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Section A :
Physique théorique.

Dense sums

by

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ABSTRACT. — We consider a uniformly bounded sequence $\{ \gamma_j (L) \}$ $(j = 1, 2, \ldots)$ of positive numbers depending on a parameter L. The numbers $\gamma_j (L)$ become « dense » when $L \to \infty$ in the sense that the weak limit of the measure

$$\underset{L \to \infty}{\text{weak limit}} \frac{\sum_{j=1}^{\infty} \delta [c - \gamma_j (L)] \gamma_j (L) \, dc}{\sum_{j=1}^{\infty} \gamma_j (L)} = \nu (dc),$$

exists (δ denotes Dirac's delta), and their sum $V (L) = \sum_{j=1}^{\infty} \gamma_j (L)$ tends to ∞ when $L \to \infty$. In particular this situation arises when $\gamma_j (L) = c \left(\frac{j}{L} \right)$ $(j = 1, 2, \ldots)$, $c (k)$ being a non negative continuous bounded function falling more rapidly than k^{-1} as $k \to \infty$.

Then defining the numbers

$$a_n^L = \sum_{j_1 < j_2 \ldots < j_n} \gamma_{j_1} (L) \gamma_{j_2} (L) \ldots \gamma_{j_n} (L),$$

which are generated by the infinite product

$$\prod_{j=1}^{\infty} [1 + z\,\gamma_j(\mathrm{L})] = \sum_{n=0}^{\infty} a_n^{\mathrm{L}} z^n,$$

we study the existence of some limits of the type

$$g\left(\frac{n}{\mathrm{V}}\right) = \lim_{\substack{n \to \infty \\ \mathrm{L} \to \infty \\ \frac{n}{\mathrm{V(L)}} = \text{const.}}} \frac{1}{\mathrm{V(L)}} \log a_n^{\mathrm{L}}; \qquad \zeta\left(\frac{n}{\mathrm{V}}\right) = \lim_{\substack{n \to \infty \\ \mathrm{L} \to \infty \\ \frac{n}{\mathrm{V(L)}} = \text{const.}}} \frac{a_n^{\mathrm{L}}}{a_{n+1}^{\mathrm{L}}}.$$

We prove that a necessary and sufficient condition for the existence of such limits is the existence of the limit measure ν, and we find simple relations among g, ζ, etc. The above problems have been suggested in connection with some recent research in quantum statistical mechanics, where they find some applications. Extensions to more general sequences $\{\gamma_j(\mathrm{L})\}$ are discussed.

1. Introduction and notations

The analysis of the mathematical structure of the thermodynamic limit of superconducting systems has recently [1] given rise to some problems which seem of interest for the pure mathematician. The following situation has been met :

Consider a function $c(k)$ defined on the positive real axis and call $\mathrm{R_L} = \left\{ k : k = \frac{2\pi}{\mathrm{L}} j,\ j = 1, 2, 3, \ldots \right\}$ a lattice on the positive axis with spacing $\frac{2\pi}{\mathrm{L}}$.

Given a positive integer n we define

$$a_n^{\mathrm{L}} = \sum_{k_1 < k_2 \ldots < k_n} c(k_1)\, c(k_2) \ldots c(k_n) = \frac{1}{n!} \sum_{k_1 \neq k_2 \ldots \neq k_n} c(k_1)\, c(k_2) \ldots c(k_n), \tag{1.1}$$

where $k_j \in \mathrm{R_L}$.

These numbers are the coefficients of the power series expansion of the following function :

$$f^{\mathrm{L}}(z) = \prod_{k \in \mathrm{R_L}} [1 + z\, c(k)] = \sum_{n=0}^{\infty} z^n a_n^{\mathrm{L}}; \qquad a_0^{\mathrm{L}} = 1. \tag{1.2}$$

The assumptions on $c(k)$ such that (1.1), (1.2) make sense will be discussed later in detail. For the sake of clarity in defining the problem, in this section we make very stringent assumptions : we assume $c(k)$ to be a continuous bounded monotonic function decreasing at least as $k^{-1-\varepsilon}$, $(\varepsilon > 0)$ as $k \to +\infty$.

The physical interpretation of the above objects suggests the investigation of the asymptotic properties of the numbers a_n^{L} as $\mathrm{L} \to \infty$, $n \to \infty$.

Simple combinatorial arguments using the decomposition of a permutation into a product of cycles [2] lead to :

$$(1.3)\qquad a_n^{\mathrm{L}} = \sum_{j_1+2j_2+\dots mj_m=n} (-1)^{j_2+j_4+j_6\dots} \frac{(\sigma_1^{\mathrm{L}})^{j_1}}{j_1\,!} \frac{(\sigma_2^{\mathrm{L}})^{j_2}}{2^{j_2} j_2\,!} \cdots \frac{(\sigma_m^{\mathrm{L}})^{j_m}}{m^{j_m} j_m\,!},$$

where [1]

$$(1.4)\qquad \sigma_p^{\mathrm{L}} = \sum_{k \in \mathrm{R_L}} c(k)^p = \frac{\mathrm{L}}{2\pi}\left(\int_0^\infty c(k)^p\, dk\right)\left[1 + \mathrm{O}\left(\frac{1}{\mathrm{L}}\right)\right].$$

It follows that

$$(1.5)\qquad a_n^{\mathrm{L}} = \frac{\mathrm{L}^n}{n!}\left(\int_0^\infty c(k)\frac{dk}{2\pi}\right)^n\left[1 + \mathrm{O}\left(\frac{1}{\mathrm{L}}\right)\right]^n - \frac{1}{2}\frac{\mathrm{L}^{n-1}}{(n-2)!}\left(\int_0^\infty c(k)\frac{dk}{2\pi}\right)^{n-2}\int_0^\infty c(k)^2\frac{dk}{2\pi} \times \left[1 + \mathrm{O}'\left(\frac{1}{\mathrm{L}}\right)\right]^{n-1} + \mathrm{O}_n(\mathrm{L}^{-2})\,\mathrm{L}^n.$$

Therefore if $\mathrm{L} \to \infty$ and n is constant we see that the asymptotic behaviour of a_n^{L} is dominated by the first term of Equation (1.5). If $\mathrm{L} \to \infty$ but $\frac{n}{\mathrm{L}}$ tends to a constant value $d < +\infty$, then this is no longer true since the number of terms contributing to the R. H. S. of Equation (1.3) is rapidly increasing with n. Let us call T. limit [2] the limit performed when $\mathrm{L} \to \infty$, $n \to \infty$, $\frac{n}{\mathrm{L}} \to d$, $0 \leq d \leq \infty$. We see that the study of the asymptotic properties of the coefficients a_n^{L} in the T. limit makes sense and is non trivial. A simple use of Stirling's formula shows that the logarithm of the first term on the R. H. S. of Equation (1.5) is asymptotically proportional to L in the T. limit.

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(1) We shall often use symbols of the type $\mathrm{O}(x)$, $\mathrm{O}'(x)$, $\mathrm{O}''(x)$, $\mathrm{O}_i(x)$, to denote functions which tend to zero as $x \to 0$.

(2) « T » stands for « Thermodynamic ».

This suggests asking whether the T. limit

$$g(d) = \mathrm{T.}\lim_{\frac{n}{\mathrm{L}} \to d} \mathrm{L}^{-1} \log a_n^{\mathrm{L}}, \tag{1.6}$$

exists.

The consideration of the « leading » term in Equation (1.5) suggests asking whether the limit

$$\zeta(d) = \mathrm{T.}\lim_{\frac{n}{\mathrm{L}} \to d} \frac{a_n^{\mathrm{L}}}{a_{n+1}^{\mathrm{L}}}, \tag{1.7}$$

exists. Another interesting question is the following : suppose we fix a point $k_0^{\mathrm{L}} \in \mathrm{R_L}$ and define

$$a_n^{\mathrm{L}}(k_0^{\mathrm{L}}) = \frac{1}{n!} \sum_{\substack{k_1, k_2 \ldots k_n \\ k_i \neq k_j}}^{k_0^{\mathrm{L}}} c(k_1)\, c(k_2) \ldots c(k_n); \qquad a_0^{\mathrm{L}}(k_0^{\mathrm{L}}) = 0, \tag{1.8}$$

where $\sum^{k_0^{\mathrm{L}}}$ means that the set $\{k_1, \ldots, k_n\}$ contains k_0^{L}; the nwe expect that $a_n^{\mathrm{L}}(k_0^{\mathrm{L}})$ and a_n^{L} have the same asymptotic properties in the T. limit. More precisely, we can ask whether the following limit exists

$$\rho_d(k_0) = \mathrm{T.}\lim_{\substack{\frac{n}{\mathrm{L}} \to d \\ k_0^{\mathrm{L}} \to k_0}} \frac{a_n^{\mathrm{L}}(k_0^{\mathrm{L}})}{a_n^{\mathrm{L}}}, \tag{1.9}$$

k_0 being a positive number. If all the above limit exist, are they uniform in d ? And can we find an « explicit » expression for their values ?

