ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 16, nº 1 (1972), p. 41-50
<http://www.numdam.org/item?id=AIHPA_1972__16_1_41_0>

Ann. Inst. Henri Poincaré,
Vol. XVI, n° 1, 1972, p. 41-50.

Section A :
Physique théorique.

Junction conditions in general relativity theory

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ABSTRACT — The purpose of this article is to make three remarks about the problem of joining two solutions of Einstein's field equations across surfaces of discontinuity.

In Section I the complete equivalence of the junction conditions proposed by Lichnerowicz and by O'Brien and Synge at non-null surfaces is demonstrated.

In Section 2 the junction of spherically-symmetric solutions with the Schwarzchild " exterior " solution is discussed.

Finally in Section 3, it is shown how the junction conditions proposed by O'Brien and Synge for null surfaces are satisfied in a natural way when " Bondi-type " coordinates are used in the study of radiation problems.

SOMMAIRE. — Le but de cet article est de faire trois remarques au sujet du problème de joindre deux solutions des équations d'Einstein à travers une hypersurface.

Dans la première partie, l'équivalence complète des conditions de raccordement proposées par Lichnerowicz et par O'Brien et Synge en hypersurfaces orientées dans le temps ou dans l'espace est montrée.

Dans la deuxième partie, le raccordement des solutions à symétrie sphérique avec la solution de Schwarzschild est discuté.

Finalement, dans la troisième partie il est montré comment une étude du problème de la radiation en utilisant des « Bondi-coordonnées » mène d'une manière naturelle à l'utilisation des conditions proposées par O'Brien et Synge pour des hypersurfaces nulles.

1. CONDITIONS OF O'BRIEN AND SYNGE AND OF LICHNEROWICZ

Metric Tensor

Let ds^2 be the metric of the four-dimensional Riemannien space-time defined by (1)

$$ds^2 = g_{ij}\, dx^i\, dx^j. \tag{1}$$

Suppose the three-dimensional non-null hypersurface S, defined by the equation

$$x^4 - a = 0 \tag{2}$$

where a is a constant, separates space-time into two regions, V^1 and V^2 defined by

$$x^4 - a \leqq 0,$$
$$x^4 - a \geqq 0, \qquad \text{respectively.}$$

Lichnerowicz [1] has suggested that suitable junction conditions are that the metric tensor, g_{ij}, and all the first order partial derivatives $\dfrac{\partial g_{ij}}{\partial x^k}$, with respect to x^k should be continuous at S.

On the other hand, O'Brien and Synge [2] proposed that g_{ij} and all the first order partial derivatives with respect to x^k, except possibly $\dfrac{\partial g_{4i}}{\partial x^4}$, $i = 1, \ldots, 4$, should be continuous at S (2).

It was pointed out by Israel [3] that any solution satisfying the conditions of Lichnerowicz can be transformed to one satisfying only the " weaker " conditions of O'Brien and Synge. It is now shown that the converse is also true, i. e. any solution satisfying the conditions of O'Brien and Synge at any non-null surface can always be transformed to one satisfying the conditions of Lichnerowicz.

Suppose then that a metric tensor g_{ij} and its partial derivatives (1) $\dfrac{\partial g_{\alpha\beta}}{\partial x^4}$ are continuous at a hypersurface S defined by equation (2). Now make the following co-ordinate transformation, which,

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(1) Latin indices $i, j, \ldots$ take values in the range 1-4, and Greek indices $\alpha, \beta, \ldots$ in the range 1-3. Also the convention of summation over repeated indices is used throughout.

(2) These conditions are equivalent to the requirement that the first and second fundamental forms be continuous at S [7].

in general, is discontinuous at S :

(3) $$x^i = \bar{x}^i$$

for points in V^1, and

(4) $$x^i = \bar{x}^i + A^i (\bar{x}^4 - a)^2$$

for points in V^2, where A^i are functions of $\bar{x}^\alpha$ only.

Notice that the defining equation for S in these new co-ordinates is

$$\bar{x}^4 - a = 0.$$

The components of the metric tensor, $\bar{g}_{ij}$, the first order partial derivatives, $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$, in the $\bar{x}^i$ co-ordinate system are now considered at points on S.

Clearly in V^1 these quantities all remain unchanged.

