relative operators $\tilde{O} = (\tilde{q}^\mu, \tilde{r}^\mu, \tilde{w}^{\mu\nu})$ given by Equations (III.83), and the Gartenhaus-Schwartz operator, $e^{i\alpha D}$, with which we want to construct relative operators O , which commute with P^μ and R^μ , by the prescription :

$$(III.130) \quad O = \tilde{O}(\infty) = \lim_{\alpha \rightarrow \infty} \tilde{O}(\alpha) = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \tilde{O} e^{i\alpha D} .$$

Now, since, by Equations (III.96), we have

$$(III.131) \quad \begin{cases} [P^\mu, \tilde{O}(\alpha)] = e^\alpha e^{-i\alpha D} [P^\mu, \tilde{O}] e^{i\alpha D}, \\ [R^\mu, \tilde{O}(\alpha)] = e^{-\alpha} e^{-i\alpha D} [R^\mu, \tilde{O}] e^{i\alpha D}, \end{cases}$$

the necessary and sufficient conditions for $\tilde{O}(\infty)$ to commute with P^μ and R^μ are that $[\tilde{P}^\mu, O] = 0$, and that $[\tilde{R}^\mu, O]$ gives a finite (or zero) result under application of the Gartenhaus-Schwartz transformation [50].

Now, let us consider the case of $\tilde{O} = \tilde{q}^\rho$, whose commutators with P^μ and R^μ are given by Equations (III.84), (III.85) and (III.91) :

$$(III.132 a) \quad [P^\mu, \tilde{q}^\rho] = 0,$$

$$(III.132 b) \quad [R^\mu, \tilde{q}^\rho] = -i P_\nu \frac{\tilde{q}^\nu g^{\mu\rho} - \tilde{q}^\mu g^{\nu\rho}}{P^2} .$$

It is clear that the second of the conditions for commutativity is not satisfied since the right-hand side of Equation (III.132 b) goes to ∞ under the Gartenhaus-Schwartz transformation and the same thing occurs for \tilde{r}^μ and $\tilde{w}^{\mu\nu}$. This is why the relative operators do not commute with the centre of-mass operators.

Note that, in the nonrelativistic case, both the conditions for commutativity are satisfied, see Equations (II.69).

4. Scattering theory

In this section, we shall elucidate the connection between the dilatation change and the time-delay ([25], [26], [27]) in a relativistic scattering process.

We first express the centre-of-mass dilatation generator, D , and the non-hermitean "orbital angular momentum" operator, $\hat{M}^{\mu\nu}$, in terms of the centre-of-mass momentum operator, P^μ , and the "modified" position operator, \hat{R}^μ , by means of Equation (III.52) :

$$(III.133) \quad \left\{ \begin{array}{l} D = \frac{1}{2} \{ P^\mu, \hat{R}_\mu \}, \\ \hat{M}^{\mu\nu} = \frac{1}{2} (\{ P^\mu, \hat{R}^\nu \} - \{ P^\nu, \hat{R}^\mu \}) \end{array} \right.$$

and then note that, for a Lorentz-invariant scattering operator, S ,

$$(III.134) \quad [P^\mu, S] = 0 = [M^{\mu\nu}, S],$$

we have, by the definitions of \hat{R}^μ and $\hat{M}^{\mu\nu}$ [Equations (III.48) and (III.44)] respectively,

$$(III.135) \quad \left\{ \begin{array}{l} [\hat{R}^\mu, S] = [R^\mu, S], \\ [\hat{M}^{\mu\nu}, S] = 0. \end{array} \right.$$

On taking the commutator of S with D and \hat{M}^{0i} [given by Equations (III.133)], we find :

$$(III.136 a) \quad [D, S] = P^0 [\hat{R}^0, S] - P^i [\hat{R}^i, S],$$

$$(III.136 b) \quad [\hat{M}^{0i}, S] = P^0 [\hat{R}^i, S] - P^i [\hat{R}^0, S] = 0,$$

where we have used the fact that $[\hat{R}^\mu, S]$ and P^ν commute [as can be seen by using the Jacobi identity, Equation (II.14 d)]. Substituting Equation (III.136 b) into Equation (III.136 a), and multiplying by S^\dagger on the left, we find

$$(III.137) \quad S^\dagger [D, S] = \frac{P^2}{P^0} S^\dagger [\hat{R}, S],$$

where $S^\dagger [\hat{R}^0, S] = S^\dagger [R^0, S]$ is just $c \times$ time-delay. Writing

