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Section A :
Physique théorique.

Generic Feynman amplitudes

by

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ABSTRACT. — We introduce certain complex parameters into a Feynman amplitude of quantum field theory to define a new function called a *generic Feynman amplitude.* These amplitudes have the same singularities in the invariant variables as the usual amplitudes, but with different local behavior; we investigate this behavior for singularities associated with contracted graphs of a type we call *normal.* Some consequences of these results for the problem of characterizing the analytic behavior of the amplitudes are pointed out.

RÉSUMÉ. — Nous introduisons certains paramètres complexes dans une amplitude de Feynman de la théorie quantique des champs. La nouvelle fonction déterminée ainsi s'appelle une *amplitude de Feynman générique.* Ces amplitudes démontrent les mêmes singularités dans les variables invariantes que les amplitudes habituelles, mais le comportement local est différent. Nous examinons ce comportement pour le cas des singularités qui sont associées avec les graphes quotients d'un type que nous appelons *normale.* Quelques conséquences de ces résultats pour le problème de caractériser la structure analytique des amplitudes sont indiquées.

1. INTRODUCTION

This paper is a contribution to the study of Feynman amplitudes, with the general aims announced in the introductory section of [*1*].

In [*2*] T. Regge suggested that the qualitative behaviour of a Feynman

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amplitude, which is given by its monodromy group, could be computed from the structure of the fundamental group of its domain, and certain local conditions on the monodromy representation (obtained from the Picard-Lefqchetz theorem and its generalizations). He was able to support this conjecture with calculations for the self energy graphs with two and three lines. Since then Regge has obtained further evidence for the conjecture in calculations carried out in collaboration with G. Ponzano [3], with G. Ponzano and the present authors [1] [4], and with the present authors [5] (1). In Section 6 of this paper we give a discussion of the ways in which the conjecture may be precisely formulated for a general amplitude and compare these formulations with the procedure actually followed in the calculations mentioned above.

Our main concern in this paper will be with the local representation conditions. As in [1] [4] [5] we consider a generalization of the customary Feynman amplitudes. The generalized amplitudes we refer to as generic. They are defined in Section 2. This generalization is considered for two reasons: first, we are thereby able to avoid divergence difficulties; second, certain degeneracies which occur in the monodromy representation of a Feynman amplitude are removed when one passes to the corresponding generic amplitude (for example the occurrence of singularities with local behaviour of logarithmic type). It should be noted that we allow the possibility that the masses of some of the intermediate particles should be zero, despite the fact that it is in building a theory of strongly interacting particles that we hope our results will be useful. This is because the structure of the monodromy ring for an amplitude with some zero masses is embedded into the monodromy ring for the corresponding amplitude with non-zero masses (see Section 3 and 6). The device of introducing generic amplitudes enables us to avoid difficulties with infrared divergencies. The analytic behaviour of the customary amplitudes may be obtained by suitable specialization.

In Section 4 we introduce the concept of a normal graph Q-defined roughly as a graph which has a leading Landau variety, which for suitable values of the masses, appears as a singularity on the physical sheet of the amplitude for any graph G which admits Q as a quotient. We then prove a local decomposition theorem for the physical sheet of the amplitude for G into a sum of a non-singular part and a singular part with definite local power behaviour in the neighbourhood of such a singularity. The relation of this theorem to the Picard-Lefschetz theorem, and its implica-

(1) A summary of Regge's present point of view is given in [29].

tions for the local representation conditions are discussed in Section 6.

The formal idea which is used in the proof of the local decomposition theorem of Section 4, and the theorem itself, are not new. Derivations of the corresponding result for the customary (i. e. non-generic) Feynman amplitudes, of varying degrees of completeness, have been given previously by a number of authors [*6*] [*7*] [*2*], beginning essentially with the original paper of Landau [*8*]. What is new in our treatment is the careful discussion of the crucial condition, the nonvanishing of the Hessian.

Section 5 is devoted to a discussion of the concept of normality, and to a proof of certain combinatorial criteria which suffice to decide that a large number of graphs are normal.

2. THE GENERIC INTEGRAL

2.1. Graph theoretical preliminaries

In this section we establish the terminology we will use in discussing Feynman graphs. We also give some simple results about the Symanzik polynomials associated with a graph; these are applied in Section 4.3.

DEFINITION 2.1.1 : A *graph* G consists of a set of lines $\{ l_j \mid j \in \Omega \}$ and a set of vertices $\{ V_k \mid k \in \Theta \}$, together with a mapping

$$i = i_1 \times i_2 : \Omega \to \Theta \times \Theta.$$

For $j \in \Omega$, $i_1(j)$ and $i_2(j)$ are called the *initial* and *final* vertices respectively, of the line l_j; collectively they are called the end points of l_j. We write $N = |\Omega|$, $n = |\Theta|$. A *tadpole* is a line l_j of G such that $i_1(j) = i_2(j)$; a *multiplet* $\{ l_j \mid j \in \chi \subset \Omega, |\chi| \geq 2 \}$ is a set of lines of G such that

$$\{ i_1(j), i_2(j) \} = \{ i_1(k), i_2(k) \},$$

for all $j, k \in \chi$. The *star* of a vertex V_k, $k \in \Theta$, is defined by

$$\text{st } k = \{ j \in \Omega \mid k \text{ is an endpoint of } l_j \}.$$

The graph G has an obvious topological realization (say in $\mathbb{R}^3$); we let c denote the number of connected components in such a realization. Finally, we let $h = N - n + c$ denote the number of loops in the graph. When several graphs are under consideration, we distinguish quantities related to G by writing Ω_G, Θ_G, and $N(G)$, $h(G)$, etc. (similarly for other quantities defined in this section).

DEFINITION 2.1.2: A *Feynman graph* is a graph G as in Def. 2.1.1 together with partitions $\Omega = \Omega^0 \cup \Omega^M$, $\Theta = \Theta^E \cup \Theta^I$. A line l_j, $j \in \Omega^0(\Omega^M)$ is called a *massless (massive)* line; a vertex V_k, $k \in \Theta^E(\Theta^I)$ is called an external (internal) vertex. We write $n^E = |\Theta^E|$, etc. (We usually refer to this Feynman graph simply as G). G is *massive* if $\Omega^M = \Omega$; G is *externally complete* if $\Theta^E = \Theta$. If G is a connected Feynman graph, we define G^∞ to be the graph (in the sense of Def. 2.1.1) with

$$\Omega_{G^\infty} = \Omega \cup \Theta^E, \ \Theta_{G^\infty} = \Theta \cup \{\infty\},$$

and

$$i_{G^\infty}(j) = \begin{cases} i(j) & \text{if} \quad j \in \Omega \\ (\infty, j) & \text{if} \quad j \in \Theta^E, \end{cases}$$

that is, G^∞ consists of G with one additional vertex V_∞, which is joined to each external vertex of G.

Note that a Feynman graph, for us, is oriented. This is for convenience; the Feynman amplitude is independent of this orientation. Note too that we do not attach external lines to the external vertices of a Feynman graph (except in the graph G^∞).

DEFINITION 2.1.3: Let G be a graph. A *subgraph* H of G is a graph as in Def. 2.1.3 such that $\Omega_H \subset \Omega_G$, $\Theta_H \subset \Theta_G$, and $i_H = i_G|_{\Omega_H}$ (note this implies $[i_{G,1}(\Omega_H) \cup i_{G,2}(\Omega_H)] \subset \Theta_H$). H is a *full subgraph* if

$$\Omega_H = \{ j \in \Omega_G \mid i_G(j) \subset \Theta_H \times \Theta_H \}.$$

If $\chi \subset \Omega_G$, we write $H(\chi)$ for the minimal subgraph of G with $\chi \subset \Omega_{H(\chi)}$. If G is a Feynman graph, a subgraph H of G is considered to be a Feynman graph with

$$\begin{aligned} \Omega_H^0 &= \Omega_G^0 \cap \Omega_H, \\ \Omega_H^M &= \Omega_G^M \cap \Omega_H, \\ \Theta_H^E &= [i_1(\Omega_G - \Omega_H) \cup i_2(\Omega_G - \Omega_H) \cup \Theta^E] \cap \Theta_H, \\ \Theta_H^I &= \Theta_H - \Theta_H^E, \end{aligned}$$

that is, $V_k \in \Theta_H$ is external if it is external in G or if it is an endpoint of some line of G not contained in H.

DEFINITION 2.1.4: Let H be a subgraph of G, with components $H_1, \ldots, H_{c(H)}$. Define $\pi_Q : \Theta_G \to 2^{\Theta^G}$ by

$$\pi_Q(k) = \begin{cases} \{k\}, & \text{if} \quad k \notin \Theta_H, \\ \Theta_{H_m}, & \text{if} \quad k \in \Theta_{H_m},\ 1 \le m \le c(H). \end{cases}$$

Define the *quotient graph* $Q = G/H$ by

$$\Omega_Q = \Omega_G - \Omega_H,$$
$$\Theta_Q = \pi_Q(\Theta_G)\ (\subset 2^{\Theta_G}),$$
$$i_Q(j) = [\pi_Q \times \pi_Q]\,[i_G(j)]\ \ (j \in \Omega_Q).$$

If G is a Feynman graph, so is Q, and we take

$$\Omega_Q^0 = \Omega_G^0 \cap \Omega_Q, \qquad \Omega_Q^M = \Omega_G^M \cap \Omega_Q;$$
$$\Theta_Q^E = \pi_Q(\Theta_G^E), \qquad \Theta_Q^I = \Theta_Q - \Theta_Q^E.$$

DEFINITION 2.1.5: Let G be any graph. G is k-connected 9, p. 205], where k is a positive integer, if for any subset $\psi \subset \Theta_G$ with $|\psi| < k$, the (unique) maximal subgraph H of G with $\Theta_H = \Theta_G - \psi$ is connected. A *path* γ of G is a sequence $l_{\gamma(1)}, \ldots, l_{\gamma(r)}$ of lines of G such that $l_{\gamma(i)}$ is not a tadpole and $\gamma(i) \neq \gamma(j)$, $1 \leq i \neq j \leq r$, $l_{\gamma(i)}$ and $l_{\gamma(i+1)}$ have precisely one common endpoint $W_i (1 \leq i < r)$, and, if $W_0 \neq W_1$ and $W_r \neq W_{r-1}$ are endpoints of $l_{\gamma(1)}$ and $l_{\gamma(r)}$, respectively, the vertices $W_0, W_1, \ldots, W_r$ are all distinct. γ is then called a path from W_0 to W_r, which passes through $W_0, \ldots, W_r$.

LEMMA 2.1.6: If G is k-connected, and $i, i' \in \Theta_G$, there exist k paths in G from V_i to V_i', such that no two paths pass through a common vertex other than V_i, V_i'.

Proof: See [9, p. 205].

DEFINITION 2.1.7: Let G be connected. A subset $T_r \subset \Omega_G$ is an *r-tree* in G if $h[H(T_r)] = 0$, $c[H(T_r)] = r$. A *co-r-tree* is the complement (in Ω) of an r-tree; a *tree* is a 1-tree (similarly for a co-tree). Note that this is not the standard definition of a tree since, for us, every vertex of G is the endpoint of some line in a tree.

We now construct the Symanzik polynomials associated with a connected graph G.

DEFINITION 2.1.8: Let $\alpha = \{\alpha_j \mid j \in \Omega\}$ be a point in $\mathbb{C}^{N(G)}$, and write

$$\alpha(\chi) = \prod_{j \in \chi} \alpha_j,$$

for any $\chi \subset \Omega$. Suppose $\psi_1, \ldots, \psi_r \subset \Theta$. Then

$$d_r(\psi_1 \mid \ldots \mid \psi_r)(\alpha) = \sum_{T_r} \alpha(\Omega - T_r)$$

where the sum is taken over all r-trees of G such that, for

$$i, i' \in \bigcup_{j=1}^{r} \psi_j$$

there is a path in $H(T_r)$ from V_i to $V_{i'}$ iff $i, i' \in \psi_j$ for some j, $1 \leq j \leq r$.

Note that $d_r(\psi_1 \mid \ldots \mid \psi_r) = 0$ if $\psi_i \cap \psi_j \neq \emptyset$, for some $i \neq j$. $d_1(\psi)$ is independent of ψ, so we write

$$d(\alpha) = d_1(\psi)(\alpha), \qquad \forall\, \psi \subset \Theta.$$

If $\psi_j = \{ i_{j1}, \ldots, i_{jk(j)} \}$. We abuse notation and write

$$d(\psi_1 \mid \ldots \mid \psi_r) = D(i_{11} \ldots, i_{1\,k(1)} \mid \ldots \mid i_{r1} \ldots i_{r\,k(r)}).$$

We now wish to relate the Symanzik polynomials to the minors of a certain matrix associated with G. The *incidence matrix* of G is defined by

$$e_{jk} = \delta_{i_2(j),k} - \delta_{i_1(j),k} \qquad (j \in \Omega,\ k \in \Theta). \tag{2.1.9}$$

Taking $\alpha \in \mathbb{C}^{N(G)}$ as above, we define the symmetric matrix

$$A_{kl}(\alpha) = \sum_{j \in \Omega} e_{jk} \alpha_l^{-1} e_{jl} \qquad (k, l \in \Theta). \tag{2.1.10}$$

The (signed) minors of A are denoted $A\binom{i}{j}$, $A\binom{i\,k}{j\,l}$, etc; by convention,

$$A\binom{i_1 \ldots i_r}{j_1 \ldots j_r} = 0 \qquad \text{if} \qquad i_k = i_l \qquad \text{or} \qquad j_k = j_l,$$

some $1 \leq k < l \leq r$.

