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Section A : *Physique théorique.*

# On a differential equation approach to quantum field theory : Scattering for Thirring's model

by

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ABSTRACT. — We consider scattering problems for Thirring's model. The latter is described by a non-linear equation in a Banach space. We derive some general properties of non-linear scattering in Banach spaces, and we apply them to the case under consideration. Wave and scattering operators turn out to be trivial, as in the formal approach to this model.

RÉSUMÉ. — Étude de la diffusion pour l'équation opérationnelle non linéaire définissant le modèle de Thirring. Après avoir discuté quelques propriétés générales de la diffusion pour les équations non linéaires dans un espace de Banach, on les applique au cas considéré. Les opérateurs d'onde et de diffusion résultant sont triviaux, ce qui s'accorde avec les résultats du traitement formel du modèle en question.

## 1. INTRODUCTION

Constructing a quantum field theory by giving a precise meaning to the equations the fields formally obey and by studying the solutions of such equations might be viewed as an alternative approach to the Hamiltonian formalism. But even when the latter proves suitable in constructing a theory satisfying the Haag-Kastler axioms and almost all the Wightman ones, as in the remarkable work carried out by Glimm and Jaffe [1], it is not always clear whether the fields not merely fulfil a non-linear relation, but really satisfy an equation of the motion, namely, whether they solve a (well posed) Cauchy problem. Therefore, as often stressed by Segal [2], the field equation approach should provide a different interpretation of the fields, which may reveal itself useful in understanding some general aspects of the theory.

As a first step towards the analysis of field equations, one may consider the equations for fields which are not distributions but functions of the space time, taking values in the space of operators on the Hilbert space of physical states. In such a frame, and without imposing any requirement about commutativity of the fields, Salusti and Tesei [3] considered the Cauchy problem for Thirring's model, and showed the existence and uniqueness of the solution. A further result has been obtained by Salusti and the present author [4]; they showed in fact that an approximate version of the canonical anticommutation relations, imposed on the Cauchy data of Thirring's model (as treated in [3]) entails — still up to an approximation — the local anticommutativity for the solutions of the field equations, that is, if

$$[\Phi(0, x), \; \Phi^{+}(0, y)]_{+} = 0 \qquad \text{for} \quad | x - y | > d$$

then

$$[\Phi(t, x), \; \Phi^{+}(u, y)]_{+} = 0 \qquad \text{for} \quad | x - y | - | t - u | > d.$$

Thus the fields above might be considered as distribution fields, regularized by convolution with a smooth function of compact support, that is approximate fields satisfying the exact equation — where by exact we simply mean the formal equation not affected by any approximation or cutoff.

In order to pursue the analysis of this approach, it should be natural to ask about decay and asymptotics of " fields " like those of References [3] and [4] : the purpose of the present work is precisely to answer this question, by investigating the scattering properties of the equation defining Thirring's model. As the latter is a non-linear equation of evolution for functions taking values in a Banach space, we cannot apply the results obtained by Strauss [5], which are valid for **C**-valued functions, and cannot be easily extended to the vector valued case. Thus we are lead to analyze the non-linear scattering in an abstract Banach space, generalizing the definitions of scattering and wave operators and proving for such quantities a number of properties which generalize some familiar results valid in the linear case.

Using these methods, we shall be able to conclude that the scattering operators exist for our version of Thirring's model. Moreover, they w ll be shown to be trivial, which agrees with the claim of the formal approach [6], asserting that this model shouldn't describe any real interaction between particles.

The present paper is organized as follows : in the next section we shall derive some general properties of the non-linear scattering in a Banach space; in the subsequent section, after recalling our set up for Thirring's model, we shall show that the wave operators exist, and applying the general results previously obtained, we shall derive the properties of the scattering operators. The triviality of the latter will follow as a by-product of an explicit evaluation.

## 2. NON-LINEAR SCATTERING IN BANACH SPACES

### 2.1. Basic Definitions

Consider the Cauchy problem

$$(1)_0 \qquad \begin{cases} \dfrac{d}{dt} u(t) + \mathrm{L}\, u(t) + \mathrm{T}\, u(t) = 0, \\ u(0) = u_0, \end{cases}$$

where, $\forall\, t \geqq 0$, $u(t)$, $u_0 \in \mathrm{X}$, X being a Banach space endowed with the norm $x \mapsto |x|$ ; L is a linear operator in X, and the "perturbation" T a (not necessarily linear) operator on the whole of X.

