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J. BROS
D. IAGOLNITZER

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Section A : *Physique théorique.*

Causality and local analyticity : mathematical study (*)

by

J. BROS and D. IAGOLNITZER

Service de Physique théorique,
Centre d'Études nucléaires de Saclay,
B. P. n° 2, 91190 Gif-sur-Yvette

ABSTRACT. — In many physical schemes, the fundamental link between causality and analyticity is the well-known Laplace transform theorem which states the equivalence between the support properties of a distribution and the corresponding analyticity properties of its Fourier-Laplace transform in certain complex domains, called " tubes ", which are *invariant under real translation.*

It turns out, however, that in a number of important physical situations, one is concerned with analyticity properties in more general domains and, on the other hand, the expression of causality can take more refined forms.

In this paper, we present a generalization of the Laplace transform theorem which uses a *non-linear* Fourier transform and which exhibits the *equivalence* between *local* analyticity properties and the corresponding " essential-support " properties of the transformed functions.

A certain class of complex domains which we call " local tubes " is especially suited to the formulation of this theorem. Using these tools, we also give a new and simple approach to the " edge-of-the-wedge " theorem. This approach allows to solve a more general problem of

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boundary values of analytic functions. (We shall call this result the generalized " edge-of-the-wedge " theorem.)

Further papers will be devoted to the physical applications of these various mathematical results, in particular, to quantum field theory and to S-matrix theory.

NOTE

The mathematical study given in this paper will be completed and developed in a coming paper which includes the collaboration of R. Stora [1].

Some of the basic mathematical ideas which are at the origin of the generalized Laplace transform theorem have been inspired by a previous work by H. P. Stapp and one of the present authors (D. I.)]2]. where applications to S-matrix theory have already been given. The use of a generalized Fourier transform to describe local analyticity there originated in the physical necessity of considering sequences of gaussian type wave packets with widths shrinking under space-time dilation. This physical necessity was first emphasized by M. Froissart and R. Omnes [3].

The mathematical part of this work which concerns the edge-of-the-wedge properties was on the other hand made possible by a previous maturation of this problem which was the result of several years of collaboration between various physicists and mathematicians such as H. Epstein, V. Glaser, J. Lascoux, B. Malgrange, A. Martineau, R. Stora, M. Zerner and one of the present authors (J. B.).

In particular, J. Lascoux and B. Malgrange played a fundamental role in showing how the edge-of-the-wedge problems were connected with the theory of hyperfunctions by Sato [4], thus leading to a deeper and more general insight of these problems.

On the other hand, a version of the " generalized edge-of-the-wedge theorem " has already been proved in certain special cases by H. Epstein and V. Glaser [5] and A. Martineau has proposed a general proof [6] which uses a completely different method from ours. (We will not try here to study the link between these two methods.)

INTRODUCTION

In the fifteen last years the role played by analyticity properties in elementary particle physics has so much increased that analyticity has become a sort of dynamical principle in itself [7], while its links with traditional physical ideas such as causality had not yet been fully understood.

In some problems of classical physics, the link between causality and analyticity was clearly exhibited through the well-known Laplace transform theorem, using the fact that the causal character of a physical quantity is then generally expressed as a certain *support* property in space time.

As a simple example, consider a one-dimensional kernel $K(t - t')$ where t and t' are time variables and which transforms an " input " wave signal $\varphi(t')$ into an " output " signal $\psi(t) = \int K(t - t') \varphi(t') dt'$; K is causal if " there is not output before input " which is equivalent to saying that $K(t)$ vanishes for $t < 0$.

The Fourier transform variable $\nu = \frac{\omega}{2\pi}$ of the time variable t is interpreted as a frequency, and through the above mentioned Laplace transform theorem, the causal character of $K(t)$ is equivalent to the fact that its frequency spectrum $\tilde{K}(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} K(t) dt$ is the boundary value of a function which is analytic for complexe values of ω in the upper half plane ($\operatorname{Im} \omega > 0$). In classical optics, a very similar situation occurs in the description of the scattering of light by atoms [8] where, due to the propagation of the light wave along the x-axis, the variable t has to be replaced by $t - \frac{x}{v}$; the analyticity of the kernel $\tilde{K}(\omega)$ is then the basis, through the use of a Cauchy integral, of the Kramers-Kronig " dispersion relation ". (This is the context in which this expression appears historically for the first time.)

We notice moreover that if a condition of " short range relaxation " on the above kernel $K(t)$, such as $|K(t)| < C e^{-\alpha t}$ for $t > 0$, is added to the expression of causality, then the kernel $\tilde{K}(\omega)$ can be analytically continued across the real region $\operatorname{Im} \omega = 0$, until the value $\operatorname{Im} \omega = \alpha$.

In quantum physics, and in particular in the theory of elementary particles, the frequency ν and the wave vector $\vec{k}$ are respectively identified (up to the Planck constant h) with the energy p_0 (1) and the momentum $\vec{p}$ of a particle, and the analogue of the kernel $K(\omega) \delta(\omega - \omega')$ [which is the Fourier transform of $K(t - t')$ with respect to t and t'] is now the set of " scattering kernels " $S_{m, n-m}(p_1, \ldots, p_m; p_{m+1}, \ldots, p_n)$

(1) In all the following we take the usual choice of units in which $\hbar = \frac{h}{2\pi} = 1$ so that p_0 is now identical with the variable $\omega = 2\pi\nu$.

where the four-momenta p_i of the incoming ($1 \leq i \leq m$) and outgoing ($m+1 \leq i \leq n$) particles are restricted to the " mass shell " manifold : $p_{i_0} = \omega_i = (\vec{p}_i^{\,2} + m_i^2)^{1/2}$ and satisfy the law of conservation $\sum_{i=1}^{m} p_i = \sum_{j=m+1}^{n} p_j$.

(One has in fact more precisely

$$S_{m,n-m}(p_1, \ldots, p_m; \ldots, p_n) = \hat{S}_{m,n-m}(p_1, \ldots, p_m; \ldots, p_n)\, \delta\left(\sum_{i=1}^{m} p_i - \sum_{m+1}^{n} p_j\right)\Bigg).$$

In the framework of general quantum field theory, the analyticity properties of the kernels $\hat{S}_{m,n-m}$ have also been investigated, in particular those of the $S_{2,2}$ scattering amplitudes [9] and some of these properties have again the form of " dispersion relations ". In this derivation, analyticity still originates in taking into account certain *support* properties through the (several variable) Laplace transform theorem; in fact the microcausality of the theory is expressed by introducing certain Green's functions of the fields with a " retarded " or " advanced " propagation character which amounts to localization of the space-time variables in (future or past) light cones. However a new physical input also plays an important role in the development of this program, namely the principle of positive total energy put in a relativistic form in the so called " spectral condition ", and the exploitation of this condition together with causality leads to very intricate problems of analytic completion.

So the analyticity properties of the kernels $\hat{S}_{m,n-m}$ and the complex domains where they are defined are still related with causal properties in space-time but this relation is there much more sophisticated than in the above mentioned problems and has not yet been fully studied [2].

The philosophy of this paper is twofold :

On the mathematical side we present a method which allows to solve a certain class of problems of analytic completion and which will lead to some applications in the program of Quantum Field Theory that we have just described above.

([2]) This accounts for the fact that in the theory of elementary particles many physicists [7] have preferred to set " maximal " analyticity properties, such as the " Mandelstam representation " as a working assumption. This can be considered as a new and attractive approach to physical intuition but we emphasize that it is not true today to state that this point of view is equivalent with an intuition based on space time causality and quantum relativistic requirements.

On the other hand this method should allow to interpret a large class of analyticity properties of the $S_{m, n-m}$ kernels or of the Green's functions of field theory in terms of " causality " properties in space-time where we now use the word " causality " in a much broader sense than above, since it should include the notions of short range forces, of resonance, the role of probability conservation expressed through unitarity, etc.

The latter point of view was expressed for the first time (3) by R. Omnes [3] who treated the problem of the equivalence between " momentum-transfer analyticity " and a " short-range-force hypothesis " in a pure S-matrix framework.

More general results of this kind were then obtained by H. P. Stapp and one of the present authors (D. I) [2] and led to the present mathematical study. However the exact shapes of the domains of analyticity (surrounding the real physical regions) were not studied precisely there, and certain notions such as the distinction between analyticity in the so-called " physical sheet " and analytic continuation into other sheets (across certain real regions) might still deserve further and more detailed investigations in terms of the above concepts formulated in space-time (4).

In the present paper, we try to give a general mathematical basis to the study of the links between analyticity and causality (in a broad sense). In various physical schemes these general results will allow to characterize in terms of " essential support " (5) properties the fact that certain fundamental kernels are analytic in bounded complex domains which contain suitable real regions Ω (but not the whole real region). We shall call this kind of property " *local analyticity* ".

In fact this important feature appears conjointly in general quantum field theory and in pure S-matrix theory (the real regions Ω are then subsets of the so called " physical regions "); and although the first theory is working with " off mass shell " quantities (the Green's functions) while the other one only uses the " on shell " scattering amplitudes, the meaning of local analyticity in both theories can now hopefully be understood in a unified way as various applications of the same mathematical theorem. This theorem which is a generalization of the Laplace transform theorem establishes equivalence between the fact that a function $f(p)$ is *locally* analytic and a property of exponential decrease for a suitable quantity which is a generalized *non linear* Fourier trans-

(3) In this connection we must also quote the pioneer work by G. Wanders [10].

(4) Investigations of this kind will extend the simple result mentioned before on the analytic continuation of the kernel $\tilde{K}(\omega)$ across the whole real region, but the problems become here much more intricate, because in particular of the unavoidable occurence of branch points (" threshold singularities ") on the real.

(5) This term will be made precise in the following.

form of $f(p)$. Compared with the Laplace transform theorem which has been mentioned above, this generalization presents two important features :

a. The introduction of a non linear Fourier transform is necessary to describe analyticity in *local* domains, while the ordinary Fourier transform can only be used for a too restrictive class of domains : those which are invariant by real translation (i. e. " tubes ").

b. The exponential fall-off properties of the transformed quantities outside a suitable " essential support " generalize the strict support properties, and will be physically interpreted as a refined expression of causality. (This feature can already occur mathematically with the ordinary Fourier transform, but it becomes crucial in the description of local analyticity.)

The physical applications of these results will be given in further papers. In particular one of the theorems which is proved here (a generalized version of the " edge-of-the-wedge " theorem) will be a useful tool in the program of field theory (*see* note added in proof).

In order to describe the contents of this paper, let us first of all fix some notations. We consider two n-dimensional spaces (x-space and ξ-space) with the Fourier transformation defined by

$$\tilde{f}(x) = (2\pi)^{-\frac{n}{2}} \int f(\xi)\, e^{-i\xi.x}\, d\xi \tag{1}$$

where

$$x = (x_1, \ldots, x_n),$$
$$\xi = (\xi_1, \ldots, \xi_n),$$

and

$$\xi.x = \sum_{i=1}^{n} \xi_i\, x_i.$$

The variables x will always be taken real while ξ will also be complexified; when it is complex, it is noted $p = \xi + i\,\eta$ with

$$p = (p_1, \ldots, p_n), \qquad \xi = (\xi_1, \ldots, \xi_n)$$

and

$$\eta = (\eta_1, \ldots, \eta_n).$$

In η-space we shall often use the polar coordinates $\eta = \rho\omega$ where $\rho = |\eta|$ and ω is a point on the $(n-1)$-dimensional unit sphere.