Consider now the values in V^2. Since g_{ij} is a tensor, it may be expressed in the $\bar{x}^i$ co-ordinate system as

(5) $$\bar{g}_{ij} = g_{mn} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j}$$

and so, on S, i. e. when $\bar{x}^4 = a$, it is easily seen that

$$\bar{g}_{ij} \approx g_{ij}$$

where the symbol $\approx$ is used to denote equality on the surface, S. Consequently, the components of the metric tensor remain continuous at S after the transformation has been performed. Now the partial derivatives $\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}$ may be found by differentiating equation (5). These quantities in V^2 may be found using equations (4) and it may be seen that they all remain unchanged at S except $\frac{\partial \bar{g}_{4i}}{\partial \bar{x}^4}$, $i = 1, \ldots, 4$, which become

$$\frac{\partial \bar{g}_{4i}}{\partial \bar{x}^4} \approx \frac{\partial g_{4i}}{\partial x^4} + 2\, g_{ij}\, A^j,$$

where all the values in these equations are evaluated when $x^4 = a$. These equations may be solved for A^i, $i = 1, \ldots, 4$, so that the values of $\frac{\partial \bar{g}_{4i}}{\partial \bar{x}^4}$ at S are equal to the corresponding values calculated in V^1, i. e. in such a way that $\frac{\partial \bar{g}_{4i}}{\partial \bar{x}^4}$ are all continuous at S. Such a solution can always be found provided that, at S,

$$\det(g_{ij}) \neq 0.$$

It has been proved, then, that the junction conditions of Lichnerowicz are exactly equivalent to those of O'Brien and Synge at non-null surfaces.

EXAMPLE. — As an example of the above procedure, consider the Schwarzschild " interior " and " exterior " solutions (discussed, for example, by Tolman [4]) where, if V^1 and V^2 are the space-time regions defined by

$$r - a \leqq 0, \qquad r - a \geqq 0,$$

respectively, the metrics are given by

$$(6) \qquad ds^2 = \left(\frac{3}{2}\sqrt{1 - \frac{2\,m}{a}} - \frac{1}{2}\sqrt{1 - \frac{2\,mr^2}{a^3}}\right)^2 dt^2 - \left(1 - \frac{2\,mr^2}{a^3}\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

in V^1, and

$$(7) \quad ds^2 = \left(1 - \frac{2\,m}{r}\right) dt^2 - \left(1 - \frac{2\,m}{r}\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

in V^2, where m and a are constants.

It is clear that these two metrics satisfy the junction conditions of O'Brien and Synge at the surface S defined by

$$(8) \qquad r - a = 0.$$

However, the partial derivative $\frac{\partial g_{44}}{\partial x^4}$, (where x^4 has been identified with r), is discontinuous at S; but transformations corresponding to equations (3) and (4) in V^1 and V^2, respectively, lead to a solution in which the metric tensor and all its partial derivatives are continuous at S. Using the above procedure, the solutions for the functions A^i are found to be

$$A^1 = A^2 = A^3 = 0,$$

$$A^4 = \frac{3\,m}{2\,a^2\left(1 - \frac{2\,m}{a}\right)}.$$

Energy-Momentum Tensor

It has been pointed out elsewhere (for example Synge [5]) that the junction conditions of Lichnerowicz at a surface S defined by equation (2) imply that the components, E_i^4, $i = 1, \ldots, 4$, of the Einstein tensor are continuous at S, since these components are independent of any second-order partial-derivatives of g_{ij} with respect to x^4. Consequently, by virtue of Einstein's field equations :

$$E_j^i = -k\,T_j^i, \tag{9}$$

it follows that the components, T_i^4, $i = 1, \ldots, 4$, of the energy-momentum tensor are also continuous at S. Now, since the transformations (3) and (4) do not alter the values of any tensor components at S, it follows that T_i^4 are continuous at S under the " weaker " conditions proposed by O'Brien and Synge.

Junction at Arbitrary Surfaces

More generally, the O'Brien-Synge junction conditions at a non-null surface of discontinuity, S, defined by the equation

$$f(x^i) = 0,$$

where f is an arbitrary function of the co-ordinates, x^i, and where S has continuous co-variant normal, $\dfrac{\partial f}{\partial x^i}$, imply the continuity at S of the following quantities :

$$g_{ij} \quad (i, j = 1, \ldots, 4), \qquad T_i^j \frac{\partial f}{\partial x^j} \quad (i = 1, \ldots, 4),$$

$$\frac{\partial g_{\alpha\beta}}{\partial x^4} - \frac{\partial g_{\alpha 4}}{\partial x^\beta} - \frac{\partial g_{4\beta}}{\partial x^\alpha} + \frac{1}{2}\left(\frac{\partial g_{44}}{\partial x^\alpha}\frac{\partial f}{\partial x^\beta} + \frac{\partial g_{44}}{\partial x^\beta}\frac{\partial f}{\partial x^\alpha}\right)\left(\frac{\partial f}{\partial x^4}\right)^{-1},$$

$$(\alpha, \beta = 1, 2, 3).$$

2. JUNCTION OF SPHERICALLY SYMMETRIC SOLUTIONS WITH VACUUM FIELD

In recent years much interest has been shown in spherically-symmetric solutions of Einstein's field equations, and in most cases the " exterior " field has been assumed to be the vacuum field, which, by Birkhoff's Theorem, must be the Schwarzschild " exterior " solution (7). The purpose here is to find necessary and sufficient conditions which must be satisfied by any spherically-symmetric solution making a satisfactory junction with the vacuum field across a surface S.

