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Section A :
Physique théorique.

A class of Lie and Jordan algebras realized by means of the canonical commutation relations

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ABSTRACT. — If one introduces the totally symmetrized monomials of the q and p as a basis in the Weyl algebra, which is the associative algebra generated by the canonical commutation relations, the polynomials of first and second degree can be given Lie and Jordan algebra structures which are isomorphic to well known matrix algebras. As an application the relation between formal real Jordan algebras, domains of positivity and symmetric spaces is used to give a classification of the second degree Hamiltonians, which is invariant under invertible linear transformations of the q and p, and has an influence on the representation theory of the solvable spectrum generating groups of these Hamiltonians described in [*1*] and [*2*]. Finally the relation of the Weyl algebra to the Clifford algebra over an orthogonal vector space is given, and the minimal embedding of an arbitrary Lie algebra into the Weyl algebra is discussed.

RÉSUMÉ. — Quand on introduit les monomes totalement symétrisés des q et p comme base de l'algèbre de Weyl, qui est l'algèbre associative générée par les relations canoniques de commutation, les polynômes de degré un et deux ont des structures d'algèbre de Lie et de Jordan qui sont isomorphes à des algèbres matricielles bien connues. Comme application, la relation entre les algèbres de Jordan formelles réelles, les domaines de positivité et les espaces symétriques est utilisée pour classifier les Hamiltoniens de degré deux. Cette classification est invariante par des transfor-

mations linéaires et inversibles des q et p. Elle a des conséquences pour la théorie des représentations des groupes solubles de ces Hamiltoniens qui sont décrits dans [1] et [2]. Finalement on donne la relation de l'algèbre de Weyl à l'algèbre de Clifford d'un espace vectoriel orthogonal et on discute l'injection minimale d'une algèbre de Lie arbitraire dans l'algèbre de Weyl.

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1. THE WEYL ALGEBRA

A real *symplectic vector space* is a pair (E, σ) of a finite dimensional vector space E over the field $\mathbb{R}$ and a non-degenerate, skew symmetric, bi-linear form σ on E. Necessarily we have $\dim(\mathrm{E}) = 2n$. Without loss of generality [1] we can choose the matrix of σ in the special form

$$\begin{pmatrix} 0 & id_n \\ -id_n & 0 \end{pmatrix} = \mathscr{I}.$$

Given the associative tensor algebra ten (E) over E, we denote by $\otimes$ its multiplication, by 1 its identity element, and by $(\sigma(x, y)1-(x\otimes y-y\otimes x))$ the two-sided ideal given by all elements $\mathrm{X}(\sigma(x, y)1-(x\otimes y-y\otimes x))\mathrm{Y}$ where $x, y \in \mathrm{E} \subset \mathrm{ten}\,(\mathrm{E})$ and $\mathrm{X}, \mathrm{Y} \subset \mathrm{ten}\,(\mathrm{E})$. Then the infinite dimensional associative *Weyl algebra* weyl (E, σ) is defined by

$$\text{(1)} \qquad \mathrm{weyl}\,(\mathrm{E}, \sigma) = \mathrm{ten}\,(\mathrm{E})/(\sigma(x, y)1 - (x \otimes y - y \otimes x)).$$

A basis of weyl (E, σ) is given by the identity element 1 and the totally symmetrized monomials of the basis elements $q_1, \ldots, q_n, p^1, \ldots, p^n$ of E. For the proofs of this statement and the following ones see [1]. Given $x_k \in \mathrm{E} \subset \mathrm{weyl}\,(\mathrm{E}, \sigma)$ we write

$$\text{(2)} \qquad \Lambda x_1 x_2 \ldots x_i = \frac{1}{i!} \sum_{\tau \in \gamma_i} x_{\tau(1)} x_{\tau(2)} \ldots x_{\tau(i)}$$

