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Properties of Non-unitary Zero Mass Induced Representations of the Poincaré Group on the Space of Tensor-valued Functions (*)

by

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ABSTRACT. — The theory of induced representations is used to discuss a class of indecomposable representations of the Poincaré group in Hilbert space with an indefinite metric which occur in theories with zero-mass particles. The formalism provides a number of further generalizations to representations with $m^2 < 0$, and to infinite-component tensor fields.

The purpose of this note is to present a concise group theoretical origin for and a proof of the quantization procedure in quantum-electrodynamics (and in linearized general relativity) using an indefinite metric. A number of detailed studies have appeared and are appearing [1] which, by direct and lengthy calculations, show how the indefinite metric comes about ⁽¹⁾. The method of induced representations provides,

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⁽¹⁾ Remark : Let H be the carrier Hilbert space of a representation of a group G , with the scalar product $[u, v]$, $u, v \in H$. A certain subclass of non-unitary representations of G in H may be defined which leave the bilinear form $[U, \Gamma v] = (u, v)$ invariant where Γ is an indefinite metric tensor. This subclass of representations may be called “ Γ -unitary” representations, or “representations of G in a Hilbert space with the indefinite metric Γ ”. This is the connection between certain non-unitary representations and the indefinite metric.

we believe, an elegant statement and a simple proof of this problem. Furthermore, the new formulation makes it possible to state a number of generalizations.

We consider the representations of the Poincaré group $P = T^4 \otimes SL(2, C)$ [i. e. semi-direct product of T^4 and $SL(2, C)$] and use the general theory of induced representations for regular semidirect product groups ([2], [3], [6]).

Let Π be an orbit in the momentum space which may be a hyperboloid, a cone, or the point $p_\mu = 0$. The stability subgroup K of an arbitrary point p of the orbit is isomorphic to a subgroup $T^4 \otimes \tilde{K}$, where \tilde{K} is a subgroup of $SL(2, C)$. The construction of induced representations of the Poincaré group is carried along the following steps:

1° Choose a representation $k \rightarrow L_k$ of K in a carrier space Φ , which conserves in Φ a bilinear form $(\varphi, \Psi)_\Phi$.

2° Form the space H of function over the orbit Π with values in Φ satisfying the condition

$$(1) \quad (\varphi, \psi_\Pi) \equiv \int_\Pi (\varphi(p), \psi(p))_\Phi d\mu(p) < \infty.$$

3° Consider the map $P \ni g = \{a, \Lambda\} \rightarrow T_g$ in H defined by

$$(2) \quad T_{\{a, \Lambda\}} \varphi(p) = L_{\tilde{k}}^{-1} \varphi(\Lambda^{-1} p),$$

where \tilde{k} is an element of K corresponding to the Mackey decomposition of the element $\{a, \Lambda\}^{-1} x_g$ and x_g is an element of P corresponding to the momentum p (*).

Then equation (2) provides a representation of P in H which conserves the scalar product (1).

If the representation $k \rightarrow L_k$ of K is irreducible and unitary then the resulting induced representation $\{a, \Lambda\} \rightarrow T_{\{a, \Lambda\}}$ given by (2) is also irreducible and unitary. Moreover this construction provides *all* irreducible unitary representations of P .

However, if we use functions $\varphi \in \Phi$ which transform under the Lorentz group in a covariant manner as vectors, spinors, tensors, etc., we are in effect using non-unitary finite-dimensional representations of $SL(2, C)$, which in some cases imply also non-unitary representations of the stability sub-group \tilde{K} of Π . For instance, in the case of massless particles

(*) Let K be a closed subgroup of a locally compact group G . Then the Mackey decomposition theorem states that there exists a Borel set X in G such that every element g in G has the unique decomposition $g = x_g k_g$, $x_g \in X$, $k_g \in K$. Because every coset gK intersects with X at one point every element $p \simeq gK$ in the quotient space $\Pi = G/K$ may be uniquely represented by the element $x_g \in X$, i. e. $p \simeq x_g k_g K = x_g K$.