LEMMA 2.1.11 : For any $i_1, \ldots, i_r, j_1, \ldots, j_r \in \Theta$,

$$\alpha(\Omega) A\binom{i_1 \ldots i_r}{j_1 \ldots j_r} = \sum_{\pi} \sigma(\pi) d_r(i, j_{\pi(1)} \mid \ldots \mid i_r j_{\pi(r)}), \tag{2.1.11}$$

the sum running over all permutations π of $\{ 1, \ldots, r \}$, with $\sigma(\pi)$ the sign of the permutation.

Proof: We give only a brief sketch of the proof. Since both sides of (2.1.11) are antisymmetric in $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$, we may assume $i_1 < \ldots < i_r, j_1 < \ldots < j_r$. Let $\chi \subset \Omega$ be any set of $h(G) + r - 1$ lines of G; and let $e\left[{}^{\ \chi}_{i_1 \ldots i_r}\right]$ denote the (unsigned) minor of e:

$$e\left[{}^{\ \chi}_{i_1 \ldots i_r}\right] = \det \{ e_{jk} \mid j \notin \chi,\ k \neq i_1 \ldots i_r \}.$$

We observe that

$$e\left[{}^{\ \chi}_{i_1\ldots i_r}\right] = \begin{cases} \pm 1 & \text{if} \quad \chi = \Omega - \mathrm{T}_r \\ 0 & \text{otherwise} \end{cases}$$

where T_r is some r-tree such that there is no path in $\mathrm{H}(\mathrm{T}_r)$ joining V_{i_j} to V_{i_k}, $1 \leq j < k \leq r$. This is proved in [10, Appendix A] for $r = 1$, and is immediately extended by applying the $r=1$ result to the graph G′ obtained from G by identifying the vertices $\mathrm{V}_{i_1}, \ldots, \mathrm{V}_{i_r}$. Now suppose $e\left[{}^{\Omega-\mathrm{T}_r}_{i_1\ldots i_r}\right] \neq 0$ and $e\left[{}^{\Omega-\mathrm{T}_r}_{j_1\ldots j_r}\right] \neq 0$; then it is easily verified that

$$e\left[{}^{\Omega-\mathrm{T}_r}_{i_1\ldots i_r}\right] = (-)^{\sum_{k=1}^{r}(i_k+j_k)} \sigma(\pi) e\left[{}^{\Omega-\mathrm{T}_r}_{j_1\ldots j_r}\right],$$

where π is the permutation of $\{1, \ldots, r\}$ such that there is a path in $\mathrm{H}(\mathrm{T}_r)$ from i_k to $j_{\pi(k)}$. Then we write $\mathrm{A} = \mathrm{B}^{\mathrm{T}}\mathrm{C}$,

where

$$\mathrm{B}_{ji} = e_{ji}$$

$$\mathrm{C}_{ji} = \alpha_j^{-1} e_{ji}$$

and apply the Cauchy-Binet theorem to calculate $\mathrm{A}\binom{i_1 \ldots i_r}{j_1 \ldots j_r}$ (see [10, Lemma A.9]).

2.2. **Definition of generic Feynman amplitudes**

Let G be a Feynman graph as in Section 2.1. Let m be a positive integer. Denote by $\mathbb{C}^m$ the vector space of dimension m over the field $\mathbb{C}$ of complex numbers and suppose that a scalar product a, b is given on $\mathbb{C}^m$ whose associated quadratic form $a, a = a^2$, is positive definite when restricted to the real subspace $\mathbb{R}^m$ of $\mathbb{C}^m$. If G is connected define

$$\mathrm{X}_{\mathrm{G}}^m = \left\{ p_j j \in \Theta^{\mathrm{E}} \mid \sum_{j\in\Theta^{\mathrm{E}}} p_j = 0 \right\} \subset \mathbb{C}^{mn^{\mathrm{E}}} \qquad (2.2.1)$$

We refer to $\mathrm{X}_{\mathrm{G}}^m$ as the space of (external) momentum vectors for the graph G in a space time of dimension m. If G is not connected define

$$\mathrm{X}_{\mathrm{G}}^m = \prod_{s=1}^{c} \mathrm{X}_{\mathrm{G}_s}^m \qquad (2.2.2)$$

the product being taken over the connected components G_s of G.

Suppose G connected. From the momentum vectors p_i, $i \in \Theta^E$, we may form invariants

$$s(\chi) = \left(\sum_{i \in \chi} p_i \right)^2 \tag{2.2.3}$$

χ a non-empty proper subset of Θ^E. These invariants satisfy the linear identities

$$s(\chi) = s(\chi') \qquad (\chi' = \Theta^E - \chi) \tag{2.2.4}$$

$$s(\chi_1 \cup \chi_2 \cup \chi_3) - s(\chi_2 \cup \chi_3) - s(\chi_3 \cup \chi_1) - s(\chi_1 \cup \chi_2) + s(\chi_1) + s(\chi_2) + s(\chi_3) = 0$$

In (2.2.4) χ_1, χ_2, χ_3 are non-empty proper disjoint subsets of Θ^E. If $m \geq (n^E - 1)$, the relations (2.2.4) are all the relations which the $s(\chi)$ must satisfy by virtue of their definition (2.2.3). Note that the relations (2.2.4) are independent of m. This motivates us to define the space of external invariants S_G of G to be the space of complex variables $s(\chi)$, labelled by non-empty proper subsets of Θ^E, which satisfy (2.2.4). Note that (2.2.4) may be used to express any $s(\chi)$ in terms of the invariants

$$s_i = s(\{ i \}), \ s_{jk} = s(\{ j, k \}) \ i, j, k \in \Theta^E - \{ i_0 \},$$

where i_0 is some fixed index in Θ^E, and that these invariants are linearly independent.

Before we can define the generic amplitude F_G for G, we must introduce two further sets of variables associated with the lines of G. For each massive line of G we introduce a complex variable z_i, called the squared mass for that line, and denote by $Z_G \simeq \mathbb{C}^{N^M}$ the space of these variables. Next for each line of G we introduce a complex variable λ_i. Finally we introduce a complex variable ν, called the space-time dimension, whose relation to the m of the preceding paragraph will be explained presently. The variables λ_i, ν will be referred to as parameters and the space of these variables denoted by Λ_G. We define $W_G = S_G \times Z_G$, the space of invariants for G, and $T_G = W_G \times \Lambda_G$. T_G will be the domain of the generic amplitude F_G.

Formally F_G is defined by the integral representation

$$F_G(s, z ; \lambda, \nu) = f_G(\lambda, \nu) \int_{\gamma_0} \frac{\Pi \alpha_i^{\lambda_i} [d(\alpha)]^{\lambda_0} \overset{(N-1)}{\eta}}{[D(\alpha, s, z)]^{\mu}} . \tag{2.2.5}$$

In (2.2.5) the integration space is the (N − 1) dimensional projective space $\mathbb{P}^{N-1}$. $(\alpha_i, i \in \Omega_G)$ are homogeneous coordinates in this space and the integration contour γ_0 is the simplex

$$\gamma_0 = \{ (\alpha) \mid \alpha_i \geq 0 \; \forall i \}. \tag{2.2.6}$$

$\overset{(N-1)}{\eta}$ is the fundamental projective differential form of degree (N − 1) [11]. $d(\alpha)$ is the Symanzik polynomial defined in Section 2.1 and $D(\alpha, s, z)$ is given by

$$D(\alpha, s, z) = D^s(\alpha, s) - D^z(\alpha, z) \tag{2.2.7}$$

$$D^s(\alpha, s) = \frac{1}{2} \sum_{\chi \subset \Theta^E} s(\chi) d_2(\chi \mid \chi') \tag{2.2.8}$$

$$D^z(\alpha, z) = \left(\sum_{i \in \Omega^M} \alpha_i z_i \right) d(\alpha) \tag{2.2.9}$$

The exponents λ_0 and μ are given by

$$\lambda_0 = \mu - \frac{\nu}{2} \tag{2.2.10}$$

$$\mu = \frac{-h(G)\nu}{2} + \sum_{i=1}^{N} \lambda_i + N. \tag{2.2.11}$$

The normalization factor $f_G(\lambda, \nu)$, which is introduced for future convenience, is given by the following product of gamma-functions

$$f_G(\lambda, \nu) = \frac{\Gamma(\mu)}{\prod_{j \in \Omega} \{ \exp(-\pi i \nu_j) \Gamma(\nu_j) \}} \tag{2.2.12}$$

where

$$\nu_j = -\lambda_j + \frac{\nu}{2} - 1. \tag{2.2.13}$$

In (2.2.5) and elsewhere in this paper a^b, $a, b \in \mathbb{C}$, $a \neq 0$, will stand for $\exp[b \log {}^+a]$. $\log {}^+a$, the principal value of $\log a$, is defined by

$$\log {}^+a = \log |a| + i \arg {}^+a$$

$\arg {}^+a$ being the value of $\arg a$ in the interval $[-\pi, +\pi]$. Note that (2.2.5) is not our final definition because the domain in T_G in which the integral is convergent may be empty.

For $\nu = m$ a positive integer (2.2.5) reduces to the parametric representation of the Feynman amplitude for the graph G obtained by taking momenta p_i from a space of dimension m with invariants given by (2.2.3) and by taking as the propagator for the line l_j

$$(q_j^2 + z_j)^{-\lambda_j - 1} \tag{2.2.14}$$

and otherwise following the Feynman rules. Thus the generic amplitude F_G differs from the amplitude introduced in [10] in order to define the method of analytic renormalization only in the replacement of the space-time dimension m by the complex variable ν. The motivation for the introduction of a complex ν, which was suggested to us by T. Regge, will be given in Section 6.

In the definition (2.2.5) G is supposed connected. If G is not connected we write the domain T_G for F_G as a product

$$T_G = \prod_{s=1}^{c} T_{G_s}$$

in the obvious way and define

$$F_G = \prod_{s=1}^{c} F_{G_s}. \tag{2.2.15}$$

In order to understand the role of the various variables in (2.2.5) the mathematical reader is advised to compare the equation with the standard integral representation for the hypergeometric function

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \tag{2.2.16}$$

cf. [12] or [13]. (The discussion of the monodromy representation of $F(a, b, c, z)$ to be found in these references is also a useful background for the corresponding discussion for F_G which we give in Section 6).

Define the Symanzik region $R_G \subset W_G$ by

$$R_G = \{ (s, z) \mid -z_i > 0 \ \forall i \in \Omega^M, s(\chi) > 0 \ \forall \text{ nonempty proper subsets } \chi \text{ of } \Theta^E \}. \tag{2.2.17}$$

We will rewrite (2.2.5) as a sum of terms each of which is well defined for $(s, z) \in R_G$ and for (λ, ν) in a certain region of Λ_G. The (λ, ν) regions for different terms may have empty intersection. However, each term can be analytically continued in (λ, ν) to give a function meromorphic in (λ, ν). The sum of these functions will then be taken to define $F_G(s, z; \lambda, \nu)$.

To obtain the splitting of (2.2.5) we proceed as in the theory of renormalization ([10] [14]) and define for each $1-1$ mapping ρ of $\{1, \ldots, N\}$ onto Ω a sector

$$\mathbb{P}_\rho^{N-1} = \{(\alpha_1, \ldots, \alpha_N) \mid 0 \leq \alpha_{\rho(1)} \leq \alpha_{\rho(2)} \leq \ldots \leq \alpha_{\rho(N)}\}. \quad (2.2.18)$$

In $\mathbb{P}_\rho^{N-1}$ we may normalize the homogeneous coordinates α_i by setting $\alpha_{\rho(N)} = 1$, and we may introduce new variables

$$x_i = \alpha_{\rho(i)}\alpha_{\rho(i+1)}^{-1} \qquad 1 \leq i \leq N-1 \quad (2.2.19)$$

$$\alpha_{\rho(i)} = \prod_{i \leq j \leq N-1} x_j$$

LEMMA 2.2.20: The integral

$$f_G(\lambda, v) \int_{\mathbb{P}_\rho^{N-1}} \left(\prod_{i=1}^{N} \alpha_i^{\lambda_i} \right) \frac{[d(\alpha)]^{\lambda_0} \overset{(N-1)}{\eta}}{[D(\alpha, s, z)]^{\mu}} \quad (2.2.21)$$

may be rewritten in the form

$$f_G(\lambda, v) \int_{\delta} \left(\prod_{i=1}^{N-1} x_i^{\tau_i} \right) g(x, s, z, \lambda, v) dx_1 \ldots dx_{N-1} \quad (2.2.22)$$

where $\delta = \{(x) \mid 0 \leq x_i \leq 1,\ 1 \leq i \leq N-1\}$ and $g(x, s, z, \lambda, v)$ is a C^∞ function of x in δ which is holomorphic in s, z for $(s, z) \in R_G$ and entire in λ, v. (The exponents τ_i are functions of λ, v which will be given explicitly in the course of the proof). The integral, convergent for $\tau_i > -1$, defines a function which is analytically continuable to a function $F_G^\rho(s, z; \lambda, v)$, holomorphic in $(s, z) \in R_G$ and meromorphic in λ, v.