(γ_i denotes the permutation group of i objects). Let the symmetrization Λ be defined first only for the monomials and then on weyl (E, σ) by linear continuation. Then Λ is a mapping of weyl (E, σ) onto itself, which does not depend on the choice of the basis of E. For all choices

$$x_1, \ldots, x_i \in \mathrm{E} \subset \mathrm{weyl}\,(\mathrm{E}, \sigma)$$

the element $\Lambda x_1 x_2 \ldots x_i$ defined by (2) is an element of the $\binom{2n+i-1}{i}$ dimensional vector space $\Lambda\mathrm{W}_i$, spanned by the totally symmetrized monomials of degree i. We have the direct vector space sum decomposition

$$\text{weyl}(E, \sigma) = \bigoplus_{i \geq 0} \Lambda W_i \tag{3}$$

with $\Lambda W_0 = \mathbb{R}1$ and $\Lambda W_1 = E$. One proves that

$$[\Lambda W_i, \Lambda W_k]_- \subset \bigoplus_{l \geq 1} \Lambda W_{i+k-2l} \quad \text{for} \quad i, k = 0, 1, 2, \ldots \tag{4}$$

(here $[\ ,\ ]_-$ denotes the commutator in weyl (E, σ)), where the summation over l can be dropped only if i or k equals zero, one or two. We have the Lie algebras $\mathbb{R}1 \oplus E$, ΛW_2, and $\mathbb{R}1 \oplus E \oplus \Lambda W_2$ in weyl (E, σ). From (3) and (4) follows that the center of the Weyl algebra consists of the multiples of the identity element only.

In the form (1) the Weyl algebra was already considered by I. Segal [3].

2. THE LIE ALGEBRA OF SYMMETRIZED POLYNOMIALS OF SECOND DEGREE

The commutation relations of the Lie algebra $(\Lambda W_2, [\ ,\ ]_-)$ are summarized in

$$[xx, zz]_- = 4\sigma(x, z)\Lambda xz \qquad x, z \in E, \tag{5}$$

since by polarizing this twice (i. e. by substituting $x \mapsto x+y$ and $z \mapsto v+z$) we get the commutation relations of the $n(2n+1)$ basis elements $q_i q_k$, $p^i p^k$, $\Lambda q_i p^k$ of ΛW_2. From $\sigma(\text{ad}(\Lambda xy)v, z) + \sigma(v, \text{ad}(\Lambda xy)z) = 0$ follows that the $2n \times 2n$ matrices ad $(Z)|_E$ (ad restricted to E, $Z \in \Lambda W_2$) are in the symplectic matrix Lie algebra, the underlying vector space of which we denote by $\mathscr{L}$. For $Z \in \Lambda W_2$, the linear mapping $Z \mapsto \text{ad}(Z)|_E$ is a Lie algebra monomorphism. A dimensional argument then shows that this mapping is an isomorphism of the Lie algebras $(\Lambda W_2, [\ ,\ ]_-)$ and $(\mathscr{L}, [\ ,\ ]_-)$. So (5) are the commutation relations of a polynomial realization of the symplectic Lie algebra.

Given any $R \in \mathscr{L}$, the $2n \times 2n$ matrix $\mathscr{I}R$ is symmetric, and conversely given any symmetric $2n \times 2n$ matrix S we have $\mathscr{I}S \in \mathscr{L}$. So the linear mapping $R \mapsto \mathscr{I}R$ is a bijection of the vector space $\mathscr{L}$ onto the $n(2n+1)$ dimensional vector space $\mathscr{A}$ of symmetric $2n \times 2n$ matrices [4, p. 911]. The composition $S \top S' = S\mathscr{I}S' - S'\mathscr{I}S$ makes $\mathscr{A}$ a Lie algebra $(\mathscr{A}, \top)$ which, because of

$$\mathscr{I}S \top S' = [\mathscr{I}S, \mathscr{I}S']_-, \tag{6}$$

is isomorphic to the symplectic Lie algebra. We may summarize the relation between the various realizations of the symplectic Lie algebra by the following diagram

$$\begin{tikzcd}(\Lambda W_2, [\ ,\ ]_-) \arrow[rr, "\mathrm{ad}(\ )|_E"] \arrow[dr, "S"'] & & (\mathscr{L}, [\ ,\ ]_-) \arrow[dl] \\ & (\mathscr{A}, \top) & \end{tikzcd}$$

where the isomorphism S will be defined below.