Consider furtherly the free problem corresponding to $(1)_0$ :

$$(2) \qquad \begin{cases} \dfrac{d}{dt} v(t) + \mathrm{L}\, v(t) = 0, \\ v(0) = v_0, \end{cases}$$

where, $\forall\, t \geqq 0$, $v(t)$, $v_0 \in \mathrm{X}$, and L is as above.

In the following we shall always suppose that (2) has a unique solution, and that L generates a group $\mathrm{S}_0(t)$ of operators on X. Thus the solution of (2) has the form $v(t) = \mathrm{S}_0(t)\, v_0$. As to $(1)_0$, we shall require something less, namely the existence and uniqueness of a mild solution [7], that is, a solution of the integral problem

$$(1) \qquad u(t) = u_0 + \int_0^t \mathrm{S}_0(t-s)\, \mathrm{T}\, u(s)\, ds$$

which is equivalent to $(1)_0$ when $u(t)$ is differentiable; we shall further suppose that $t \mapsto \mathrm{S}(t)$, as defined by $u(t) = \mathrm{S}(t)\, u_0$, is a group of operators on X.

We may then ask :

*a.* whether, given a solution $u(t)$ of the perturbed problem (1) with Cauchy datum $u_0$, there exists Cauchy data $v_\pm$ of the corresponding free problem (2) such that the corresponding solutions $v_\pm(t)$ become asymptotic to $u(t)$ as $t \to \pm\infty$;

*b.* whether, given a solution $v(t)$ of the free problem (2) with Cauchy datum $v_0$, there exists Cauchy data $u_\pm$ of the perturbed problem (1) such that the corresponding solutions $u_\pm(t)$ become asymptotic to $v(t)$ as $t \to \pm\infty$.

We then introduce the operators on X,

$$\tilde{W}(t) = S_0(-t)\,S(t),$$
$$W(t) = S(-t)\,S_0(t)$$

and their limits

$$\tilde{W}_\pm u = \lim_{t\to\pm\infty} \tilde{W}(t)\,u,$$
$$W_\pm u = \lim_{t\to\pm\infty} W(t)\,u.$$

The latter operators are of course defined on the elements of X where the limits exist. Then it is evident that the answer to (*a*) is positive if and only if $u_0 \in D(\tilde{W}_\pm)$ (and in such case $v_\pm = \tilde{W}_\pm u_0$); analogously, the answer to (*b*) is positive if and only if $v_0 \in D(W_\pm)$ (and in such case $u_\pm = W_\pm v_0$).

We next define — whenever the definition makes sense —

$$\tilde{\mathbf{S}} = \tilde{W}_+ \tilde{W}_-^{-1},$$
$$\mathbf{S} = W_+ W_-^{-1}.$$

The latter operators map $D(\tilde{W}_-)$ in $D(\tilde{W}_+)$, $D(\tilde{W}_-)$ in $D(\tilde{W}_+)$, respectively.

In the classical case, namely that of potential scattering with X = Hilbert space; $i\,L$, $i\,T$, $i\,L + i\,T$ linear selfadjoint operators (to be interpreted as the free Hamiltonian, the perturbation, and the complete Hamiltonian), $\mathbf{S}$ is the scattering operator in the Heisenberg picture [8], and it is a constant of the motion as it commutes with the complete Hamiltonian [9]. On the other hand, $\tilde{\mathbf{S}}$ doesn't commute with the total Hamiltonian but with the free one, and it is not a constant of the motion; nevertheless, it reveals itself useful to compute observable quantities [10]. In general, the problems one faces concern : (i) the domain and the range of the operators $\tilde{W}_\pm$, $W_\pm$, and (ii) the existence and invertibility of $\mathbf{S}$, $\tilde{\mathbf{S}}$. These items are strongly connected, because, as an immediate consequence of the definitions, we have

PROPOSITION 0. — **S** *is well defined as an operator from* $R(W_-)$ *to* $R(W_+)$ *if and only if* $W_-$ *is one-to-one and* $D(W_-) \subset D(W_+)$; *its inverse* $\mathbf{S}^{-1}$ *is well defined if and only if* $W_+$ *is one-to-one and* $D(W_+) \subset D(W_-)$. *Thus* **S** *is invertible if and only if* $W_+$, $W_-$ *are one-to-one and invertible and* $D(W_+) = D(W_-)$. *Analogously,* $\tilde{\mathbf{S}}$, *as an operator from* $R(\tilde{W}_-)$ *to* $R(\tilde{W}_+)$ *is invertible if and only if* $\tilde{W}_+$, $\tilde{W}_-$ *are one-to-one and* $D(\tilde{W}_+) = D(\tilde{W}_-)$.