The domains which generalize in several complex variables the upper half-plane $\eta_1 = \operatorname{Im} p_1 > 0$, or strips $a < \eta_1 < b$, are called " *tubes* " [11]. In the space $\mathbf{C}^n$ of n complex variables $p_i = \xi_i + i\,\eta_i\ (1 \leq i \leq n)$, a tube T_B is defined as the set of all points p such that $\eta = \operatorname{Im} p$ belongs to a given domain ([6]) B; B is a domain in the n-real dimensional η-space and is called the basis of the tube T_B. (The tube T_B thus defined is clearly invariant by *real* translations, since there is no restriction on the values of $\xi = \mathbf{Re}\, p$.) We shall also use the closures of B and T_B which will be denoted respectively $\overline{B}$ and $\overline{T}_B$ (more generally $\overline{D}$ will denote the closure of the open set D).

In section 1, we recall the usual Laplace transform theorem in its simplest form : equivalence between support properties of a function $\tilde{f}(x)$ in a convex cone and analyticity of its Laplace transform $f(\xi + i\eta)$ in a suitable tube with conical basis.

Section 2 will describe the extension (well-known for mathematicians) of this theorem to the case of tubes with more general (possibly bounded) bases B; there the support property in x-space is replaced by a condition of exponential decrease outside a suitable domain $\tilde{B}$ which will be called " essential support ".

A general class of bounded complex domains in p-space will be introduced in section 3; they will be called " local tubes " and defined with help of a " localizing " analytic function $\Phi(p)$. Ordinary tubes are reobtained as a limiting case ($\Phi(p) \to 0$).

In section 4 we define a generalized Fourier transform $\mathcal{F}_\Phi$ in which the linear exponent $-i\xi.x$ of formula (1) is replaced by a non linear function of ξ, involving the function Φ of the class introduced in section 3.

In section 5, the Parseval and inversion formulae of the Fourier transformation are generalized to the case of the transformation $\mathcal{F}_\Phi$.

This allows to prove in section 6 a generalized Laplace-transform theorem which states the equivalence between analyticity of functions in a local tube with localizing function Φ, and essential support properties of the $\mathcal{F}_\Phi$ transforms.

In section 7, we show how this theorem allows to obtain a very precise and intuitive presentation of the " edge-of-the-wedge theorem " and to prove a generalized version of this theorem.

We finally wish to point out that this paper has been written as far as possible in the language of classical mathematics in order to be accessible to the physicist; in the same spirit we have voluntarily neglected certain mathematical features or developments of our results which we prefer to reserve for another paper [1]; the latter will be presented more specifically to be read by mathematicians.

([6]) i. e. a connected open set.

1. THE LAPLACE TRANSFORM THEOREM FOR A TUBE WITH CONICAL BASIS

Being given a cone C with apex at the origin in η-space, we shall associate with C a closed cone $\tilde{C}$ in *x-space* which is the set of all points x such that $\eta . x \geq 0$ for all points η in C (*fig.* 1); we shall call $\tilde{C}$, the "*dual cone*" of C.

It can be verified that the dual $\tilde{\tilde{C}}$ of $\tilde{C}$, taken now in η-*space*, is the closed convex hull of C.

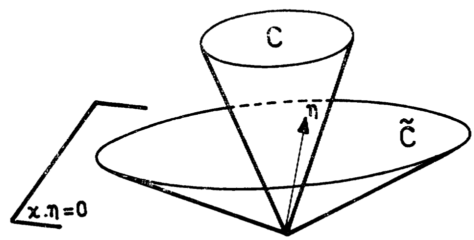

Fig. 1

We now recall the classical Laplace transform theorem [12] which we first state for simplicity in the case when $f(p)$ [and $\tilde{f}(x)$] are infinitely differentiable functions which decrease rapidly [7] at infinity as well as all their derivatives (according to Schwartz's notations [12] we say that f and $\tilde{f}$ belong to the space $\mathcal{S}$).

For such functions f and $\tilde{f}$, the two following properties are equivalent :

(i) $f(\xi)$ is the boundary value of a function $f(\xi + i\eta)$ analytic in a tube T_C, where C is a convex open cone.

$f(p)$ is moreover infinitely differentiable in the closure of T_C and has a rapid [7] decrease at infinity in this region.

(ii) $\tilde{f}(x)$ has its support in the dual cone $\tilde{C}$ of C.

The proof that (ii) implies (i) is easily obtained by writing the inverse Fourier formula :

$$f(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{i\xi x} \tilde{f}(x) \, dx, \tag{2}$$

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[7] By rapid decrease at infinity we always mean that there exists for every positive integer N a bound of the form $\frac{C_N}{(1 + |\xi|)^N}$ in real directions, or $\frac{C_N}{(1 + |\xi| + |\eta|)^N}$ in complex directions.

and investigating the convergence conditions of the right-hand side when ξ is replaced by $\xi + i\eta$ (8).

The proof that (i) implies (ii) is obtained by replacing the integration contour of formula (1) (i. e. the real ξ-space) by the set of all $\xi + i\eta$ with a fixed η taken inside the cone C (this is allowed because f is analytic in T_C).

Since $|\eta|$ can be chosen arbitrarily large, the contour can be moved to infinity as soon as $\eta . x < 0$. This proves the vanishing of $\tilde{f}(x)$ outside $\tilde{C}$.

In the above argument it appeared that

$$\tilde{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\eta \in C} e^{-i(\xi + i\eta).x} f(\xi + i\eta)\, d\xi;$$

this is equivalent to saying that $\tilde{f}(x)\, e^{-\eta.x}$ is the Fourier transform of the function $f_\eta(\xi) = f(\xi + i\eta)$ for all points η inside C (9).

This remark allows to define the correspondence between $f(p)$ and $\tilde{f}(x)$ in the more general case when $f(p)$ has not necessarily a limit in the usual sense on the boundary of T_C and can in fact have an arbitrary behaviour near this boundary.

It is useful in many physical contexts to consider situations in which the notion of a " *boundary value* " *in the sense of tempered distributions* can be defined ([6], [13]).

One says that a function $f(p)$ analytic in T_C admits a tempered distribution f_0 as its boundary value on the real ξ-space if for all test functions $\varphi(\xi)$ in $\mathcal{S}$, one has

$$\langle f_0, \varphi \rangle = \lim \int f(\xi + i\eta)\, \varphi(\xi)\, d\xi,$$

when $\eta \to 0$ inside the cone C.

It has been proved that this condition is equivalent [13] to the fact that $|f(p)|$ is bounded by a fixed inverse power of $|\eta|$ (with polynomially increasing coefficients in ξ) when η tends to zero.

More generally, we say that a function analytic in T_C (or in any domain) is slowly increasing in this domain if $|f(p)|$ is bounded by a polynomial in $|p|$ at infinity and by an inverse power of the distance to the boundary of T_C, when p comes close to this boundary.

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(8) The rapid decrease properties of $f(p)$ at infinity inside T_C are obtained by repeating the argument for all the functions $(-ip)^m f(p)$ which are the Fourier transforms of $\dfrac{d^m \tilde{f}(x)}{dx^m}$.

(9) The Fourier transform can also be considered in cases when $f_\eta(\xi)$ is no longer integrable, but for instance has a polynomial increase at infinity; $\tilde{f}(x)$ then becomes a tempered distribution.

It is then interesting for the application to quantum field theory to quote the following version of the Laplace transform theorem in the case when $\tilde{f}(x)$ is no longer a function but a tempered distribution. There is equivalence between the two properties :

(i) $\tilde{f}$ is a tempered distribution in x-space with support $\tilde{C}$.

(ii) $f(\xi + i\eta)$ is analytic and slowly increasing in the tube T_C, and its boundary value f_0 on the real admits $\tilde{f}$ as its Fourier transform in the sense of tempered distributions.

2. THE LAPLACE TRANSFORM THEOREM FOR MORE GENERAL TUBES

In the following the tube T_B is always chosen such that the closure of its basis B in η-space contains the origin [10]. Moreover we shall assume for simplicity that B is " star-shaped " with respect to the origin : we mean here that B can be described in polar coordinates $\eta = (\rho, \omega)$ by an inequality $\rho < r(\omega)$.

The function $r(\omega)$ can have finite or infinite values and is not necessarily continuous everywhere, but lower continuity is required.

When the origin lies on the boundary of B, we shall say that we have a " wedge situation "; in this case $r(\omega)$ has a certain support which is the intersection of the unit sphere with a (connected) cone C_B.

If one takes care of excluding the origin in the case of wedge situation, the sets B that we just described are open sets.

With the basis B we associate in x-space its *polar set* $\tilde{B}$, defined as the intersection of all half-spaces with equations

$$x.\omega + \frac{1}{\rho} \geqq 0,$$

for all points $\eta = (\rho, \omega)$ in B; $\tilde{B}$ is a closed convex set.

Remarks :

(i) If B is a bounded set, the origin in x-space lies in the interior of $\tilde{B}$, whereas it lies on its boundary if $r(\omega)$ can take infinite values.

An example of the latter case occurs in section 1, where the basis B is the cone C and $\tilde{B}$ is identical with the dual cone $\tilde{C}$ of C.

(ii) If B is in a wedge situation [with support of $r(\omega)$ in a cone C_B], then $\tilde{B}$ is unbounded and admits as an asymptotic cone the dual cone $\tilde{C}_B$ of C_B (see *fig.* 2).

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[10] The case when the closure of B does not contain the origin could be treated in a similar way, but it is without interest in the framework of this paper.

(iii) For any set B, the convex envelope $\hat{B}$ of B has the same polar set $\tilde{B}$ as B.

We now introduce the notion of " *essential support* " in x-space : we shall say that a function $\tilde{f}(x)$ admits $\tilde{B}$ as its essential support if it satisfies for every x *outside* $\tilde{B}$ an exponential bound of the type :

(3) $$|\tilde{f}(x)| < C_\varepsilon \, e^{-\beta(x)(1-\varepsilon)},$$

for every $\varepsilon > 0$.

In this inequality, $\beta(x)$ is a positive and possibly infinite quantity which is defined as follows : for every x, we call $\hat{x}$ the (unique) point

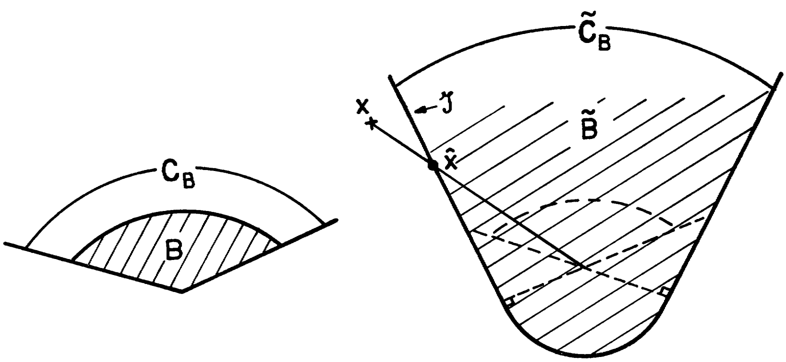

Fig. 2

on the boundary $\mathfrak{J}$ of $\tilde{B}$ which lies on the segment joining x to the origin; then we put $\beta(x) = \dfrac{|x|}{|\hat{x}|}$.

The level surfaces of the function $\beta(x)$ are clearly obtained from $\mathfrak{J}$ by dilation with respect to the origin and for that reason we shall also call $\mathfrak{J}$ the " *indicatrix of decrease* " of the function f.