Section 2 recalls and generalizes the results of [1], and is meant to provide the basic technical tools for the proof of the main theorem of the paper which is presented in Section 3.

The methods of Section 2 are combinatorial techniques used in [3], [1]; they give results stronger than usual since it is possible to give a closed form to $\lim_{\mathrm{L} \to \infty} \mathrm{L}^{-1} \log f^{\mathrm{L}}(z)$ [*see* Eq. (1.2)]. Clearly

$$\lim_{\mathrm{L} \to \infty} \mathrm{L}^{-1} \log f^{\mathrm{L}}(z) = \int_0^{\infty} \frac{dk}{2\pi} \log[1 + zc(k)]. \tag{1.10}$$

Section 3 can be read independently from Section 2 provided one accepts theorem 1 and inequality (2.14). This Section is devoted to the study of the following general problem : the questions raised in Equations (1.6), (1.7), (1.8), (1.9) can be regarded as concerning the asymptotic behaviour of sums over a set of points which becomes « uniformly dense » on the positive real axis. A similar but more general problem of « dense sums » can be formulated as follows : let L denote a real number and give for each L a sequence of positive numbers $\{\gamma_j(\mathrm{L})\}$ $(j = 1, 2, \ldots)$; suppose

$$(1.11) \qquad \sup_{\mathrm{L}, j} \gamma_j(\mathrm{L}) \leqq \Lambda < +\infty; \qquad \mathrm{V}(\mathrm{L}) = \sum_{i=1}^{\infty} \gamma_j(\mathrm{L}) < +\infty.$$

Consider then the infinite product

$$(1.12) \qquad f^{\mathrm{L}}(z) = \prod_{j=1}^{\infty} [1 + z\gamma_j(\mathrm{L})] = \sum_{n=0}^{\infty} z^n a_n^{\mathrm{L}},$$

which implicitely defines the a_n^{L}. Clearly if we choose $\gamma_j(\mathrm{L}) = c\left(\frac{2\pi j}{\mathrm{L}}\right)$ we obtain Equation (1.2), and V (L) becomes equal to σ_1^{L} [*see* Eq. (1.4)], which is asymptotically proportional to L.

Assume that $\mathrm{V}(\mathrm{L}) \to \infty$ as $\mathrm{L} \to \infty$, and let us investigate the asymtotic properties of the coefficients a_n^{L} in the general case. Intuitively we expect that the role before played by L will now be played by V (L). Indeed we shall show that a necessary and sufficient condition in order that Equations (1.6), (1.7), (1.9) hold [with L replaced by V (L)] is that, defining a measure $\mu_{\mathrm{L}}(dc)$ over the positive real axis as

$$(1.13) \qquad \mu_{\mathrm{L}}(dc) = \sum_{i=1}^{\infty} \delta[c - \gamma_j(\mathrm{L})] \frac{dc}{\mathrm{V}(\mathrm{L})},$$

(δ denotes the Dirac's δ), the following limit exists :

$$(1.14) \qquad \underset{\mathrm{L} \to \infty}{\text{weak lim}}\, c\,\mu_{\mathrm{L}}(dc) = \nu(dc).$$

In other words, for any continuous bounded function $\varphi(c)$ we must have

$$(1.15) \qquad \lim_{\mathrm{L} \to \infty} \int_0^{\infty} \varphi(c)\, c\, \mu_{\mathrm{L}}(dc) = \int_0^{\infty} \varphi(c)\, \nu(dc).$$

Notice that, due to the identity $\int_0^{\infty} c\,\mu_{\mathrm{L}}(dc) = 1$, there are always (3) convergent subsequences of measures even if the limit (1.14) does not exist.

The fact that Equation (1.14) is a necessary and sufficient condition for the existence of T. limits like T. $\lim \frac{1}{\mathrm{V}(\mathrm{L})} \log a_n^{\mathrm{L}}$ constitutes a strong generalization of the analogous result obtained in [1] concerning the case of a bounded monotonic function. As will be seen in para-

(3) Notice that because of (1.11) the integration is over $[0, \Lambda]$.

graph 4, given an arbitrary non-negative piecewise continuous function c, it is possible to group together points of the k-axis at which c assumes the same value, by means of the relation $\gamma_j(\mathrm{L}) = c\left(\frac{2\pi j}{\mathrm{L}}\right)$. Since the points $\gamma_j(\mathrm{L})$ become « dense » when $\mathrm{L} \to \infty$ in such a way that the limit (1.14) exists, we obtain the above mentioned generalization of the results of [1]. Finally, the following extension of Theorem 1 of [1] is obtained : we prove that all the T. limits (1.6) through (1.9) exist even when c is singular for $k \to 0$, provided $\log[1 + c(k)]$ is integrable on $[0, \varepsilon]$ for some $\varepsilon > 0$.

2. This Section is devoted to the proof of the existence of the limits [(1.6) à (1.9)]. Since part of the techniques used are standard in modern statistical mechanics, and part of the details can be found in [1], we shall omit the more elementary steps of the proof.

Theorem 1. — *Let $c(k)$ be a non-negative decreasing function defined on $(0, +\infty)$ and suppose :*

(i) $\log[1 + c(k)]$ *is integrable on* $[0, \varepsilon]$ *for some* $\varepsilon > 0$;

(ii) $\sum_{\substack{k > k^* \\ k \in \mathrm{R_L}}} c(k) < +\infty$ *for some* L *and for* $k^* > 0$.

Then the limits (1.6), (1.7) exist and the limit (1.9) exists in all points k_0 where c is continuous. Furthermore, defining

$$d^* = \inf \{ d : g(d) = -\infty \},$$

all the above limits are uniform in d for d belonging to any interval $[d_1, d_2]$, where $0 < d_1 < d_2 < d^*$.

Proof. — Let us prove first the following inequality

$$a_n^{2\mathrm{L}} \geq \sum_{n_1 + n_2 = n} a_{n_1}^{\mathrm{L}} a_{n_2}^{\mathrm{L}}. \tag{2.1}$$

Since the lattice $\mathrm{R_{2L}} = \left\{ k : k = \frac{\pi}{\mathrm{L}} j, j = 1, 2, 3, \ldots \right\}$ can be decomposed as $\mathrm{R_L} \cup \mathrm{R'_L}$, where $\mathrm{R'_L}$ is obtained simply shifting the points of $\mathrm{R_L}$ of an amount $\frac{\pi}{\mathrm{L}}$ to the left, we can write

$$a_n^{2\mathrm{L}} = \sum_{n_1=0}^{n} \sum_{\substack{k_1 < k_2 \ldots < k_{n_1} \\ k_i \in \mathrm{R_L}}} c(k_1)\, c(k_2) \ldots \times c(k_{n_1}) \sum_{\substack{k'_1 < k'_2 \ldots < k'_{n-n_1} \\ k'_i \in \mathrm{R'_L}}} c(k'_1)\, c(k'_2) \ldots c(k'_{n-n_1}). \tag{2.2}$$

Due to the decreasing character of c, when we substitute in the last summation R_L in the place of R'_L, the value of the sum decreases. Setting $n - n_1 = n_2$, we obtain inequality (2.1). In particular for $n = 2m$, and picking only the term with $n_1 = n_2 = m$, we have

$$a_{2m}^{2L} > (a_m^L)^2. \tag{2.3}$$

Let us set now $G(L, n) = \frac{1}{L} \log a_n^L$, and consider the sequences

$$L_i = 2^i L_0; \qquad n_i = 2^i d L_0 \quad \text{for } i = 1, 2, \ldots \tag{2.4}$$

where $L_0 > 0$ and dL_0 is integer. From (2.3) it follows that the sequence $\{G(L_i, n_i)\}$ is strictly increasing. Let us now assume that $\sup_k c(k) < \infty$. Then, since $\sum_{k \in R_L} c(k) < \infty$, c is summable ([4]) on $[0, +\infty)$ and we have

$$a_n^L < \frac{\left(\sum_{k \in R_L} c(k)\right)^n}{n!} \leqq \frac{L^n}{n!}\left(\int_0^\infty c(k) \frac{dk}{2\pi}\right)^n. \tag{2.5}$$