In the classical case, one considers $W_\pm$ as strong limits on X ([8], [9], [10], [11]), that is, one requires $D(W_\pm) = X$ (1). Such a wave operator $W_\pm$, whenever it exists, is isometric (and therefore one-to-one); thus **S** turns out to be well-defined and unitary from $R(W_-)$ on $R(W_+)$. When $R(W_\pm) = X$ (and therefore $W_\pm$ is unitary on X), then $W_\pm = W_\pm^{-1}$, and both **S** and $\tilde{\mathbf{S}}$ are well-defined and unitary on X. In the latter case, the theory is said to be complete.

In the non linear case, the condition $D(W_\pm) = X$ is true only under very special circumstances [12], and the completeness is even more hard to be established.

## 2.2. Properties of the scattering operators

Referring to the definitions of the 1st paragraph, we furtherly precise our hypothesis concerning $S_0(t)$ and $S(t)$ :

(i) $\{ S_0(t) \}_{t \in \mathbf{R}}$ is a group of linear contractions (with generator L);

(ii) the group $\{ S(t) \}$ is such that :

(ii-1) $| S(t) |_B \leq 1$, where $| S(t) |_B = \sup_{w \in X} \frac{| S(t) w |}{| w |}$;

(ii-2) $| S(t) |_L^r \leq f(r, t)$, where $f$, as a function of $r$, is bounded on the compacts of $\mathbf{R}_+$, for any $t$, and

$$| S(t) |_L^r = \sup_{\substack{u, v \in X \\ u \neq v \\ |u|, |v| \leq r}} \frac{| S(t) u - S(t) v |}{| u - v |} \qquad (r \geq 0).$$

A typical situation where (ii) is verified occurrs if

(ii)′ T is locally Lipschitz continuous (i. e., $| T |_L^r \leq N(r) < +\infty$) and $T \in K_B(X)$ [13] :

in fact, in such case, (ii-1) is true, and

$$| S(t) |_L^r \leq \exp(t N(r)) \quad [14].$$

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(1) We refer here to the case where both the free and the complete Hamiltonian have a pure absolutely continuous spectrum [9].

We then have :

LEMMA 1. — *If, for any* $u \in D(\tilde{W}_{\pm})$,

$$\lim_{t \to \pm\infty} | S(-t) |_L^{|u|} | \tilde{W}(t) u - \tilde{W}_{\pm} u | = 0,$$

*then*

$$R(\tilde{W}_{\pm}) \subset D(W_{\pm}), \qquad \textit{and} \qquad W_{\pm} \tilde{W}_{\pm} = 1 |_{D(\tilde{W}_{\pm})}.$$

*Proof.* — Consider

$$\begin{aligned} | W(t) \tilde{W}_{\pm} u - u | &= \lim_{s \to \pm\infty} | W(t) \tilde{W}(s) u - u | \\ &= \lim_{s \to \pm\infty} | S(-t) S_0(t) S_0(-s) S(s) u \\ &\qquad - S(-t) S_0(t) S_0(-t) S(t) u | \\ &\leqq \lim_{s \to \pm\infty} S(-t) |_L^r | W(s) u - W(t) u | \\ &\leqq | S(-t) |_L^r | W_{\pm} u - W(t) u |, \end{aligned}$$

where

$$r = \sup(| W(s) u |, | W(t) u |) = | u |.$$

Therefore, if the hypothesis is satisfied, the limit of $| W(t) W_{\pm} u - u |$ exists and it is zero.

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In a similar way one proves.