Remark. — If $\mathfrak{J}$ contains the origin where it admits a tangent cone, then $\beta(x)$ becomes infinite when x is outside this cone, and therefore $f(x)$ vanishes in this region. The case considered in section 1 gives an example of this situation.

For any tube T_B with open convex basis B, the Laplace transform theorem now displays a correspondence [11] between the functions $f(\xi + i\eta)$ which are analytic in p-space in the tube T_B (and sufficiently

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[11] Rigorous statements of equivalence of this kind can be obtained if one prescribes the exact functional spaces to which the boundary values of $f(\xi + i\eta)$ belong (on the various parts of the boundary of T_B) and correspondingly more refined decrease properties of $\tilde{f}(x)$.

decreasing at infinity) and functions $\tilde{f}(x)$ in x-space which have $\tilde{B}$ as their essential support.

As it has been described in section 1, the correspondence between f and $\tilde{f}$ is still defined by saying that $\tilde{f}(x)\, e^{-\eta . x}$ is the Fourier transform of the function $f_\eta(\xi) \equiv f(\xi + i\eta)$ where this is true now for all points η in B.

The idea of the proof is then the following : starting from the analyticity of f in T_B, the relevant exponential bounds (3) on $f(x)$ are obtained by optimizing on all η's in B the boundedness condition for the product $f(x)\, e^{-\eta . x}$. (The case of section 1 is reobtained by this procedure); here again the converse is proved by investigating the convergence condition of the integral $\int f(x)\, e^{i(\xi + i\eta) x}\, dx$.

Remark. — The Laplace transform theorem gives a direct proof of the following well-known *tube theorem* [11] : any function f which is analytic in a tube T_B with arbitrary basis B (and sufficiently regular at infinity) can be analytically continued in the tube $T_{\hat{B}}$ whose basis $\hat{B}$ is the convex envelope of B. In fact the argument which allows to prove that the essential support B of f is the polar set of B still holds when B is not convex; since B and $\hat{B}$ have the same polar set, any function f analytic in T_B corresponds to an $\tilde{f}(x)$ with essential support $\tilde{B} = \widetilde{(\hat{B})}$, and by the converse of the above theorem f is also analytic in $T_{\hat{B}}$.

3. LOCAL TUBES

In sections 1 and 2, we have described classes of functions which are analytic in tubes; we shall now introduce more general complex domains which include tubes as a limit case; we will call them " *local tubes* ".

A local tube will be defined by means of two elements.

(i) A bounded domain B in an n-dimensional real space which we assume for simplicity to be described as in section 2 by an inequality $\rho < r(\omega)$.

Here again B will be called the *basis* of the local tube.

(ii) An analytic function $\Phi(p)$ with the following properties :

a. $\Phi(\bar{p}) = \overline{\Phi(p)}$ for any p in the domain of Φ.

b. The set of all real points ξ which satisfy $0 \leq \Phi(\xi) < 1$ is an open bounded set Ω whose closure is compact inside the analyticity domain of Φ (in complex p-space).

c. The origin $\xi = 0$ belongs to Ω and is a critical point for Φ ($\nabla \Phi(0) = 0$); moreover we assume for simplicity that Φ has no other critical point ([12]) inside Ω so that the set of level surfaces $\Phi(\xi) = c$ $(0 \leqq c \leqq 1)$ is topologically equivalent to the set of nested spheres with equations $\sum_{i=1}^{n} \xi_i^2 = c$; in particular $\Phi(\xi) = 0$ implies $\xi = 0$.

The simplest example of a function Φ which we shall sometimes refer to is in fact the function $\Phi(\xi) = \xi^2 = \sum_{i=1}^{n} \xi_i^2$.

Let us now consider the set $\mathcal{E}$ of points $p = \xi + i\eta$ (with $\eta = |\eta|\,\omega$) in the domain of analyticity of Φ such that :

$$(4) \qquad |\eta| + r(\omega)\,(\mathrm{Re}\,\Phi(\xi + i\eta) - 1) < 0.$$

We notice that the open set Ω always belong to $\mathcal{E}$ [$|\eta| = 0$ implies $\Phi(\xi) < 1$ since $r(\omega) \neq 0$].

If the connected component of $\mathcal{E}$ which contains Ω is *bounded* and has a compact closure inside the domain of Φ, we define the *local tube* $T_{B,\Phi}$ as the interior of this component.

A more technical restriction on the domains $T_{B,\Phi}$ which we consider will be imposed in the last part of this section.

In the unbounded case one can also derive results which are similar to those described below, but since we are only concerned here with possible applications to local problems, we will restrict ourselves to the case of bounded local tubes.

The only real points which belong to the closure of $T_{B,\Phi}$ are those of $\overline{\Omega}$ and that is why we say that Φ is a " *localizing function* " in the open set Ω; Ω plays the same role for $T_{B,\Phi}$ as the whole real ξ-space for a *tube* T_B. However, we note that B and Ω are not sufficient to specify the domain $T_{B,\Phi}$ since a large family of functions Φ are localizing in the same open set Ω.

Remark. — By putting $\Phi \equiv 0$ in equation (4), one reobtains the equation of the tube T_B of section 2. In fact if $\Phi(\xi)$ $(\not\equiv 0)$ is analytic on the whole real ξ-space, then the family of local tubes $T_{B,\lambda\Phi}$ with λ real, satisfies the inclusion property $T_{B,\lambda\Phi} \subset T_{B,\lambda'\Phi}$ for all $\lambda > \lambda'$, and when λ tends to zero, $T_{B,\lambda\Phi}$ tends to the tube T_B.

As in section 2, we will consider two cases according to whether B contains the origin or only admits it as a boundary point.

In the former case, Ω is contained in $T_{B,\Phi}$, whereas in the latter case it only lies on its boundary; we thus have again a " wedge situation ", $T_{B,\Phi}$ is contained in the tube whose basis is the cone C_B (*see* section 2).

([12]) More general situations could also be considered but would not bring essentially new features.

The shape of the sets $\mathcal{E}$ and $T_{B,\Phi}$ in an arbitrary imaginary direction ω is typically illustrated by taking the following one-dimensional example where $\Phi(p) = p^2$; $\{ r(\omega) \}$ is a set of two numbers r_+, r_-, Ω is the interval $-1 < \xi < 1$ and the set $\mathcal{E}$ is defined by the equations :

$$\begin{aligned} \eta + r_+ (\xi^2 - \eta^2 - 1) < 0 \qquad (\eta \geq 0), \\ -\eta + r_- (\xi^2 - \eta^2 - 1) < 0 \qquad (\eta \leq 0). \end{aligned}$$

The situation in the upper half-plane is represented on figure 3; two cases occur :

(i) The region $\mathcal{E}$ is connected but unbounded for $r_+ > \frac{1}{2}$.

(ii) for $r_+ < \frac{1}{2}$ the region $\mathcal{E}$ is disconnected and $T_{B,\Phi}$ only corresponds to its lower part.

In order to generalize this intuitive picture to the case of an arbitrary localizing function Φ, we need to introduce the following notions. Being given a localizing function Φ, one can associate with every point (ρ, ω) in $\mathbf{R}^n$ ([13]), the n real-dimensional manifold $\widehat{\Gamma}_{\rho\omega}$ in C^n which is defined by the equations :

$$(5) \qquad \begin{cases} \eta = |\eta| \omega, \\ \psi_\rho (\xi, \eta) = |\eta| + \rho (\text{Re}\, \Phi (\xi + i\eta) - 1) = 0. \end{cases}$$

For any fixed direction ω, and any sufficiently small value of $\rho \geq 0$, one shows that this manifold $\widehat{\Gamma}_{\rho\omega}$ contains a connected bounded component which has the same boundary as Ω; we shall call this component the cycle ([14]) $\Gamma_{\rho,\omega}$. All the cycles $\Gamma_{\rho,\omega}$ (with ω fixed) are contained in the $(n + 1)$ dimensional manifold $p = \xi_\perp + \lambda\omega$, where λ is a *complex* variable and $\xi_\perp$ denotes a set of $(n - 1)$ *real* variables in the hyperplane orthogonal to ω; now we shall say that a cycle $\Gamma_{\rho\omega}$ is *admissible* if for any fixed value of $\xi_\perp$, its section in the λ-plane has no critical point and can be obtained through a continuous distortion of cycles $\Gamma_{\rho',\omega}$ with $0 \leq \rho' \leq \rho$ (starting from the section of Ω on the real axis, when ρ' starts from 0).

One can show that a cycle $\Gamma_{\rho\omega}$ is admissible provided that at all the points of $\Gamma_{\rho\omega}$ and of all cycles $\Gamma_{\rho'\omega}$ with $0 \leq \rho' \leq \rho$ the derivatives $\omega . \nabla_\xi \psi_{\rho'} (\xi, \eta)$ and $\omega . \nabla_\eta \psi_{\rho'} (\xi, \eta)$ do not vanish simultaneously; this is clearly fulfilled for ρ small enough, but for any given ω, there is a critical value $\rho = r_\Phi (\omega)$ such that the cycle $\Gamma_{\rho,\omega}$ ceases to be admissible at this value.

([13]) ρ, ω are the polar coordinates of this point.
([14]) In the sense of " relative cycle modulo the boundary of Ω ".

Using the property a of Φ, one can easily show that :

$$\omega . \nabla_\xi \psi_\rho (\xi, \eta) - i\, \omega . \nabla_\eta \psi_\rho (\xi, \eta) = - i + \rho\omega . \nabla_p \Phi (p), \tag{6}$$

and what we have to keep in mind is that on all admissible cycles $\Gamma_{\rho\omega}$ $(0 \leqq \rho < r_\Phi (\omega))$ the complex valued vector (6) does not vanish; on each limit cycle $\rho = r_\Phi (\omega)$ it vanishes at least once.

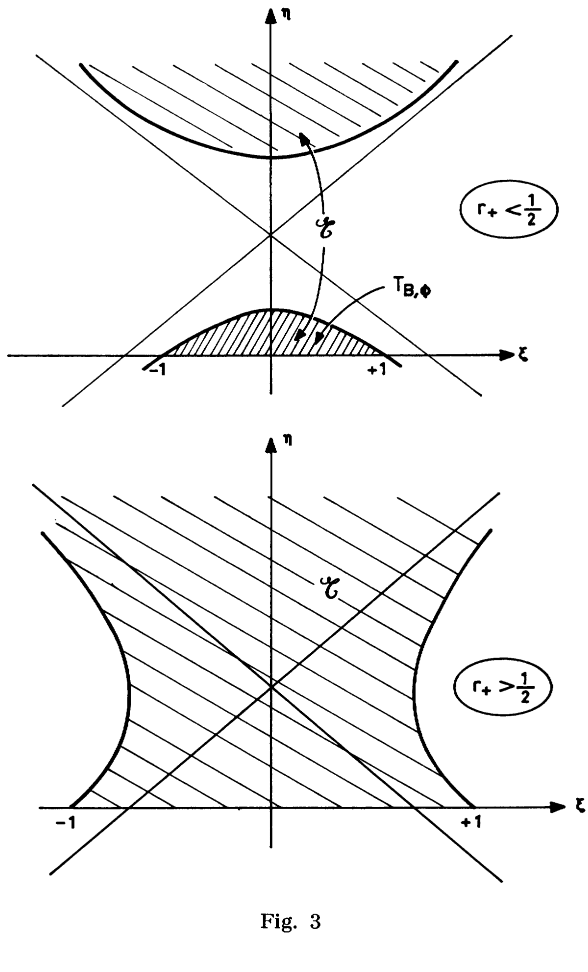

Fig. 3

We shall define the *admissible set* B_Φ associated with Φ as the set of all points ρ, ω in $\mathbf{R}^n$ such that the cycle $\Gamma_{\rho,\omega}$ be admissible; B_Φ obviously contains the origin and is given in polar coordinates by the formula :

$$0 \leqq \rho < r_\Phi (\omega).$$

The introduction of the cycles $\Gamma_{\rho,\omega}$ allows the following alternative definition for a " *local tube* " $T_{B,\Phi}$: $T_{B,\Phi}$ is the union of the cycles $\Gamma_{\rho,\omega}$ associated with all the points (ρ, ω) in the basis B.