Hence using Stirling's inequality $\log n! \geqq n \log n - n$, we obtain the bound

$$G(L_i, n_i) \leqq d\left[1 + \log \frac{1}{d} \int_0^\infty c(k) \frac{dk}{2\pi}\right], \quad \forall i. \tag{2.6}$$

Therefore the $\lim_{i \to \infty} G(L_i, n_i) = g(d)$ exists, and it is easy to prove, using the arbitrarity of L_0, that the same result holds when we take the T. limit of more general sequences than (2.4). Inequality (2.1) can be generalized as (*see* [1])

$$a_n^{L_1 + L_2} \geqq \sum_{n_1 + n_2 = n} a_{n_1}^{L_1} a_{n_2}^{L_2} \tag{2.7}$$

and it can be proved without difficulty that this inequality implies the concavity of $g(d)$:

$$g[\alpha d_1 + (1 - \alpha) d_2] \geqq \alpha g(d_1) + (1 - \alpha) g(d_2); \qquad 0 \leqq \alpha \leqq 1. \tag{2.8}$$

([4]) We have

$$\int_{\frac{2\pi}{L}}^\infty c(k)\, dk \leqq \frac{2\pi}{L} \sum_{k \in R_L} c(k).$$

It follows the continuity of $g(d)$ as well as the uniformity in d of the limit (1.6) for $d \in [d_1, d_2]$, $0 < d_1 < d_2 < d^*$.

The limit

$$p(z) = \lim_{L \to \infty} \frac{1}{L} \log f^L(z) = \int_0^\infty \log[1 + zc(k)] \frac{dk}{2\pi}, \tag{2.9}$$

exists for all $z > 0$ since $\log[1 + zc(k)] < zc(k)$. On the other hand the uniformity of the convergence of $G(L, n)$ to $g(d)$ allows us to apply the maximum term (or saddle point) method to find a relation between $p(z)$ and $g(d)$:

$$\begin{aligned} p(z) &= \lim_{L \to \infty} \frac{1}{L} \max_{\frac{n}{L} < d^*} \log a_n^L z^n = \max_{0<d<d^*} [d \log z + g(d)] \\ &= d(z) \log z + g[d(z)] \quad (^5), \end{aligned} \tag{2.10}$$

where we have called $d(z)$ the value of d such that $d \log z + g(d)$ is maximum. Indeed there exists only one such value since p is everywhere differentiable (*see* [1]).

Differentiating the expression $d \log z + g(d)$ with respect to d, we we obtain

$$g'[d(z)] + \log z = 0. \tag{2.11}$$

Hence from Equations (2.9), (2.10) we obtain by differentiation

$$\frac{dp}{dz} = \frac{dd}{dz}\left[\log z + \frac{dg}{dd}\right]_{d=d(z)} + \frac{d(z)}{z} = \frac{d(z)}{z} = \int_0^\infty \frac{c(k)}{1 + z\,c(k)} \frac{dk}{2\pi}, \tag{2.12}$$

i. e. :

$$d(z) = \int_0^\infty \frac{z\,c(k)}{1 + z\,c(k)} \frac{dk}{2\pi}, \tag{2.13}$$

which implies in particular that $g(d)$ is differentiable in d infinitely many times and is analytic in d in a neighborhood of $0 < d_1 < d < d_2 < d^*$.

Let us prove (6) the existence of the limit (1.7) : from a direct computation (*see* [1]) or also using the theory of entire functions (*see* [4]) one can prove that

$$\frac{a_n^L}{a_{n-1}^L} > \frac{a_{n+1}^L}{a_n^L} \frac{n+1}{n} > \frac{a_{n+1}^L}{a_n^L}. \tag{2.14}$$

(5) It is easy to verify that a_n^L vanishes if c has compact support and $\frac{n}{L}$ is large enough. From this follows the bound $\frac{n}{L} < d^*$.

(6) This procedure is similar to that used by Dobrushin and Minlos (*see* [3]) for proving the continuity of pressure.

By repeated application of this inequality we find

$$\frac{a^{L}_{n+\alpha}}{a^{L}_{n}} = \frac{a^{L}_{n+\alpha}}{a^{L}_{n+\alpha-1}} \frac{a^{L}_{n+\alpha-1}}{a^{L}_{n+\alpha-2}} \cdots \frac{a^{L}_{n+1}}{a^{L}_{n}} < \left(\frac{a^{L}_{n}}{a^{L}_{n-1}}\right)^{\alpha}, \tag{2.15}$$

where α is any positive integer. Therefore

$$\frac{a^{L}_{n}}{a^{L}_{n-1}} > \left(\frac{a^{L}_{n+\alpha}}{a^{L}_{n}}\right)^{\frac{1}{\alpha}} = \exp \frac{L}{\alpha} \frac{1}{L} (\log a^{L}_{n+\alpha} - \log a^{L}_{n}). \tag{2.16}$$

Letting $L \to \infty$, $\frac{n}{L} \to d$, $\frac{\alpha}{L} \to \varepsilon$ we find

$$\liminf_{\substack{L \to \infty \\ \frac{n}{L} \to d}} \frac{a^{L}_{n}}{a^{L}_{n-1}} \geq \exp \frac{g(d+\varepsilon) - g(d)}{\varepsilon}; \qquad \varepsilon > 0. \tag{2.17}$$

Analogously from

$$\frac{a^{L}_{n}}{a^{L}_{n-\alpha}} = \frac{a^{L}_{n}}{a^{L}_{n-1}} \cdots \frac{a^{L}_{n-\alpha+1}}{a^{L}_{n-\alpha}} > \left(\frac{a^{L}_{n}}{a^{L}_{n-1}}\right)^{\alpha}, \tag{2.18}$$

we get

$$\limsup_{\substack{L \to \infty \\ \frac{n}{L} \to d}} \frac{a^{L}_{n}}{a^{L}_{n-1}} \leq \exp \frac{g(d) - g(d-\varepsilon)}{\varepsilon}; \qquad \varepsilon > 0. \tag{2.19}$$

Letting now $\varepsilon \to 0$ and using the differentiability of $g(d)$ we find

$$\text{T.}\lim \frac{a^{L}_{n}}{a^{L}_{n-1}} = \exp g'(d) = \frac{1}{z(d)}, \tag{2.20}$$

where $z(d)$ is the inverse function of $d(z)$ [*see* Eq. (2.11)].

From the uniformity of the convergence of $G(L, n)$ to $g(d)$ it follows the uniformity of the convergence of (2.17), (2.19), and then of the T. limit (2.20), for $d \in [d_1, d_2]$, $0 < d_1 < d_2 < d^*$.

Let us now consider the limit (1.9); from the definition (1.8) of $a^{L}_{n}(k_0)$ it follows that

$$\sum_{n=0}^{\infty} a^{L}_{n}(k_0) z^n = z\, c(k_0) \prod_{\substack{k \in R_L \\ k \neq k_0}} [1 + z\, c(k)]. \tag{2.21}$$

Therefore

$$\frac{\sum_{n=0}^{\infty} a^{L}_{n}(k_0) z^n}{\sum_{n=0}^{\infty} a^{L}_{n} z^n} = \frac{z\, c(k_0)}{1 + z\, c(k_0)}. \tag{2.22}$$

From the existence of the $\text{T.}\lim \frac{1}{L} \log a_n^L z^n$ it follows that $a_n^L z^n$, considered as function of n, is strongly peaked around the value $n = d(z) L$ for L large. The same result holds for $a_n^L(k_0) z^n$. Let us now take the T. limit of both sides of (2.22) keeping k_0 fixed $\Big($this is possible for instance letting $L \to \infty$ along the sequence $L_i = 2^i L_0$, $L_0 > 0$, and assuming $k_0 \in \bigcup_{i=1}^{\infty} R_{L_i}\Big)$. Since only terms with $n \sim d(z) L$ are dominating in each summation, it is not surprising that

$$\text{T.}\lim_{\frac{n}{L}=d} \frac{a_n^L(k_0)}{a_n^L} = \frac{z(d)\, c(k_0)}{1 + z(d)\, c(k_0)}. \tag{2.23}$$

A rigorous proof of (2.23) has been given in [1] and therefore is omitted. The proof follows from (2.14), (2.20) and implies also the uniformity in d. On the other hand we will show later that

$$\left| \frac{a_n^L(k)}{a_n^L} - \frac{a_n^L(k_0)}{a_n^L} \right| \leq \frac{1}{n} \frac{a_{n-1}^L}{a_n^L} \,|\, c(k) - c(k_0) \,|. \tag{2.24}$$