$$P^2 \rightarrow \frac{E_{cm}^2}{c^2},$$

$$S \rightarrow \exp(2i\delta(E_{cm})), \quad \hat{R}^0 \rightarrow -i \left(\frac{\partial}{\partial p^0} \right) = -ic^2 \left(\frac{p^0}{E_{cm}} \right) \left(\frac{\partial}{\partial E_{cm}} \right).$$

where E_{cm} is the centre-of-mass energy, we find :

$$(III.138) \quad S^\dagger[\hat{R}, S] = \frac{2c^2 p^0}{E_{cm}} \frac{d\delta}{dE_{cm}}.$$

So Equation (III.137) becomes

$$(III.139) \quad S^\dagger[D, S] = 2 E_{cm} \frac{d\delta}{dE_{cm}},$$

which should be compared with the corresponding nonrelativistic formula, Equation (II.91). [The factor 2 difference in the two equations comes from the absence of a factor 1/2 in the energy-time part of the nonrelativistic dilatation generator, Equation (II.25), which itself comes from the fact that the nonrelativistic time dilatation $t' = \lambda^2 t$.] Once again, the scattering process is dilatation-invariant for zero time-delay or zero centre-of-mass energy.

5. Supersuperselection rules

We mentioned in Section III.1.C that, in constructing localised states, the integration over p is limited to positive p^2 and positive p^0 [see Equation (III.65)] and that this was an example of a “supersuperselection rule”. We now discuss this concept in greater generality.

By a supersuperselection rule, we mean that our Hilbert space, \mathcal{H} , of states splits up into two ⁽²³⁾ subspaces, \mathcal{H}_1 and \mathcal{H}_2 , such that the states in \mathcal{H}_1 are physically realizable but the states in \mathcal{H}_2 , though mathematically well-defined, are not physically realizable. Note that the existence of a supersuperselection rule implies the existence of a superselection rule [52] i. e. the superposition principle cannot hold between \mathcal{H}_1 and \mathcal{H}_2 .

In the case of the Weyl group, our Hilbert space, \mathcal{H} , splits up into three subspaces :

$$(III.140) \quad \begin{cases} \mathcal{H}_1 : & p^2 > 0, & p^0 > 0, \\ \mathcal{H}_2 : & p^2 > 0, & p^0 < 0, \\ \mathcal{H}_3 : & p^2 < 0 \end{cases}$$

and the states in \mathcal{H}_1 are physically realizable whilst those in \mathcal{H}_2 and \mathcal{H}_3 are not. The supersuperselection rule between $\mathcal{H}_1 \oplus \mathcal{H}_2$ and \mathcal{H}_3 merely says that it is impossible to have physical particles travelling faster than light i. e. it is the correct model-independent (i. e. S-matrix theoretic) formulation of the “microscopic causality” of quantum field theory.

⁽²³⁾ The generalization to, more than two subspaces is straightforward.

We suggest the existence of other supersuperselection rules in nature, e. g. the nonobservability of quarks and magnetic monopoles might be due to supersuperselection rules for electric charge and baryon number for the former (i. e. only states with integer Q and B are physically realizeable) and for magnetic charge for the latter (i. e. only states with zero magnetic charge are physically realizeable). There might even be a connection between the electric and magnetic charge supersuperselection rules. In fact, this connection seems quite likely in view of the connection between monopoles and electric charge quantization established by Dirac [53].

IV. CONCLUSION

By considering the Weyl group and its nonrelativistic analogue, we have succeeded in constructing time and position operators for both relativistic and nonrelativistic quantum mechanics, and have shown that the irreducible representation of these groups describe virtual particles. It seems to us that areas where our work may have possible applications are : the significance of the time operators for measurement theory, and the connection between analyticity and causality in relativistic and nonrelativistic quantum mechanics. We hope to look into these topics in the near future.

APPENDIX A

Extensions of Lie groups and Lie algebras

In this appendix, we shall give a very brief review of the theory of central extensions of Lie groups and Lie algebras and their connection with unitary ray representations, and then apply these results, in Appendices A-I and A-II to the two groups discussed in this paper. The standard work on unitary ray representations of continuous groups is, of course, the classic paper of Bargmann [13], and more detailed reviews of Bargmann's work have been given by Hamermesh ([54], Chapter 12), Lévy-Leblond ([16], Section III. A), and the present author [55].