($\Pi = \{ p; p^2 = 0 \}$), the stability subgroup K has the form

$$(3) \quad K = T^1 \otimes \tilde{E}(2),$$

where $\tilde{E}(2)$ is a covering group of the Euclidian group $E(2)$. The group in equation (3) is connected and solvable. Hence, every finite-dimensional irreducible representation of K is one-dimensional by Lie's theorem. An arbitrary n -dimensional representation of K in (3) may be, moreover, reduced again by Lie's theorem, to a triangular form

$$(4) \quad k \rightarrow L_k = \begin{pmatrix} \chi_1(k) & & 0 \\ \star & \ddots & \\ & & \chi_n(k) \end{pmatrix},$$

where $\chi_i(k)$, $i = 1, \dots, n$, are the complex characters of K . Consequently an arbitrary n -dimensional representation of K is either a direct sum of one-dimensional irreducible representations, or is indecomposable. In addition, if we demand that the representation $k \rightarrow L_k$ is a faithful finite-dimensional representation, which conserves a bilinear form $(\varphi, \Psi)_\Phi$ in the carrier space Φ , then, because K is non-compact, the bilinear form $(., .)_\Phi$ must be necessarily indefinite. By virtue of equation (1), the indefinite form $(\varphi, \Phi)_\Phi$ makes the scalar product $(., .)_H$ in H also indefinite. Consequently, the representation defined by (2) is also non-unitary, but Γ -unitary. Note that this result holds for any choice of the bilinear form $(\varphi, \Psi)_\Phi$ in the carrier space Φ of the representation of K .

This is the idea of our proof. The precise statement, proof and generalizations now follow :

THEOREM. — *Every representation of the Poincaré group P with $m = 0$, on the space of tensor-valued functions is non-unitary. Each representation is realized as a Γ -unitary representation in the Hilbert space H of tensor-valued functions, with domain on the momentum cone, by the formula*

$$(5) \quad T_{\{a, \Lambda\}} \varphi_{\mu_1 \dots \mu_N}(p) = e^{i p a} D_{\mu_1 \dots \mu_N}^{-1} \mu'_1 \dots \mu'_N(\tilde{k}) \varphi^{\mu'_1 \dots \mu'_N}(\Lambda^{-1} p),$$

where $\tilde{k} \rightarrow D(\tilde{k})$ is a finite-dimensional indecomposable representation of $\tilde{E}(2)$ obtained by the reduction of the representation $\bigotimes_1^N (D^{00} \oplus D^{10})$ of $SL(2, C)$ to $\tilde{E}(2)$; \tilde{k} is the element of $\tilde{E}(2)$ obtained from the Mackey decomposition $P = XK, X \subset P, K = T^1 \otimes \tilde{E}(2)$, of the product

$$\{ a, \Lambda \}^{-1} x_g = \tilde{x} \tilde{k}, \quad \tilde{x} \in X, \quad \tilde{k} \in \tilde{E}(2),$$

where x_g is the unique element of the Poincaré group P characterizing the coset

$$p \simeq g K = x_g k_g K = x_g K.$$

The indefinite invariant scalar product in H is given by the formula

$$(\varphi, \psi)_H = \int_{\{p^2=0\}} \frac{d^3 p}{P_0} \bar{\varphi}^{\mu_1 \dots \mu_N}(p) \psi^{\mu_1 \dots \mu_N}(p).$$