3. THE JORDAN ALGEBRA OF SYMMETRIZED POLYNOMIALS OF SECOND DEGREE

The anti-commutator of two elements of ΛW_2 is not in ΛW_2 again. To make ΛW_2 a Jordan algebra (for the notion and description of Jordan algebras see [5]), we use the above diagram: the symmetric matrices form a Jordan algebra under anticommutation

$$\frac{1}{2}(AB + BA) = [A, B]_+.$$

The composition

$$R \perp R' = \frac{1}{2}(R\mathscr{I}R' + R'\mathscr{I}R)$$

makes the vector space $\mathscr{L}$ a Jordan algebra $(\mathscr{L}, \perp)$ too, which because of

(7) $$\mathscr{I}R \perp R' = [\mathscr{I}R, \mathscr{I}R']_+$$

is isomorphic to $(\mathscr{A}, [\ ,\ ]_+)$. Its identity element is $-\mathscr{I}$.

To get a Jordan algebra composition on ΛW_2, which is isomorphic to $(\mathscr{L}, \perp)$, we polarize $x \mapsto x + y$ and $z \mapsto v + z$ in

$$2\ \mathrm{ad}\,(xx)|_E \perp \mathrm{ad}\,(zz)|_E = \mathrm{ad}\,(4\sigma(x, \mathscr{I}z)\Lambda xz|_E \qquad x, z \in E.$$

Defining then on the generators of ΛW_2

(8) $$\frac{1}{2}\{\Lambda xy, \Lambda vz\}_+ = \sigma(x, \mathscr{I}v)\Lambda yz + \sigma(x, \mathscr{I}z)\Lambda yv + \sigma(y, \mathscr{I}v)\Lambda xz + \sigma(y, \mathscr{I}z)\Lambda xv,$$

and continuating linearly on ΛW_2, we get, because of

(9) $$\mathrm{ad}\,(\{Z, Z'\}_+)|_E = \mathrm{ad}\,(Z)|_E \perp \mathrm{ad}\,(Z')|_E \qquad Z, Z' \in \Lambda W_2,$$

the desired polynomial realization of the Jordan algebra of symmetric matrices. Its identity element is just the *Hamiltonian* of the *harmonic oscillator*

$$\mathrm{H}_0 = \frac{1}{2}\sum_{i=1}^{n}(q_i q_i + p^i p^i)$$

with ad $(\mathrm{H}_0)|_{\mathrm{E}} = -\mathscr{I}$. The relation (8) is the Jordan analog of the commutation relations [*1*; (65)] of the Lie algebra $(\Lambda \mathrm{W}_2, [\ ,\ ]_-)$. The above diagramm is valid for the Jordan algebras as well if one substitutes the Lie brackets by the corresponding Jordan compositions.

4. REALIZATION OF $gl(n, \mathbb{R})$ AND $u(n)$ IN $\Lambda \mathrm{W}_2$

Sometimes in physics a different notation is used [*8*] [*9*] [*10*]. To relate this one to ours we introduce the formal row $(q_1, \ldots, q_n, p^1, \ldots, p^n) = z^{\mathrm{T}}$, z being the corresponding column. Using then matrix multiplication, for every $\mathrm{Z} \in \Lambda \mathrm{W}_2$,

$$\frac{1}{2} z^{\mathrm{T}} \mathrm{S}(\mathrm{Z}) z = \mathrm{Z} \tag{10}$$

defines a $2n \times 2n$ matrix S(Z), which is symmetric and actually equal to $\mathscr{I}$ ad $(\mathrm{Z})|_{\mathrm{E}}$. The linear mapping $\mathscr{L} \to \Lambda \mathrm{W}_2$