Let \mathcal{G} be an n -parameter Lie group, with general element G . Then in quantum mechanics we are interested in unitary ray representations of \mathcal{G} in Hilbert space, i. e. unitary operators $U(G)$ which satisfy Equation (II.15) (reproduced here for convenience) :

$$(A.1) \quad U(G') U(G) = e^{i\xi(G', G)} U(G'G),$$

where the exponent $\xi(G', G)$ is a real function of G' and G . It is clear that the operators $e^{i\theta} U(G)$ where θ is a real number, form a true unitary

representation of a group, the *local group* ([13], Section 2 d) with elements $\tilde{G} = (\theta, G)$ and multiplication law

$$(A.2) \quad \tilde{G}' \tilde{G} = (\theta', G') (\theta, G) = (\theta' + \theta + \xi(G', G), G' G).$$

Using Equation (A.2), we see that $(\theta_0, \mathbf{1})$ where $\mathbf{1}$ denotes the unit element of \mathcal{G} , commutes with a general element (θ, G) of the local group :

$$(A.3) \quad (\theta, G) (\theta_0, \mathbf{1}) = (\theta_0, \mathbf{1}) (\theta, G),$$

since $\xi(\mathbf{1}, G) = \xi(\mathbf{1}, \mathbf{1}) = \xi(G, \mathbf{1})$, as can be seen by putting $G' = \mathbf{1}$ in Equation (A.1) and then multiplying on the left by $U(\mathbf{1})$, and then putting $G = \mathbf{1}$ in Equation (A.1) and multiplying in the right by $U(\mathbf{1})$. Thus Equation (A.3) tells us that the local group is a *central extension* of \mathcal{G} by a one-dimensional group. In terms of Lie algebras, this means that, if the Lie algebra of \mathcal{G} is

$$(A.4) \quad [\alpha_i, \alpha_j] = ic_{ijk} \alpha_k,$$

where $\{c_{ijk}\}$ are the structure constants of \mathcal{G} and $\alpha_i (i = 1, \dots, n)$ are the elements of its Lie algebra, then the Lie algebra of the local group, and its representation by hermitean operators in Hilbert space, is given by ([13], Sections 4 g and 4 l) :

$$(A.5) \quad \begin{cases} [A_i, A_j] = ic_{ijk} A_k + i \beta_{ij} A_0, \\ [A_i, A_0] = 0 \quad (i, j, k = 1, \dots, n), \end{cases}$$

where the $A_i (i = 0, 1, \dots, n)$ are hermitean operators, and

$$\beta_{ij} = \Xi(\alpha_i, \alpha_j) = -\beta_{ji},$$

where $\Xi(\alpha_i, \alpha_j)$ is the *infinitesimal exponent* of the Lie algebra. To discover whether the Lie algebra (A.4) has non-trivial extensions (i. e. extensions other than those isomorphic to the direct sum where $\beta_{ij} = 0$ for all i, j), we apply the Jacobi identity, Equation (II.14 d), to the extended Lie algebra (A.5) as a consistency condition, and it is easily seen that this procedure gives the set of equations :

$$(A.6) \quad d\Xi(\alpha_i, \alpha_j, \alpha_k) \equiv \Xi([\alpha_i, \alpha_j], \alpha_k) + \Xi([\alpha_j, \alpha_k], \alpha_i) + \Xi([\alpha_k, \alpha_i], \alpha_j) = 0.$$

So if Equation (A.6) tells us that all the $\Xi(\alpha_i, \alpha_j)$ are zero, or if the extended Lie algebra, (A.5), can be transformed into the trivial extension (direct sum) :

$$(A.7) \quad \begin{cases} [A'_i, A'_j] = ic_{ijk} A'_k, \\ [A'_i, A_0] = 0 \quad (i, j, k = 1, \dots, n), \end{cases}$$

by making the trivial redefinition of A_i :

$$(A.8) \quad A_i \rightarrow A'_i = A_i + \lambda_i A_0 \quad (i = 1, \dots, n),$$

where $\{\lambda_i\}$ are real constants ⁽²⁴⁾, then the Lie algebra (A.4), and therefore the group \mathcal{G} , has only trivial central extensions. If, however, not all the $\Xi(\alpha_i, \alpha_j)$ are zero, and the extended Lie algebra cannot be transformed into the trivial direct sum in the afore-mentioned way, then the Lie algebra (A.4), and therefore the group \mathcal{G} , has non-trivial extensions, and there exist unitary ray representations of \mathcal{G} for which the phase factor in Equation (A.1) is non-trivial.