In the following, we shall always impose that the basis B has a compact closure inside the admissible set B_Φ; this means that $\overline{T}_{B,\Phi}$ is a union of cycles $\Gamma_{\rho\omega}$ which are all admissible and implies that the modulus of the complex vector (6) has a strictly positive lower bound in the tube $T_{B,\Phi}$. This property will reveal crucial for the argument given in section 6.

4. A CLASS OF NON LINEAR FOURIER TRANSFORMATIONS F_Φ

Being given an analytic function Φ with the properties described in section 3, we shall associate with every distribution $f(\xi)$ suitably chosen a generalized Fourier transform $F(x, x_0)$ in the $n + 1$ *dimensional* real space of the variables $x = (x_1, \ldots, x_n)$ and x_0 by the formula :

$$F(x, x_0) = \mathcal{F}_\Phi (f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\xi x - \Phi(\xi) x_0} f(\xi)\, d\xi. \tag{7}$$

To be more precise, we consider a bounded connected closed set $\hat{\Omega}$ [15] which contains the closure of Ω in its interior $(\Omega \subset\subset \hat{\Omega})$ and such that the function Φ is defined and analytic at all points of $\hat{\Omega}$.

We then consider the class of distributions $f(\xi)$ which have their support in $\hat{\Omega}$, and denote by $E_{\hat{\Omega}}$ the class of their Fourier transforms :

$$\tilde{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\xi x} f(\xi)\, d\xi;$$

$\tilde{f}(x)$ is an entire function which is polynomially bounded on real x-space.

Then for any fixed value of x_0, $F(x, x_0)$ is the Fourier transform of $f(\xi)\, e^{-\Phi(\xi) x_0}$ and therefore also belongs to $E_{\hat{\Omega}}$.

An alternative way of considering the transformation $\mathcal{F}_\Phi$ is obtained by imbedding the real ξ-space into the $n + 1$-real dimensional space of the variables $(\xi = \xi_1, \ldots, \xi_n; z)$ and introducing the closed piece of analytic manifold $\mathcal{M}$ which is the set of all points (ξ, z) such that $z = \Phi(\xi)$ with ξ in $\hat{\Omega}$.

We then see that $\mathcal{F}_\Phi$ is the ordinary $n + 1$ dimensional Fourier-Laplace transformation for distributions with support $\mathcal{M}$ and which

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(15) Be careful that here the notation ^ does *not* mean the convex closure.

have the form $f(\xi)\,\delta(z - \Phi(\xi))$ in a neighborhood of $\mathcal{M}$:

$$(8) \qquad F(x, x_0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\xi.x - zx_0} f(\xi)\,\delta(z - \Phi(\xi))\, d_n\xi\, dz.$$

Let us now introduce the (possibly infinite-order) differential operator $\Phi\left(i\frac{\partial}{\partial x}\right)$ defined for every function $\tilde{g}(x)$ in the class $E_{\hat{\Omega}}$ by

$$(9) \qquad \Phi\left(i\frac{\partial}{\partial x}\right)\tilde{g}(x) = \mathcal{F}(\Phi f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{-i\xi.x}\, g(\xi)\,\Phi(\xi)\, d\xi.$$

Since $\Phi(\xi)$ is differentiable at all points of $\hat{\Omega}$, Φg is again a distribution with support $\hat{\Omega}$ and $\Phi\left(i\frac{\partial}{\partial x}\right)\tilde{g}(x)$ is therefore a well-defined function in the class $E_{\hat{\Omega}}$.

With this definition, we immediately see on formula (7) or (8) that $F(x, x_0)$ is a solution of the equation

$$(10) \qquad D_{\mathcal{M}}\, F(x, x_0) \equiv \left(\frac{\partial}{\partial x_0} + \Phi\left(i\frac{\partial}{\partial x}\right)\right) F(x, x_0) = 0.$$

We emphasize that the operator $\frac{\partial}{\partial x_0} + \Phi\left(i\frac{\partial}{\partial x}\right)$ will only be here considered as acting on differentiable functions of x and x_0 which (for any value of x_0) belong to the class $E_{\hat{\Omega}}$ (in the variables x); this operator is thus canonically associated with the set $\mathcal{M}$ and therefore denoted by $D_{\mathcal{M}}$.

If Φ is a polynomial of degree m, (10) reduces to an ordinary partial differential equation of order m (with constant coefficients); in the typical case where $\Phi = \xi^2$, (10) is the heat equation. In the general case, the essential properties of this parabolic equation still hold true. In particular, the solution $F(x, x_0)$ with Cauchy data $F(x, o) = \tilde{f}(x)$ is uniquely determined by formula (7).

5. THE CLOSED DIFFERENTIAL FORM W (F, G) AND THE GENERAL PARSEVAL AND INVERSION FORMULAE FOR F_Φ

Let us consider for a moment the case when $f(\xi)$ is a (differentiable) function. Then f can be recovered from its transform $F(x, x_0)$ by using the inverse Fourier formula :

$$(11) \qquad f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{x_0\Phi(\xi)} \int e^{i\xi.x}\, F(x, x_0)\, d_n x,$$

where the right hand side integral can be performed equivalently on any hyperplane $x_0 = \text{const}$.

In the next section we will need an extension of this inversion formula in which the integration will be performed on a more general hypersurface Σ of the (x, x_0)-space and which will be valid provided that $F(x, x_0)$ has special decrease properties.

We shall now present this type of extension in its natural framework which is the Parseval formula.

In fact, the inverse Fourier formula can be considered as a special case of the Parseval formula :

$$\int \overline{(\tilde{g})}(x) \tilde{f}(x)\, dx = \int \bar{g}(\xi) f(\xi)\, d\xi, \tag{12}$$

since the latter is valid for any couple (f, g) in duality : by taking $g(\xi) = \delta(\xi - \xi_0) = \delta_{\xi_0}$, i. e. $\tilde{g}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-i\xi_0 . x}$ one reobtains the inverse Fourier formula.

The extension that we give was inspired by a result which concerns the Klein-Gordon equation and is well-known in relativistic quantum physics. We mean the following form of Parseval formula :

$$\begin{aligned} &\int \bar{g}(p) f(p) \delta(p^2 - m^2)\, d^4 p \\ &= i \int_{x_0 = \text{const}} d^3 \vec{x} \left[\overline{G}(\vec{x}, x_0) \left(\frac{\partial}{\partial x_0} F(\vec{x}, x_0) \right) - \left(\frac{\partial}{\partial x_0} \overline{G}(\vec{x}, x_0) \right) F(\vec{x}, x_0) \right] \\ &= i \int_{\Sigma} \sum_{\mu} d\sigma_{\mu}(x) \left[\overline{G}(x) \frac{\partial}{\partial x_{\mu}} F(x) - \left(\frac{\partial}{\partial x_{\mu}} \overline{G}(x) \right) F(x) \right], \end{aligned} \tag{13}$$

for functions $f(p)$ and $g(p)$ in duality.

In this formula, the first integral represents the (Lorentz invariant) scalar product on the mass shell hyperboloïd H_m and $F(x)$, $G(x)$ are the associated solutions of the *Klein-Gordon* equation.

In the third term, the integrand is a *closed differential* form W and in view of Stokes theorem ([16]) its integral on the (space-like) hypersurface Σ is independent of Σ (for suitable decrease properties of F and G at infinity).

([16]) $\int_{\partial\Omega} W = \int_{\Omega} dW$ where $\partial\Omega$ is the boundary of Ω.

In the present case, we similarly consider a couple of functions $f(x)$, $g(x)$ in $E_{\hat{\Omega}}$ which are in duality and for which we have the usual Parseval formula (12).

Let $F(x, x_0)$ and $G(x, x_0)$ be the solutions of the equation (10) with respective Cauchy data $f(x)$, $g(x)$ on the hyperplane $x_0 = 0$. Then we are able to construct a differential form $W(F, G)$ of degree n in the [$(n + 1)$-dimensional] (x, x_0)-space such that :

(i) $W(F, G)$ is *closed* ($dW = 0$).

(ii) The restriction of $W(F, G)$ to any hyperplane $x_0 =$ const. is of the form :

$$F(x, x_0)\,\overline{G}(x, -x_0)\, dx_1 \wedge \ldots \wedge dx_n$$

[which reduces to $\tilde{f}(x)\,\overline{(\tilde{g})}(x)\, dx_1 \wedge \ldots \wedge dx_n$ when $x_0 = 0$].

In view of Stokes theorem, we therefore obtain that, for couples of solutions (F, G) having suitable decrease properties at infinity, the following generalized form of Parseval formula [analogous to (13)] holds :

$$(14) \qquad \int \bar{g}(\xi)\, f(\xi)\, d\xi = \int_{\Sigma} W(F, G),$$

for a certain class of admissible hypersurfaces Σ.

If we take for f a regular function $f(\xi)$ and for $g(\xi)$ the measure δ_ξ, then G is equal to $e^{-i\xi x - \Phi(\xi) x_0}$ and the formula (14) gives an inversion of formula (7) under the following general form

$$(15) \qquad (2\pi)^{\frac{n}{2}} f(\xi) = \int_{\Sigma} W(F, e^{-i\xi x - \Phi(\xi) x_0}).$$

Let us now give the expression of the differential form $W(F, G)$; we first introduce n differentiable functions $\rho_k(\xi, \xi')$ of $2n$ variables by the identity

$$(16) \qquad \Phi(\xi) - \Phi(\xi') = \sum_{k=1}^{n} (\xi_k - \xi'_k)\, \rho_k(\xi, \xi').$$

The existence of such functions is ensured [14] if Φ is differentiable. Moreover if Φ is analytic in a natural domain Δ, ρ_k is analytic in $\Delta \times \Delta$; [for Δ we can take here the domain of all points $p = \xi + i\eta$ such that $\operatorname{Re} \Phi(p) < 1 + \varepsilon$].

We note that the functions $\rho_k(\xi, \xi')$ are good multipliers for the distributions $f(\xi)\, g(\xi')$ whose support lies in $\hat{\Omega} \times \hat{\Omega}$ (since $\hat{\Omega}$ is contained in Δ).

Therefore it is meaningful to define the action of an operator noted $\rho_k\left(i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial x'}\right)$ on the product $F(x, x_0)\,\overline{G}(x', -x'_0)$ by the formula

$$(17)\quad \rho_k\left(i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial x'}\right) F(x, x_0)\,\overline{G}(x', -x'_0) = \frac{1}{(2\pi)^n}\int e^{-i(\xi x-\xi' x')-(\Phi(\xi)x_0-\Phi(\xi')x'_0)}\,\rho_k(\xi, \xi')\,f(\xi)\,\bar{g}(\xi')\,d\xi\,d\xi'.$$

We now put

$$(18)\qquad W(F, G) = \left[i\sum_{k=1}^{n}(-1)^k \rho_k\left(i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial x'}\right) \times F(x, x_0)\,\overline{G}(x', -x'_0)\Big|_{\substack{x=x'\\ x_0=x'_0}} \times dx_0 \wedge dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n \right] + F(x, x_0)\,\overline{G}(x, -x_0)\,dx_1 \wedge \ldots \wedge dx_n,$$

where the notation $\widehat{dx_k}$ indicates that this factor is omitted.