Assume now that k_0 is a continuity point for c. Then Equations (2.23), (2.24) and the uniformity of the T. limit (2.20) imply Equation (2.23) also for k_0 not belonging to the set $\bigcup_{i=1}^{\infty} R_{L_i}$. The T. limit (1.9) exists and is clearly uniform in $d \in [d_1, d_2]$, $0 < d_1 < d_2 < d^*$, and in k. It remains only to prove the inequality (2.24) : From the definition (1.8) we have

$$\begin{aligned} a_n^L(k_0) &= \frac{1}{n!} c(k_0) \sum_{\substack{k_i \neq k_j \\ k_i \neq k_0}} c(k_1)\, c(k_2) \dots c(k_{n-1}) \\ &= \frac{1}{n!} \Bigg[c(k)\, c(k_0) \sum_{\substack{k_i \neq k_j \\ k_i \neq k_0, k}} c(k_1)\, c(k_2) \dots c(k_{n-2}) \\ &\qquad + c(k_0) \sum_{\substack{k_i \neq k_j \\ k_i \neq k_0, k}} c(k_1)\, c(k_2) \dots c(k_{n-1}) \Bigg]. \end{aligned} \tag{2.25}$$

Subtracting the analogous relation obtained interchanging k_0 and k we obtain

$$\begin{aligned} & a_n^L(k_0) - a_n^L(k) \\ &= \frac{1}{n} [c(k_0) - c(k)] \frac{1}{(n-1)!} \sum_{\substack{k_i \neq k_j \\ k_i \neq k_0, k}} c(k_1)\, c(k_2) \dots c(k_{n-1}), \end{aligned} \tag{2.26}$$

from which (2.24) follows. Therefore our theorem is proved under the hypothesis $\sup_k c(k) < \infty$.

Let us now drop this restriction, so that c can be a non-summable function. We notice that inequality (2.7) still holds, while (2.6) has to be replaced by a different upper bound since c might be not-summable. If we could find such an upper bound the proofs given before would hold unchanged.

Let $d = \frac{n}{L}$ and $0 \leq \xi < 1$; define $\varkappa = 2\pi d \xi$ and

$$(2.27) \qquad \begin{cases} c^{(0)}(k) \begin{cases} = 0 & \text{for } k > \varkappa, \\ = c(k) & \text{for } k \leq \varkappa; \end{cases} \\ \\ c^{(1)}(k) \begin{cases} = c(k) & \text{for } k > \varkappa, \\ = 0 & \text{for } k \leq \varkappa. \end{cases} \end{cases}$$

We denote by $a_n^{0\,L}$, $a_n^{1\,L}$ the expressions obtained from a_n^L replacing c respectively by $c^{(0)}$, $c^{(1)}$. Clearly from (1.1) it follows that

$$(2.28) \qquad a_n^L = \sum_{k=0}^{n} a_k^{0\,L}\, a_{n-k}^{1\,L}.$$

Notice that $a_k^{0\,L} = 0$ for $k > n\xi$ since $n\xi$ is the maximum number of distinct k's that fit in $[0, \varkappa]$. Furthermore we can write

$$(2.29) \qquad \begin{aligned} a_h^{0\,L} &= \sum_{k_1 < k_2 \ldots < k_h \leq \varkappa} c(k_1)\, c(k_2) \ldots c(k_h) \\ &= \sum_{k_1 < k_2 \ldots < k_h \leq \varkappa} \exp \sum_{i=1}^{h} \log c(k_i) \\ &\leq \sum_{k_1 < k_2 \ldots < k_h \leq \varkappa} \exp \sum_{k \leq \varkappa} \log [c(k) + 1] \\ &\leq \binom{n\xi}{h} \exp \frac{L}{2\pi} \int_0^{\varkappa} \log [c(k) + 1]\, dk, \end{aligned}$$

$$(2.30) \qquad a_{n-h}^{1\,L} \leq \frac{\left(L \int_{\varkappa}^{\infty} c(k) \frac{dk}{2\pi} \right)^{n-h}}{(n-h)!},$$

$\left(c \text{ is summable on } [\varkappa, +\infty] \text{ since it is decreasing and } \sum_k c(k) < +\infty\right)$.

Therefore, using Stirling's inequality $(n-h)! \geq \left[\frac{(n-h)}{e}\right]^{n-h}$,

$$(2.31) \qquad a_n^{\mathrm{L}} \leq \left[\sum_{h=0}^{n\xi} \binom{n\xi}{h} \left(\frac{\mathrm{L}\, e \int_{\varkappa}^{\infty} c(k) \frac{dk}{2\pi}}{n-h}\right)^{n-h}\right]$$

$$\times \exp \frac{\mathrm{L}}{2\pi} \int_0^{\varkappa} \log\left[c(k)+1\right] dk$$

$$\leq \left[\sum_{h=0}^{n\xi} \binom{n\xi}{h} \left(\frac{\mathrm{L}\, e \int_{\varkappa}^{\infty} c(k) \frac{dk}{2\pi}}{n(1-\xi)}\right)^{n-h}\right]$$

$$\times \exp \frac{\mathrm{L}}{2\pi} \int_0^{\varkappa} \log\left[c(k)+1\right] dk$$

$$= \left(\exp \frac{\mathrm{L}}{2\pi} \int_0^{\varkappa} \log\left[c(k)+1\right] dk\right)$$

$$\times \left(\frac{e\,\mathrm{L}}{n(1-\xi)} \int_{\varkappa}^{\infty} c(k) \frac{dk}{2\pi}\right)^{n} \left(1 + \frac{n(1-\xi)}{e\,\mathrm{L} \int_{\varkappa}^{\infty} c(k) \frac{dk}{2\pi}}\right),$$

which gives a bound for $G(\mathrm{L}, n)$ of the desired form.

The inequalities (2.14), (2.24) still hold since they are purely algebraic. Therefore the limits (1.7), (1.9) exist and have the same uniformity properties as before. The theorem is proved.

An interesting example is given by taking $c(k) = k^{-\rho}$, $\rho > 1$. This example has been studied in detail in [5]. The coefficients a_n^{L} have the simple L-dependence $a_n^{\mathrm{L}} = \left(\frac{\mathrm{L}}{2\pi}\right)^{n\rho} b_n$, where b_n does not depend on L. The existence of the above limits allows extablishing simple properties of the coefficients b_n, like for instance

$$(2.32) \qquad \lim_{n\to\infty} \log b_n^{\frac{1}{n}} n^{\rho} = \frac{g(d)}{d} + \rho \log 2\pi d, \quad \forall\, d,$$

$$(2.33) \qquad \lim_{n\to\infty} \frac{b_n}{b_{n-1}} n^{\rho} = (2\pi d)^{\rho} \exp \frac{dg}{dd}, \quad \forall\, d,$$

where $g(d) = d\rho \left[1 - \log\left(2\, d\rho \sin \frac{\pi}{\rho}\right)\right]$; of course the right hand sides of (2.32), (2.33) do not depend on d.

3. Let us now consider the second problem mentioned in the introduction, i. e. the problem concerning the asymptotic properties of general « dense sums ». We shall prove the following theorem :

THEOREM 2. — *Let* $\{\gamma_j(L)\}$ *be a sequence of non-negative numbers depending on a parameter* L. *Suppose* :

(i) *The limit* (1.14) *exists*;

(ii) $\sup_{j,L} \gamma_j(L) < \Lambda < +\infty$;

(iii) $\sum_j \gamma_j(L) = V(L) < +\infty$;

(iv) $\lim_{L \to \infty} V(L) = +\infty$.

Then defining the coefficients a_n^L by formula (1.12), the limits for $L \to \infty, \frac{n}{V(L)} \to d$ (that we shall still call T. limits)

$$\text{T.} \lim \frac{1}{V(L)} \log a_n^L = g(d), \tag{3.1}$$

$$\text{T.} \lim \frac{a_{n-1}^L}{a_n^L} = z(d), \tag{3.2}$$

exist for all $d \geq 0$ if and only if the limit (1.14) exists; if they exist they are uniform in d for $d \in [d_1, d_2]$, $0 < d_1 < d_2 < d^*$ where d^* is the greatest lower bound of the set of d's such that $g(d) = -\infty$.