$$\mathrm{ad}\,(\mathrm{Z})|_{\mathrm{E}} \to \frac{1}{2} z^{\mathrm{T}} \mathscr{I}\ \mathrm{ad}\,(\mathrm{Z}) z \tag{11}$$

gives the inverse Lie algebra isomorphism of the isomorphism ad $(\ )|_{\mathrm{E}}$ described in 2, and the linear mapping $\mathscr{A} \to \Lambda \mathrm{W}_2$

$$\mathrm{S}(\mathrm{Z}) \mapsto \frac{1}{2} z^{\mathrm{T}} \mathrm{S}(\mathrm{Z}) z \tag{12}$$

gives the inverse Jordan algebra isomorphism of that one described in 3.

The matrix Lie algebras $gl(n, \mathbb{R})$ resp. $u(n)$ are embedded into the symplectic Lie algebra matrices of $\mathscr{L}$ by

$$\begin{pmatrix} -\mathrm{G}^{\mathrm{T}} & 0 \\ 0 & \mathrm{G} \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} \mathrm{L} & -\mathrm{K} \\ \mathrm{K} & \mathrm{L} \end{pmatrix}; \tag{13}$$

here G is an arbitrary, L an skew symmetric, K a symmetric $n \times n$ matrix (i. e. $\mathrm{L} + i\mathrm{K}$ is skew hermitic). (11) defines isomorphisms of these algebras (and of their sub-Lie-algebras) into sub-vector-spaces of $\Lambda \mathrm{W}_2$, spanned by the polynomials $\Lambda q_i p^k$ resp. $q_i p^k - q_k p^i$ and $q_i q_k + p^i p^k$ [*8*] [*9*] [*10*].

For G skew symmetric or vanishing K in (13), (11) gives a mapping of the orthogonal matrix Lie algebra in n dimensions onto the vector space of the $q_i p^k - q_k p^i$. More general we get by (11) an isomorphism of any matrix sub-Lie-algebra from $\mathscr{L}$ onto some sub-Lie-algebra of ΛW_2. Analogous results hold for the Jordan algebras and the mapping (12).

5. CLASSIFICATION OF SECOND DEGREE HAMILTONIANS

For the following we need some facts on Jordan algebras. Let $[\ ,\ ]_+$ denote the product of a Jordan algebra J, and $L(a)b = [a, b]_+$ for $a, b \in J$, $P(a) = 2L(a)^2 - L(a^2)$. Suppose J is special, i. e. J can be embedded into an associative algebra with the product being the anti-commutator. Then $P(a)b = aba$. If J has an identity element e, the set of invertible elements of J (an element a of J is invertible iff det $P(a) \neq 0$, [5] [*11*] with the multiplication

$$a \cdot b = P(a)b^{-1} = ab^{-1}a$$

is a symmetric manifold [*11*, p. 68], i. e. a pair of a manifold Inv (J) and a composition Inv (J) × Inv (J) → Inv (J), denoted by a dot, fulfilling the following identities

(a) $a \cdot a = a$

(b) $a \cdot (a \cdot b) = b$

(c) $a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c)$

(d) every a has a neighbourhood U such that $a \cdot b = b$ implies $b = a$ for all b in U.

The tangential space in e of the « pointed » symmetric manifold (Inv (J), ., e) can be identified with J [*11*, p. 81].