In order to verify that W is *closed*, we just compute

$$dW(F, G) = \left[i\sum_{k=1}^{n}\left(\frac{\partial}{\partial x_k} + \frac{\partial}{\partial x'_k}\right)\rho_k\left(i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial x'}\right) + \left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x'_0}\right)\right] \times F(x, x_0)\,\overline{G}(x', -x'_0)\Big|_{\substack{x=x'\\ x_0=x'_0}} dx_0 \wedge dx_1 \wedge \ldots \wedge dx_n.$$

To show that the coefficient of $dx_0 \wedge \ldots \wedge dx_n$ vanishes we just note that it is the generalized $\mathcal{F}_\Phi \times \mathcal{F}_\Phi$ transform (restricted at $x = x'$, $x_0 = x'_0$) of

$$\left[\sum_{k=1}^{n}(\xi_k - \xi'_k)\,\rho_k(\xi, \xi') - (\Phi(\xi) - \Phi(\xi'))\right] f(\xi)\,\bar{g}(\xi'),$$

which is equal to zero in view of equation (16).

Writing now formula (18) in the case when

$$G(x, x_0) = e^{-i\xi x-\Phi(\xi)x_0} \qquad \text{(for a given value of } \xi)$$

we obtain :

$$(19)\quad W(F, e^{-i\xi x-\Phi(\xi)x_0}) = i\,e^{i\xi x+\Phi(\xi)x_0}\left[\sum_{k=0}^{n}(-1)^k F_k(x, x_0, \xi)\,dx_0 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n\right]$$

where we have put

$$(20) \qquad F_k(x, x_0, \xi') = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \rho_k(\xi, \xi') f(\xi) e^{-i\xi x - \Phi(\xi) x_0} d\xi.$$

For $k = 0$ we give a meaning to this equation by putting $\rho_0(\xi, \xi') = -i$ in such a way that by formula (7) we have $F_0(x, x_0, \xi) = -i F(x, x_0)$.

We notice that formula (20) defines F_k not only for ξ' real, but for any $p' = \xi' + i\eta'$ in the complex domain Δ where Φ and ρ_k are analytic. Therefore F_k is also an analytic function of p' in Δ for all values of x and x_0.

By putting the expression (19) into formula (15) we obtain for appropriate classes of hypersurfaces Σ the general inversion formulae for $\mathcal{F}_\Phi$:

$$(21) \quad (2\pi)^{\frac{n}{2}} f(\xi) = i \int_\Sigma e^{i\xi x + \Phi(\xi) x_0} \times \left[\sum_{k=0} (-1)^k F_k(x, x_0, \xi)\, dx_0 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n \right].$$

Remarks :

(i) When Φ is a polynomial, the kernels F_k can be chosen to be polynomials in ξ, whose coefficients are finite combinations of derivatives of $F(x, x_0)$ [as in our pedagogical example (13)].

(ii) The construction of the differential form W (F, G) depends on the set of functions ρ_k *which is not uniquely determined* [17]; in fact W (F, G) is defined up to an exact differential form (whose integral on Σ is equal to zero). This point will be developed in [1].

6. THE GENERALIZED LAPLACE-TRANSFORM THEOREM FOR LOCAL TUBES

We shall now use the transformation $\mathcal{F}_\Phi$ of section 4 to represent the functions which are analytic in a *local tube* $T_{B,\Phi}$ by means of solutions $F(x, x_0)$ of the equation (10) which have exponential decrease properties outside a certain *essential support.*

In section 2, we had introduced in x-space the polar set $\tilde{B}$ of B; $\tilde{B}$ was the essential support of every function whose Laplace transform was analytic in the tube T_B with basis B. Similarly we shall now associate with the basis B of a local tube $T_{B,\Phi}$ a set S_B in (x, x_0)-space which will again play the role of " essential support " : S_B is the convex

(17) We are indebted to Dr. Brüning who pointed out this fact to one of us.

cone with apex at the origin in (x, x_0)-space whose basis is the set of points $x \in \tilde{B}$, $x_0 = 1$; for convenience this basis will still be denoted $\tilde{B}$.

An example of a set S_B is exhibited on figure 4 with the same set B as in figure 2.

Finally, it will be useful to introduce also polar coordinates in (x, x_0) space; we shall put

$$x_0 = u_0 \tau,$$
$$x = u \tau,$$

where $\tau \equiv \tau[(x, x_0)] = \sqrt{x^2 + x_0^2}$ and $v = (u, u_0)$ is a point on the unit sphere in (x, x_0) space; we shall call " *indicatrix associated with* B "

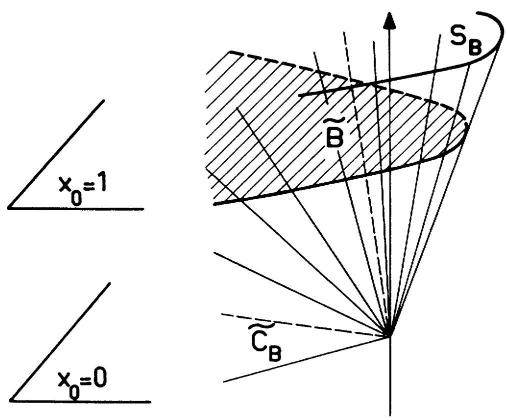

Fig. 4

the intersection of the boundary ∂S_B of S_B with this unit sphere and we shall denote it $\mathcal{I}_B$. For every point $v = (u, u_0)$ on $\mathcal{I}_B$ we note that there exists a point ρ, ω on the boundary of B (i. e. $\rho = r(\omega)$) such that $u_0 = -r(\omega)\, u.\omega$; and since B has been taken bounded, we have (for all points of $\mathcal{I}_B$) the inequality

(22) $$u_0 \leq r_{\max} |u|,$$

with $r_{\max} = \sup_\omega r(\omega)$.

Given an analytic function Φ which is a localizing function in a real open set Ω in ξ-space, we now choose once for all a closed bounded set $\hat{\Omega}$ as in section 4 $(\Omega \subset \subset \hat{\Omega})$.

We will here consider functions $f(\xi)$ which are infinitely differentiable and have their supports in $\hat{\Omega}$. (We shall say that f belongs to the space $\mathcal{D}_{\hat{\Omega}}$.) With such a function the transformation $\mathcal{F}_\Phi$ associates a function $F(x, x_0)$ [*see* formula (7)] which is solution of equation (10) :

$$D_{\mathcal{M}} F(x, x_0) = 0,$$

where we recall that $D_{\mathcal{M}}$ is completely determined by the set $\mathcal{M} : z = \Phi(\xi),\ \xi \in \hat{\Omega}$. Moreover one can easily check that $F(x, x_0)$ is bounded in the whole half-space $x_0 \geqq 0$, and that for every fixed $x_0 \geqq 0$ it is rapidly decreasing in $|x|$.

We shall now consider the class of functions f in $\mathcal{O}_{\hat{\Omega}}$ whose restriction to the open set Ω is the boundary value of a function which is analytic in a local tube $T_{B,\Phi}$; for simplicity we still denote by $f(\xi + i\eta)$ this analytic function [at the points ξ of Ω, $f(\xi)$ is analytic in the only case when the basis B contains the origin; if not, we are in a wedge situation and $f(\xi)$ is only the boundary value of $f(\xi + i\eta)$].

We shall now prove that the $\mathcal{F}_\Phi$-transform of any function f in this class admits S_B as its essential support and conversely.

More precisely one can state the following equivalence theorem which we present with a certain specification of regularity conditions on the boundary of $T_{B,\Phi}$:

THEOREM. — *There is equivalence between the two following properties :*

(i) *The function $f(\xi)$ belongs to $\mathcal{O}_{\hat{\Omega}}$ and its restriction to Ω is the boundary value of a function $f(\xi + i\eta)$ which is analytic in a local tube $T_{B,\Phi}$ with convex basis* B; *moreover $f(\xi + i\eta)$ extends to an infinitely differentiable function defined on the whole closure of $T_{B,\Phi}$ ($\overline{T}_{B,\Phi}$ is compact in $\Delta = \{p, \operatorname{Re}\Phi(p) < 1 + \varepsilon\}$).*

(ii) *The $\mathcal{F}_\Phi$-transform $F(x, x_0)$ belongs to $\mathcal{S}_x$ for every fixed value of x_0, and all the associated kernels $F_k(x, x_0, \xi)$ $(0 \leqq k \leqq n)$ defined by equation* (20) *satisfy the following bounds*

$$|F_k(x, x_0, p)| \leqq \frac{C_N}{1 + \tau^N} e^{-x_0} \quad [18], \tag{23}$$

at all points (x, x_0) of the half-space $x_0 \geqq 0$ which lie outside S_B (or on its boundary) and for all values of p in Δ.

These bounds hold for all positive integers N *and the constants C_N are independent of x, x_0 and p when these variables vary in the above domains.*

a. Proof that (i) *implies* (ii). — As a first step we shall establish the majorization formula (23) without the rapid decrease factor $\dfrac{1}{1 + \tau^N}$. To this purpose, we rewrite the integral formula (20) (with ξ' replaced

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[18] We recall that $\tau \equiv \tau[(x, x_0)] = \sqrt{x^2 + x_0^2}$.

by $p' = \xi' + i\eta'$) under the following form

$$(24) \qquad (2\pi)^{\frac{n}{2}} F_k(x, x_0, p') = F'_k(x, x_0, p') + F''_k(x, x_0, p')$$

with

$$(25) \qquad F'_k(x, x_0, p') = \int_{\Omega} f(\xi)\, \rho_k(\xi, p')\, e^{-i\xi x - \Phi(\xi) x_0}\, d\xi,$$

$$(26) \qquad F''_k(x, x_0, p') = \int_{\complement \Omega} f(\xi)\, \rho_k(\xi, p')\, e^{-i\xi x - \Phi(\xi) x_0}\, d\xi$$

$\left(\text{here } \complement\Omega \text{ is the complementary set of } \Omega \text{ in real } \xi\text{-space}\right)$.

Since $\Phi(\xi) > 1$ outside Ω and Φ is regular in the support $\hat{\Omega}$ of f, the integral (26) is trivially bounded in modulus by $C'' e^{-x_0}$ in the whole half space $x_0 \geq 0$; the constant C'' is given by

$$C'' = \sup_{\xi \in \mathbb{R}^n} |f(\xi)| \cdot \sup_{\substack{0 \leq k \leq n \\ p,\, p' \in p,\, p' \in \Delta}} |\rho_k(p, p')| \cdot V_{\hat{\Omega} \setminus \Omega}$$

where $V_{\hat{\Omega} \setminus \Omega}$ is the volume of the integration set $\hat{\Omega} \setminus \Omega$.

We now concentrate on the integral (25); in order to derive an exponential bound e^{-x_0} for this quantity, it is crucial to use the analyticity of $f(p)$ [and of $\rho_k(p, p')$] in the domain $T_{B,\Phi}$. In fact this will allow to make a suitable distortion of the integration contour Ω inside $T_{B,\Phi}$, before majorizing the integrand of formula (25).

Here we shall use the fact that $\bar{T}_{B,\Phi}$ is a union of admissible cycles $\Gamma_{\rho,\omega}$ and the definition of an admissible cycle given at the end of section 3.