Proof. — The idea underlying the proof is to construct, by using the measure $\nu(dc)$, a monotonic function $k(c)$ with inverse $c(k)$ such that the set $\left\{c\left(\frac{2\pi}{V(L)}j\right)\right\}$ $j = 1, 2, \ldots$ essentially reproduces the set $\{\gamma_j(L)\}$. Then it will be shown that replacing $\gamma_j(L)$ with $c\left(\frac{2\pi}{V(L)}j\right)$ in the definition of the a_n^L the limits (3.1), (3.2), if existent, are not affected and then use theorem 1 to prove their existence.

Let us first assume that $\inf_{j,L} \gamma_j(L) > \xi > 0$. Let N be the set of points (atoms) in the support (contained in $[\xi, \Lambda)$) of $\nu(dc)$ which have non zero measure. N is countable since ν is normalized.

If E denotes an interval, then

$$\int_E \mu_L(dc) = \frac{\{\text{number of } \gamma_j \text{ contained in E}\}}{V(L)} = \frac{N_I(E)}{V(L)} < +\infty. \tag{3.3}$$

This formula suggests the definition of the monotonic function

$$k(c) = 2\pi \int_c^\infty \frac{\nu(d\gamma)}{\gamma}. \tag{3.4}$$

Let $c(k)$ be the inverse of $k(c)$ [this function has discontinuities where $k(c)$ is constant but is otherwise unambiguosly defined]; $k(c) = 0$ for $c \geqq \Lambda$ and $c(k) = 0$ for $k > k(0)$, but $c(k(0)) \geqq \xi > 0$.

The set of points where $c(k)$ has discontinuities is at mostd enumerable; to each of these points k_j, $(j = 1, 2, \ldots)$ there corresponds an interval with endpoints c_j, c'_j. Call M this denumerable set of c' s. Divide the interval $[\xi, \Lambda)$ into the union of a finite number of semiopen intervals $[\delta_i, \delta_{i+1})$ (for $i = 1, 2, \ldots, \tau_\varepsilon$) such that $\delta_i \notin M \cup N$ for $i = 1, 2, \ldots, \tau_\varepsilon$ and $\log \frac{\delta_{i+1}}{\delta_i} < \varepsilon$. Then by the Alexandroff theorem (*see* [6]), and using that $\lim_{L \to \infty} \mu_L(\{\delta_i\}) = 0$, the weak convergence of $c\,\mu_L(dc)$ to $\nu(dc)$ implies that

$$\lim_{L \to \infty} \int_{[\delta_i, \delta_{i+1})} \mu_L(dc) = \int_{[\delta_i, \delta_{i+1})} \frac{\nu(dc)}{c}. \tag{3.5}$$

It follows that the number of $\gamma_j(L)$ belonging to $[\delta_i, \delta_{i+1})$ is asymptotically given by

$$\begin{aligned} N_L([\delta_i, \delta_{i+1})) &= V(L) \int_{[\delta_i, \delta_{i+1})} \mu_L(dc) \\ &= V(L)\left[\int_{[\delta_i, \delta_{i+1})} \frac{\nu(dc)}{c} + 0_i\left(\frac{1}{V(L)}\right)\right]. \end{aligned} \tag{3.6}$$

On the other hand in the interval $[k(\delta_{i+1}), k(\delta_i)]$ fall a number

$$\tilde{N}_L([\delta_i, \delta_{i+1})) = \frac{V(L)}{2\pi}[k(\delta_i) - k(\delta_{i+1})] \pm \Delta; \qquad 0 \leqq \Delta \leqq 1, \tag{3.7}$$

of points of the set $\left\{\frac{2\pi}{V(L)}\, j\right\}$ $(j = 1, 2, \ldots)$, and since

$$k(\delta_i) - k(\delta_{i+1}) = 2\pi \int_{[\delta_i, \delta_{i+1})} \frac{\nu(dc)}{c}, \tag{3.8}$$

we see that $\lim_{L \to \infty} \frac{N_L}{\tilde{N}_L} = 1$ for each interval $[\delta_i, \delta_{i+1})$ such that $k(\delta_i) - k(\delta_{i+1}) > 0$.

We shall now construct a new lattice of points $\left\{\frac{2\pi}{V^-(L)}\, j\right\}$ $(j = 1, 2, \ldots)$ with $V^-(L) \leqq V(L)$ in such a way that :

(1) $\lim_{L \to \infty} \frac{V(L)}{V^-(L)} = 1$;

(2) the number $\tilde{N}_L^-$ of points of the lattice falling in δ_i, $[\delta_{i+1})$ is not greater than $N_L([\delta_i, \delta_{i+1}))$;

(3) no points of the lattice is a discontinuity point of c.

Let us set $b_i = \frac{k(\delta_i) - k(\delta_{i+1})}{2\pi}$ for $i = 1, 2, \ldots, \tau_\varepsilon$. We define

$$\text{(3.9)} \qquad V^-(L) = V(L)\left[1 - \eta_L \max_{i,\, b_i > 0} \frac{\left|O_i\left(\frac{1}{V(L)}\right)\right| + \frac{1}{V(L)}}{b_i}\right],$$

where $2 > \eta_L > 1$. Relation (1) is clearly satisfied. Let us check relation (2). We have

$$N_L - \tilde{N}_L^- = V(L)\, b_i + V(L)\, O_i\left(\frac{1}{V(L)}\right)$$
$$- V(L)\left[1 - \eta_L \max_{i,\, b_i>0} \frac{\left|O_i\left(\frac{1}{V(L)}\right)\right| + \frac{1}{V(L)}}{b_i}\right] b_i \pm \Delta,$$

where $|\Delta| < 1$ and $\eta_L > 1$. Hence for L large enough

$$N_L - \tilde{N}_L^- \geqq V(L)\, O_i\left(\frac{1}{V(L)}\right) + V(L)\left|O_i\left(\frac{1}{V(L)}\right)\right| + 1 \pm \Delta \geqq 0.$$

It remains to prove that it is possible to choose η_L in such a way that also condition (3) is verified. Clearly there exists a real number A such that all the $\frac{A}{k_j}$ ($j = 1, 2, \ldots$ and k_j denotes again a discontinuity point for c) are irrationals; and any lattice with spacing $r\,A$ (r rational) has empty intersection with the set $\{k_j\}$.

We define $\tilde{a}_n^L$ and $\tilde{a}_n^{L-}$ through Equation (1.1) by letting k run through the set of indeces $\frac{2\pi}{V(L)}\, j$, $\frac{2\pi}{V^-(L)}\, j$, respectively. Finally let a_n^{L*} be the number

$$\text{(3.10)} \qquad a_n^{L*} = \sum_{j_1 < j_2 \ldots < j_n} \gamma_{j_1}^*(L)\, \gamma_{j_2}^*(L) \ldots \gamma_{j_n}^*(L),$$

where the $\gamma_j^*(L)$ are the elements of the countable set

$$\text{(3.11)} \qquad \{\gamma_j^*(L)\} = \bigcup_{h,k=1}^{\infty} \left\{c\left(\frac{2\pi}{V^-(L)}\, h\right)\right\} \cup \{\gamma'_k(L)\},$$

and the points $\gamma'_k(L)$ have been chosen in such a way that in each interval $[\delta_i, \delta_{i+1})$ the number of points belonging to the sets $\{\gamma_j^*(L)\}$ and $\{\gamma_j(L)\}$ respectively are equal. This choice is clearly possible (and can be performed in an infinite number of ways) since $N_L \geqq \tilde{N}_L^-$ in each interval $[\delta_i, \delta_{i+1})$ [*see* condition (2)]. Therefore we can extablish a one-to-one mapping between the points $\gamma_j^*(L)$ and $\gamma_j(L)$ belonging to the same interval $[\delta_i, \delta_{i+1})$. Since the intervals $[\delta_i, \delta_{i+1})$ have been chosen so that $\log \frac{\delta_i}{\delta_{i+1}} < \varepsilon$, we find

$$e^{-\varepsilon} < \frac{\delta_i}{\delta_{i+1}} \leq \frac{\gamma_j^*(L)}{\gamma_j(L)} \leq \frac{\delta_{i+1}}{\delta_i} < e^{\varepsilon},$$

and hence, using the inequality

$$\min_\alpha \frac{x_\alpha}{y_\alpha} \leq \frac{\sum_\alpha x_\alpha}{\sum_\alpha y_\alpha} \leq \max_\alpha \frac{x_\alpha}{y_\alpha} \qquad \text{for} \quad x_\alpha \geq 0,\ y_\alpha \geq 0$$

we obtain

$$e^{-\varepsilon n} \leq \frac{a_n^{L*}}{a_n^L} \leq e^{\varepsilon n}. \tag{3.12}$$

Furthermore, from Equation (3.11) it follows that

$$a_n^{L*} = \sum_{k=0}^{n} \tilde{a}_{n-k}^{L-}\, a_k^{L'}, \tag{3.13}$$

where the meaning of $a_k^{L'}$ is obvious.