The question arises: Does a given solution, which has zero components T_i^4 of the energy-momentum tensor at a surface S, necessarily also satisfy the junction conditions required of the metric tensor at S ?

Suppose then, that in a region V^1 of space-time there is a spherically-symmetric distribution of matter represented by the following metric

$$ds_1^2 = D\,dt^2 + E\,dr\,dt - A\,dr^2 - B\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{10}$$

where A, B, D and E are functions of r and t only. Further suppose that this distribution joins on to an empty space region, V^2, across a non-null surface S defined by the equation [3]

$$r - a = 0. \tag{11}$$

The metric in V^2 may be expressed in terms of some co-ordinates $(T, R, \bar{\theta}, \bar{\varphi})$ in the form

$$ds_2^2 = e^\alpha\,dT^2 - e^{-\alpha}\,dR^2 - R^2\,(d\bar{\theta}^2 + \sin^2\bar{\theta}\,d\bar{\varphi}^2) \tag{12}$$

where

$$e^\alpha = 1 - \frac{2\,m}{R},$$

and m is a constant.

Make the following transformation in V^2

$$T = \psi(r, t), \qquad R = \eta(r, t), \qquad \bar{\theta} = \theta, \qquad \bar{\varphi} = \varphi \tag{13}$$

where $\psi(r, t)$, and $\eta(r, t)$ are functions of r and t only. The metric ds_2^2 in V^2 now takes the form:

$$ds_2^2 = \left[e^\alpha\left(\frac{\partial\psi}{\partial t}\right)^2 - e^{-\alpha}\left(\frac{\partial\eta}{\partial t}\right)^2\right]dt^2 + 2\left[e^\alpha\,\frac{\partial\psi}{\partial t}\,\frac{\partial\psi}{\partial r} - e^{-\alpha}\,\frac{\partial\eta}{\partial t}\,\frac{\partial\eta}{\partial r}\right]dr\,dt$$
$$- \left[e^{-\alpha}\left(\frac{\partial\eta}{\partial r}\right)^2 - e^\alpha\left(\frac{\partial\psi}{\partial r}\right)^2\right]dr^2 - \eta^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{14}$$

From this and equation (10) the continuity of g_{ij} at S may be expressed as four equations between the functions ψ and η (and their derivatives) at S. Let ψ_0, η_0, ψ_1 and η_1 denote the values of ψ, η, $\frac{\partial\psi}{\partial r}$ and $\frac{\partial\eta}{\partial r}$ at S respectively, then the four equations may be solved for these quantities

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(3) The co-ordinates t, θ, φ and r are here identified with x^1, x^2, x^3 and x^4 respectively.

giving:

$$(15)\quad \left\{\begin{aligned}
&\eta_0 \approx B^{\frac{1}{2}} \\
&\frac{\partial\psi_0}{\partial t} \approx \left(1 - \frac{2m}{B^{\frac{1}{2}}}\right)^{-1}\left[D\left(1 - \frac{2m}{B^{\frac{1}{2}}}\right) + \left(\frac{\partial B}{\partial t}\right)^2 \frac{1}{4B}\right]^{\frac{1}{2}}, \\
&\psi_1 \approx \frac{E}{2D}\left(1 - \frac{2m}{B^{\frac{1}{2}}}\right)^{-1}\left[D\left(1 - \frac{2m}{B^{\frac{1}{2}}}\right) + \left(\frac{\partial B}{\partial t}\right)^2 \frac{1}{4B}\right]^{\frac{1}{2}} \\
&\qquad + \frac{1}{2}\left(1 - \frac{2m}{B^{\frac{1}{2}}}\right)^{-1}\frac{\partial B}{\partial t}\left(\frac{A}{BD} + \frac{E^2}{4BD^2}\right)^{\frac{1}{2}}, \\
&\eta_1 \approx \frac{E}{4BD^{\frac{1}{2}}}\frac{\partial B}{\partial t} \\
&\qquad + \left[D\left(1 - \frac{2m}{B^{\frac{1}{2}}}\right) + \left(\frac{\partial B}{\partial t}\right)^2 \frac{1}{4B}\right]^{\frac{1}{2}}\left(\frac{4AD + E^2}{4D^2}\right)^{\frac{1}{2}},
\end{aligned}\right.$$

where all of these quantities are evaluated with $r = a$.