A Jordan algebra is called *formal real* if the bi-linear form trace $L([a, b]_+)$ is positiv definit. For such a Jordan algebra the connectivity component of Inv (J) containing e is just the *domain of positivity* Pos (J), which by definition is exp (J) [*12*; p. 168]. The *structure group* of J Struc (J) is the group of all those invertible linear transformations W of J with

$$P(Wa) = WP(a)W^{\#}$$

where $W^{\#}$ is uniquely determined by W. For invertible a we have $P(a) \in$ Struc (J). The subgroup of Struc (J) leaving Pos (J) invariant is called Aut Pos (J), and the automorphism group of J is given by the set of all elements $A \in$ Struc (J) with $Ae = e$. We have the inclusions

$$\text{Aut (J)} \subset \text{Aut Pos (J)} \subset \text{Struc (J)}.$$

Let us now give the results for the special case of the formal real Jordan algebra $(\mathscr{A}, [\ ,\ ]_+)$ of symmetric $2n \times 2n$ matrices. Inv $(\mathscr{A})$ is given by all invertible symmetric matrices, Pos $(\mathscr{A})$ by all positiv definit ones. Every $S \in$ Pos $(\mathscr{A})$ can be decomposed into $S = Q^T Q$ for some $Q \in Gl(2n, \mathbb{R})$ and conversely we have $Q^T Q \in$ Pos $(\mathscr{A})$ for all $Q \in Gl(2n, \mathbb{R})$. This shows that Pos $(\mathscr{A})$ itself is a symmetric manifold under the dot product. We get the various other connectivity components of Inv $(\mathscr{A})$ by multiplying Pos $(\mathscr{A})$ by one of those 2^{2n} diagonal matrices I_i having only ± 1 on their diagonals. These matrices form a discrete symmetric space $C(2^{2n})$ under the dot product (the axiom (d) has to be dropped) and we have

$$\text{Inv}\,(\mathscr{A}) = \text{Pos}\,(\mathscr{A}) \otimes C(2^{2n}),$$

where $\otimes$ denotes a semidirect product of symmetric spaces, Pos $(\mathscr{A})$ being the ideal. If Nul $(\mathscr{A})$ denotes the set of zero divisors of $(\mathscr{A}, [\ ,\ ]_+)$ we have the decomposition

$$(14) \qquad \mathscr{A} = \text{Nul}\,(\mathscr{A}) \cup \text{Pos}\,(\mathscr{A}) \cup I_1\,\text{Pos}\,(\mathscr{A}) \cup \ldots \cup -\text{Pos}\,(\mathscr{A})$$

of $\mathscr{A}$.

The automorphism group of $\mathscr{A}$ is given by the set of transformations $\mathscr{A} \mapsto R^T \mathscr{A} R$ for orthogonal R, Aut Pos $(\mathscr{A})$ is given by the same transformations with $R \in Gl\,(2n, \mathbb{R})$, and Struc $(\mathscr{A})$ is given by all $W \in Gl\,(\mathscr{A})$ such that $\pm W \in$ Aut Pos $(\mathscr{A})$. In addition every $W \in$ Aut Pos $(\mathscr{A})$ has the special representation $P(R_1)P(R_2)P(S)$ with $S \in$ Pos $(\mathscr{A})$ and R_1, R_2 orthogonal, $R_1{}^2 = R_2{}^2 = id_{2n}$. Aut Pos $(\mathscr{A})$ acts transitively on Pos $(\mathscr{A})$ and leaves the decomposition (14) invariant.

Let us now transfer the decomposition (14) by means of the isomorphism (12) to ΛW_2. Given $M \in$ Struc (ΛW_2), we apply S to the defining relation of Struc (ΛW_2), and get because of $S \circ P(Z) = P(S(Z)) \circ S$ for all $Z \in \Lambda W_2$, with the definition

$$W(M)S(Z) = S(MZ)$$

a monomorphism $M \mapsto W(M)$, Struc $(\Lambda W_2) \to$ Struc $(\mathscr{A})$. The above special representation of Aut Pos $(\mathscr{A})$ (and therefore of Struc $(\mathscr{A})$ too) then shows that it is even surjectiv. This proves $S(MZ) = \pm W^T S(Z) W$ for all $M \in$ Struc (ΛW_2) and some $W \in Gl\,(2n, \mathbb{R})$. Applying (10) to this result we have

$$MZ = \pm \frac{1}{2} (Wz)^T S(Z) Wz$$

This establishes the invariance of the decomposition (14) for ΛW_2 under exactly the transformations of Aut Pos (ΛW_2), which are those induced by the invertible linear transformations of E.