On the other hand for each interval $[\delta_i, \delta_{i+1})$ we have

$$N_L - \tilde{N}_L^- = (N_L - \tilde{N}_L) + (\tilde{N}_L - \tilde{N}_L^-) = O_i^-\left(\frac{1}{V(L)}\right) V(L),$$

because $N_L - \tilde{N}_L = O_i\left(\frac{1}{V(L)}\right) V(L)$ since

$$\lim_{L\to\infty} \frac{N_L}{\tilde{N}_L} = 1 \qquad \text{and} \qquad \tilde{N}_L - \tilde{N}_L^- = \tilde{O}_i\left(\frac{1}{V(L)}\right) V(L),$$

since $\lim_{L\to\infty} \frac{V(L)}{V^-(L)} = 1$. Taking into account that all $\gamma'_k(L)$ are not greater than Λ, we obtain the bound

$$a_n^{L'} = \sum_{k_1<k_2\ldots<k_n} \gamma'_{k_1}(L)\,\gamma'_{k_2}(L)\ldots\gamma'_{k_n}(L) \tag{3.14}$$

$$\leq \frac{\left[\sum_k \gamma'_k(L)\right]^n}{n!} = \frac{\left[\Lambda V(L)\, O'\left(\frac{1}{V(L)}\right)\right]^n}{n!},$$

where

$$\mathrm{O}'\left(\frac{1}{\mathrm{V}(\mathrm{L})}\right)=\sum_{i=1}^{\tau_\varepsilon} \mathrm{O}_i^-\left(\frac{1}{\mathrm{V}(\mathrm{L})}\right).$$

Therefore

$$1\leq\frac{a_n^{\mathrm{L}*}}{\tilde{a}_n^{\mathrm{L}-}}=\sum_{k=0}^{n}\frac{\tilde{a}_{n-k}^{\mathrm{L}-}}{\tilde{a}_n^{\mathrm{L}-}}\,a_k^{\mathrm{L}'}\leq\sum_{k=0}^{n}\left(\frac{\tilde{a}_{n-1}^{\mathrm{L}-}}{\tilde{a}_n^{\mathrm{L}-}}\right)^k\frac{\left[\Lambda\,\mathrm{V}(\mathrm{L})\,\mathrm{O}'\left(\frac{1}{\mathrm{V}(\mathrm{L})}\right)\right]^k}{k\,!},$$

where in the last step the inequality (2.14) has been used. Extending the summation up to infinity, we obtain

$$(3.15)\qquad 1\leq\frac{a_n^{\mathrm{L}*}}{\tilde{a}_n^{\mathrm{L}-}}\leq\exp\left[\frac{\tilde{a}_{n-1}^{\mathrm{L}-}}{\tilde{a}_n^{\mathrm{L}-}}\,\Lambda\,\mathrm{V}(\mathrm{L})\,\mathrm{O}'\left(\frac{1}{\mathrm{V}(\mathrm{L})}\right)\right].$$

Since the limits $\mathrm{T}.\lim\limits_{\frac{n}{\mathrm{V}^-(\mathrm{L})}\to d}\frac{1}{\mathrm{V}^-(\mathrm{L})}\log\tilde{a}_n^{\mathrm{L}-}=g(d)$ and $\mathrm{T}.\lim\limits_{\frac{n}{\mathrm{V}^-(\mathrm{L})}\to d}\frac{\tilde{a}_{n-1}^{\mathrm{L}-}}{\tilde{a}_n^{\mathrm{L}-}}$ exist and are uniform in d and independent on ε (*see* Theorem 1), it follows that

$$(3.16)\qquad \mathrm{T}.\lim_{\frac{n}{\mathrm{V}(\mathrm{L})}\to d}\frac{1}{\mathrm{V}(\mathrm{L})}\log a_n^{\mathrm{L}*}=g(d),$$

uniformly for $d\in[d_1,\,d_2]$, $0<d_1<d_2<d^*$. On the other hand from Equation (3.12) we have

$$(3.17)\qquad \limsup_{\substack{\mathrm{V}(\mathrm{L})\to\infty\\ \frac{n}{\mathrm{V}(\mathrm{L})}\to d}}\frac{1}{\mathrm{V}(\mathrm{L})}\log\frac{a_n^{\mathrm{L}*}}{a_n^{\mathrm{L}}}-\liminf_{\substack{\mathrm{V}(\mathrm{L})\to\infty\\ \frac{n}{\mathrm{V}(\mathrm{L})}\to d}}\frac{1}{\mathrm{V}(\mathrm{L})}\log\frac{a_n^{\mathrm{L}*}}{a_n^{\mathrm{L}}}\leq\varepsilon\,d,$$

so that finally we obtain

$$(3.18)\qquad \mathrm{T}.\lim_{\frac{n}{\mathrm{V}(\mathrm{L})}\to d}\frac{1}{\mathrm{V}(\mathrm{L})}\log a_n^{\mathrm{L}}=g(d),$$

and by combining (3.17) with the uniformity and ε independence of the limit (3.16) (at fixed ε), we deduce the uniformity in $d\in[d_1,\,d_2]$ of the limit (3.18).

It is now clear the reason for introducing the spacing $\frac{2\pi}{\mathrm{V}^-(\mathrm{L})}$: if we had chosen V (L) in the place of $\mathrm{V}^-(\mathrm{L})$, in some interval $[\delta_i,\,\delta_{i+1})$ we would have $\mathrm{N}_\mathrm{L}>\tilde{\mathrm{N}}_\mathrm{L}$, in other intervals $\mathrm{N}_\mathrm{L}<\tilde{\mathrm{N}}_\mathrm{L}$, so that it would have been impossible to use a formula of the type (3.15).

Having proved the existence of the limit (3.18) we can easily obtain the remaining results; we can write

$$(3.19)\quad \int \frac{\log(1+zc)}{c}\,\nu(dc) = \lim_{L\to+\infty}\int \log(1+zc)\,\mu_L(dc)$$
$$= \lim_{L\to+\infty}\frac{1}{V(L)}\sum_{j=1}^{\infty}\log[1+z\gamma_j(L)]$$
$$= \lim_{L\to+\infty}\frac{1}{V(L)}\log\prod_{j=1}^{\infty}[1+z\gamma_j(L)]$$
$$= \lim_{L\to+\infty}\frac{1}{V(L)}\log\left(\sum_{n=0}^{\infty}a_n^L z^n\right)$$
$$= \max_{0\leq d<d^*}[g(d)+d\log z],$$

where in the last step we have applied the saddle-point method (*see* [1], formulae (70), (71)]. The maximum is attained in only one point $d(z)$; $g(d)$ is differentiable and we have

$$(3.20)\qquad g'[d(z)]+\log z = 0,$$

$$(3.21)\qquad d(z) = z\int\frac{1}{1+zc}\,\nu(dc),$$

$$(3.22)\qquad \mathrm{T.}\lim_{\frac{n}{V(L)}\to d}\frac{a_{n-1}^L}{a_n^L} = z(d);\qquad 0<d<d^*.$$

Formulae (3.20) through (3.22) can be proved by the same procedure used for proving Theorem 1.

We drop now the condition $\inf_{L,j}\gamma_j(L)>\xi>0$, assuming only that the ν-measure of the origin is zero. In the following we will see that even this assumption is not needed.