So, it can be seen that the values of ψ_0, η_0, ψ_1 and η_1, are all determined uniquely (except for an arbitrary constant in ψ_0) by the condition that the metric tensor must be continuous at S. This condition does not restrict the form of (10) in any way. However, it is now shown how the continuity of $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ and of T_i^4 at S does restrict the form of (10).

The continuity of T_i^4, $i = 1, \ldots, 4$, at S implies that the following equations hold

$$T_i^4 \approx 0 \qquad (i = 1, \ldots, 4),$$

since V^2 is a vacuum region. These equations imply that the components E_i^4, $i = 1 \ldots 4$, of the Einstein tensor should be zero at S. The components E_i^4, $i = 1 \ldots 4$, may be calculated in terms of the metric (10), and it can be seen that E_2^4 and E_3^4 are zero identically because of the spherical symmetry of the metric (10). However, when E_1^4 and E_4^4 are equated to zero on S the following equations result:

$$(16)\quad \frac{4D}{B(4AD + E^2)}\left\{ - \frac{\partial^2 B}{\partial r\,\partial t} + g\frac{\partial^2 B}{\partial t^2} + \frac{\partial g}{\partial t}\frac{\partial B}{\partial t} + \frac{1}{2B}\left(\frac{\partial B}{\partial r} - g\frac{\partial B}{\partial t}\right)\frac{\partial B}{\partial t} + \frac{2D}{(4AD + E^2)} \times\left(\frac{E}{2D}\frac{\partial E}{\partial t} - \frac{E^2}{4D^2}\frac{\partial D}{\partial t} + \frac{\partial A}{\partial t}\right)\left(\frac{\partial B}{\partial r} - g\frac{\partial B}{\partial t}\right) + \frac{1}{2D}\frac{\partial B}{\partial t}\left(\frac{\partial D}{\partial r} - g\frac{\partial D}{\partial t} - 2\frac{\partial g}{\partial t}D\right)\right\} \approx 0;$$

$$(17)\quad \frac{\beta}{B}\left\{-1-\frac{1}{D}\frac{\partial^2 B}{\partial t^2}+\frac{1}{2D^2}\frac{\partial B}{\partial t}\frac{\partial D}{\partial t}+\frac{1}{4BD}\left(\frac{\partial B}{\partial t}\right)^2 + \frac{D}{(4AD+E^2)B}\left(\frac{\partial B}{\partial r}-g\frac{\partial B}{\partial t}\right)^2+\frac{2}{(4AD+E^2)} \times\left(\frac{\partial B}{\partial r}-g\frac{\partial B}{\partial t}\right)\left(\frac{\partial D}{\partial r}-g\frac{\partial D}{\partial t}-2\frac{\partial g}{\partial t}D\right)\right\}\approx 0$$

where

$$(18)\qquad g(t)=\frac{E}{2D}$$

and all the quantities in these equations are evaluated at $r=a$.

Assuming that the components g_{ij} have been made continuous at S by the transformation (13) in V^2, it can be seen from equations (16), (17) and (18) that the continuity of $\frac{\partial g_{\alpha\beta}}{\partial x^4}$, $\left(\text{i. e. of } \frac{\partial B}{\partial r} \text{ and } \frac{\partial D}{\partial r}\right)$, at S implies the continuity of E^4_1 and E^4_4 (and therefore of T^4_1 and T^4_4) at S, verifying the previous remarks. However, it may also be seen that the converse is not true, i. e. the continuity of T^4_1 and T^4_4 at S does not necessarily imply the continuity of $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ $\left(\text{i. e. of } \frac{\partial B}{\partial r} \text{ and } \frac{\partial D}{\partial r}\right)$, at S.

This last remark may be demonstrated in the following way. The continuity of T^4_1 and T^4_4 at S yields equations (16) and (17) which involve $\frac{\partial B}{\partial r}$ and $\frac{\partial D}{\partial r}$. Since these equations depend non-linearly on $\frac{\partial B}{\partial r}$ and $\frac{\partial D}{\partial r}$ and, furthermore, involve $\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial r}\right)$, it is clear that, in general, unique solutions for $\frac{\partial B}{\partial r}$ and $\frac{\partial D}{\partial r}$ cannot be obtained from them.