The Hamiltonians of the non-relativistic free particle (which was treated in [*1*, § 10]) and the relativistic free particle (which was treated in [*2*]) are elements of Nul (ΛW_2), whereas the Hamiltonian of the harmonic oscillator is in Pos (ΛW_2). We expect that the representation theory of the spectrum generating solvable groups described in [*1*] and [*2*] differs according to which part of the decomposition (14) for ΛW_2 the Hamiltonian in question belongs to; for instance in the sense that the representations are labeled by a continuous variable if the Hamiltonian is in Nul (ΛW_2), and by a discrete one if it is in Inv (ΛW_2).

6. THE RELATION OF THE WEYL ALGEBRA TO THE CLIFFORD ALGEBRA

The vector space $E \oplus \mathbb{R}1$ can be made a Jordan algebra too: since $\sigma(\mathscr{I}x, y)$ is the Euclidean bi-linear form on E, the definition

$$\{ x, y \}_+ = [x, \mathrm{ad}\,(H_0)y]_- \quad , \quad \{ x, 1 \}_+ = x \tag{15}$$

gives the $2n + 1$ dimensional *Clifford Jordan algebra* [*12*; p. 171], the universal enveloping algebra of which is the *Clifford algebra*

$$\mathrm{ten}\,(E)/(x \otimes y + y \otimes x - 2\sigma(\mathscr{I}x, y)1)$$

over the Euclidean vector space E [*13*, p. 367] [*14*]. Most results of this article have their analog in the Clifford algebra, though its associative multiplication is not related to that one of weyl (E, σ); for instance the Clifford algebra has zero divisors contrary to the Weyl algebra.

7. MINIMAL REALIZATION OF LIE ALGEBRAS IN THE WEYL ALGEBRA

Since every finite dimensional Lie algebra has a faithful finite dimensional representation (theorem of Ado) we get by (13) and the mapping (11) an embedding of every Lie algebra into ΛW_2. But in general this embedding is not minimal. Take for instance the conformal Lie algebra in n dimensions. It is realized in weyl (E, σ) by the $\frac{1}{2}n^2 + \frac{3}{2}n + 1$ generators

$$p^i, \qquad \sum_{s=1}^{n} q_s p^s, \qquad q_i p^k - q_k p^i, \qquad \Lambda \sum_{s=1}^{n} (q_s q_s p^i - 2 q_i q_s p^s), \tag{16}$$

and isomorphic to some noncompact form of so $(n + 2, \mathbb{R})$. The smallest faithful representation of the conformal Lie algebra is therefore $n + 2$ dimensional. Using then (13) and (11) for the embedding into ΛW_2, we need a $2(n + 2)$ dimensional symplectic vector space. But (16) shows that already a $2n$ dimensional symplectic vector space suffices if we use $E \oplus \Lambda W_2 \oplus \Lambda W_3$. So, given an arbitrary real Lie algebra $\mathscr{L}$ we may state the problem in the following form: let $m(\mathscr{L})$ denote the minimal integer n with

(I) $n = \dim E$

(II) $\mathscr{L}$ is isomorphic to some sub-Lie-algebra of weyl (E, σ).

In this form the problem was stated first by M. Koecher [*6*, p. 363], see also [*7*], though for a different polynomial algebra. The example of the conformal Lie algebra shows that $m(\mathscr{L})$ can be smaller than the smallest dimension of a (symplectic) representation space.

Let us remark that there is a natural isomorphism of weyl (E, σ) into an infinite dimensional Lie algebra of transformations of E, such that exactly the elements of ΛW_2 are mapped onto linear transformations of E, whereas the other elements are mapped onto non-linear ones. Thus the above problem of minimalizing the dimension of E corresponds to the dropping of the linearity requirement of the transformations. We intend to come back to the question of non-linearity elsewhere.

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