LEMMA 1'. — *If for any* $u \in D(W_{\pm})$,

$$\lim_{t \to \pm\infty} | S(t) |_L^{|u|} | W(t) u - W_{\pm} u | = 0,$$

*then*

$$R(W_{\pm}) \subset D(\tilde{W}_{\pm}) \qquad \textit{and} \qquad \tilde{W}_{\pm} W_{\pm} = 1 |_{D(W_{\pm})}.$$

Putting together the two preceding results, we have :

COROLLARY 2. — *If the hypothesis of both the Lemmata* 1 *and* 1' *are satisfied, then*

$$\tilde{W}_{\pm} = W_{\pm}^{-1},$$

*and, in particular,* $W_{\pm}$, $\tilde{W}_{\pm}$ *are one-to-one.*

Then, recalling Proposition 0, we may express the relationship between the operators W and **S** :

PROPOSITION 3. — *If the hypothesis of Corollary 2 are satisfied, and*

$$D(W_+) = D(W_-) = Y,$$

*then* **S** *is well-defined and invertible from* $R(W_-)$ *to* $R(W_+)$. *If furtherly*

$$R(W_+) = R(W_-) = \tilde{Y},$$

*then* **S** *is well-defined and invertible on* $\tilde{Y}$, *and* $\tilde{\mathbf{S}}$ *is well-defined and invertible on* Y.

By this way, the classical formulation is extended to the non linear case. Yet the conditions of Proposition 3 are not easily verified in the general case [for instance, Brodsky [12], working in a Hilbert space, is able to characterize only some subsets of $R(W_+)$]; for this reason we consider subsets $Z_\pm$, $\tilde{Z}_\pm$ of $D(W_\pm)$, $D(\tilde{W}_\pm)$, respectively, and ask whether suitable restrictions of **S**, $\tilde{\mathbf{S}}$ may be defined as invertible operators.

After introducing the notations

$$\tilde{W}_{\tilde{Z}_\pm} = \tilde{W}_\pm \big|_{\tilde{Z}}, \qquad W_{Z_\pm} = W_\pm \big|_{Z_\pm}$$

we state the following results, quite analogous to the former lemmata 1 and 1′ :

LEMMA 4. — *If for any* $u \in \tilde{Z}_\pm$,

$$\lim_{t \to \pm\infty} | S(-t) |_L^{|u|} | \tilde{W}(t) u - \tilde{W}_\pm u | = 0,$$

*then*

$$R(\tilde{W}_{\tilde{Z}_\pm}) \subset D(W_\pm), \qquad \textit{and} \qquad W_\pm \tilde{W}_{\tilde{Z}_\pm} = 1 \big|_{\tilde{Z}_\pm}.$$

LEMMA 4′. — *If for any* $u \in Z_\pm$,

$$\lim_{t \to \pm\infty} | S(t) |_L^{|u|} | W(t) u - W_\pm u | = 0,$$

*then*

$$R(W_{Z_\pm}) \subset D(\tilde{W}_\pm), \qquad \textit{and} \qquad \tilde{W}_\pm W_{Z_\pm} = 1 \big|_{Z_\pm}.$$

As an immediate consequence, we have :

COROLLARY 5. — *If the hypothesis of both the Lemmata* 4, 4′ *are satisfied, and furtherly if* $R(\tilde{W}_{\tilde{Z}_\pm}) \subset Z_\pm$ *and* $R(W_{Z_\pm}) \subset \tilde{Z}_\pm$, *then* $W_{Z_\pm} = \tilde{W}_{\tilde{Z}_\pm}^{-1}$.

Then it is easy to generalize Proposition 3 as follows :

PROPOSITION 6. — *Assuming the hypothesis of Corollary* 5, *if* $Z_+ = Z_- = Z$, $\mathbf{S}_Z$, *defined as* $\mathbf{S}_Z = W_{Z_+} W_{Z_-}^{-1}$, *is well-defined and invertible from* $\tilde{Z}_-$ *to* $\tilde{Z}_+$; *if furtherly* $\tilde{Z}_- = \tilde{Z}_+ = \tilde{Z}$, *then* $\mathbf{S}_Z$ *is well-defined and invertible on* $\tilde{Z}$, *and* $\tilde{\mathbf{S}}_{\tilde{Z}} = \tilde{W}_{\tilde{Z}_+} \tilde{W}_{\tilde{Z}_-}^{-1}$ *is well-defined and invertible on* Z.