I. THE GROUP OF INHOMOGENEOUS GALILEI TRANSFORMATIONS AND THE DILATATION $(t', \mathbf{x}') = (\lambda^2 t, \lambda \mathbf{x})$

We now apply these ideas to the Lie algebra given by Equations (II.11). Since the Galilei group Lie algebra is a subalgebra of Equations (II.11), we can immediately say from the work of Bargmann ([13], Section 6 *f*) that the infinitesimal exponents of this subalgebra satisfy [either directly from Equation (A.6) or by using in addition the trivial redefinition technique (A.8)] :

$$(A.9 a) \quad \begin{aligned} \Xi(\mathcal{J}_i, \mathcal{J}_j) &= \Xi(\mathcal{J}_i, \mathcal{X}_j) = \Xi(\mathcal{J}_i, \mathcal{K}_j) = \Xi(\mathcal{K}_i, \mathcal{K}) = \Xi(\mathcal{K}_i, \mathcal{K}_j) \\ &= \Xi(\mathcal{J}_i, \mathcal{X}) = \Xi(\mathcal{X}_i, \mathcal{X}_j) = \Xi(\mathcal{X}_i, \mathcal{X}) = 0, \end{aligned}$$

$$(A.9 b) \quad \Xi(\mathcal{K}_i, \mathcal{X}_j) = m \delta_{ij},$$

i. e. that the Galilei Lie algebra has one non-trivial central extension ([17], [12]). Our task now is to discover whether the Lie brackets in (II.11) involving \mathcal{O} still allow this extension, and whether there are any other non-trivial extensions. The only possible equation of Equations (A.6) which could affect the extension (A.9 *b*) is

$$(A.10) \quad d\Xi(\mathcal{K}_i, \mathcal{X}_j, \mathcal{O}) = 0$$

and this gives no information about $\Xi(\mathcal{K}_i, \mathcal{X}_j)$. Hence the non-trivial central extension (A.9 *b*) is allowed. Using Equations (A.9 *a*), it is immediately seen that

$$(A.11) \quad \begin{cases} d\Xi(\mathcal{J}_i, \mathcal{J}_j, \mathcal{O}) = 0 & \rightarrow \Xi(\mathcal{O}, \mathcal{J}_k) = 0, \\ d\Xi(\mathcal{K}_i, \mathcal{J}_i, \mathcal{O}) = 0 & \rightarrow \Xi(\mathcal{O}, \mathcal{K}_k) = 0, \\ d\Xi(\mathcal{X}_i, \mathcal{J}_j, \mathcal{O}) = 0 & \rightarrow \Xi(\mathcal{O}, \mathcal{X}_k) = 0. \end{cases}$$

There exists no equation of the form (A.6) which gives information on $\Xi(\mathcal{O}, \mathcal{X})$ since there is no Lie bracket, apart from $[\mathcal{O}, \mathcal{X}] = -2i\mathcal{X}$,

⁽²⁴⁾ The condition that such a set $\{\lambda_i\}$ satisfies is $c_{ijk} \lambda_k = \Xi(\alpha_i, \alpha_j)$ for all i, j .

which has ω or \mathcal{K} on the right-hand side. However, the extension

$$(A.12) \quad [D, H] = -2iH - i\Xi(\omega, \mathcal{K})1,$$

is clearly trivial since the redefinition

$$(A.13) \quad H \rightarrow H' = H + \frac{1}{2}\Xi(\omega, \mathcal{K})1,$$

brings us back to Equations (II.16). Hence the Lie algebra (II.11) has a non-trivial central extension of the form (II.16) and, as discussed in Section II.1.A, it is this extension which describes virtual nonrelativistic particles. The actual calculation of the exponent $\xi(G', G)$ in Equation (A.1) corresponding to this extension was carried out in Section II.1.B.