Let $\varepsilon>0$ denote any number such that $\nu(\{\varepsilon\})=0$. We divide the set $\{\gamma_j(L),\, j=1,2,\ldots\}$ in two subsets : the set of γ_j such that $\gamma_j\geq\varepsilon$ and the set of γ_j such that $\gamma_j<\varepsilon$. Then defining $a_n^{L,1}$ in the usual way but considering only the $\gamma_j\geq\varepsilon$ and $a_n^{L,2}$ by considering the $\gamma_j<\varepsilon$, we have

$$(3.23)\qquad a_n^L = \sum_{k=0}^{n} a_{n-k}^{L,1}\,a_k^{L,2}.$$

Therefore, by the inequality (2.14) we can write

$$(3.24) \qquad 1 \leqq \frac{a_n^{\mathrm{L}}}{a_n^{\mathrm{L},1}} = \sum_{k=0}^{n} \frac{a_{n-k}^{\mathrm{L},1}}{a_n^{\mathrm{L},1}} a_k^{\mathrm{L},2} \leqq \sum_{k=0}^{\infty} \left(\frac{a_{n-1}^{\mathrm{L},1}}{a_n^{\mathrm{L},1}}\right)^k \frac{\left[\sum_{\gamma_j<\varepsilon} \gamma_j(\mathrm{L})\right]^k}{k!}$$
$$= \exp\left[\frac{a_{n-1}^{1,\mathrm{L}}}{a_n^{1,\mathrm{L}}} \sum_{\gamma_j<\varepsilon} \gamma_j(\mathrm{L})\right]$$

and since

$$(3.25) \qquad \sum_{\gamma_j<\varepsilon} \gamma_j(\mathrm{L}) = \mathrm{V}(\mathrm{L}) \int_0^{\varepsilon} c\, \mu_{\mathrm{L}}(dc),$$

we have, by the Alexandroff theorem [6] applied to $[0, \varepsilon)$:

$$(3.26) \qquad \lim_{\mathrm{L}\to\infty} \frac{1}{\mathrm{V}(\mathrm{L})} \sum_{\gamma_j<\varepsilon} \gamma_j(\mathrm{L}) = \int_0^{\varepsilon} \nu(dc).$$

Therefore from (3.24) it follows that

$$(3.27) \qquad 0 \leqq \mathrm{T.}\lim_{\frac{n}{\mathrm{V(L)}}=d} \frac{1}{\mathrm{V}(\mathrm{L})} \log a_n^{\mathrm{L}} - g^{\varepsilon}(d) \leqq z^{\varepsilon}(d) \int_0^{\varepsilon} \nu(dc),$$

where we have denoted by g^{ε} and z^{ε} the quantities analogous to g and z obtained considering only the $\gamma_j \geqq \varepsilon$. Since $z^{\varepsilon}(d) = \exp \frac{dg^{\varepsilon}}{dd}$ is continuous when $\varepsilon \to 0$ and $\int_0^{\varepsilon} \nu(d\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ by assumption, it follows that the limit (3.18) exists also in this case, and is uniform. The existence of the limit (3.22) follows easily applying the same techniques as before.

We now drop the condition that the ν-measure of the origin is zero. We set :

$$(3.28) \qquad \gamma_{\max}(\mathrm{L}) = \sup_j \gamma_j(\mathrm{L}) = \max_j \gamma_j(\mathrm{L}).$$

We consider first the particular case when $\lim_{\mathrm{L}\to\infty} \gamma_{\max}(\mathrm{L}) = 0$, and we keep the conditions (i) through (iv) of Theorem 2. Then we shall prove that the limit

$$(3.29) \qquad \mathrm{T.}\lim_{\frac{n}{\mathrm{V(L)}}\to d} \frac{1}{\mathrm{V}(\mathrm{L})} \log a_n^{\mathrm{L}} = g_0(d),$$

exists and is given by

$$(3.30) \qquad g_0(d) = d - d \log d.$$

Let us prove Equation (3.30). We call « elementary mapping » a mapping τ of the sequence $\{\gamma_j(L)\}$ into another sequence in such a way that : (1) Only two γ' s, say γ_α and γ_β ($\gamma_\alpha > \gamma_\beta$), are modified. 2) $\gamma_\alpha \xrightarrow{\tau} \gamma_\alpha + \Delta$, $\gamma_\beta \xrightarrow{\tau} \gamma_\beta - \Delta$ $(\Delta > 0)$. Clearly these elementary mappings leave V (L) invariant. Suppose now for simplicity that $\dfrac{V(L)}{\gamma_{max}} = n_{max}$ is an integer; (otherwise only minor modifications are needed). It is clear that by means of an appropiate sequence of elementary mappings it is possible to transform $\{\gamma_j(L)\}$ into a sequence $\{\gamma'_j(L)\}$ having n_{max} elements equal to γ_{max}, all others being zero. Let a'^{L}_n denote the expression $\sum_{j_1<j_2\ldots<j_n} \gamma'_{j_1}(L)\ldots\gamma'_{j_n}(L)$. a'^{L}_n can be evaluated explicitly, and gives

$$a'^{L}_n = \gamma_{max}(L)^n \binom{n_{max}}{n}. \tag{3.31}$$

Let us now verify that under an elementary mapping the value of a^L_n decreases. We can write

$$a^L_n = \frac{1}{n!}\Bigg[\gamma_\alpha\gamma_\beta \sum_{\substack{j_i\neq j_k\\ j_i\neq\alpha,\beta}} \gamma_{j_1}(L)\ldots\gamma_{j_{n-2}}(L) + \gamma_\alpha \sum_{\substack{j_i\neq j_k\\ j_i\neq\alpha,\beta}} \gamma_{j_1}(L)\ldots\gamma_{j_{n-1}}(L) + \gamma_\beta \sum_{\substack{j_i\neq j_k\\ j_i\neq\alpha,\beta}} \gamma_{j_1}(L)\ldots\gamma_{j_{n-1}}(L) + \sum_{\substack{j_i\neq j_k\\ j_i\neq\alpha,\beta}} \gamma_{j_1}(L)\ldots\gamma_{j_n}(L)\Bigg]. \tag{3.32}$$

Under an elementary mapping, the variations of the second and third term of Equation (3.32) cancel, the fourth term does not change and the first term decreases since

$$(\gamma_\alpha + \Delta)(\gamma_\beta - \Delta) = \gamma_\alpha\gamma_\beta - \Delta(\gamma_\alpha - \gamma_\beta + \Delta) < \gamma_\alpha\gamma_\beta. \tag{3.33}$$

Therefore $a^L_n \geq a'^{L}_n$; on the other hand $a^L_n \leq \dfrac{V(L)^n}{n!}\cdot$ Hence

$$\frac{1}{V(L)}\log a'^{L}_n \leq \frac{1}{V(L)}\log a^L_n \leq \frac{1}{V(L)}\log\frac{V(L)^n}{n!}. \tag{3.34}$$

Using Stirling's formula, and the condition $\lim_{L\to\infty}\gamma_{max}(L) = 0$, it is easy to deduce from Equation (3.31) that $\mathrm{T.}\lim_{\frac{n}{V(L)}\to d} \dfrac{1}{V(L)} \log a'^{L}_n = g_0(d)$.

Since also $\dfrac{1}{V(L)}\log\dfrac{V(L)^n}{n!}$ tends to the same limit, we obtain the result (3.30).

Consider now the general case

$$(3.35) \qquad \nu(dc) = \alpha\delta(c)\,dc + (1-\alpha)\,\nu'(dc); \qquad 0 < \alpha < 1,$$

where the ν'-measure of the origin is zero. Suppose ε_m is a positive sequence such that $\lim_{m\to\infty} \varepsilon_m = 0$ and such that $\nu'(\{\varepsilon_m\}) = 0$ for $m = 1, 2, \ldots$ (i. e. such that ν' has no atoms at the points ε_m). From Equation (1.14) and the Alexandroff theorem it follows that

$$(3.36) \qquad \frac{1}{V(L)} \sum_{\gamma_j(L)<\varepsilon_m} \gamma_j(L) = \alpha + (1-\alpha)\int_0^{\varepsilon_m} \nu'(dc) + O_m\left(\frac{1}{L}\right).$$

Notice that $\lim_{m\to\infty}\int_0^{\varepsilon_m} \nu'(dc) = 0$. For given m, there exists an $L(m)$ such that for $L > L(m)$, $O_m\left(\frac{1}{L}\right) < \frac{1}{m}$, $\lim_{m\to\infty} L(m) = +\infty$, and the sequence $L(m)$ is increasing. We call $m(L)$ the inverse function of $L(m)$. Clearly $m(L) \to \infty$ as $L \to \infty$; therefore $\lim_{L\to\infty} \varepsilon_{m(L)} = 0$. Let now $\xi_j(L)$ denote the characteristic function (thought as a function of j) :

$$(3.37) \qquad \xi_j(L) \begin{cases} = 1 & \text{if} \quad \gamma_j(L) < \varepsilon_{m(L)}, \\ = 0 & \text{if} \quad \gamma_j(L) \geqq \varepsilon_{m(L)}. \end{cases}$$

We decompose the sequence $\{\gamma_j(L)\}$ into the union of the two sequences

$$(3.38) \qquad \gamma_j^0(L) = \gamma_j(L)\,\xi_j(L); \qquad \gamma_j^1(L) = \gamma_j(L)\,[1 - \xi_j(L)].$$