Proof: For $1 \leq i \leq N$, define $G_\rho^i \subset G$ to be the subgraph

$$H(\{\rho(1), \ldots, \rho(i)\})$$

of G (cf. (2.1.3)) and define inductively a tree $T_\rho^i \subset G_\rho^i$ by

$$\begin{aligned} T_\rho^1 &= \{\rho(1)\} \\ T_\rho^{i+1} &= T_\rho^i && \text{if} \quad h(G_\rho^{i+1}) = h(G_\rho^i) + 1 \\ &= T_\rho^i \cup \{\rho(i+1)\} && \text{if} \quad h(G_\rho^{i+1}) = h(G_\rho^i) \end{aligned}$$

Define also integers i_ρ, j_ρ, k_ρ

$$\begin{aligned} i_\rho &= \max\{i \mid \rho(i) \in \Omega^M\} \\ j_\rho &= \max\{j \mid \text{not every pair of external vertices may be joined by a path in } G_\rho^{j-1}\} \\ k_\rho &= \max\{i_\rho, j_\rho\} \end{aligned}$$

and the variable

$$\begin{array}{lll} \zeta_\rho = -\, z_{\rho(i_\rho)} & \text{if} & i_\rho > j_\rho \\ \zeta_\rho = s(\chi_\rho) & \text{if} & i_\rho < j_\rho \\ \zeta_\rho = s(\chi_\rho) - z_{\rho(i_\rho)} & \text{if} & i_\rho = j_\rho \end{array}$$

where χ_ρ is the set of external vertices lying in (either) one of the two components of $T^N_\rho - l_{j_\rho}$.

We now assert that for $\alpha \in \mathbb{P}^{N-1}_\rho$ and $x(\alpha)$ given by (2.2.19)

$$d(\alpha(x)) = \prod_{i=1}^{N-1} x_i^{h(G^i_\rho)} \{ 1 + \sigma_1(x) \} \tag{2.2.23}$$

$$D(\alpha(x), s, z) = \prod_{1 \le i < k_\rho} x_i^{h(G^i_\rho)} \prod_{k_\rho \le i \le (N-1)} x_i^{h(G^i_\rho)+1} \{ \zeta_\rho + \sigma_2(x, s, z) \} \tag{2.2.24}$$

where $\sigma_1(x)$, $\sigma_2(x, s, z)$ are sums of monomials in the x_i with positive coefficients for $(s, z) \in R^G$.

Proof : Let T be any tree in G. Then for each i, $1 \le i \le N - 1$ the corresponding cotree $T' = \Omega - T$ intersects G^i_ρ in at least $h(G^i_\rho)$ lines (otherwise we should have $h(T \cap G^i_\rho) \neq 0$). The cotree product $\alpha(T')$, when expressed as a monomial in the x_i, is therefore divisible by

$$\prod_{i=1}^{N-1} x_i^{h(G^i_\rho)}.$$

T' intersects G^i_ρ in precisely $h(G^i_\rho)$ lines for each i, $1 \le i \le N - 1$, iff $T = T^N_\rho$. This completes the proof of (2.2.23).

$$D(\alpha(x), s, z) = D^s(\alpha(x), s) + D^z(\alpha(x), z)$$

Each term in D^s is defined by a 2-tree T_2 each of whose components has a non-empty set of external vertices (2.2.8) and (2.1.7). The corresponding co-2-tree $T'_2 = \Omega - T_2$ intersects G^i_ρ, for each i, $1 \le i \le N - 1$, in at least $h(G^i_\rho)$ lines since $h(T_2 \cap G^i_\rho) = 0$. Further for $i \ge j_\rho$ T'_2 intersects G^i_ρ in at least $h(G^i_\rho) + 1$ lines since the external vertices all lie in the same component of G^i_ρ but are separated in $G^i_\rho \cap T_2$. The co-2-tree product $\alpha(T'_2)$, when expressed as a monomial in the x_i, is therefore divisible by

$$\prod_{1 \le i < j_\rho} x_i^{h(G^i_\rho)} \prod_{j_\rho \le i \le N-1} x_i^{h(G^i_\rho)+1}.$$

It is equal to this product iff $T_2 = T_\rho^N - l_j$. Each term in D^z is by (2.2.9) and (2.2.23) evidently divisible by

$$\prod_{1 \le i < i_\rho} x_i^{h(G_\rho^i)} \prod_{i_\rho \le i \le N-1} x_i^{h(G_\rho^i)+1}$$

and the quotient is independent of x iff the term is $\alpha_{i_\rho} z_{i_\rho} \alpha(T_\rho^{N'})$. By comparing the dominant therms in D^s and D^z we evidently obtain (2.2.24).

The transformed integral (2.2.22) is obtained directly from (2.2.23), (2.2.24) and the transformation equations (2.2.19). Explicitly

$$\tau_i = -1 + \sum_{j \in G_\rho^i} (\lambda_j + 1) - \frac{v}{2} h(G_\rho^i) \qquad (1 \le i < k_\rho) \qquad (2.2.25)$$

$$\tau_i = -1 + \sum_{j \in G_\rho^i} (\lambda_j + 1) - \frac{v}{2} h(G_\rho^i) - \mu \quad (k_\rho \le i \le N-1) \quad (2.2.26)$$

$$g(x, s, z, \lambda, v) = (1 + \sigma_1(x))^{\lambda_0} (\zeta_\rho + \sigma_2(x, s, z))^{-\mu} \qquad (2.2.27)$$

The final statement of the lemma is immediate from (2.2.22), and the result of Gel'fand-Šilov [*15*] that x^τ as a distribution is a meromorphic function of τ, with simple poles for τ a negative integer.

We are now ready to give a precise definition of the generic Feynman amplitude F_G.

DEFINITION 2.2.28 : For $(s, z) \in R_G$ and $(\lambda, v) \in \Lambda_G$ the *physical sheet* of F_G is the single valued function, holomorphic in s and z, meromorphic in λ, v defined by

$$F_G^{\text{phys}} (s, z ; \lambda, z) = \sum_\rho F_G^\rho(s, z ; \lambda, v) \qquad (2.2.29)$$

the summation being over the $1 - 1$ mappings ρ of $\{ 1, \ldots, N \}$ onto Ω. The generic *Feynman amplitude* F_G is the multi-valued analytic function defined on $T_G = W_G \times \Lambda_G$ by analytic continuation of F_G^{phys} .

Remark 2.2.30 : The definition 2.2.28 is not convenient for the investigation of the analytic continuation of F_G in (s, z). For generic λ, v it is possible to obtain a representation for F_G as an integral over a closed contour, to which the methods of [*16*] are applicable. From this representation it follows that the singularity set of F_G is an algebraic variety $L_G(s, z)$ (which does not depend on λ, v) together with the poles in λ, v noted already.

DEFINITION 2.2.31 : L_G is the *Landau variety* for G.

Remark 2.2.32 : The information about the poles of F_G in λ, ν given by (2.2.29) and the formulae (2.2.25), (2.2.26) is not the best possible i. e. Not every pole of one of the distributions x^τ used in the proof of Lemma (2.2.20) is a pole of F_G. The problem of obtaining the precise set of poles in λ, ν of F_G is essentially the problem of investigating the conditions under which a Feynman integral has ultraviolet or infrared divergences (recall that we allow zero masses for any or all of the lines of G). For the ultraviolet divergences the reader may consult [*10*], where the minimal poles are obtained in the case of a massive graph and for the infrared divergences [*17*]. The following lemmas will be of use to us.

LEMMA 2.2.33 : Let G be a connected Feynman graph, $\chi \subset \Omega$ a set of lines, non one of which is a tadpole, such that $h(H(\Omega - \chi)) = h(G) - |\chi|$. Let $\pi : \{1, \ldots, N - |\chi|\} \to \Omega - \chi$ be $1 - 1$. A $1 - 1$ mapping

$$\rho : \{1, \ldots, N\} \to \Omega$$

is called compatible with π if

$$\rho^{-1}(i) < \rho^{-1}(j) \leftrightarrow \pi^{-1}(i) < \pi^{-1}(j), \qquad \forall i, j \in \Omega - \chi.$$

Then there is a region $\Lambda_\pi \subset \Lambda_G$ such that for $(s, z) \in R_G$, $(\lambda, \nu) \in \Lambda_\pi$, and ρ compatible with π, the integral of the form (2.2.22) which defines $F_G^\rho(s, z ; \lambda, \nu)$ is absolutely convergent.

Proof : The integral (2.2.22) is absolutely convergent iff the complex numbers $\tau_i (= \tau_i^\rho)$ satisfy $\operatorname{Re} \tau_i^\rho > -1$, $1 \leq i \leq N - 1$. According to (2.2.25), (2.2.26) these conditions are explicitly

$$\sum_{j \in G_\rho^i} (\operatorname{Re} \lambda_j + 1) - \frac{\operatorname{Re} \nu}{2} h(G_\rho^i) > 0, \ 1 \leq i < k_\rho, \tag{2.2.34}$$

$$-\sum_{j \notin G_\rho^i} (\operatorname{Re} \lambda_j + 1) + \frac{\operatorname{Re} \nu}{2} (h(G) - h(G_\rho^i)) > 0, \qquad k_\rho \leq i \leq N - 1. \tag{2.2.35}$$

We choose the region Λ_π to be the set of all λ, ν satisfying

$$(h(G) - |\chi| + 1)\frac{\operatorname{Re} \nu}{2} > \operatorname{Re} \lambda_{\pi(1)} + 1 > \left(h(G) - |\chi| + \frac{|\chi|}{|\chi| + 1}\right)\frac{\operatorname{Re} \nu}{2}$$

$$1 > \frac{\operatorname{Re} \nu}{2} > \operatorname{Re} \lambda_j + 1 > \left(\frac{|\chi|}{|\chi| + 1}\right)\frac{\operatorname{Re} \nu}{2} > 0 \qquad j \in \chi$$

$$0 > \operatorname{Re} \lambda_j + 1 > \varepsilon \qquad j \notin \chi, \qquad j \neq \pi(1),$$

for suitably chosen small positive ε. Conditions (2.2.34) and (2.2.35) then follow by noting

(*a*) if $\Omega_{G^i_\rho} \subset \chi$, $h(G^i_\rho) \leq i - 1$ (because χ contains no tadpoles);
(*b*) if $|\Omega_{G^i_\rho} \cap \chi| = r$, $h(G^i_\rho) \leq h(G) + r - |\chi|$ (because $h(H(\Omega - \chi)) = h(G) - |\chi|$).

We note that λ, $\nu \in \Lambda_\pi$ also satisfy

$$\text{Re}\ \nu_j > 0, \qquad (j \in \chi) \tag{2.2.36}$$

$$\text{Re}\ \mu < 1. \tag{2.2.37}$$

LEMMA 2.2.38 : If G is a *massive* connected Feynman graph, there exists a region $\Lambda^0 \subset \Lambda_G$ such that the integral (2.2.5) is absolutely convergent for all $(\lambda, \nu) \in \Lambda^0$, $(s, z) \in R_G$. Further, we may choose Λ^0 such that for any quotient graph $Q = G/S$ for which $N(Q) > 1$,

$$\text{Re}\ \nu_Q = h(Q)\frac{\text{Re}\ \nu}{2} - \sum_{i \in \Omega_Q} \text{Re}\ \lambda_i + \left(\frac{N(Q) - 1}{2}\right) - 1 > 0.$$

Proof: Omitted, comparable to 2.2.33.

2.3. Multiplets of massless lines

Let G be a Feynman graph, and let $\{ l_j \mid j \in \chi \}$ be a multiplet in G such that $\chi \subset \Omega^0$. Let G′ be the Feynman graph obtained from G by replacing the lines l_j, $j \in \chi$ by a single massless line l_a. In this section we relate the Feynman amplitudes of G and G′.