As we have done there, we restrict ourselves to the manifold $p = \xi_{\perp} + \lambda\omega$ and use the analyticity of $f(p)$ in the complex variable λ for any fixed value of $\xi_{\perp}$. Then the definition of admissible cycles allows us to distort Ω into any cycle $\Gamma_{\rho,\omega}$ contained in $\bar{T}_{B,\Phi}$ and to write, for any point (ρ, ω) in $\bar{B}$:

$$(27) \qquad F'_k(x, x_0, p') = \int_{\Gamma_{\rho,\omega}} f(p)\, \rho_k(p, p')\, e^{-ipx - \Phi(p) x_0}\, d_n p.$$

By using the formulae (5) it is easy to see that for any point p in $\Gamma_{\rho,\omega}$:

$$|e^{-ipx - \Phi(p) x_0}| = e^{-x_0 + |\eta| \left(\omega \cdot x + \frac{x_0}{\rho}\right)}.$$

Therefore in the whole closed half-space $H_{\rho\omega}$ of (x, x_0) space with equation

$$\omega . x + \frac{x_0}{\rho} \leqq 0 \tag{28}$$

we obtain the majorization

$$| F'_k (x, x_0, p') | < C' e^{-x_0}, \tag{29}$$

where we have put

$$C' = V \sup_{p, \in \bar{T}_{B,\Phi}} | f(p) | . \sup_{\substack{0 \leqq k \leqq n \\ p, p' \in p, p' \in \Delta}} | \rho_k (p, p') | \tag{30}$$

and V is the upper bound of the volumes of all cycles $\Gamma_{\rho\omega}$ in $\bar{T}_{B,\Phi}$.

But in view of the definitions of S_B and of the polar $\tilde{B}$ of B (*see* section 3) we notice that the union of the half spaces $H_{\rho\omega}$ associated with all points (ρ, ω) in B covers precisely the closure of the complementary set of S_B in (x, x_0) space.

Putting together the majorizations for $| F'_k |$ and $| F''_k |$ we obtain

$$| F_k (x, x_0, p') | \leqq C_0 e^{-x_0}, \tag{31}$$

where $C_0 = \frac{1}{(2\pi)^{\frac{n}{2}}} \sup (C', C'')$, this bound being valid in the closure of the complementary set of S_B inside the half space $x_0 \geqq 0$; and for all values of p' in in Δ.

In order to derive the majorization formulae (23) for all successive values of the integer N we shall integrate the expression of $F_k (x, x_0, p)$ by part and use the properties of analyticity (and regularity at the boundary) of the successive derivatives of $f(p)$ [and of $\Phi(p)$] in $\bar{T}_{B,\Phi}$.

Let us start with the following expression of F_k :

$$(2\pi)^{\frac{n}{2}} F_k (x, x_0, p') = \int_{(\complement \Omega) \cup \Gamma_{\rho\omega}} f(p) \rho_k (p, p') e^{-ipx - \Phi(p) x_0} d_n p \tag{32}$$

where we have put together the expressions (26) and (27) of F''_k and F_k respectively ($\Gamma_{\rho\omega}$ being any admissible cycle).

Choosing an arbitrary direction ω_1, we can always rewrite (32) as follows

$$(2\pi)^{\frac{n}{2}} F_k (x, x_0, p') = - \int_{\complement \Omega \cup \Gamma_{\rho\omega}} d_n p \frac{f(p) \rho_k (p, p')}{i \omega_1 . x + (\omega_1 . \nabla_p \Phi(p)) x_0} \omega_1 . \nabla_p (e^{-ipx - \Phi(p) x_0}) \tag{33}$$

and provided that the denominator of the integrand does not vanish on the integration contour, partial integration is allowed and yields

$$(34)\quad (2\pi)^{\frac{n}{2}} F_k(x, x_0, p') = \int_{\complement \Omega \cup \Gamma_{\rho\omega}} d_n p\, \omega_1 . \nabla_p \left(\frac{f(p)\, \rho_k(p, p')}{i\,\omega_1 . x + (\omega_1 . \nabla_p \Phi(p))\, x_0} \right) . e^{-ip.x - \Phi(p).x_0}$$

$$= \frac{1}{\tau[(x, x_0)]} \int_{\complement \Omega \cup \Gamma_{\rho\omega}} d_n p\, \omega_1 . \nabla_p \times \left(\frac{f(p)\, \rho_k(p, p')}{i\,\omega_1 . u + \omega_1 . \nabla_p \Phi(p)\, u_0} \right) . e^{-ipx - \Phi(p)x_0}.$$

We shall below divide the set of points $(u, u_0 \geq 0)$ of the unit sphere which lie outside S_B (or on its boundary) into two parts and show that in each of them ω_1 can be chosen such that the denominator $i\,\omega_1 . u + (\omega_1 . \nabla_p \Phi(p))\, u_0$ never vanishes and has a uniform lower bound in modulus.

The argument which led to the majorization (31) of F_k can then be identically applied to the integral of the right hand side of (34) and therefore will yield the bound (23) for $N = 1$. By an obvious recursion over N one would obtain similarly all the bounds of the formula (23).

We first consider the set of points (u, u_0) which satisfy an inequality $u_0 \geq \varepsilon$ for a given strictly positive number ε.

For any fixed point (x, x_0) outside S_B or on its boundary there always exists at least one point (ρ, ω) in $\bar{B}$ such that the point (x, x_0) belongs to the *boundary* of the half space $H_{\rho\omega}$ defined in formula (28), i. e. we have

$$(35)\qquad x_0 = -\rho\omega . x$$

so that

$$u_0 = -\rho\omega . u.$$

We choose to integrate (34) precisely on the contour $\Gamma_{\rho\omega}\left(U \complement \Omega\right)$ associated with this point (ρ, ω) and moreover we choose $\omega_1 = \omega$. Then, in view of (35) the denominator $i\,\omega_1 . u + (\omega_1 . \nabla_p \Phi(p))\, u_0$ can be written

$$(36)\qquad i\,\omega_1 . u + (\omega_1 . \nabla_p \Phi(p))\, u_0 = -\frac{u_0}{\rho}(i - \rho\omega . \nabla_p \Phi(p)).$$

Now we recall as it was observed at the end of section 3 that the vector $i - \rho\omega . \nabla_p \Phi(p)$ [*see* (6)] certainly does not vanish on the admissible cycle $\Gamma_{\rho,\omega}$; it obviously does not vanish either on the contour $\complement\Omega$ since $\nabla_p \Phi(p)$ is real there.

A uniform lower bound of the modulus of (36) easily follows since $u_0 \geqq \varepsilon$.

We next consider the set of points $(u, 0 \leqq u_0 \leqq \varepsilon)$.

The quantity $(\omega_1 . \nabla_p \Phi(p)) u_0$ is then bounded in modulus by $\varepsilon \max_{p \in \overline{T}_{B,\Phi}} |\nabla_p \Phi(p)|$ independently of ω_1. If ε is chosen sufficiently small, it is thus always possible to find [for every value of $v = (u, u_0)$] a direction ω_1 such that $i\,\omega_1 . u + (\omega_1 . \nabla_p \Phi(p)) u_0$ does not vanish and has a uniform lower bound. We take for instance ω_1 in the direction of u.

Since $u^2 + u_0^2 = 1$ one obtains immediately :

$$| i\,\omega_1 . u + (\omega_1 . \nabla_p \Phi(p)) u_0 | > (1 - \varepsilon^2)^{1/2} - \varepsilon \max_{p \in \overline{T}_{B,\Phi}} | \nabla_p \Phi(p) |$$

$$> 0 \text{ (for } \varepsilon \text{ small enough).}$$

b. Proof that (ii) *implies* (i). — Conversely we shall now show that the bounds (23) allow to prove that $f(\xi)$ is the boundary value in Ω of a function $f(p)$ which is analytic in $T_{B,\Phi}$ and regular on its boundary. [The fact that $f(\xi)$ is infinitely differentiable on the real simply comes from the rapid fall off of $F(x, o)$.]

To this purpose we first show that as a consequence of the bounds (23) the boundary ∂S_B of the cone S_B is an admissible surface Σ in equation (21) when ξ is inside $\overline{\Omega}$.

We recall that in view of Stokes' theorem and of the fact that the differential form W is closed, the difference between the integrals of W on the hyperplane $x_0 = 0$ and on the surface ∂S_B is the limit when $R \to \infty$ of the integral of W on the surface defined by the conditions $|x| = R$, $x_0 \geqq 0$ and (x, x_0) in the complementary set of S_B. An example of such a surface is shown in figure 5 (with the cone S_B of figure 4).

But in view of the bounds (23) (where τ can be replaced by $|x| = R$) the integral on this surface is bounded in modulus by the following expression

$$\frac{C'_N}{1 + R^N} \int_0^{R\,r_{\max}} e^{(\Phi(\xi) - 1)x_0}\, dx_0 .$$

In the integral over x_0, the upper bound $R\,r_{\max}$ is a consequence of the fact that all the points of the integration surface satisfy the bound $0 \leqq x_0 \leqq r_{\max} |x|$ in view of formula (22) (see also *fig.* 5).

This quantity clearly tends to zero when $R \to \infty$ if N is chosen large enough, since $\Phi(\xi) \leq 1$ in $\overline{\Omega}$; therefore the expression (21) of $f(\xi)$ is valid when one takes ∂S_B for the integration surface Σ.

We now show that this integral defines a function which is analytic in the local tube $T_{B,\Phi}$, infinitely differentiable on its boundary, and coincides on $\overline{\Omega}$ with the function $f(\xi)$. We shall still call this function $f(p)$.

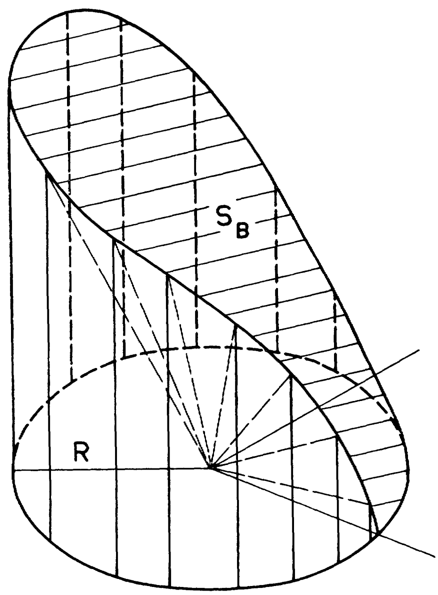

Fig. 5

We use the polar coordinates τ, $v = (u, u_0)$ in (x, x_0) space that we have already introduced, and we consider $f(p)$ as defined by the following integral [19] :

$$(37)\quad I(p) = \int_{\partial_B} \sum_{k=0}^{n} (-1)^k \sum_{\substack{l= \\ l \neq k}}^{n} \times (-1)^l u_l\, du_0 \wedge \dots \wedge \widehat{du_l} \wedge \dots \wedge \widehat{du_k} \wedge \dots \wedge du_n \times I_k(v, p, p)$$

where

$$(38)\quad I_k(v, p, p') = \int_0^{+\infty} \tau^{n-1} F_k(\tau, v, p')\, e^{\tau[ip\cdot u + \Phi(p) u_0]}\, d\tau.$$

[here we have used the notation $F_k(\tau, v, p')$ for $F_k(x(\tau, v), x_0(\tau, v), p')$].

(19) Which reduces to $(2\pi)^{\frac{n}{2}} f(\xi)$ when p is real in $\overline{\Omega}$ (in view of eq. (21).