## 3. THIRRING'S MODEL

In this section we shall apply the previously given formulation to Thirring's model [6], [15]. Following the treatement of References [3], [4], this model is defined through equations like $(1)_0$, (1), where $X = C_0(\mathbf{R}; B) \oplus C_0(\mathbf{R}; B)$ endowed with the norm

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto |\, u \,| = \sup_{x \in \mathbf{R}} (|\, u_1(x) \,|_B + |\, u_2(x) \,|_B),$$

$C_0(\mathbf{R}; B)$ is the Banach algebra of the continuous functions on **R**, bounded at infinity and taking values in B, B being the Banach algebra $\mathcal{L}(H)$ (norm $b \mapsto |\, b \,|_B$) of the linear continuous operators on the Hilbert space H; L is the linear closed operator

$$\begin{pmatrix} -\dfrac{d}{dx} & 0 \\ 0 & \dfrac{d}{dx} \end{pmatrix}$$

and T the non-linear operator defined by

$$T u = T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i \begin{pmatrix} u_2^* u_2 u_1 \\ u_1^* u_1 u_2 \end{pmatrix}.$$

It is known [3] that L generates a group on X, $S_0(t)$, and that $L + T$ as well generates a group $S(t)$, satisfying (1); moreover, $S_0(t)$ and $S(t)$ fulfil the conditions (i), (ii) [in particular, T satisfies (ii)′ with $N(r) \leq 3 r^2$].

We want to prove the following result :

THEOREM 7. — *The subset* K *of* X *defined by* $K = C_0^k(\mathbf{R}; B) \oplus C_0^k(\mathbf{R}, B)$, $C_0^k$ *being the space of continuous functions of compact support, satisfies the hypothesis of Proposition* 6 *with* $Z = \tilde{Z} = K$; *therefore both* $\mathbf{S}_K$ *and* $\tilde{\mathbf{S}}_K$ *are well-defined and invertible on* K.

*Moreover,* $\tilde{W}_\pm$, $W_\pm$ *exist as well, and*

$$\tilde{W}_\pm = W_\pm^{-1} = 1.$$

*Thus* $\mathbf{S}$, $\tilde{\mathbf{S}}$ *exist and are equal to the identity.*

In order to prove this theorem, we must show :

$$(a) \qquad K \subset D(\tilde{W}_\pm), \quad K \subset D(W_\pm);$$

$$(b) \qquad \tilde{W}_\pm K \subset K, \quad W_\pm K \subset K;$$

$$(c) \qquad \begin{cases} \lim_{t \to \pm\infty} | S(-t) |_L^{|u|} \, | W(t) u - W_\pm u | = 0, & \forall\, u \in K, \\ \lim_{t \to \pm\infty} | S(t) |_L^{|u|} \quad | W(t) u - W_\pm u | = 0, & \forall\, u \in K. \end{cases}$$

The proof of the above items (*a*), (*b*), (*c*) depends essentially on the following Lemma :

LEMMA 8. — *If* $u$*, as a function from* $\mathbf{R}$ *to* $B \oplus B$*, vanishes for* $|x| > c$*, then :*

(i) *the function* $(S_0(-s)\, TS(s)\, u)(x)$ *vanishes for any* $|x| > c$, $\forall\, s$;

(ii) *the* $(C_0(\mathbf{R}; B) \oplus C_0(\mathbf{R}; B))$*-valued function* $S_0(-s)\, TS(s)\, u$ *vanishes for* $|s| > c$;

(iii) *the* $(B \oplus B)$*-valued function* $(S_0(-t-s)\, TS(s)\, S_0(t)\, u)(x)$ *vanishes for* $|x| > c$, $\forall\, t$, $\forall\, s \in [0, t]$;

(iv) *the* $(C_0(\mathbf{R}; B) \oplus C_0(\mathbf{R}; B))$*-valued function* $S_0(-t-s)\, TS(s)\, S_0(t)\, u$*, vanishes for* $|t+s| > c$, $\forall\, t$, $\forall\, s \in [0, t]$*, and therefore vanishes for any* $t$ *and any* $s \in [0, t]$.