THEOREM 2.3.1. — Let $g: \Lambda_G \to \Lambda_{G'}$ be defined by $g(\lambda, \nu) = (\lambda', \nu')$, where

$$\nu' = \nu \tag{2.3.2}$$

$$\lambda'_j = \lambda_j,\ j \notin \chi \tag{2.3.3}$$

$$\lambda'_a = \sum_{j \in \chi} \lambda_j - \left(\frac{\nu}{2} - 1\right)(|\chi| - 1). \tag{2.3.4}$$

(Note that (2.3.4) is equivalent to $\nu_a = \sum_\chi \nu_j$). Then

$$F^{\text{phys}}_G(s, z; \lambda, \nu) = F^{\text{phys}}_{G'}(s, z; \lambda', \nu'). \tag{2.3.5}$$

Proof. — If the lines l_j, $j \in \chi$, are tadpoles, both sides of (2.3.5) vanish; we may therefore exclude this case. We also assume $|\chi| = 2$ (say $\chi = \{b, c\}$), since repeated application of this case may be used to prove the theorem for any value of $|\chi|$. Now let $\pi: \{1, \ldots, N-2\} \to \Omega - \chi$ be $1-1$. By Lemma 2.2.33 there is a region $\Lambda_\pi \subset \Lambda_G$ such that $(\lambda, \nu) \in \Lambda_\pi$ implies that the integral (2.2.5) is absolutely convergent when integrated over the region

$$\begin{aligned} \alpha_j \geq 0, \qquad j \in \chi \\ 0 \leq \alpha_{\pi(1)} \leq \ldots \leq \alpha_{\pi(N-2)}. \end{aligned} \tag{2.3.6}$$

(Actually, Lemma 2.2.33 does not apply unless the graph $H(\Omega_G - \chi)$ is connected, but, if it is not, a separate proof of the existence of Λ_π is easily given). Thus (normalizing $\alpha_{\pi(N-2)} = 1$) we may define

$$F_G^\pi = f_G(\lambda, \nu) \int_0^\infty d\alpha_b d\alpha_c \int_0^1 d\alpha_{\pi(N-3)} \cdots \int_0^{\alpha_{\pi(2)}} d\alpha_{\pi(1)} \frac{\Pi \alpha_i^{\lambda_i} d(\alpha)^{\lambda_0}}{D(\alpha, s, z)^\mu} \tag{2.3.7}$$

and continuing F_G^π out of the region Λ_π,

$$F_G(s, z; \lambda, \nu) = \sum_\pi F_G^\pi(s, z; \lambda, \nu).$$

Now in (2.3.7) we make the change of variables

$$\alpha_a = \frac{\alpha_b \alpha_c}{\alpha_b + \alpha_c}$$

$$\tau = \frac{\alpha_b}{\alpha_b + \alpha_c}.$$

Then (2.3.7) becomes

$$F_G^\pi(s, z; \lambda, \nu) = f_G(\lambda, \nu) \int_0^1 d\tau (1-\tau)^{\nu_b - 1} \tau^{\nu_c - 1}$$

$$\times \int_0^\infty \alpha_a^{\lambda_b + \lambda_c - \frac{\nu}{2} + 1} \int_0^1 d\alpha_{\pi(N-3)} \cdots \int_0^{\alpha_{\pi(2)}} d\alpha_{\pi(1)} \frac{\left(\prod_{\Omega-\chi} \alpha_i^{\lambda_i}\right) d_{G'}(\alpha)^{\lambda_0}}{D_{G'}(\alpha, s, z)^\mu} \tag{2.3.8}$$

since

$$d_G(\alpha) = d_{G'}(\alpha)(\alpha_b + \alpha_c),$$
$$D_G(\alpha, s, z) = D_{G'}(\alpha, s, z)(\alpha_b + \alpha_c).$$

The τ integral in (2.3.8) gives a factor

$$\frac{\Gamma(\nu_b)\Gamma(\nu_c)}{\Gamma(\nu_b + \nu_c)}$$

and, using (2.2.12), we see that

$$F_G^{\pi}(s, z; \lambda, \nu) = F_{G'}^{\pi}(s, z; \lambda', \nu') \tag{2.3.9}$$

where $F_{G'}^{\pi}$ is defined in the obvious way. Continuing (2.3.9) out of Λ_π, and summing over π, proves the theorem.

3. ZERO MASS SINGULARITIES

THEOREM 3.1. — Let G be a connected Feynman graph. Let $\omega \in \Omega_M$ be such that the Feynman graph G_ω, obtained from G by deleting one line l_ω, is connected. Then $\{ z_\omega = 0 \}$ is a singularity of the generic Feynman amplitude $F_G(s, z; \lambda, \nu)$. More precisely, the physical sheet F_G^{phys} admits in the neighbourhood of $z_\omega = 0$ the decomposition

$$F_G^{phys}(s, z; \lambda, \nu) = z_\omega^{\nu_\omega} H_G(s, z; \lambda, \nu) + K_G(s, z; \lambda, \nu) \tag{3.2}$$

where H_G and K_G are holomorphic in z_ω in the neighbourhood of $z_\omega = 0$. Denote by G^ω the Feynman graph obtained from G by deleting the line l_ω from the set of massive lines. Then, with the obvious identifications of the spaces T_{G_ω}, T_{G^ω} with subspaces of T_G,

$$T_{G_\omega} = \{ (s, z; \lambda, \nu) \mid z_\omega = \lambda_\omega = 0 \} \subset T_G,$$
$$T_{G^\omega} = \{ (s, z; \lambda, \nu) \mid z_\omega = 0 \} \subset T_G,$$

we have

$$H_G(s, z; \lambda, \nu) \mid_{T_{G_\omega}} = \frac{-\Gamma(-\nu_\omega)}{\Gamma(\nu_\omega)} F_{G_\omega}^{phys} \tag{3.3}$$

$$K_G(s, z; \lambda, \nu) \mid_{T_{G^\omega}} = F_{G^\omega}^{phys}. \tag{3.4}$$

Proof. — If l_ω is a tadpole, the theorem holds trivially with H_G independent of z_ω and $K_G \equiv 0$. In the following proof we may therefore exclude this case. We apply Lemma 2.2.33 with $\chi = \{ \omega \}$ to produce a region $\Lambda_\pi \subset \Lambda_G$ for each $1 - 1$ π: $\{ 1, \ldots, N - 1 \} \to \Omega - \{ \omega \}$. We then define, for $(s, z) \in R_G$,

$$F_G^{\pi} = \Sigma F_G^{\rho},$$

(ρ compatible with π), where the concept of compatability is defined in Lemma 2.2.33.

LEMMA 3.5. — For $(\lambda, \nu) \in \Lambda_\pi$, $F_G^{\pi}(s, z)$ can be analytically continued in z_ω around $z_\omega = 0$ (the remaining s, z variables being held fixed). The discontinuity of F_G^{π} around $z_\omega = 0$ (i. e. the difference between the function obtained by continuation of F_G^{π} along a loop circling $z_\omega = 0$ anticlockwise and F_G^π itself) can be analytically continued in z_ω clockwise around zero to small positive z_ω values. For z_ω small and positive it is given by the sum of absolutely convergent integrals

$$\text{disc } F_G^\pi = \Sigma \, (\exp 2\pi i\mu - 1) J_G^\rho \tag{3.6}$$

(ρ compatible with π, $k_\rho = \rho^{-1}(\omega) > j_\rho$). Here J_G^ρ is defined by replacing δ in (2.2.22) by

$$\delta' = \{ (x) \mid g(x, s, z, \lambda, \nu) \leq 0 \} \cap \delta. \tag{3.7}$$

Proof. — From (2.2.27) we see that we may choose a sufficiently small neighbourhood U of $z_\omega = 0$ in the z_ω plane so that for $z_\omega \in U$ and the remaining s, z variables held fixed (the radius of U depends on their values)

$$\delta(\rho) \cap \{ g(x, s, z, \lambda, \nu) = 0 \} \neq \emptyset \tag{3.8}$$

iff $k_\rho = \rho^{-1}(\omega) > j_\rho$ (so that $\zeta_\rho = - z_\omega$). Also (3.8) holds only for z_ω real and positive and then the set $\{ g(x, s, z, \lambda, \nu) = 0 \}$ intersects the faces of the cube $\delta(\rho)$ transversely. The analytic continuation of F_G^ρ in z_ω along a path circling $z_\omega = 0$ is thus made trivially (since no distortion of the contour is required) until z_ω approaches the real axis. The discontinuity in (3.6) is the difference between the functions $F_G^{\rho^-}$, $F_G^{\rho^+}$ obtained by taking clockwise and anticlockwise paths. Now

$$g(x, s, z, \lambda, \nu)_+^{-\mu} = g(x, s, z, \lambda, \nu)_-^{-\mu} = \exp[-\mu \log^+ g] \quad \text{for} \quad g > 0 \tag{3.9}$$

$$\left.\begin{aligned} g_+^{-\mu} &= \exp[-\mu \log^+ g] \\ g_-^{-\mu} &= \exp[-\mu \log^+ g] \exp 2\pi i\mu \end{aligned}\right\} \quad \text{for} \quad g < 0$$

Since (2.2.37) holds the integral J_G^ρ is absolutely convergent for $(\lambda, \nu) \in \Lambda_\pi$. In making this assertion we have used also the transverse intersection property noted above, which guarantees that to prove absolute convergence of J_G^ρ we have only to examine the exponents of the factors in the integrand. From (3.9) we have

$$\text{disc } F_G^\rho = (\exp 2\pi i\mu - 1) J_G^\rho$$

for ρ such that $k_\rho = \rho^{-1}(\omega) > j_\rho$. Otherwise $\text{disc } F_G^\rho = 0$. Summing over the ρ compatible with π we obtain (3.6).

LEMMA 3.10. — For any 1:1 mapping $\pi : \{ 1, \ldots, N-1 \} \to \Omega - \{ \omega \}$ and $(\lambda, \nu) \in \Lambda_\pi$

$$\text{disc } F_G^\pi = z_\omega^{\nu_\omega} H^\pi(s, z; \lambda, \nu) (\exp 2\pi i\nu_\omega - 1) \tag{3.11}$$

with H^π holomorphic in the neighbourhood of z_ω, and

$$H^\pi |_{T_{G\omega}} = F_{G_\omega}^\pi. \tag{3.12}$$

Proof. — We may write the sum of integrals (3.6) in the form of a single integral

$$\operatorname{disc} F_G^\pi = (\exp 2\pi i\mu - 1) f_G(\lambda, \nu) \int_{\delta'} \left(\prod_{i=1}^{N-1} x_i^{\tau_i} \right) g(x, s, z, \lambda, \nu) dx_1 \ldots dx_{N-1} \quad (3.13)$$

where $\delta' = \{ (x) \mid 0 \leq x_i \leq 1, 1 \leq i \leq N-2, x_{N-1} \geq 0, g(x, s, z, \lambda, \nu) \leq 0 \}$ and the variables x_i are defined by (2.2.19) with ρ given by

$$\rho(j) = \pi(j) \qquad 1 \leq j \leq N-1 \qquad \rho(N) = \omega$$

For this ρ we have $\zeta_\rho = -z_\omega$. For z_ω positive and $0 \leq x_i \leq 1$, $1 \leq i \leq N-2$, $-z_\omega + \sigma_2(x, s, z)$ is a quadratic polynomial in x_{N-1} with a negative constant term $-z_\omega$ and positive linear and quadratic terms. Thus it has a unique positive zero in x_{N-1} which we denote by $z_\omega v = v(x_1, \ldots, x_{N-2}, s, z) z_\omega$. We regard (3.13) as an iterated integral, integrating first over x_{N-1}, then over $x_1, \ldots, x_{N-2}$ and make the change of variable

$$y = x_{N-1}/v z_\omega. \quad (3.14)$$

Since $-\mu + \tau_{N-1} + 1 = \nu_\omega$ (from (2.2.11), (2.2.13), (2.2.25)) we obtain (3.11) with

$$H^\pi(s, z; \lambda, \nu) = (\exp 2\pi i\mu - 1) f_G(\lambda, \nu) (\exp 2\pi i \nu_\omega - 1)^{-1}$$

$$\times \int dx_1 \ldots dx_{N-2} \left(\prod_{i=1}^{N-2} x_i^{\tau_i} \right) v^{\tau_{N-1}+1} \int_0^1 dy y^{\tau_{N-1}} (1-y)^{-\mu} h(x, y, s, z, \lambda, \nu) \quad (3.15)$$

where

$$h = \{ 1 + \sigma_1(x) \}^{\lambda_0} \left. \frac{(-1 + \sigma_2(x, s, z) z_\omega^{-1})^{-\mu}}{(1-y)^{-\mu}} \right|_{x_{N-1} = v z_\omega y} \quad (3.16)$$

Now $h = \{ 1 + \sigma_1(x) \}^{\lambda_0} \{ 1 + B(x, s, z) y \}^{-\mu} \exp(-i\pi\mu)$ where B is a certain polynomial, positive in the region of integration. Hence h is uniformly bounded in $x_1, \ldots, x_{N-2}$ in the region of integration. v can be written in the form

$$v = \prod_{k_\pi \leq i \leq N-2} x_i v' \quad (3.17)$$

where v' is bounded away from zero uniformly in $x_1, \ldots, x_{N-2}$ for z_ω in the neighbourhood of zero. Since

$$\tau_{N-1} + 1 = \prod_{j=1}^{N-1} (\lambda_j + 1) - (h-1)\frac{\nu}{2} = \mu',$$

the value of μ for the graph G_ω, the change in the exponent of x_i for $k_\pi \leq i \leq N - 2$ obtained when (3.17) is substituted into (3.15) is just the replacement of τ_i by τ_i', the value of τ_i for G_ω (for other values of i, $\tau_i' = \tau_i$). Now in Λ_π we also have $\operatorname{Re} \tau_i' + 1 > 0$ for $1 \leq i \leq N - 2$. It follows that H^π is holomorphic in z_ω in the neighbourhood of $z_\omega = 0$ and that $H^\pi|_{z_\omega = 0}$ is given by the integral (3.15) with $z_\omega = 0$ in the integrand. Now by combinatorial arguments, which we will not detail here since more general considerations of the same kind (but expressed in terms of the α's rather that the scaled variables x) will be given in Section 4.3, it may be shown that

$$(1 + \sigma_1(x))|_{x_{N-1}=0} = 1 + \sigma_1'(x)$$
$$v'|_{z_\omega = 0} = \zeta_\pi + \sigma_2'(x, s, z).$$

Also $B|_{z_\omega = 0} = 0$. Thus when z_ω is set equal to zero in (3.17) the y integration may be carried out explicitly to give a product of Γ functions. Taking into account the definition of the normalization factor $f_G(\lambda, \nu)$ (2.2.12) and making use of the identity $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$ we find that the resulting integral over $x_1, \ldots, x_{N-2}$ is just the integral defining $F^\pi_{G_\omega}$. This completes the proof of Lemma 3.10.