Proof. — Let Φ be the linear space of all tensors $\varphi^{\mu_1 \dots \mu_N}$ of order N which carries a finite-dimensional representation $k \rightarrow L_k$ of K . Every representation of P in the space of tensor valued functions may be obtained by induction from a corresponding tensor representation of $SL(2, C)$. Consequently we take the representation $k \rightarrow L_k$ in the form

$$(6) \quad k = (a, \tilde{k}) \rightarrow L_k = e^{-i\hat{p}a} D_{\tilde{k}}, \quad \hat{p} = \omega(1, 0, 0, 1),$$

where $\tilde{k} = D_{\tilde{k}}$ is a finite-dimensional faithful indecomposable representation of $\tilde{E}(2)$ obtained by the reduction of a finite-dimensional representation $\bigotimes_1^N (D^{00} \oplus D^{10})$ of $SL(2, C)$ to the subgroup $\tilde{E}(2)$. The representation (6) of K conserves the following sesquilinear form in Φ :

$$(7) \quad (\varphi, \psi)_\Phi \equiv \bar{\varphi}^{\mu_1 \dots \mu_N} \psi^{\mu_1 \dots \mu_N}.$$

Clearly, the form (7) is indefinite.

Now let H be the space of all functions on P with values in Φ satisfying the conditions

$$(8) \quad \left\{ \begin{array}{l} 1^\circ \quad (\varphi(g), \Psi)_\Phi \text{ is a measurable function on } P \text{ for all } \Psi \in \Phi, \\ 2^\circ \quad \varphi(gk) = L_k^{-1} \varphi(g), \quad g \in P, \\ 3^\circ \quad \int_{P/K} (\varphi, \varphi)_\Phi d\mu(g) < \infty, \quad g = gK. \end{array} \right.$$

Then the action of $g_0 \rightarrow T_{g_0}$ of P in H is given by the formula

$$T_{g_0} \varphi(g) = \varphi(g_0^{-1} g).$$

Let

$$g = x_g k_g, \quad k_g \in K, \quad x_g \in X \subset P$$

be the Mackey decomposition $P = XK$ for the Poincaré group implied by the subgroup K . Then the action of the operators T_{g_0} in the space Φ of tensor functions with domain $C_0 = K/P$ is given by

$$(9) \quad T_{\{a, \Lambda\}} \varphi(p) = e^{ip a} D_{\tilde{k}}^{-1} \varphi(\Lambda^{-1} p),$$

where \tilde{k} is the element of $\tilde{E}(2)$ corresponding to the Mackey decomposition $\{a, \Lambda\}^{-1} x_g = \tilde{x} \tilde{k}$, $x_g \simeq p$ is the unique element of G corresponding to p ,

$$p \simeq g K = x_g k_g K = x_g K, \quad x_g \in X.$$

Because the invariant bilinear form (7) is indefinite, (8.3°) is also indefinite. Moreover, because K is noncompact any conserved form $(\varphi, \Psi)_\Phi$ for a faithful representation (6) of K must be indefinite.

Remarks and Generalizations. — 1° In the massive case, $m^2 > 0$, the little group K is $T^1 \otimes SU(2)$. The finite-dimensional tensor representations of $SL(2, C)$ reduced with respect to $SU(2)$ yield representations of $SU(2)$ equivalent to a direct sum of *unitary* irreducible representations. Hence the induced representations (2) of the Poincaré group P in this case will be also unitary.

2° As a special case, consider the representations of the Poincaré group in the space of vector functions $\varphi_\mu(p)$. They are reducible *non-unitary* but Γ -unitary representations which occur immediately in the quantization, by correspondence principle, of the classical electromagnetic field. In order to obtain the *unitary* representations corresponding to free physical photons with helicities ± 1 one projects out the two redundant components of the vector $\varphi_\mu(p)$. We may achieve this by imposing the Lorentz condition

$$(10) \quad p^\mu \varphi_\mu(p) = 0$$

and the gauge condition

$$\varphi_\mu(p) \rightarrow \varphi_\mu(p) + p_\mu \lambda(p),$$

where $\lambda(p)$ is a scalar function. Let H_1 be the subspace of H consisting of functions $\varphi_\mu(p)$ satisfying (8). Functions of the form $p_\mu \lambda(p)$ do satisfy (10). Let us introduce the equivalence relation in H defined by

$$\overset{1}{\varphi}_\mu(p) \approx \overset{2}{\varphi}_\mu(p) \quad \text{if} \quad \overset{1}{\varphi}_\mu - \overset{2}{\varphi}_\mu = p_\mu \lambda(p) + f_\mu(p),$$

where $p^\mu f_\mu(p) = 0$. Then in the quotient space H/H_1 a unitary representation of the Poincaré group is realized [1].