Before proving Lemma 8, let us deduce from it Theorem 7. In fact, for $u \in K$ with $u(x) = 0$ for $|x| > c$,

$$| W(t_1) u - W(t_2) u | \leq \int_{t_2}^{t_1} | S_0(-s)\, TS(s)\, u |\, ds,$$

and the last quantity converges to zero as $t_1$, $t_2$ go to infinity, because, applying Lemma 8, we know that

$$\int_0^\infty | S_0(-s)\, TS(s)\, u |\, ds < \infty.$$

Thus we have shown that :

$$K \subset D(\tilde{W}_\pm).$$

Putting for short $f(s, t) = S_0(-t-s)\,TS(s)\,S_0(t)\,u$, we have :

$$|W(t_1)u - W(t_2)u| = \left|\int_0^{t_1} f(s, t_1)\,ds - \int_0^{t_2} f(s, t_2)\,ds\right|$$
$$= \left|\int_0^{\pm c} f(s, t_1)\,ds - \int_0^{\pm c} f(s, t_2)\,ds\right|,$$

which is zero for $|t_1|$, $|t_2|$ larger than $c$. Thus $K \subset D(W_\pm)$, and the proof of $(a)$ is complete.

As to $(b)$, notice that

$$\tilde{W}_\pm u = u + \int_0^{\pm\infty} S_0(-s)\,TS(s)\,u\,ds = u + \int_0^{\pm c} S_0(-s)\,TS(s)\,u\,ds,$$

where we made use of Lemma 8 again. Using once more Lemma 8 [in particular (ii)], we get $(\tilde{W}_\pm u)(x) = 0$ for $|x| > c$, which proves that $\tilde{W}_\pm K \subset K$. Analogously,

$$W_\pm u = u + \lim_{t\to\pm\infty} \int_0^{t} f(s, t)\,ds = u + \lim_{t\to\pm\infty} \int_0^{c} f(s, t)\,ds$$
$$= u + \lim_{t\to\pm\infty} \int_0^{c} f(s, t)\,ds = u,$$

where we used Lemma 8, (iii) and (iv). Thus $W_\pm K \subset K$, and we have also proved that

$$W_{K_\pm} = 1.$$

This completes the proof of $(b)$. Finally, in order to prove $(c)$, it is easy to convince ourselves that both the functions $|\tilde{W}(t)u - \tilde{W}_\pm u|$ and $|W(t)u - W_\pm u|$ have compact support (in the variable $t$) whenever $u \in K$. The argument to be used is of the same type of those exploited in proving $(a)$ and $(b)$.

As $W_{K_\pm} = 1$, we may extend by continuity $W_\pm$ to the whole of X, obtaining $W_\pm = 1$ (²). Applying Corollary 5, we find that $\tilde{W}_{K_\pm} = 1$, thus $\tilde{W}_\pm = 1 = W_\pm^{-1}$, and $\mathbf{S} = \tilde{\mathbf{S}} = 1$.

We are then left with the proof of Lemma 8 :

Let us define :

$$u^{(k)}(t) = S_0(t)\,u + \int_0^{t} S_0(t-s)\,Tu^{(k-1)}(s)\,ds \qquad (k > 0),$$
$$u^{(0)}(t) = S_0(t)\,u.$$

(²) This extension is peculiar to the case under consideration, because, in the general case, $W_\pm$ cannot be easily shown to be uniformly bounded on K.

We then have :

SUBLEMMA 9. — *For any* $k \in \mathbf{N}$, $t \in \mathbf{R}$, *there are functions of* $C_0(\mathbf{R}; B)$, *namely*

$$q_1^{(k)}(t, .), \quad q_2^{(k)}(t, .),$$

*such that*

$$(u_1^{(k)}(t))(x) = q_1^{(k)}(t, x)\, u_1(x + t),$$
$$(u_2^{(k)}(t))(x) = q_2^{(k)}(t, x)\, u_2(x - t).$$

*Proof.* — The claim is obviously true for $k = 0$, with $q_1^{(0)} = q_2^{(0)} = 1$.

Let us next suppose that the required property is valid for a given $k$, and show that the same holds for $k + 1$. fact,

$$\begin{aligned}(u_1^{(k+1)}(t))(x) &= u_1(x + t) + i \int_0^t ds\, u_2^*(x - 2s + t) \\ &\quad \times q_2^{(k)*}(s, x - s + t)\, q_2^{(k)}(s, x - s + t)\, u_2(x - 2s + t) \\ &\quad \times q_1^{(k)}(s, x - s + t)\, u_1(x + t) \\ &= \Big(1 + i \int_0^t ds\, u_2^*(x - 2s + t) \\ &\qquad \times q_2^{(k)*}(s, x + t - s)\, q_2^{(k)}(s, x - s + t) \\ &\qquad \times u_2(x - 2s + t)\, q_1^{(k)}(s, x - s + t)\Big)\, u_1(x + t).\end{aligned}$$

As a similar argument applies to $(u_2^{(k+1)}(t))(x)$, Sublemma 9 is proved. Let us next prove :

SUBBLEMMA 10. — *If* $u(x) = 0$ *for* $|x| > c$, *then, for any* $k$, $(S_0(-t)\, T\, u^{(k)}(t))(x)$ *vanishes for* $|x| > c$ *and for* $|t| > c$.