LEMMA 3.18. — $F^\pi_G - z_\omega^{\nu_\omega} H^\pi_G = K^\pi_G$ is, for $(\lambda, \nu) \in \Lambda_\pi$, holomorphic in z_ω in the neighbourhood of 0 and $K^\pi_G|_{z_\omega = 0} = F^\pi_{G^\omega}$.

Proof. — By construction of H^π_G, K^π_G is single valued in z_ω. Since for (λ, ν) in Λ_π, $\nu_\omega > 0$ (2.2.36)) and the convergence conditions $1 + \tau_i'' > 0$ for the integral $F^\rho_{G^\omega}$ are satisfied for any ρ compatible with π, it is easily checked that K^π_G is also bounded in the neighbourhood of $z_\omega = 0$, hence holomorphic, and its restriction to $z_\omega = 0$ is $F^\pi_{G^\omega}$ as asserted.

To complete the proof of Theorem 3.1 we note that both sides of the decomposition $F^\pi_G = z_\omega^{\nu_\omega} H^\pi_G + K^\pi_G$ may be continued in (λ, ν) outside Λ_π to give meromorphic functions (by the Gel'fand-Šilov technique) and that the decomposition continues to hold. Summing over π we obtain (3.2).

4. NORMAL LANDAU SINGULARITIES

4.1. The definition of a normal Landau singularity

DEFINITION 4.1.1. — Let Q be a massive Feynman graph (2). Denote by π the natural projection

$$\pi : W_Q \times \mathbb{P}^{N_Q - 1} \to W_Q,$$

(2) Throughout this and the following section we restrict ourselves to such graphs.

and consider the restriction π_1 of π to the set U

$$U = \left\{ (\alpha, s, z) \,|\, \frac{\partial D}{\partial \alpha_i} = 0,\ i \in Q;\ d(\alpha) \neq 0 \right\} \subset W_Q \times \mathbb{P}^{N-1}.$$

If $\pi_1(U) \subset W_Q$ is of complex codimension 1 in W_Q we call the closure of $\pi_1(U)$ the *leading Landau variety* of Q, and write $\overline{\pi_1(U)} = L_Q^l$. In this case Q will be said to be *normal.*

Remark 4.1.2. — For $(\alpha, s, z) \in U$ we have $D = 0$ since D is homogeneous in α, and hence

$$\frac{\partial}{\partial \alpha_i}\left(\frac{D}{d}\right) = 0, \qquad i \in Q.$$

These equations give

$$z_i = \frac{\partial}{\partial \alpha_i}\left(\frac{D^s}{d}\right), \qquad i \in Q, \tag{4.1.2}$$

and conversely (4.1.2) implies $(\alpha, s, z) \in U$. (4.1.2) shows that the set U has dimension equal to one less than the dimension of W_Q, and so $\pi_1(U)$ is of codimension ≥ 1 in W_Q for any Q.

Remark 4.1.3. — (4.1.2) may be regarded as giving a rational parametrization of L_Q^l. Thus L_Q^l is a rational irreducible algebraic variety.

LEMMA 4.1.4. — For generic $(s, z) \in L_Q^l$, $\pi_1^{-1}((s, z))$ is a single point. The coordinates α_i of $\pi_1^{-1}((s, z))$ are rational functions on L_Q^l.

Proof. — Since $(s, z) \in L_Q^l$ is generic it is a non-singular point of L_Q^l. Let $(\alpha, s, z) \in \pi_1^{-1}((s, z))$. Then the 1-form

$$\frac{D^s(ds, \alpha)}{d(\alpha)} - \sum_{i \in Q} \alpha_i dz_i$$

is a normal 1-form to L_Q^l at (s, z). But L_Q^l has a unique normal 1-form at (s, z) (up to a factor), so $\alpha \in \mathbb{P}^{N_Q - 1}$ is uniquely determined by (s, z). This proves the first part of the Lemma. The second part then follows from elimination theory (see e. g. [*18*]).

Remark 4.1.5. — Note that it is essential in Lemma 4.1.4 that the masses z_i be regarded as variables. For Q the crossed square graph (fig. 1), which will be seen from the results of Section 5 to be normal, C. Risk [*19*] has shown that a particular section $L_Q^l \cap \{ z = z^0,\ s_i = s_i^0,\ i \in \Theta^E \}$ in which the internal and external masses are fixed is a reducible variety.

having two irreducible components C_1 and C_2. For a generic point $(s, z) \in C_2$, $\pi_1^{-1}((s, z))$ consists of two points. This result does not contradict Lemma 4.1.4 since it may be shown that as $z^1 \to z^0$, $s_i^1 \to s_i^0$ the generic section $L_Q^l \cap \{ z = z^1, s_i = s_i^1, i \in \Theta^E \}$ degenerates into $C_1 C_2^2$ i. e. C_2 appears with multiplicity two.

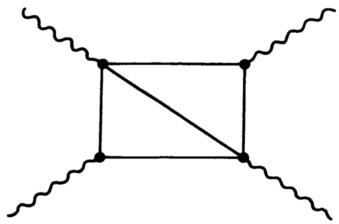

FIG. 1. — The crossed square graph.

LEMMA 4.1.6. — If Q is normal, Q is 2-connected (cf. (2.1.5)).

Proof. — If Q is not 2-connected we may write $Q = S_1 \cup S_2$ where the subgraphs S_1 and S_2 have at most one common vertex and no common lines. Given $(\alpha, s, z) \in U$ and $\lambda \in \mathbb{C}$ we define $\alpha(\lambda)$ by

$$\alpha(\lambda)_i = \lambda \alpha_i \qquad i \in S_1 \qquad \alpha(\lambda)_i = \alpha_i \qquad i \in S_2.$$

Then it is easily checked that $(\alpha(\lambda), s, z) \in U$ for all λ. But this implies that for no $(s, z) \in \pi_1(U)$ is $\pi_1^{-1}((s, z))$ a single point, so by Lemma 4.1.4 Q is not normal.

LEMMA 4.1.7. — If Q is a normal graph and (s, z) a generic point of L_Q^l and (α', s, z) the point $\pi_1^{-1}((s, z))$, the Hessian matrix

$$H_{ij} = \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \left(\frac{D}{d} \right) \bigg|_{\alpha = \alpha'}$$

has rank $N - 1$. Conversely, if H has rank $N - 1$ for generic (s, α), Q is normal.

Proof. — Rank $H = N - 1$ is just the condition that π_1 be non-singular at (α', s, z). The lemma thus follows from our definition by an application of Sard's theorem [20].

4.2. A criterion for a Feynman graph to be normal

In this section we begin by recalling an important property of the function $D^*(\alpha, s, z) = D(\alpha, s, z)/d(\alpha)$, and we use this property to derive a necessary and sufficient condition for a Feynman graph Q to be normal, which is more amenable than the Hessian condition of Lemma 4.1.6. This condition will be used in Section 5 to shown that an externally complete graph Q (cf. (2.1.2)) is normal, and to obtain certain sufficient conditions for the normality of Q in the case in which Q is not externally complete.

Denote by $Y_Q^m(p)$ the set

$$Y_Q^m = \left\{ q_i,\ i \in \Omega_Q \,\middle|\, \sum_i e_{is} q_i = \begin{cases} 0 & \text{if } s \notin \Theta^E \\ -p_s & \text{if } s \in \Theta^E \end{cases} \right\} \tag{4.2.1}$$

where $p \in X_Q^m$ (cf. (2.2.1)), and we suppose $m \geq (n^E(Q) - 1)$.

LEMMA 4.2.2. — The function

$$D'(\alpha, q) = \sum_{i \in \Omega_Q} \alpha_i (q_i^2 - z_i) \tag{4.2.3}$$

has for fixed α, $d(\alpha) \neq 0$, and $p \in X_Q^m$ and variable $q \in Y_Q^m(p)$ an extremum equal to $D^*(\alpha, s, z)$ which occurs for

$$q_i = g_i(\alpha, p). \tag{4.2.4}$$

The vectors $g_i(\alpha, p)$ are given by the following combinatorial rule: choose some vertex $s \in \Theta^E$. Then $g_i(\alpha, p)$ has an expansion as a sum of the vectors p_u, $u \in \Theta^E - \{ s \}$:

$$g_i(\alpha, p) = \sum_{u \in \Theta^E - \{s\}} \frac{1}{d} g_{ius}(\alpha) p_u \tag{4.2.5}$$

where

$$g_{ius}(\alpha) = \sum_{\text{all trees T}} \varepsilon_{ius}(T) \alpha(\Omega - T). \tag{4.2.6}$$

In (4.2.6) the function $\varepsilon_{ius}(T)$ is defined by

$$\varepsilon_{ius} = 0 \quad \text{unless} \quad i \in T$$

and the path joining u to s in T passes through i,

$$\varepsilon_{ius} = +1$$

if the path from u to s in T passes through i in the direction given by the orientation of i

$$\varepsilon_{ius} = -1 \quad \text{otherwise.}$$

If $\alpha_i > 0$ for all i and the variables z are real and the p_u imaginary, and $D'(\alpha, q)$ is considered as a function of imaginary q, $D^*(\alpha, s, z)$ is a minimum.

Remark 4.2.7. — The first part of the lemma is well-known (see e. g. [*21*]). The rule for the construction of $g_i(\alpha, p)$ does not seem to have been explicity stated before.

Proof. — Note that $D'(\alpha, q)$ is a quadratic form on $Y^m_Q(p)$. It is an extremum iff the loop equation

$$\sum_{i \in l} \alpha_i q_i = 0 \tag{4.2.8}$$

holds for any loop l of Q. For given α these are linear equations on q and for $d(\alpha) \neq 0$ and linearly independent p_u $(u \neq s)$, they determine $q = g(\alpha, p)$ uniquely as a linear combination of the vectors p_u with coefficients depending on α. It may then be verified that $g_i(\alpha, p)$ is given explicitly by (4.2.5), (4.2.6).

For any Feynman graph Q we define matrices $T_Q(\alpha)$, $T^*_Q(\alpha)$ by

$$T_{i,jus} = \frac{\partial}{\partial \alpha_i} g_{jus}(\alpha) \tag{4.2.9}$$

$$T^*_{i,jus} = \frac{\partial}{\partial \alpha_i}\left(\frac{1}{d} g_{jus}(\alpha)\right). \tag{4.2.10}$$

Here the row index i is in Ω_Q, and the column index j is in $\Omega_Q \times \Theta^E \times \Theta^E$.

LEMMA 4.2.11. — If $d(\alpha) \neq 0$, the conditions rank $T = N$ and rank $T^* = N - 1$ are equivalent.

Proof.

$$T = dT^* + T_1 \tag{4.2.12}$$

with $T_{1i,jus} = \left(\frac{1}{d}\frac{\partial d}{\partial \alpha_i}\right) g_{jus}$. T_1 has rank 1 so (4.2.12) implies rank $T^* \geq$ rank $T - 1$. Now rank $T^* \leq N - 1$ since α_i is always a null vector for T^* (by Euler's theorem since $\frac{1}{d} g_{jus}$ is homogeneous of degree 0 in α). Thus rank $T = N \rightarrow$ rank $T^* = N - 1$. To prove the converse suppose that rank $T^* = N - 1$ and that β_i is a null-vector of T. The g_{jus} satisfy the identities

$$\sum_j g_{jus} e_{jv} = -\delta_{uv} d \qquad (v \neq s). \tag{4.2.13}$$

They imply that β_i satisfies $\sum_i \beta_i \frac{\partial d}{\partial \alpha_i} = 0$, i. e. that $T_1\beta = 0$. Hence $T^*\beta = 0$. Since rank $T^* = N - 1$ this gives $\beta = c\alpha$ for some constant c. But then $0 = \sum_i \beta_i \frac{\partial d}{\partial \alpha_i} = c\, h(Q)d$. Since $d \neq 0$ this implies $c = 0$ *i. e.* $\beta = 0$ so rank $T = N$.