In general, in order to separate out unitary irreducible representations corresponding to a massless particle of spin J , we may also utilize the following connection between the canonical wave function $\chi^\lambda(p)$ of the massless particle of helicity λ , and the wave function $\Psi_{\alpha\beta}^{a,b}(p)$ which transforms according to a finite-dimensional irreducible representation (a, b) of $SL(2, C)$:

$$(11) \quad \chi^\lambda(p) = a_0 \sum_{\alpha\beta} D_{\alpha_0\beta_0\alpha\beta}^{a,b} [h_0(p)^{-1}] \psi_{\alpha\beta}^{a,b}(p), \quad \lambda = \alpha_0 + \beta_0,$$

where $\{D_{\alpha_0\beta_0\alpha\beta}^{ab}(\Lambda)\}$ is the matrix of the irreducible representation (a, b) of $SL(2, C)$ and the Lorentz transformation $h_0(p)$ satisfies: $h_0(p)a = p$, where $a = (1, 0, 0, 1)$. We have the invariant scalar product

$$(\chi^\lambda, \chi^\lambda) = \int d^\mu(p) \bar{\chi}^\lambda(p) \chi^\lambda(p);$$

hence the irreducible unitary representation $[0, \lambda]$ of the Poincaré group in the space H of wave functions $\chi^\lambda(p)$ given by (11). In order now to apply equation (11) it remains to decompose the tensor product $\bigotimes_{i=1}^N (D^{00} \oplus D^{10})$ into the irreducible representations (a, b) of $SL(2, C)$.

We remark that the form of vector-valued representations of P on the space H of functions $\varphi_\mu(p)$ arises from the quantization of the classical coupling $\varphi_\mu J^\mu$ of the electro-magnetic field to a matter current J^μ . If we want a theory of physical photons we may obtain directly the unitary representations of P realized in the space of two-dimensional vector functions $\varphi_\alpha(p)$, $\alpha = \pm 1$. This may be achieved by taking the representation $\tilde{k} \rightarrow D_{\tilde{k}}$ of \tilde{E}_2 in the form

$$(12) \quad \tilde{k} = (\vec{a}, r) \rightarrow D_{\tilde{k}} = \begin{pmatrix} e^{ir} & 0 \\ 0 & e^{ir} \end{pmatrix}, \quad \vec{a} \in T^2, \quad r \in U(1).$$

(We have taken this two-dimensional reducible representation because of the parity doubling in order to have both of the helicities ± 1 .) However, it is not known how to write the coupling of the physical photons to particles and the form of the Coulomb field, for example, in the space of the unitary representations of P .

3° The representations of the Poincaré group for massless spin 2 particles (e. g. gravitons) will have the same properties. We may obtain the irreducible unitary representations for massless spin 2 particles starting from the one-dimensional representation of the subgroup \tilde{K} of the little group K :

$$\tilde{k} = (\vec{a}, r) \rightarrow D_{\tilde{k}} = e^{i2r}, \quad \vec{a} \in T^2, \quad r \in U(1).$$

However, again it is not known how to write the coupling of gravitons to matter using this Hilbert space of states. Hence, from classical considerations one starts from a reducible representation $\bigotimes_1^2 (D^{00} \oplus D^{10})$ of $SL(2, C)$ restricts it to $\tilde{E}(2)$ and induces to the Poincaré group P . The representation so obtained will be only Γ -unitary and reducible. To get rid of redundant components we can use again the technique of projection operators given in equation (11).