The bound (23) (which is still valid on ∂S_B) allows to obtain :

$$(39)\quad |\tau^{n+1} F_k(\tau, v, p')\, e^{\tau[ip.u+\Phi(p)u_0]}| \leqq C_N \frac{\tau^{n-1}}{1+\tau^N} e^{\tau(-\eta u+(\mathrm{Re}\,\Phi(p)-1)u_0)}.$$

In order to prove the absolute convergence of the integrand in equation (38) at a given point $p \in \Delta$ and for a given value of v on $\mathcal{J}_B$, it is thus clearly sufficient to verify that

$$(40)\qquad -\eta u + (\mathrm{Re}\,\Phi(p) - 1)\, u_0 \leqq 0,$$

since N can be taken arbitrarily large.

With the given point v, let us associate the half space π_v defined as the set of all points (ρ, ω) which satisfy the inequality

$$(41)\qquad \omega . u + \frac{u_0}{\rho} \geqq 0.$$

Then taking the equation (5) into account we notice that the set $\mathcal{R}_v$ of points p where (40) is satisfied is exactly the union of all the manifolds $\hat{\Gamma}_{\rho,\omega}$ associated with all the points (ρ, ω) in the half space π_v (*see* section 3).

The factor $-\eta u + (\mathrm{Re}\,\Phi(p) - 1)\, u_0$ stays strictly negative when p is in the interior of $\mathcal{R}_v$ and can vanish on its boundary. This immediately implies that $I_k(v, p, p)$ is an analytic function of p inside $\mathcal{R}_v \cap \Delta$ and is continuous on its boundary and moreover that its bound is independent of v. [The regularity of all the derivatives of I_k on the boundary of $\mathcal{R}_v$ would be proved in exactly the same way since these derivatives have expressions which are completely similar to the equation (37).]

According to equation (37), $I(p)$ is an integral of the functions $I_k(v, p, p)$ over the points v in the indicatrix $\mathcal{J}_B$. Since this indicatrix is compact and the functions I_k are uniformly bounded in v, $I(p)$ is regular at all points p which lie in the intersection of all the sets $\mathcal{R}_v$ when v varies over $\mathcal{J}_B$, and it is analytic in the interior of this region.

Since the intersection of all half-spaces π_v is the polar of $\tilde{B}$, i. e. $\bar{B}$ [20], the closed local tube $\bar{T}_{B,\Phi} = \bigcup_{\rho,\omega \in \bar{B}} \Gamma_{\rho,\omega}$ thus belongs to the intersection [21] of all the regions $\mathcal{R}_v$, therefore we have proved that $f(p)$ is analytic in the local tube $T_{B,\Phi}$ and regular in its closure.

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(20) In view of the convexity of B.

(21) More precisely, it is a connected component of this intersection, the whole intersection being $\bigcap_{\rho,\omega \in \bar{B}} \hat{\Gamma}_{\rho\omega}$.

We shall now use the theorem which we have just proved to give a characterization of the functions $f(p)$ which are analytic in a local tube $T_{B,\Phi}$ and regular in its closure :

For every such $f(p)$, the boundary value of $f(\xi)$ on the real set $\overline{\Omega}$ always admits extensions to the whole ξ-space which are infinitely differentiable and have their supports in $\hat{\Omega}$; this is because the closed set $\overline{\Omega}$ has a sufficiently regular boundary (22).

Such an extension of $f(\xi)$ will be called *admissible*, and in view of the above theorem, any admissible extension of f has an $\mathcal{F}_\Phi$-transform $F(x, x_0)$ and associated kernels $F_k(x, x_0, p)$ whose essential support is the cone S_B.

If we now consider two admissible extensions of $f(p)$ we have $f_1(\xi) - f_2(\xi) = 0$ on $\overline{\Omega}$ and therefore their $\mathcal{F}_\Phi$-transforms and associated kernels $F_k^{(1)}$, $F_k^{(2)}$ satisfy the following bounds

$$(42) \qquad | F_k^{(1)}(x, x_0, p) - F_k^{(2)}(x, x_0, p) | < \frac{C_N}{1 + \tau^N} e^{-x_0}$$

for all points x_0 in the half space $x_0 \geq 0$.

[To show this, we apply the argument given for majorizing the expression (26).]

Conversely one shows that every solution of the equation $D_{\mathcal{M}} F = 0$ which is bounded by e^{-x_0} in the half space $x_0 \geq 0$ is the $\mathcal{F}_\Phi$-transform of a function f which vanishes in $\overline{\Omega}$: to see this, one notices that for ξ in $\overline{\Omega}$ the inverse $\mathcal{F}_\Phi$-formula can be applied when one chooses the integration surface to be any plane $x_0 = C > 0$; then letting C tend to infinity, we see that such an integral is necessarily equal to zero.

So it is proved that to all the admissible extensions of any function $f(p)$ analytic in $T_{B,\Phi}$ and regular in $\overline{T}_{B,\Phi}$ corresponds an *equivalence class* of solutions $F(x, x_0)$ of the equation $D_{\mathcal{M}} F = 0$ which have the essential support S_B; all the solutions in the given class are obtained from one of them by adding to it any solution which is bounded by $\frac{C_N}{1 + \tau^N} e^{-x_0}$ in the whole half-space $x_0 \geq 0$.

We next want to point out the following geometrical fact which appeared in the course of the proof of (ii) $\to$ (i) : the interior of each region $\mathcal{R}_v$ is a natural domain of holomorphy (23) in C^n, since it is bounded

(22) For this kind of results see for instance the papers by B. Malgrange : *Le théorème de préparation* and Whitney's theorem [15].

(23) We recall that a natural domain of holomorphy (or " holomorphy domain ") in $\mathbf{C}^n$ is a domain D such that there exists at least one function f which is defined and analytic in D and cannot be analytically continued across any part of the boundary of D [11]; for $n \geq 2$, this property is not true for an arbitrary domain in $\mathbf{C}^n$.

by the analytic hypersurface [10] with equation

$$\operatorname{Re}\,(ipu + (\Phi\,(p) - 1)\,u_0) = 0.$$

Therefore any connected component of the intersection of all these domains is also a holomorphy domain. As we saw, one of these components coincides with the local tube $T_{B,\Phi}$ if (and only if) B is convex.

So we have proved :

THEOREM. — *A necessary and sufficient condition for a local tube* $T_{B,\Phi}$ *to be a holomorphy domain* ([23]) *is that its basis* B *be convex.*

Let us now consider a local tube $T_{B,\Phi}$ whose basis B is not convex. Any function $f(p)$ which is analytic in $T_{B,\Phi}$ and regular in $\overline{T}_{B,\Phi}$ has an admissible extension whose associated kernels $F_k(x, x_0, p)$ have their essential support in S_B.

But in view of the proof that (ii) implies (i), the inverse formula (37) defines $f(p)$ as an analytic function in the local tube $T_{\hat{B},\Phi}$ whose basis $\hat{B}$ is the (open) convex hull of B. So we have proved :

THEOREM. — *The holomorphy envelope* [11] *of a local tube* $T_{B,\Phi}$ *(for the class of functions which are holomorphic in* $T_{B,\Phi}$ *and regular in* $\overline{T}_{B,\Phi}$*) is the local tube* $T_{\hat{B},\Phi}$ *whose basis* $\hat{B}$ *is the convex hull of* B.

This theorem is a generalization of the " tube theorem " which we already quoted in section 2.

We are now finished with what we wanted to present here about the generalized Laplace transform method but before we go to its application to the edge of the wedge theorem in section 7, we want to add a few words about certain features of our problem which we omitted to study here.

As it was announced at the end of the introduction these features will be fully developed in a coming paper [1] and are the following.

Extensions of the generalized Laplace transform to the case when the boundary value of $f(p)$ on the real is no more regular can be given; one will be able to give a version in the case when these boundary values are distributions and another one in the most general case of hyperfunctions. One will also get rid of the regularity condition in the theorem about the holomorphy envelope of a local tube $T_{B,\Phi}$.

We also hope that this method will make simpler some aspects of the theory of hyperfunctions [4] and will have some connections with the problems considered by Hörmander in [16].

Finally we also reserve for [1] (although it is a very simple consequence of what has been done in the present section) the proof of a Cauchy-Fantappié integral representation for the functions which are analytic in a local tube $T_{B,\Phi}$ and regular in its closure.

7. THE GENERALIZED EDGE-OF-THE-WEDGE THEOREM

The " edge-of-the-wedge " problems are generalizations to the theory of functions of several complex variables of the following simple result in one variable.

If two functions $f_1(z)$, $f_2(z)$ are analytic respectively in the upper and lower half planes and have boundary values which coincide on a certain open interval Ω, then there exists a single analytic function $f_{12}(z)$ which coincides with f_1 and f_2 in their respective domains and is analytic at all the points of Ω.

In the space $\mathbf{C}^n$ an analogous theorem has been proved by using standard methods of several complex variables theory (such as the disk theorem). It is known under the traditional name of the " edge-of-the-wedge " theorem and various versions of it have been presented about ten years ago by various physicists ([24]).

It turns out that the introduction of local tubes allows to state a very neat version of this theorem, which we are going to present now. Our proof is made very simple by using the results of section 6 and this method allows a generalization of the theorem which we present afterwards.

In all the following, we consider functions $f_i(p)$ which are analytic respectively in local tubes $T_{B_i,\Phi}$ and we assume for simplicity that they are infinitely differentiable in the closures of the domains $T_{B_i,\Phi}$; Φ is a given localizing function in an open set Ω (the same for all the considered local tubes) and we are mainly interested in the case when Ω belongs to the boundary of each of the domains $T_{B_i,\Phi}$ (" wedge " situations). Each B_i is assumed to be convex.

One also introduces for any couple (i, j) the local tube $T_{B_{ij},\Phi}$ whose basis B_{ij} is the convex hull of the union of B_i and B_j.

In the (usual) edge-of-the-wedge theorem, one is concerned with two functions f_1, f_2 respectively analytic in the local tubes $T_{B_1,\Phi}$ and $T_{B_2,\Phi}$.

The statement is the following : if f_1 and f_2 have the same boundary value $f(\xi)$ on the (closed, real) set $\overline{\Omega}$, then there exists an analytic function $f_{12}(p)$ which coincides with f_1 (resp. f_2) in the domain $T_{B_1,\Phi}$ (resp. $T_{B_2,\Phi}$), which is analytic in the local tube $T_{B_{12},\Phi}$ and is regular in its closure [it therefore coincides with $f(\xi)$ in $\overline{\Omega}$].

([24]) For complete references, *see* Streater and Wightman [13]. The most general of these versions and the closest to the one which we prove here has been given by Epstein [17].

Remarks :

(i) This theorem can be considered as a refined version of the theorem on the holomorphy envelope of a local tube (*see* the end of section 6); and since $\overline{T}_{B_{12},\Phi}$ is in general larger than the union of $\overline{T}_{B_1,\Phi}$ and $\overline{T}_{B_2,\Phi}$, it really provides a common analytic continuation f_{12} for the couple (f_1, f_2). This clearly shows why this theorem is not trivial as soon as the number of variables is larger than one.

(ii) Two geometrical situations can be distinguished (see *fig.* 6).