Denote by $J(\alpha)$ the Jacobian map of the map $\alpha \to g(\alpha)$, evaluated at $\alpha \in \mathbb{P}^{N-1}$ i. e. the linear map $J(\alpha): \mathbb{P}^{N-1} \to Y_Q^m(0)$ given by

$$(J(\alpha)\beta) = \sum_{ij} \frac{\partial}{\partial \alpha_j} g_i(\alpha)\beta_j. \tag{4.2.14}$$

LEMMA 4.2.15.

$$\sum_{i,j} H_{ij}(\alpha)\beta_i\beta_j = -\sum_i \alpha_i((J(\alpha)\beta)_i)^2 \tag{4.2.16}$$

Proof. — According to Lemma 4.2.2

$$D^*(\alpha) = D'(\alpha, g(\alpha)). \tag{4.2.17}$$

Hence

$$\frac{\partial D^*}{\partial \alpha_i} = \left(\frac{\partial D'}{\partial \alpha_i} + \sum_l \frac{\partial D'}{\partial q_l}\frac{\partial g_l}{\partial \alpha_i}\right)\Bigg|_{q=g} \tag{4.2.18}$$

$$\frac{\partial^2 D^*}{\partial \alpha_i \partial \alpha_j} = \left(\frac{\partial^2 D'}{\partial \alpha_i \partial \alpha_j} + \sum_l \frac{\partial^2 D'}{\partial \alpha_i \partial q_l}\frac{\partial g_l}{\partial \alpha_j} + \sum_l \frac{\partial^2 D'}{\partial \alpha_j \partial q_l}\frac{\partial g_l}{\partial \alpha_i}\right.$$
$$\left. + \sum_{l,s} \frac{\partial^2 D'}{\partial q_l \partial q_s}\frac{\partial g_l}{\partial \alpha_i}\frac{\partial g_s}{\partial \alpha_j} + \sum_l \frac{\partial D'}{\partial q_l}\frac{\partial^2 g_l}{\partial \alpha_i \partial \alpha_j}\right)\Bigg|_{q=g} \tag{4.2.19}$$

Since D^* is an extremum of D' with respect to variations of q the second term in (4.2.18) is zero. The first term in (4.2.19) is zero as also is the last. By differentiating the equation

$$\sum_l \frac{\partial D'}{\partial q_l}\frac{\partial g_l}{\partial \alpha_i}\Bigg|_{q=g} = 0$$

with respect to α_j we find that the second and third terms in (4.2.19) are each equal to minus the fourth. From the resulting formula for $\frac{\partial^2 D^*}{\partial \alpha_i \partial \alpha_j}$ (4.2.16) follows immediately.

THEOREM 4.2.20. — The Feynman graph Q is normal iff for generic $\alpha \in \mathbb{P}^{N-1}$ the matrix T_Q has rank N.

Proof. — According to Lemma 4.2.11 we may consider instead of the condition rank $T_Q = N$ the equivalent condition rank $T_Q^* = N - 1$. If for some s, α, rank $H = N - 1$, it is immediate from (4.2.16) and (4.2.5) that rank $T^* = N - 1$. Conversely if rank $T^* = N - 1$ for some α we may choose α with $\alpha_i > 0$, $\forall_i$, and rank $T^*(\alpha) = N - 1$, and we may choose imaginary momenta p_a. Then the corresponding Hessian form

$$\sum_{i,j} H_{ij}(\alpha, s)\beta_i\beta_j$$

is non-negative, and β is a null vector for this form iff, $\forall_i$, $(J(\alpha)\beta)_i = 0$. But if the p_a have been chosen so that their span has dimension $n(Q)^E - 1$ the expansion

$$0 = (J(\alpha)\beta)_i = \sum_{u \in \Theta^E - \{s\}} \sum_j \beta_j T^*_{j,ius}(\alpha) p_u$$

implies $\sum_j \beta_j T^*_{j,ius} = 0$, $\forall_{ius}$. Since the choice of s is arbitrary β is a null vector for T^*. This gives rank H = rank $T^* = N - 1$, which completes the proof

4.3. **Limiting values of Symanzik functions**

Let G be a Feynman graph, S a subgraph of G with components $S_1, \ldots, S_{c(s)}$, and $Q = G/S$. In this section we prove a lemma relating the d and D functions of these graphs. The reader uninterested in combinatorics may omit the proof of Lemma 4.3.4 without loss of continuity.

There is a natural projection map $\xi' : X_G^m \to X_Q^m$ given by

$$(\xi' p)_i = \sum_{j \in \pi_Q^{-1}(i)} p_j, \qquad i \in \Theta_Q^E, \tag{4.3.1}$$

where π_Q is the map constructed in Definition 2.1.4, and a corresponding map $\xi : W_G \to W_Q$ given by $\xi = \xi^{(s)} \times \xi^{(z)}$, with

$$\xi_i^{(z)} = z_i, \qquad i \in \Omega_Q^M, \tag{4.3.2}$$

$$\xi^{(s)}(\chi) = s[\pi_Q^{-1}(\chi)], \qquad \chi \subsetneq \Theta_Q^E, \qquad \chi \neq \emptyset.$$

For $\beta \in \mathbb{C}^{N(Q)}$ and $p \in X_Q^m$ we have defined in (4.2.5) momenta $g_i(\beta, p) \in Y_Q^m$.

These momenta are external to the graph S; it is therefore natural to define maps ζ_i': $\mathbb{C}^{N(Q)} \times X_G^m \to X_{S_i}^m$ by

$$[\zeta_i'(\beta, p)]_j = p_j + \sum_{k \in \Omega_Q} e_{kj} g_k(\beta, \xi' p), \qquad j \in \Theta_{S_i}^E, \tag{4.3.3}$$

and a corresponding map ζ_i: $\mathbb{C}^{N(Q)} \times W_G \to W_{S_i}$ with

$$\zeta_i = \zeta_i^{(s)} \times \zeta_i^{(z)},$$
$$[\zeta_i^{(z)}]_j = z_j, \qquad (j \in \Omega_{S_i}^m),$$

and with $\zeta_i^{(s)}(\chi)$ given by summing (4.3.3) over χ and squaring (more precisely, 2.2.3 defines a map ψ_G: $X_G^m \to S_G$; $\zeta_i^{(s)}$ is defined by factoring: $\psi_{S_i}\zeta_i' = \zeta_i^{(s)}[id_{\mathbb{C}^{N(Q)}} \times \psi_G]$). We now state the main result of this section.

LEMMA 4.3.4. — Take γ_i, x, $\beta_j \in \mathbb{C}$, with $i \in \Omega_S$, $j \in \Omega_Q$. Define

$$\varphi\colon \mathbb{C}^{N(G)+1} \to \mathbb{C}^{N(G)}$$

by $\varphi(\gamma, x, \beta) = \alpha$, with

$$\alpha_i = \gamma_i, \qquad (i \in \Omega_S),$$
$$\alpha_j = x^{-1}\beta_j, \qquad (j \in \Omega_Q).$$

Then

a) $$x^{h(Q)} d_G(\varphi(\gamma, x, \beta))\,|_{x=0} = d_Q(\beta) \prod_1^{c(S)} d_{S_i}(\gamma)$$

b) $$x D^*(\varphi(\gamma, x, \beta), s, z)\,|_{x=0} = D_Q^*(\beta, \xi(s, z))$$

c) $$\frac{\partial}{\partial x}[x D^*(\varphi(\gamma, x, \beta), s, z)]\,|_{x=0} = \sum_{i=1}^{c(S)} D_{S_i}^*(\gamma, \xi(s, z)).$$

Proof. — We may assume that G is externally complete, since the result for arbitrary G is a specialisation of this case. We will let T_r, T_r^i, T_r^Q denote r-trees in G, S_i, and Q respectively. Suppose $\chi \subset \Theta_{S_j}$ with $\chi \neq \phi$, $\chi \neq \Theta_{S_j}$. Then we define $\mathcal{I}(j, \chi)$ to be the set of those 2-trees T_2^j which appear in the definition 2.1.8 of $d_{S_j,2}(\chi \mid \Theta_{S_j} - \chi)$. Suppose now that $\mathcal{I}(j, x)$ is not empty, i. e., $\exists\, T_2^j \in \mathcal{I}(j, \chi)$; we wish to define a new quotient graph $Q(j, \chi)$ of G by collapsing each subgraph S_i, $i \neq j$, and collapsing each component of $H(T_2^j)$. Explicitly,

$$Q(j, \chi) = H[(\Omega_G - \Omega_{S_j}) \cup T_2^j)/H\Big[\bigcup_{i \neq j} \Omega_{S_i} \cup T_2^j\Big]. \tag{4.3.5}$$

(4.3.5) is easily seen to be independent of the choice of $T_2^j \in \mathcal{I}(j, \chi)$.

Now we clearly have, for any T_r,

$$|T_r \cap \Omega_{S_i}| \leq N(S_i) - h(S_i) \qquad (i = 1, \ldots, c(s)) \qquad (4.3.6)$$

and hence, since $\Sigma h(S_i) + h(Q) = h(G)$,

$$|T_r \cap \Omega_Q| \geq N(Q) - h(Q) - r + 1. \qquad (4.3.7)$$

It is now straightforward to verify

i) For any T_r such that $|T_r \cap \Omega_Q| = N(Q) - h(Q) - r + 1$,

$$T_r = T_r^Q \cup \bigcup_{i=1}^{c(S)} T^i,$$

for some T_r^Q, T^i; conversely, any T_r^Q, T^i give such an r-tree T_r.

ii) For any T_r such that $|T_r \cap \Omega_Q| = N(Q) - h(Q) - r + 2$,

$$T_r = T_r^{Q(j,\chi)} \cup T_2^j \cup \bigcup_{\substack{i=1 \\ i \neq j}}^{c(S)} T^i,$$

for some j, χ, $T_r^{Q(j,\chi)}$, $T_2^j \in \mathscr{I}(j, \chi)$, and T^i $(i \neq j)$. Conversely, any such union yields an r-tree T_r with $|T_r \cap \Omega_Q| = N(Q) - h(Q) - r + 2$.

We now proceed with the proof of the lemma.

a) $$x^{h(Q)} d_G(\varphi(\gamma, x, \beta)) = \sum_T x^{h(Q)} \alpha(\Omega_Q - T) \prod_{i=1}^{c(S)} \alpha(\Omega_{S_i} - T)$$
$$= \sum_T \beta(\Omega_Q - T) \prod_{i=1}^{c(s)} \gamma(\Omega_{S_i} - T) x^{h(Q) - |\Omega_Q - T|}$$

The restriction to $x = 0$ restricts the sum to those T for which $|\Omega_Q - T| = h(Q)$.

i) Then yields the desired conclusion.

b) The argument here is similar. We have, from (2.2.7),

$$xD^*(\varphi(\gamma, x, \beta), s, z) = \frac{\frac{1}{2} \sum_{\psi} d_2(\psi \mid \psi') s(\psi) x^{h(Q)+1}}{x^{h(Q)} d} - \sum_{\Omega_Q} \beta_i z_i - x \sum_{\Omega_S} \gamma_i z_i.$$

The result is immediate for the z-dependent terms. When we put $x = 0$, we restrict the sum which occurs in d_2 to run over those T_2 which satisfy $|\Omega_Q - T_2| = h(Q) + 1$. If we again apply *i*), we may decompose such a T_2 into trees in the S_i and a 2-tree in Q. Together with *a*), this completes the proof.

c) In this proof it is convenient to choose a distinguished vertex $k(j) \in \Theta_{S_j}$. We have

$$\frac{\partial}{\partial x}[x D^*(\varphi(\gamma, x, \beta), s, z)]\,|_{x=0}$$

$$= (x^{h(Q)}d)^{-2}\left\{d\frac{\partial}{\partial x}(x^{h(Q)+1}D^s) - D^s\frac{\partial}{\partial x}(x^{h(Q)}d)\right\}\Bigg|_{x=0} - \sum_{\Omega_S}\gamma_i z_i. \quad (4.3.8)$$

Again, the z-dependent terms are trivial. With the first term we argue as above, noting that the contribution of $\frac{\partial}{\partial x}(x^{h(Q)+1}D^s)$ which survives for $x = 0$ comes from those 2-trees satisfying $|\Omega_Q - T_2| = h(Q)$, and similarly for $\frac{\partial}{\partial x}(x^{h(Q)}d)$. Using *ii*), we can then rewrite (4.3.8) in the form

$$d_Q^{-2}\sum_{j=1}^{c(S)} d_{S_j}^{-1}\sum_{\substack{\psi \subsetneq \Theta_Q \\ k(j)\in\psi}} d_{S_j,2}(\psi\,|\,\Theta_{S_j} - \psi)\Bigg\{d_Q\sum_{\substack{\chi \subsetneq \Theta_{Q(j,\psi)} \\ \psi'\in\chi}} d_{Q(j,\psi),2}(\chi\,|\,\Theta_{Q(j,\psi)} - \chi)$$

$$\times\, s[\pi^{-1}_{Q(j,\psi)}(\Theta_{Q(j,\psi)} - \chi)] - d_{Q(j,\psi)}\sum_{\substack{\chi \subsetneq \Theta_Q \\ \Theta_{S_j}\in\chi}} d_{Q,2}(\chi\,|\,\Theta_Q - \chi)s(\pi_Q^{-1}(\chi))\Bigg\} \quad (4.3.9)$$

where $\psi' = \Theta_{S_j} - \psi$, and we recall from 2.1.4 that vertices in a quotient graph of G are labeled by subsets of Θ_G.