II. THE WEYL GROUP

We now ask whether the Weyl group Lie algebra (III.8) has any non-trivial central extensions. Since the inhomogeneous Lorentz group Lie algebra is a sub-algebra of (III.8), we can immediately say from the work of Bargmann ([13], Section 6 *d*) that the infinitesimal exponents of this subalgebra can be chosen to be zero :

$$(A.14) \quad \Xi(\mathcal{N}^{\mu\nu}, \mathcal{N}^{\rho\sigma}) = \Xi(\mathcal{N}^{\mu\nu}, \mathcal{P}^\sigma) = \Xi(\mathcal{P}^\mu, \mathcal{P}^\nu) = 0,$$

i. e. that the inhomogeneous Lorentz group has no non-trivial extensions. Using Equations (A.14) in Equations (A.6), we see that

$$(A.15) \quad \left\{ \begin{array}{l} d\Xi(\mathcal{N}^{0i}, \mathcal{N}^{0j}, \omega) = 0 \rightarrow \Xi(\omega, \mathcal{N}^{ij}) = 0, \\ d\Xi(\mathcal{N}^{0i}, \mathcal{N}^{ij}, \omega) = 0 \rightarrow \Xi(\omega, \mathcal{N}^{0j}) = 0, \\ d\Xi(\mathcal{N}^{0i}, \mathcal{P}^0, \omega) = 0 \rightarrow \Xi(\omega, \mathcal{P}^i) = 0, \\ d\Xi(\mathcal{N}^{0i}, \mathcal{P}^j, \omega) = 0 \rightarrow \Xi(\omega, \mathcal{P}^0) = 0, \end{array} \right.$$

so that the Weyl group has no non-trivial central extensions, a result first proved by Ottoson [56]. It does however have trivial central extensions, one of which

$$(A.16) \quad \Xi(\mathcal{N}^{0i}, \mathcal{P}^j) = \Xi(\omega, \mathcal{P}^0) \mathcal{G}^{ij} = -mc \mathcal{G}^{ij},$$

is isomorphic to the direct sum (III.11) which describes virtual relativistic particles.

APPENDIX B

Integrals for the relativistic Gartenhaus-Schwartz transformation

As mentioned in Section III.3, this Appendix is concerned with the calculation of the set of operators $O = (q^\mu, r^\mu, w^{\mu\nu})$ from the set

$\tilde{O} = (\tilde{q}^\mu, \tilde{r}^\mu, \tilde{w}^{\mu\nu})$, defined in Equations (III.83), by Equation (III.97), using Equations (II.77), reproduced here for convenience :

$$(B.1) \quad \left\{ \begin{array}{l} 0 = \lim_{\alpha \rightarrow \infty} e^{-i\alpha D} \tilde{O} e^{i\alpha D}, \\ \tilde{O}(\alpha) = e^{-i\alpha D} \tilde{O} e^{i\alpha D}, \\ \frac{d\tilde{O}(\alpha)}{d\alpha} = -i e^{-i\alpha D} [D, \tilde{O}] e^{i\alpha D}, \\ 0 = \tilde{O}(\infty) = \tilde{O} + \int_0^\infty \frac{d\tilde{O}(\alpha)}{d\alpha} d\alpha, \end{array} \right.$$

where $D = (1/2) \{ P^\mu, \tilde{R}_\mu \}$ [Equation (III.92)] and P^μ and \tilde{R}^μ are given by Equations (III.77) and (III.81) respectively.

The transformation of \tilde{q}^μ is clearly trivial since both P^μ and \tilde{R}^μ commute with \tilde{q}^ν , Equations (III.85), so we immediately obtain Equation (III.98 a). For \tilde{r}^μ , we find

$$(B.2) \quad \frac{d\tilde{r}^\mu(\alpha)}{d\alpha} = -e^{-i\alpha D} \left(\frac{W_1^{\mu\rho} q_\rho}{P_1^2} + \frac{W_2^{\mu\rho} q_\rho}{P_2^2} \right) e^{i\alpha D},$$

where we have used Equations (III.85) and the facts that

$$W_1^{\mu\rho} P_{1\rho} = 0 = W_2^{\mu\rho} P_{2\rho}.$$

Using the same sets of equations we find

$$(B.3) \quad \frac{dW_1^{\mu\nu}(\alpha)}{d\alpha} = e^{-i\alpha D} \left(\frac{P_1^\mu W_1^{\nu\rho} q_\rho - P_1^\nu W_1^{\mu\rho} q_\rho}{P_1^2} \right) e^{i\alpha D}$$

and similarly for $W_2^{\mu\nu}$ with $1 \leftrightarrow 2$. Clearly to evaluate the integrals in Equation (B.1) for $\tilde{O} = \tilde{r}^\mu$, $W_1^{\mu\nu}$ and $W_2^{\mu\nu}$ we need $W_1^{\mu\rho}(\alpha) q_\rho$. Contracting Equation (B.3) with q_ν , we find

$$(B.4) \quad \frac{dW_1^{\mu\nu}(\alpha) q_\nu}{d\alpha} = - \frac{P_1(\alpha) \cdot q W_1^{\mu\nu}(\alpha) q_\nu}{P_1^2(\alpha)},$$

where

$$(B.5) \quad P_1^\mu(\alpha) = \frac{M_1 P^\mu e^{-\alpha}}{M_1 + M_2} + q^\mu.$$