4° Our formulation suggests the following generalizations :

(a) For the space-like representations of the Poincaré group, $m^2 < 0$, the little group K is $T^4 \otimes SU(1,1)$. The reduction of the finite-dimensional representations of $SL(2, C)$ with respect to $SU(1,1)$ yields non-unitary faithful representations of K , hence the induced representations of P are also non-unitary but Γ -unitary.

(b) For the null-representations of P ,

$$p_\mu = 0 \quad (m^2 = 0), \quad K = T^4 \otimes SL(2, C),$$

the induced representations are again non-unitary (Γ -unitary) for finite dimensional representations of $SL(2, C)$.

(c) We can also immediately extend the theorem to infinite-component fields. Here one starts with *unitary* infinite-dimensional representations of $SL(2, C)$, for example. Thus, we consider the space of tensor valued functions $\varphi_A(p)$, where A now has an infinite range determined by the representation of $SL(2, C)$. Again we distinguish various little groups. For $m^2 > 0$, we have an infinitely reducible *unitary* representation of P . For $m = 0$, $\tilde{K} = \tilde{E}(2)$, the unitary representations of $SL(2, C)$ restricted to $\tilde{E}(2)$ are now *unitary* and infinite-dimensional. Hence the induced representations of P will be unitary by virtue of equations (1) and (2). These are the so-called "continuous spin" representations of the Poincaré group. It is interesting that finite-dimensional tensors of $SL(2, C)$ give rise to zero-mass particles with a single value of helicity, whereas the infinite-dimensional tensors of $SL(2, C)$ give rise to zero-mass "particles" with all values of the helicity $0, \pm 1, \pm 2, \dots$

It is evident that this technique can be readily applied to other stability subgroups, as well as to non-unitary infinite-dimensional representations of $SL(2, C)$.

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REFERENCES

- [1] R. SHAW, *Nuovo Cim.*, vol. 37, 1965, p. 1086; J. BERTRAND, *Nuovo Cim.*, vol. 1 A, 1971, p. 1; L. BRACCI, Pisa preprint (July 1971); S. N. GUPTA, *Proc. Phys. Soc.*, A, vol. 63, 1950, p. 691; K. BLEULER, *Helv. Phys. Acta*, vol. 23, 1950, p. 567; H. E. MOSES, *J. math. Phys.*, vol. 5, 1968, p. 16 and references therein; W. LANGBEIN, *Comm. Math. Phys.*, vol. 5, 1967, p. 73; S. WEINBERG, *Brandeis Lectures*, vol. 2, 1964, p. 405 (Prentice-Hall, 1965); A. S. WIGHTMAN and L. GÄRDING, *Ark. Fys.*, vol. 28, 1965, p. 129; F. STROCCHI, *Phys. Rev.*, vol. 162, 1967, p. 1429; H. P. DÜRR and E. RUDOLPH, *Nuovo Cim.*, vol. 65 A, 1970, p. 423.

- [2] G. MACKEY, *see* Appendix in E. SEGAL, *Mathematical Problems in Relativistic Physics*, 1965. (M. I. T. Press, 1966).
- [3] The extension of Imprimitivity Theorem for representations of Poincaré group in topological vector spaces allows us to give a classification of non-unitary representation. *See* M. FLATO and D. STERNHEIMER, *Proceedings of French-Swedish Colloquium on Mathematical Physics*, Goteborg, 1971.
- [4] E. P. WIGNER, *Annals of Mathematics*, vol. 40, 1939, p. 149.
- [5] J. M. G. FELL, *Acta Math.*, vol. 114, 1965, p. 267.
- [6] R. RACZKA, *Theory of Unstable Particles*. Part I : *Relativistic Quantum Mechanics of Isolated Particles*, preprint, University of Colorado, Boulder, 1971.

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