Consequently, the continuity of T^4_i, $i=1\ldots 4$, at S are necessary conditions, but not sufficient to ensure that $\frac{\partial g_{\alpha\beta}}{\partial x^4}$, $\alpha, \beta = 1, 2, 3$ are continuous at S.

Now let $\frac{\partial B}{\partial r}$ at S be put equal to the corresponding value in V^2 calculated from (14), i. e.

$$2\,\eta_0\,\eta_1\approx\frac{\partial B}{\partial r}.$$

When substitution is made from (15) this equation gives

$$\left(1-\frac{2m}{B^{\frac{1}{2}}}\right)(E^2+4AD)B+A\left(\frac{\partial B}{\partial t}\right)^2+E\frac{\partial B}{\partial t}\frac{\partial B}{\partial r}\approx D\left(\frac{\partial B}{\partial r}\right)^2,$$

where all of these quantities are calculated at S. It can be seen that, if this condition is satisfied, then equation (17) determines the value of $\frac{\partial D}{\partial r}$ at S uniquely.

These results may be expressed in the following way:

THEOREM. — *Necessary and sufficient conditions for the metric* (10) *to join onto the vacuum field at a non-null surface,* S, *defined by equation* (11) *are:*

(19 *a*) $$T^4_4 \approx 0,$$

(19 *b*) $$\left(1 - \frac{2m}{B^{\frac{1}{2}}}\right)(E^2 + 4\,AD)\,B + A\left(\frac{\partial B}{\partial t}\right)^2 + E\,\frac{\partial B}{\partial t}\,\frac{\partial B}{\partial r} \approx D\left(\frac{\partial B}{\partial r}\right)^2.$$

For example, if the region V^1 is filled with a perfect fluid, then it is not only necessary that the fluid pressure be zero at S, as required by (19 *a*), but (19 *b*) must also be satisfied.

3. NULL SURFACE CONDITIONS

O'Brien and Synge [2] proposed that, at a null surface S, defined by equation (2), the components of the metric tensor, and the following combinations of the partial derivatives of the metric tensor should be continuous at S

(20) $$g^{\alpha\beta}\,\frac{\partial g_{\alpha\beta}}{\partial x^4}, \qquad g^{4\alpha}\,\frac{\partial g_{\alpha\beta}}{\partial x^4}$$

where $\alpha, \beta = 1, 2, 3$.

It is interesting to note that these conditions take a particularly simple form in terms of the following metric (introduced by Bondi *et al.* [6])

(21) $$ds^2 = (V\,r^{-1}\,e^{2\beta} - U^2\,r^2\,e^{2\gamma})\,du^2 + 2\,e^{2\beta}\,du\,dr + 2\,U\,r^2\,e^{2\gamma}\,du\,d\theta - r^2\,(e^{2\gamma}\,d\theta^2 + e^{-2\gamma}\sin^2\theta\,d\varphi^2)$$

where U, V, β and γ are functions of the co-ordinates u, r and θ. For this metric a null surface S is defined by $u = \text{constant}$.

The continuity of g_{ij} at S implies that the function U, V, β and γ are all continuous at S.

Identifying r, θ, φ and u with x^1, x^2, x^3 and x^4 respectively, the contravariant components of the metric tensor are given by

$$g^{ij} = \begin{pmatrix} V\,e^{-2\beta}\,r^{-1} & U\,e^{-2\beta} & 0 & e^{-2\beta} \\ U\,e^{-2\beta} & -\,e^{-2\gamma}\,r^{-2} & 0 & 0 \\ 0 & 0 & -\,e^{-2\gamma}\sin^{-2}\theta & 0 \\ e^{-2\beta} & 0 & 0 & 0 \end{pmatrix}$$

from which it can easily be seen that all the quantities (20) are identically zero.

Thus none of the first order derivatives with respect to u of U, V, β or γ need to be continuous at S in order to satisfy these O'Brien-Synge conditions. On the other hand, none of the field equations, when expressed in terms of the above metric (*see* [6]), involve any second order derivatives with respect to u, so the field equations reduce to first order differential equations with respect to u, for which the O'Brien-Synge conditions yield satisfactory boundary conditions. Of course, the solutions of these equations are not, in general, unique — indeed they give rise to the " news function ".

However, it should be noticed that, if all the partial derivatives $\frac{\partial g_{ij}}{\partial x^k}$ were to be made continuous at S, then the boundary conditions for the field equations would be over-prescribed — a situation which would always occur at null surfaces no matter what the form of the metric.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Dr. C. Gilbert for his guidance and encouragement during this work.

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(*Manuscrit reçu le* 12 *juillet* 1971)