Integrating Equation (B.4) we find

$$(B.6) \quad \begin{aligned} W_1^{\mu\nu}(\alpha) q_\nu &= W_1^{\mu\nu} q_\nu \exp\left(-\int_0^\alpha \frac{P_1(\alpha') \cdot q}{P_1^2(\alpha')} d\alpha'\right) \\ &= W_1^{\mu\nu} q_\nu \left(\frac{P_1^2}{e^{2\alpha} |P_1^2(\alpha)|} \right)^{1/2}. \end{aligned}$$

The integrals (B.1) involving Equations (B.2) and (B.3) are now straightforward :

$$(B.7) \left\{ \begin{aligned} r^\mu &= \tilde{r}^\mu - (P_1^2)^{1/2} W_1^{\mu\rho} q_\rho \int_0^\infty \frac{dx}{e^x |P_1^2(x)|^{3/2}} \\ &\quad - (P_2^2)^{1/2} W_2^{\mu\rho} q_\rho \int_0^\infty \frac{dx}{e^x |P_2^2(x)|^{3/2}}, \\ W_1^{\mu\nu}(\infty) &= W_1^{\mu\nu} + \left((P_1^2)^{1/2} W_1^{\nu\rho} q_\rho \int_0^\infty \frac{P_1^\mu(x) dx}{e^x |P_1^2(x)|^{3/2}} - \mu \leftrightarrow \nu \right) \end{aligned} \right.$$

and the final results are Equations (III.98 b) and (III.98 c).

A similar procedure gives \bar{R}^μ , defined by Equation (III.107). Defining $\tilde{R}^\mu(x)$ by

$$(B.8) \quad \tilde{R}^\mu(x) = e^{-i\alpha D} \tilde{R}^\mu e^{i\alpha D}$$

and using Equations (B.1) and (III.82), we obtain

$$(B.9) \quad \frac{d\tilde{R}^\mu(x)}{dx} = -e^{-i\alpha D} \left(\frac{M_1 W_1^{\mu\nu} q_\nu}{(M_1 + M_2) P_1^2} - \frac{M_2 W_2^{\mu\nu} q_\nu}{(M_1 + M_2) P_2^2} \right) e^{i\alpha D} + e^{-i\alpha D} \tilde{R}^\mu e^{i\alpha D},$$

which, on using Equations (B.6) and (B.1) gives :

$$(B.10) \quad \bar{R}^\mu = \lim_{\alpha \rightarrow \infty} e^{-\alpha} \tilde{R}^\mu(x) = \tilde{R}^\mu - \frac{M_1 (P_1^2)^{1/2} W_1^{\mu\nu} q_\nu}{M_1 + M_2} \int_0^\infty \frac{e^{-2\alpha} dx}{|P_1^2(x)|^{3/2}} + \frac{M_2 (P_2^2)^{1/2} W_2^{\mu\nu} q_\nu}{M_1 + M_2} \int_0^\infty \frac{e^{-2\alpha} dx}{|P_2^2(x)|^{3/2}}$$

and, on evaluating the integrals, we obtain Equation (III.108).

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Note added in proof. — A review of the problems encountered in constructing a relativistic position operator, together with an extensive bibliography on the subject, has been given by Kálnay [57]. Inönü [58] has reviewed the general theory of the contraction of Lie groups and their representations.

Throughout Section III, we have studied only tardyons, i. e. relativistic particles with positive on-mass-shell mass-squared. A tachyon, i. e. a particle with negative on-mass-shell mass-squared, would be described by the trivial extension

$$\Xi(\varrho^0, \varrho^i) = -\Xi(\partial \tilde{x}^{0i}, \varrho^0) = \lambda^i$$

which is the analogue of Equation (A.16). If we choose the standard frame, as in Section III.2, to be the one in which P^μ has the form $(0, (-P^2)^{1/2}, 0, 0)$, then only the $i = 1$ component of λ^i is non-zero; Luxons, i. e. particles with zero on-mass-shell mass-squared, are described by the zero on-mass-shell mass limit of tardyons.

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