To complete the proof we must identify the expression in { } in (4.3.9) with $d_Q^2\zeta_j^{(s)}(\psi)$. To take into account the linear relations on the $s(\chi)$ it is more convenient to work with the p variables. Define

$$p(\eta) = \sum_{i\in\eta} p_i, \qquad (\eta \subset \Theta_G).$$

Then using (2.2.3) the term in { } in (4.3.9) becomes

$$d_Q\sum_{u,v\in\Theta_{Q(j,\psi)}}[d_{Q(j,\psi),2}(uv\,|\,\psi')d_Q - d_{Q(j,\psi)}d_{Q(j,\psi),3}(\psi\,|\,\psi'\,|\,uv)]p(u).p(v). \quad (4.3.10)$$

From (4.3.3) and (4.2.5) one finds easily

$$d_Q^2 \zeta_j^{(s)}(\psi) = \left[\sum_{u \in \Theta_{Q(j,\psi)}} d_{Q(j,\psi),2}(u\psi \mid \psi') p(u) \right]^2. \qquad (4.3.11)$$

We now compare the coefficients of $p(u)\cdot p(v)$ in (4.3.10) and (4.3.11). Writing $A = A_{Q(j,\psi)}$ (see (2.1.10)), noting that $d_Q = d_{Q(j,\psi),2}(\psi \mid \psi')$, and using Lemma 2.1.11, we find we must show that, for any $u, v \in \Theta_{Q(j,\psi)}$,

$$A\begin{pmatrix}\psi' & \psi\\ \psi' & \psi\end{pmatrix} A\begin{pmatrix}\psi' & u\\ \psi' & v\end{pmatrix} - A\begin{pmatrix}\psi' & u\\ \psi' & \psi\end{pmatrix} A\begin{pmatrix}\psi' & v\\ \psi' & \psi\end{pmatrix}$$
$$= A\begin{pmatrix}\psi'\\ \psi'\end{pmatrix} A\begin{pmatrix}\psi' & \psi & u\\ \psi' & \psi & v\end{pmatrix}. \qquad (4.3.12)$$

But (4.3.12) is simply Jacobi's theorem for the symmetric matrix $\{ A_{rs} \mid r, s \neq \psi' \}$. This completes the proof.

4.4. The local decomposition theorem

Let G be a massive Feynman graph, $Q = G/S$ a quotient graph of G with corresponding subgraph S. Suppose that Q is normal with leading Landau variety L_Q^l. Then, if ξ denotes the map $W_G \to W_Q$ introduced in Section 4.3, $\xi^{-1}(L_Q^l)$ is a subvariety of W_G. When no confusion is likely to arise we denote this subvariety simply by L_Q^l. We assert then that $L_Q^l \subset L_G$. To prove this we will need a convenient path from the Symanzik region to a point of L_Q^l. Such a path is given by the following Lemma 4.4.1, due to Landau and Cutkosky [6] [8] (3).

LEMMA 4.4.1. — Let G be a massive Feynman graph with normal quotient graph Q. Then there exists a path δ: $[0, 1] \to W_G$ such that

1) $\delta(0) \in R_G$.

2) F_G^{phys} can be analytically continued in s, z along $\delta([0, 1])$. The analytic continuation is obtained directly from the definition of F_G^{phys} as an integral (2.2.5), without any need to distort the contour of integration (Recall that, according to Lemma 2.2.38, there exists for a massive graph G a region $\Lambda^0 \subset \Lambda_G$ such that for $(s, z) \in R_G$, $(\lambda, v) \in \Lambda^0$ (2.2.2) is absolutely convergent).

We denote this analytic continuation also by F_G^{phys}.

3) $\delta(1) \in L_Q^l$, and L_Q^l is the only singularity of F_Q^{phys} in the neighbourhood of this point.

(3) It should, however, be noted that the proof given in [6] is incomplete.

Proof. — Choose external momenta p_u^0 which are purely imaginary and span a space of dimension $n(G)^E - 1$ and denote by $s^0(\chi)$ the corresponding invariants. Then we define

$$s(\chi)(t) = (2t - 1)s^0(\chi) \qquad t \in [0, 1]$$

so that $s(\chi)(0) = - s^0(\chi) > 0$ as required by 1). The mass variables z_i will remain constant along the path

$$z_i(t) = z_i^0 \qquad t \in [0, 1].$$

To construct suitable z_i^0 we begin by choosing an open set V contained in the interior of the region $\gamma_{0Q} \{ (\alpha) \mid \alpha_i \geq 0,\ i \in \Omega_Q \}$. The map π_1 introduced in (4.1.1) maps the set

$$U(V) = \left\{ (\alpha, s^0, z) \mid \frac{\partial D}{\partial \alpha_i} = 0,\ i \in Q,\ \alpha \in V \right\} \subset W_Q \times \mathbb{P}^{N(Q)-1}$$

onto an open set $K \subset L_Q^l(s^0)$, the section of L_Q^l defined by $s = s^0$ (in the induced topology). Since Q is normal we may assume (by replacing V by a smaller open set if necessary) that π restricted to U(V) is non-singular, so that there exists a map $\eta : K \to P^{N(Q)-1}$ of K onto V such that

$$(\eta(s^0, z), s^0, z) \in U(V) \quad \text{and} \quad \pi_1(\eta(s^0, z), s^0, z) = (s^0, z).$$

Denote by $f_Q^l(s, z)$ an irreducible polynomial in (s, z) such that

$$L_Q^l = \{ (s, z) \mid f_Q^l(s, z) = 0 \}.$$

From the definition of L_Q^l in Section 4.1 it is easy to check that for no $i \in \Omega_Q$ is $\frac{\partial}{\partial z_i} f_Q^l \equiv 0$ so that the set

$$\left\{ (s, z) \mid f_Q^l(s, z) = 0 \quad \frac{\partial}{\partial z_i} f_Q^l = 0 \quad \text{for some} \quad i \in \Omega_Q \right\}$$

is of complex codimension 1 in L_Q^l, and correspondingly the intersection of this set with $s = s^0$ is for generic s^0 of codimension 1 in $L_Q^l(s^0)$. Thus, by changing s^0 a little and replacing V by a smaller open set if necessary, we may assume $\frac{\partial}{\partial z_i} f_Q^l(s^0, z) \neq 0$ for all $i \in \Omega_Q$ and $(s^0, z) \in K$.

Now for any $(s^0, z) \in K$ and $\alpha \in \gamma_{0Q}$ we have by Lemma 4.2.2

$$D_Q^*(\alpha, s^0, z) \geq D_Q'(\alpha, g(p^0, \eta(s^0, z))) = 0 \tag{4.4.2}$$

since $g_i^2(p^0, \eta(s^0, z)) - z_i = 0$ for all $i \in \Omega_Q$. We next show that by changing z_Q a little, if necessary, we may arrange that (4.4.2) holds with equality only for points in the interior of γ_{0Q}. We proceed inductively to remove the zeros of D_Q^* from the faces of the simplex γ_{0Q}. Suppose then that it is known that $D_Q^*(\alpha, s^0, z) > 0$ for $\alpha \in \gamma_{0Q} \cap \{ (\alpha) \mid \alpha_i = 0 \text{ for some } i \in \chi \}$ where χ is a subset of Ω_Q. We begin the induction with $\chi = \emptyset$ so that at the beginning of the induction this statement holds vacuously and induces on $|\chi|$. Let j be an index $\notin \chi$. We have to show that z_Q can be changed a little to give $D_Q^*(\alpha, s^0, z) > 0$ also for $\alpha \in \gamma_{0Q} \cap \{ (\alpha) \mid \alpha_j = 0 \}$. Since

$$D_Q^*(\alpha, s^0, z) = D_Q^s(\alpha, s^0)/d_Q(\alpha) - \sum_{i \in Q} \alpha_i z_i,$$

it is only necessary to decrease the z_i, $i \neq j$, by an arbitrarily small amount. Since $\frac{\partial}{\partial z_j} f_Q^l(s^0, z) \neq 0$ for $(s^0, z) \in K$ it is then possible to change z_j appropriately so that we still have $(s^0, z) \in K$. Since each of the faces γ_{0Q}^i, $i \in \chi$, previously considered are compact sets on which the function $D_Q^*(\alpha, s^0, z)$ is positive, $D_Q^*(\alpha, s^0, z)$ will have a positive minimum on the union of these faces. This minimum is a continuous function of z so if the change in z which we have made to remove the zeros on γ_{0Q}^j is sufficiently small it will not reintroduce zeros on the faces γ_{0Q}^i, $i \in \chi$. This completes the proof of the induction step, and hence of the assertion that we can choose $(s^0, z^0) \in L_Q^l$ so that in (4.4.2) equality holds for $z = z^0$ only for points in the interior of γ_{0Q}.

Finally we must choose the z_i^0, $i \notin \Omega_Q$. From (4.2.5) it is easy to show that $\frac{\partial}{\partial \alpha_i}\left(\frac{D^s}{d}\right) = g_i^2(\alpha, p^0)$ is continuous in α on γ_{0G}. Hence we may choose the z_i^0, $i \notin \Omega_Q$, sufficiently large and negative that, $\forall i \notin \Omega_Q$ and $\alpha \in \gamma_{0G}$, $\frac{\partial}{\partial \alpha_i} D^*(\alpha, s^0, z^0) > 0$. Since $D_Q^*(\alpha, s^0, z^0)$ is the boundary value of $D^*(\alpha, s^0, z^0)$ as $\alpha_i \to 0$ for $i \notin \Omega_Q$ and (4.4.2) holds, $D^*(\alpha, s^0, z^0) > 0$ for $\alpha \in \gamma_{0G} \cap \{ (\alpha) \mid \alpha_i \neq 0 \text{ for some } i \notin \Omega_Q \}$.

We have now completed the construction of the path δ. The verification of (1)-(3) is immediate.

Next we need an elementary lemma on determinants. We omit the proof.

LEMMA 4.4.3. — Let A be a $N \times N$ symmetric matrix of rank $N - 1$, (α_i) a non-zero null vector for A. Then

$$\Delta(A) = \alpha_i^{-2} \det A\binom{i}{i} \qquad (4.4.4)$$

is independent of i.

We are now ready to state the main theorem of this section.

THEOREM 4.4.5. — Let G be a massive Feynman graph with normal quotient graph $Q = G/S$. Then $L_Q^l \subset L_G$ and in the neighbourhood of a point $(s^0, z^0) \in L_Q^\rho$ constructed as in Lemma 4.4.1 we have a local decomposition

$$F_G^{phys}(s, z) = (R_Q^l)^{\nu_Q} H_G(s, z) + K_G(s, z) \tag{4.4.6}$$

where $R_Q^l = 0$ is a local equation of L_Q^l. H_G and K_G are holomorphic in (s, z) in the neighbourhood of (s^0, z^0). Moreover there is a rational map η: $L_Q^l \to W_S$ such that

$$H_G(s, z; \lambda, \nu)\,|_{L_Q^l} = (\text{certain factors})\, F_S^{phys}(\eta(s, z)). \tag{4.4.7}$$

The explicit form and the meaning of the factors appearing in (4.4.7) will be given in the course of the proof. The exponent ν_Q in (4.4.6) is given by

$$\nu_Q = h(Q)\frac{\nu}{2} - \sum_{i \in Q_Q} \lambda_i + \left(\frac{N(Q) - 1}{2}\right) - 1. \tag{4.4.8}$$

Proof. — Without loss of generality we may assume that G is connected. Consider first the case $N(Q) = 1$ so that $Q = \{ l_\omega \}$ for some $\omega \in \Omega_G$. If $h(Q) = 1$, $G - l_\omega$ is connected and the present theorem reduces to Theorem 3.1. If $h(Q) = 0$, $G - l_\omega = G_1 \cup G_2$ is the disjoint union of two connected components. Denote by χ the set of $j \in \Theta^E$ such that $V_j \in G_1$. Then with the convention that $s(\chi) = 0$ if $\chi = \emptyset$ or Θ^E it is easy to show that

$$F_G^{phys}(s, z) = f_Q(\lambda, \nu)(s(\chi) - z_\omega)^{-1-\lambda_\omega} F_{G-l_\omega}(s, z), \tag{4.4.9}$$

so that (4.4.6) holds with $K_G \equiv 0$. Thus the theorem is proved in the case $N(Q) = 1$ and we will therefore assume $N(Q) > 1$. Then according to Lemma 2.2.38 there is a region $\Lambda^0 \subset \Lambda_G$ such that for $(\lambda, \nu) \in \Lambda^0$ the integral (2.2.5) is absolutely convergent for $(s, z) \in R_G$, and also Re $\nu_Q > 0$. We will prove the theorem for $(\lambda, \nu) \in \Lambda^0$. After this has been done we obtain the theorem for all $(\lambda, \nu) \in \Lambda_G$ by analytic continuation of both sides of (4.4.6) in (λ, ν).