*Proof.* — By Sublemma 9, we may write :

$$(S_0(-t)\, T\, u^{(k)}(t))(x) =$$
$$\begin{pmatrix} u_2^*(x - 2t)\, q_2^{(k)*}(t, x - t)\, q_2^{(k)}(t, x - t)\, u_2(x - 2t)\, q_1^{(k)}(t, x - t)\, u_1(x) \\ u_1^*(x + 2t)\, q_1^{(k)*}(t, x + t)\, q_1^{(k)}(t, x + t)\, u_1(x + 2t)\, q_2^{(k)}(t, x + t)\, u_2(x) \end{pmatrix}.$$

Therefore this expression vanishes for $|x| > c$ and for $2|t| > c + x$, therefore for $|x| > c$ and for $|t| > c$, which proves Sublemma 10.

As an immediate consequence of Sublemma 10, the $C_0(\mathbf{R}; B) \oplus C_0(\mathbf{R}; B)$-valued function $S_0(-t)\, T\, u^{(k)}(t)$ vanishes for $|t| > c$, for any $k$.

Then the items (i) and (ii) of Lemma 8 follow: in fact, it can be shown [4] that a constant $h$ exist such that :

$$\lim_{k \to \infty} | u^{(k)}(t) - u(t) | = 0 \qquad \text{for} \quad t < h(| u |^{-2});$$

this implies that (i) and (ii) hold for $t < h(| u |^{-2})$, moreover, the estimate $| S(t) u | \leq | u |$ and the group property of $S(t)$ ensure that (i) and (ii) hold for any $t$ [4].

In order to prove the items (iii) and (iv) of Lemma 8, we proceed in a similar way; after defining

$$u^{(k)}(t, s) = S_0(t+s) u + \int_0^s S_0(s-r) T u^{(k-1)}(t, r) \, dr \qquad (k > 0),$$

$$u^{(0)}(t, s) = S_0(t+s) u,$$

we prove :

Sublemma 9′. — *For any $k \in \mathbf{N}$, $t, s \in \mathbf{R}$, there are functions of* $C_0(\mathbf{R}; B)$ :

$$p_1^{(k)}(t, s, .), \quad p_2^{(k)}(t, s, .)$$

such that :

$$(u_1^{(k)}(t, s))(x) = p_1^{(k)}(t, s, x) \, u_1(t+s+x),$$
$$(u_2^{(k)}(t, s))(x) = p_2^{(k)}(t, s, x) \, u_2(-t-s+x).$$

The proof is the same as for Sublemma 9, because :

$$(u_1^{(k+1)}(t, s))(x) = u_1(x+t+s) + i \int_0^s dr \, u_2^*(x+t+r-2s)$$
$$\times \, p_2^{(k)*}(t, r, x+s-r) \, u_2(x+t+r-2s)$$
$$\times \, p_2^{(k)}(t, r, s+x-r) \, p_1^{(k)}(t, r, x-s+r) \, u_1(x+t+s).$$

As a consequence, we have :

Sublemma 10′. — *If $u(x) = 0$ for $| x | > c$, then, for any $k$,*

$$(S_0(-t-s) T u^{(k)}(t, s))(x) = 0$$

*for $| x | > c$ and for $| t+s | > c$.*

The proof mimics that of Sublemma 10, because

$$(S_0(-t-s) T u^{(k)}(t, s))_1(x) =$$
$$= u_2^*(x-2s-2t) \, p_2^{(k)*}(t, s, x-t-s) \, p_2^{(k)}(t, s, x-t-s) \, u_2$$
$$\times (x-2s-2t) \, p_1(t, s, x+t+s) \, u_1(x).$$

Then, repeating the arguments mentioned above, the proof of Lemma 8 can be easily completed.

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