Clearly defining the measures

$$(3.39) \qquad \begin{cases} \nu_L^{(0)}(dc) = \sum_{j=1}^{\infty} c\,\delta(c - \gamma_j^0(L))\,\frac{dc}{V(L)}; \\ \nu_L^{(1)}(dc) = \sum_{j=1}^{\infty} c\,\delta(c - \gamma_j^1(L))\,\frac{dc}{V(L)}, \end{cases}$$

we have, using (3.36) and $O_{m(L)}\left(\frac{1}{L}\right) < \frac{1}{m(L)}$:

$$(3.40) \qquad \operatorname*{weak\,lim}_{L\to\infty} \nu_L^{(0)}(dc) = \alpha\delta(c)\,dc; \qquad \operatorname*{weak\,lim}_{L\to\infty} \nu_L^{(1)}(dc) = (1-\alpha)\,\nu'(dc).$$

We denote by $a_n^{0\mathrm{L}}$, $a_n^{1\mathrm{L}}$ the expressions obtained from a_n^{L} replacing $\{\gamma_j(\mathrm{L})\}$ respectively by $\{\gamma_j^0(\mathrm{L})\}$ and $\{\gamma_j^1(\mathrm{L})\}$. We have, as usual :

$$a_n^{\mathrm{L}} = \sum_{k=0}^{n} a_k^{0\mathrm{L}}\, a_{n-k}^{1\mathrm{L}}. \tag{3.41}$$

In order to find the $\mathrm{T.}\lim\limits_{\frac{n}{\mathrm{V(L)}}=d} \frac{1}{\mathrm{V(L)}} \log a_n^{\mathrm{L}}$ we use the maximum term method. Since $\lim\limits_{\mathrm{L}\to\infty} \varepsilon_{m(\mathrm{L})} = 0$ we van apply Equation (3.30). Therefore

$$\mathrm{T.}\lim_{\frac{k}{\mathrm{V(L)}}=\delta} \frac{1}{\mathrm{V(L)}} \log a_k^{0\mathrm{L}} = \delta - \delta \log \frac{\delta}{\alpha}. \tag{3.42}$$

Since $\nu'(\{0\}) = 0$ we know that the

$$\mathrm{T.}\lim_{\frac{n}{\mathrm{V(L)}(1-\alpha)}=d} \left[\frac{1}{\mathrm{V(L)}(1-\alpha)} \log a_n^{1\mathrm{L}}\right] = g^{(1)}(d)$$

exists. Therefore

$$\mathrm{T.}\lim_{\frac{n-k}{\mathrm{V(L)}}=d-\delta} \frac{1}{\mathrm{V(L)}} \log a_{n-k}^{1\mathrm{L}} = (1-\alpha)\, g^{(1)}\left(\frac{d-\delta}{1-\alpha}\right). \tag{3.43}$$

It follows from Equation (3.41) that

$$\mathrm{T.}\lim_{\frac{n}{\mathrm{V(L)}}=d} \frac{1}{\mathrm{V(L)}} \log a_n^{\mathrm{L}} = \max_{0<\delta<d} \left[\delta - \delta \log \frac{\delta}{\alpha} + (1-\alpha)\, g^{(1)}\left(\frac{d-\delta}{1-\alpha}\right)\right], \tag{3.44}$$

and the maximum can be determined by differentiating the R. H. S. of Equation (3.44) with respect to δ. The remaining results [Eq. (3.21), (3.22)] follow easily.

Finally we notice that the conditions of convergence of $c\,\mu_{\mathrm{L}}(dc) \to \nu(dc)$ as $\mathrm{L}\to\infty$ is not only sufficient for the results of Theorem 2 to hold but also necessary. In fact the measures $c\,\mu_{\mathrm{L}}$ are positive and normalized measures on the compact $[0, \Lambda]$. Therefore they form a weakly sequentially compact set, and if there were two different subsequences converging respectively to ν_1 and ν_2 $(\nu_1 \neq \nu_2)$, the limits

$$\frac{1}{\mathrm{V(L)}} \log \sum_{n=0}^{\infty} a_n^{\mathrm{L}} z^n \to \int_0^{\Lambda} \nu_{1,2}(dc)\, \frac{\log(1+zc)}{c},$$

would differ over the two sequences and would yield different *g (d)'s*. $\left(\text{Expanding the logarithm in powers of } z \text{ we find that the transform} \int_0^\Lambda \nu(dc) \frac{\log(1+zc)}{c} \text{ uniquely determines } \nu \right)$.

4. Further results and concluding remarks

1. In Theorem 2 we have not included the natural generalization of the formula (1.9). This generalization is provided by the following Theorem :

THEOREM 3. — *Let c be a point internal to an interval* (c_1, c_2) *where* $k(c)$ *is strictly decreasing and continuous. Then the above limit is uniform in d for* $0 < d < d^*$ *and has the value*

$$\text{T.}\lim_{\substack{\frac{n}{V(L)} = d \\ \gamma_j(L) \to c}} \frac{a_n^L[\gamma_j(L)]}{a_n^L} = \frac{z(d)\,c}{1 + z(d)\,c}. \tag{3.45}$$

This result can be somewhat generalized to include also points which are continuity points for both $c(k)$ and $k(c)$. However we omit the proofs related to these results since they easily reduce to the ones presented for Theorem 1.

2. The results obtained in paragraph 3 allow us to drop the hypothesis of monotonicity of the function c that was used in proving the Theorem 1.

Indeed let $c(k)$ be a non-negative piecewise continuous function defined for $k > 0$; we suppose that $\Lambda = \sup_k c(k) < +\infty$ and $\sup_{k \geq k_1 > 0} c(k)\, k^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$ and some k_1.

We set $\gamma_j(L) = c\left(\frac{2\pi}{L} j\right)$ for $j = 1, 2, \ldots$ and we consider the measure

$$\nu_L(dc) = \frac{\sum_{j=1}^{\infty} c\,\delta[c - \gamma_j(L)]\,dc}{\sum_{j=1}^{\infty} \gamma_j(L)}. \tag{3.46}$$

Suppose now that $\varphi(c)$ is a continuous bounded function defined on the interval $[0, \Lambda]$. We have

$$\int_0^\Lambda \varphi(c)\,\nu_L(dc) = \frac{\sum_{j=1}^{\infty} c\left(\frac{2\pi}{L} j\right) \varphi\left[c\left(\frac{2\pi}{L} j\right)\right]}{\sum_{j=1}^{\infty} c\left(\frac{2\pi}{L} j\right)}. \tag{3.47}$$

Hence the weak limit for $L \to \infty$ of ν_L exists and defines a measure ν such that

$$\int_0^\Lambda \varphi(c)\,\nu(dc) = \frac{\int_0^\infty c(k)\,\varphi[c(k)]\,dk}{\int_0^\infty c(k)\,dk}. \tag{3.48}$$

Therefore the numbers $\gamma_j(L)$ are such that Equation (1.14) is satisfied, and we can apply Theorem 2 : the limits (3.1), (3.2), (3.45) exist and this implies the existence of the analogous limits (1.6) through (1.9). In this way a strong generalization of Theorem 1 has been obtained.

3. Theorem 2 has been proved under the condition $\sup_{j,L} \gamma_j(L) < +\infty$. Of course it is possible to generalize this condition and replace it by additional assumptions on the measures μ_L and their limits as $L \to \infty$. These new assumptions can be found by considering the set $\{\gamma_j(L)\}$ divided in two parts :

$$J^+ = \{ j : \gamma_j(L) \geq 1 \}, \tag{3.49}$$

$$J^- = \{ j : \gamma_j(L) < 1 \}. \tag{3.50}$$

Then we define

$$V(L) = \sum_{j \in J^+} \log[1 + \gamma_j(L)] + \sum_{j \in J^-} \gamma_j(L) \tag{3.51}$$

and we require that the measures

$$\nu_L(dc) = c\,\mu_L(dc) = \frac{\sum_j c\,\delta[\gamma_j(L) - c]\,dc}{V(L)}, \tag{3.52}$$

converge weakly to a mesure $\nu(dc)$ over all the bounded closed intervals $[0, \Lambda]$, $\Lambda < +\infty$, and that $\sup_L \int_1^\infty \log(1+c)\frac{\nu_L(dc)}{c} < +\infty$.

We do not reproduce here the purely technical calculations necessary to prove this generalization of Theorem 2.

4. Finally we want to mention that it remains an open question whether or not some of the results that we have obtained still hold when the function c [or the numbers $\gamma_j(L)$] are allowed to be negative.

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