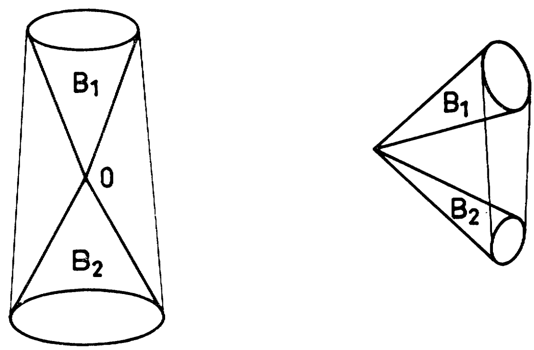

Fig. 6

If B_1 and $(-B_2)$ intersect each other, B_{12} contains the origin and therefore the points of Ω are points of analyticity for f_{12}. (This is typically the case when $B_1 = -B_2$.)

If B_1 and $(-B_2)$ have an empty intersection, then $B_{1,2}$ does not contain the origin and the points of Ω remain boundary points for $f_{1,2}$.

Proof. — In the case when $T_{B_1,\Phi}$, $T_{B_2,\Phi}$ are respectively replaced by two tubes T_{B_1}, T_{B_2} and the open set Ω by R^n, a very intuitive proof can be given through using the Laplace transform theorem of section 2.

The proof that we give for the general cases of local tubes is based on the same idea and makes use of the generalized Laplace transform theorem of section 6.

The proof goes as follows : one chooses an arbitrary admissible C^∞-extension of $f(\xi)$ (*see* section 6) and then one considers its $\mathcal{F}_\Phi$-transform $F(x, x_0)$ and the associated kernels $F_k(x, x_0, p)$.

Since $f(\xi)$ is the boundary value of $f_1(p)$ [resp. $f_2(p)$], these kernels have their essential support in (x, x_0)-space inside S_{B_1} [resp. S_{B_2}]. Therefore their actual essential support is the intersection of S_{B_1} and S_{B_2}.

We notice now that the intersection of the bases B_1 and B_2 of S_{B_1} and S_{B_2} in the hyperplane $x_0 = 1$ is in fact the polar set of B_{12} and the edge-of-the-wedge theorem is then simply obtained through the equivalence theorem of section 6.

For the sake of rigour, we note that the following fact is used in the end of the argument : two analytic functions in a *common* local tube $T_{B,\Phi}$ which have the same boundary values on the real domain Ω are identical in $T_{B,\Phi}$ (this can be proved easily by taking one-dimensional sections and applying the " Schwarz symmetry principle ").

The idea of the generalized edge-of-the-wedge theorem which we shall now present is issued both from considerations of quantum field theory [where some special problems of boundary values occur ([25])], and from the cohomological scheme of the theory of hyperfunctions ([26]) by Sato [4]. We shall state this theorem as follows :

THEOREM. — *Let $f_1(p) \dots f_l(p)$ be l functions analytic respectively in local tubes $T_{B_i,\Phi}$ $(i = 1, \dots, l)$ and regular in the closures of these domains.*

If their boundary values $f_i(\xi)$ in $\overline{\Omega}$ satisfy the condition

$$\sum_{i=1}^{l} f_i(\xi) = 0 \tag{43}$$

then there exists a set of functions $f_{ij}(p)$ $(i, j = 1, 2, \dots, l;\ i \neq j)$ with the following properties :

a. for every couple (i, j), $f_{ij}(p)$ is analytic in the local tube $T_{B_{ij},\Phi}$ and regular in its closure;

b.

$$f_i(\xi) = \sum_{\substack{j=1 \\ j\neq i}}^{l} f_{ij}(\xi) \quad (\textit{for every } \xi \textit{ in } \overline{\Omega}); \tag{44}$$

c.

$$f_{ij}(p) = -f_{ji}(p) \quad (\textit{for every } p \textit{ in } T_{B_{ij},\Phi}). \tag{45}$$

Remarks :

(i) The edge-of-the-wedge theorem is reobtained as the special case $l = 2$ [one just has to change f_2 into $-f_2$ to transform the relation (43) into the coincidence condition $f_1 = f_2$ of the edge of the wedge theorem].

([25]) This was realized in the course of a collaboration of one of the authors (J. B.) with H. Epstein, V. Glaser and R. Stora.

([26]) Here one has to emphasize the fundamental role played by J. Lascoux and B. Malgrange to make these concepts accessible to the mathematical physicist.

(ii) Applying the above corollary of the edge-of-the-wedge theorem to the function $f_i(p) - \sum_{j=1}^{l} f_{ij}(p)$, one sees that formula (44) also holds for p complex inside the local tube $T_{B_i,\Phi}$. But we notice that the right hand side of this equation is analytic in the domain $\bigcap_{\substack{1 \leq j \leq l \\ j \neq i}} T_{B_{ij},\Phi}$ which is in general larger than $T_{B_i,\Phi}$ (see *fig.* 7). Therefore the generalized edge-of-the-wedge theorem implies in general *the existence of analytic continuation* for each function f_i.

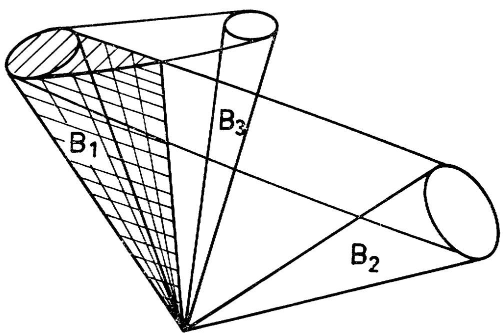

Fig. 7. — $B_{12} \cap B_{13}$ has been represented hatched and is clearly larger than B_1.

Proof of the theorem. — In the case of tubes T_{B_i} (i. e. $\Omega = R^n$), it was here again recognized several years ago [27] that the Laplace transform theorem was a simple tool to prove this property.

We shall use similarly the $\mathcal{F}_\Phi$-transform and reobtain the case $\Omega = R^n$ as a special case, since all the results of section 6 remain valid in the limit $\Phi \to 0$.

Our first step will be to show that in $\overline{\Omega}$ it is possible to write

$$f_1(\xi) = \sum_{j=2}^{l} f_{1j}(\xi), \tag{46}$$

where each function f_{1j} is analytic in $T_{B_{1j},\Phi}$ and regular in $\overline{T}_{B_{1j},\Phi}$.

To this purpose we consider for $1 \leq j \leq l$ the $\mathcal{F}_\Phi$-transforms $F^{(j)}(x, x_0)$ of admissible C^∞-extensions of $f_j(\xi)$ and all the associated kernels $F_k^{(j)}$

[27] In particular by V. Glaser [5]. An alternative proof of the tube version was also given by A. Martineau [6] who indicates that his method can still be used in the local case.

introduced by formula (20). Obviously these C^∞-extensions of the functions $f_j(\xi)$ can be performed in such a way that the relation (43) holds everywhere in ξ-space [for instance, one chooses the extensions of $f_2, \ldots, f_n$ and then defines the extension of f_1 through formula (43)]. Let us now rewrite the condition (43) under the form

$$f_1(\xi) = -\sum_{j=2}^{l} f_j(\xi),$$

which implies the following identities in (x, x_0) space (where $k = 0, 1, \ldots, n$) :

$$\text{(47)} \qquad F_k^{(1)}(x, x_0) = -\sum_{j=2}^{l} F_k^{(j)}(x, x_0).$$

In view of the analyticity properties of the functions $f_1(p) \ldots f_l(p)$, the essential support of the right-hand side of (47) is contained in the union of the sets S_{B_j} for all values of j such that $2 \leqq j \leqq l$, while the left-hand side of (47) has its essential support contained in S_{B_1}. So the actual essential support of all the functions $F_k^{(1)}(x, x_0)$ is the (non convex) cone

$$\text{(48)} \qquad S_{B_1} \cap \left(\bigcup_{j=2}^{l} S_{B_j} \right) = \bigcup_{j=2}^{l} (S_{B_1} \cap S_{B_j}).$$

The integral (21) which allows to re-express $f_1(p)$ in terms of the $n + 1$ associated kernels $F_k^{(1)}(x, x_0)$ can then be performed on the boundary Σ_1 of the set (48). But if one considers the covering of Σ_1 by the closed sets $S_{B_1} \cap S_{B_j}$, it is always possible to make a partition ([28]) of Σ_1 into $l - 1$ piece Σ_{1j} in such a way that each piece Σ_{1j} be contained in the boundary of the convex cone $S_{B_1} \cap S_{B_j}$. For $2 \leqq j \leqq l$, we define the function f_{1j} as the restriction of the integral (21) to the surface Σ_{1j}. Then the geometrical study made in section 6 shows that f_{1j} is analytic in the local tube $T_{B_{ij}, \Phi}$ and regular in its closure, and we thus have made our first step which was the decomposition (46).

Similarly each function f_i has an analogous decomposition. However the second step to perform is the proof that the functions f_{ij} of all these decompositions can always be constructed in order to satisfy the anti-symmetry relations (45).

This will be done by using a recurrent procedure which we owe to R. Stora [18]. The theorem is true for $l = 2$ (edge-of-the-wedge theorem) and we suppose it holds for $(l - 1)$ functions f_i.

([28]) With a certain degree of arbitrariness which we shall not analyze here.

In the case of l functions $f_i(\xi)$ we first write

$$(48') \qquad f_1 = \sum_{j=2}^{l} f_{1j},$$

as described above.

We then define the functions $f_{j1} = -f_{1j}$ (49) and the $l-1$ functions

$$(50) \qquad g_j = f_j + f_{1j} \qquad (j = 2, \ldots, l)$$

which satisfy, in view of (48') :

$$(51) \qquad \sum_{j=2}^{l} g_j = \sum_{i=1}^{l} f_i = 0.$$

Since $f_{1j}(p)$ is analytic in $T_{B_{1j},\Phi}$ (which contains $T_{B_j,\Phi}$) and regular in its closure, g_j is analytic in $T_{B_j,\Phi}$ and regular in its closure. So the $(l-1)$ functions g_j fulfill all the properties which allow to apply our recurrent assumption, and we can thus write

$$(52) \qquad g_j = \sum_{\substack{k \neq j \\ k=2,\ldots,l}} f_{jk} \qquad \text{for} \quad j = 2, \ldots, l,$$

with

$$(53) \qquad f_{jk} = -f_{kj}.$$

Putting together the formulae (52), (49), (50), we obtain :

$$(54) \qquad f_j = \sum_{\substack{k \neq j \\ 1 \leq k \leq l}} f_{jk} \qquad \text{for} \quad 1 \leq j \leq l,$$

which achieves our proof, since all the functions f_{jk} $(1 \leq j \neq k \leq l)$ satisfy the antisymmetry relations (49) and (53).

As a final remark, we must emphasize that all these results, as those of section 6, could have been presented under more general assumptions concerning the boundary values of the functions $f_i(p)$; one could have taken distributions, or more generally hyperfunctions. There is no essential new difficulty in doing this and it will be done in the already announced forth-coming paper [1].

Moreover, our method allows to handle without new difficulty the special configurations where the local tubes $T_{B_i,\Phi}$ are no longer open sets in $\mathbf{C}^n$, but open sets in lower-dimensional linear manifolds : one thus reobtains theorems of the same kind as the Malgrange-Zerner theorem or " flattened tube theorem " [9] (this will also be done in [1]).

Note added in proof. — Since the first publication of this work in 1971, some applications of it to quantum Field Theory have already been achieved and published : *see* J. Bros, H. Epstein and V. Glaser, *Helv. Phys. Acta.*, vol. 45, 1972, p. 149; *see* also J. Bros and D. Iagolnitzer in *Proceedings of the* 1972 *Moscow International Conf. on Math. Methods in Q. F. T. and quantum Statistics.* Concerning the applications to S-matrix theory, *see* also the book by D. Iagolnitzer : *Introduction to* S-*matrix theory.*

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