If δ is the path constructed in Lemma 4.4.1 we refer to a real point (s, z) in the neighbourhood of $(s^0, z^0) \in L_Q^l$ as a point *below* L_Q^l if it can be joined in this neighbourhood to a point of δ without crossing L_Q^l; otherwise it will be said to be *above* L_Q^l. Then the analogue of Lemma 3.11 holds:

LEMMA 4.4.10. — For $(\lambda, \nu) \in \Lambda^0$, $F_G^{phys}(s, z)$ can be analytically continued around L_Q^l. The discontinuity of F_G^{phys} around L_Q^l (i. e. the difference between the function obtained by continuation of F_G^{phys} along a loop circling L_Q^l anticlockwise and the function F_G^{phys} itself) can be analytically continued around L_Q^l along a clockwise path to real (s, z) values just above L_Q^l. For (s, z) real and just above L_Q^l it is given by

$$\text{disc } F_G^{phys} = (\exp 2\pi i\mu - 1)J_G. \qquad (4.4.11)$$

Here J_G is the absolutely convergent integral defined by replacing γ_0 in (2.2.5) by

$$\gamma_0' = \{ (\alpha) \mid D(s, z, \alpha) \leq 0 \} \cap \gamma_0. \qquad (4.4.12)$$

We omit the proof since it is essentially a copy of the proof of Lemma 3.11 (without the complications caused by the need to transform to the x variables in that case).

Returning to the proof of the main theorem, we now make some changes of integration variables in J_G, which will enable the local power behaviour of J_G to be displayed explicity.

We single out a line $l_\omega \in Q$ and set

$$\begin{aligned} \alpha_i &= \gamma_i, & i &\in \Omega_S, \\ \alpha_i &= x^{-1}, & i &= \omega, \\ \alpha_i &= x^{-1}\beta_i, & i &\in \Omega_Q^\omega = \Omega_Q - \{ \omega \}. \end{aligned} \qquad (4.4.13)$$

Since Q is supposed normal $\pi_1^{-1}((s^0, z^0))$ consists of just one point (α^0, s^0, z^0). In the transformed variables this point is given by $x = 0$ and $\beta_i = \beta_i^0 = \alpha_i^0/\alpha_\omega^0$, $i \in \Omega_Q^\omega$. According to Lemmas 4.1.6 and 4.2.15 and the construction of (s^0, z^0) in Lemma 4.4.1, the Hessian $H_{ij} = \dfrac{\partial^2 D_Q^*}{\partial\alpha_i \partial\alpha_j}$ is for $\alpha = \alpha^0$ non-negative and of rank $N - 1$. We may therefore choose ω so that the corresponding submatrix H^ω defines a positive definite quadratic form

$$H^\omega(u) = \sum_{i,j} H_{ij}u_i u_j \qquad i, j \in \Omega_Q^\omega.$$

Then

$$J_G = f_G \int_{\gamma_{0S}'} \prod_{i\in\Omega_S} \gamma_i^{\lambda_i}(\eta(\gamma)) \int_{\gamma_{0Q}'} \prod_{i\in\Omega_Q^\omega} \beta_i^{\lambda_i} \frac{d_n^{-\nu/2}}{[D_n^*]^\mu} x^\rho d\beta dx \qquad (4.4.14)$$

where

$$\begin{aligned} \gamma_{0Q}' &= \{ (x, \beta) \mid D_n(s, z, x, \beta, \gamma) \leq 0,\ \beta_i \geq 0,\ i \in \Omega_Q^\omega,\ x \geq 0 \} \\ d_n &= x^{h(Q)}d \\ D_n^* &= xD^* \\ \rho &= -\sum_{i\in\Omega_Q} (\lambda_i + 1) - h(Q)\frac{\nu}{2} + \mu - 1 \end{aligned}$$

We may define uniquely holomorphic functions $\bar{\beta}_i^0 = \bar{\beta}_i^0(s, z)$ of s, z in the neighbourhood of (s^0, z^0) by the conditions

$$\left.\frac{\partial D_Q^*}{\partial \beta_i}\right|_{\beta=\bar{\beta}^0} = 0, \qquad i \in \Omega_Q^{\omega}, \tag{4.4.15}$$
$$\bar{\beta}_i^0(s^0, z^0) = \beta_i^0.$$

This follows from the implicit function theorem. We then write

$$R_Q^l(s, z) = - D_Q^*(\bar{\beta}^0, s, z) \tag{4.4.16}$$

and note that $R_Q^l = 0$ is a local equation for L_Q^l (since $\frac{\partial}{\partial z_i} R_Q^l = \beta_i^0 \neq 0$, for $(s, z) \in L_Q^l$).

Now we make a further change of variables

$$x = kx', \tag{4.4.17}$$
$$\beta_i = \bar{\beta}_i^0 + v_i x' + \sum_{j \in \Omega_Q^{\omega}} A_{ij} y_j, \qquad i \in \Omega_Q^{\omega}.$$

The coefficients k, v_i, A_{ij} are so chosen that

$$D_n^*(x', y) = D_n^* |_{x=0,\beta=\bar{\beta}^0} \left[1 - \sum_{j \in \Omega_Q^{\omega}} y_j^2 - x' + S(x', y) \right]$$
$$= - R_Q^l \left[1 - \sum_{j \in \Omega_Q^{\omega}} y_j^2 - x' + S(x', y) \right] \tag{4.4.18}$$

since by Lemma 4.3.4 *b*) $D_n^* |_{x=0} = D_Q^*$. $S(x', y)$ is a polynomial in x', y with coefficients (depending on s, z, γ) which vanish for $R_Q^l = 0$, with zero derivatives up to the first order in x' and the second order in y at $x' = y = 0$. The equations which give the coefficients k, v_i, A_{ij} are

$$k = - \left.\frac{D_n^*}{\dfrac{\partial D_n^*}{\partial x}}\right|_{x=0,\beta=\bar{\beta}^0} \tag{4.4.19}$$

$$v_i = - k \sum_{j \in \Omega_Q^{\omega}} (H_{\omega}^{-1})_{ij} \left.\frac{\partial^2 D_n^*}{\partial \beta_j \partial x}\right|_{x=0,\beta=\bar{\beta}^0} \tag{4.4.20}$$

$$\sum_{j,k \in \Omega_Q^{\omega}} A_{jl} A_{km} (H_{\omega})_{lm} |_{\beta=\bar{\beta}^0} = (- D_n^*) |_{x=0,\beta=\bar{\beta}^0} \delta_{lm}. \tag{4.4.21}$$

We will need only to compute the Jacobian $\frac{\partial(x, \beta)}{\partial(x', y)} = k \det A$. (4.4.21) gives for det A the equation

$$\det A = (R_Q^l)^{\frac{N(Q)-1}{2}} (\det H^{\omega})^{-\frac{1}{2}} |_{\beta = \bar{\beta}^0} \tag{4.4.22}$$

where we have again used Lemma 4.3.4 *b*) to write $D_n^* |_{x=0} = D_Q^*$. From Lemma 4.3.4 *b*) and *c*) we have

$$k = (R_Q^l)\left(\sum_i D_{S_i}^*(\gamma, \zeta_i(\bar{\beta}^0, s, z))\right)^{-1} \tag{4.4.23}$$

where the summation is over the connected components S_i of S, and the maps ζ_i are defined immediately preceeding Lemma 4.3.4. Finally Lemma 4.3.4 *a*) gives us

$$d_n(x', y) = d_Q(\bar{\beta}^0)\prod_i d_{S_i}(\gamma) + T(x', y) \tag{4.4.24}$$

where the coefficients of the polynomial T vanish for $R_Q^l = 0$, and we have made the convention that $\bar{\beta}_\omega^0 = 1$.

After the second change of variables we have

$$J_G = (R_Q^l)^{\nu_Q} H_G(s, z) (\exp 2\pi i\nu_Q - 1) (\exp 2\pi i\mu - 1)^{-1} \tag{4.4.25}$$

with

$$H_G = \frac{2if_G \sin \pi\mu}{(\exp 2\pi i\nu_Q - 1)} \int_{\gamma_{0S}} \prod_{i \in \Omega_s} \gamma_i^{\lambda_i} \eta(\gamma) \int_{\gamma'} \prod_{i \in \Omega_Q^0} \frac{\beta_i^{\lambda_i} d_n^{-\nu/2} x'^{\rho} (\det H^{\omega})^{-\frac{1}{2}} dx' dy}{\left(\sum_i D_{S_i}^*\right)^{\mu} S \left(1 - \sum_j y_j^2 - x' + S\right)^{\mu}}$$

where $\gamma' = \left\{(x', y) \mid x' \geq 0,\ 1 - \sum_j y_j^2 - x' + S \geq 0\right\}$. $H_G(s, z)$ is holomorphic in s, z in the neighbourhood of (s^0, z^0). This is not quite manifest in (4.4.26) since the integrand contains $(R_Q^l)^{\frac{1}{2}}$ e. g. in S. However, one notes that a change in the determination of the square root followed by the change $y' = -y$ of integration variable leaves the integral invariant so that the integral is single valued in the neighbourhood of L_Q^l. It is also bounded (we will write down its value for $R_Q^l = 0$ below) so the singularity $R_Q^l = 0$ is removable i. e. $H_G(s, z)$ is holomorphic in s, z as asserted,

If $R_Q^l = 0$ the integral over γ' reduces to

$$\int_{\gamma'} \frac{x'^{\rho} dx' dy}{\left(1 - \sum_j y_j^2 - x'\right)^{\mu}} = \frac{\Gamma(\mu_S)\Gamma(1 - \mu)\pi^{(N(Q)-1)/2}}{\Gamma(1 - \mu_Q + (N(Q) - 1)/2)} \tag{4.4.27}$$

after a short calculation. Here we have used the fact that

$$\rho + 1 = \mu_S \qquad \mu_S + \mu_Q = \mu.$$

The integral over γ_{0S} gives us a product of Feynman amplitudes for the graphs S_i

$$f_S \int_{\gamma_{0S}} \prod_{i \in \Omega_S} \gamma_i^{\lambda_i} \frac{\prod_k [d_{S_k}(\gamma)]^{-\nu/2}}{\left(\sum_k D^*_{S_k}\right)^{\mu_S}} \eta(\gamma) = \prod_{k=1}^{c(S)} F^{phys}_{S_k}(\zeta_k(\beta^0, s, z))$$

To restore the symmetry which was lost when we singled out the index $\omega \in \Omega_Q$ we make use of Lemma 4.4.3 to write

$$\det H^\omega = \Delta(H)\,|_{\beta=\beta^0}$$

and we insert a power of $\beta^0_\omega = 1$ to give an expression which is homogeneous of degree zero in β^0. The normalization $\beta^0_\omega = 1$ can then be dropped, and we have finally a symmetric expression. This is

$$H_G\,|_{L^l_Q} = \frac{f_Q(\lambda, \nu) \exp(-i\pi\nu_Q)\pi^{(N(Q)+1)/2}}{\Gamma(\mu_Q)\sin(\pi\nu_Q)\Gamma(1-\mu_Q+(N(Q)-1)/2)}$$
$$\times \prod_{i\in\Omega_Q} (\beta_i^0)^{\lambda_i} [d_Q(\beta^0)]^{-\nu/2} [\Delta(H)]^{-\frac{1}{2}}_{\beta=\beta^0} \prod_{k=1}^{c(S)} F^{phys}_{S_k}(\zeta_k(\beta^0, s, z)). \quad (4.4.28)$$

From the theory of elimination [*18*], it follows that the β_i^0, $i \in \Omega_Q$, regarded as functions defined on L^l_Q are rational so that (4.4.28) has the form (4.4.7).

The meaning of the factors in (4.4.28) is the following: they become singular at points on L^l_Q, which are effective intersections of L^l_Q with other components of L_G or effective self-intersections of L^l_Q (cuspidal points). For example if $\beta_i^0 = 0$, (s, z) is a point of intersection of L^l_Q with $L^l_{Q'}$, where Q' is a quotient of Q in which l_i is contracted. If $d_Q(\beta^0)(s, z) = 0$ (s, z) is a point of intersection of L^l_Q with some anomalous Landau or second kind singularity. If $\Delta(H)(\beta^0)(s, z) = 0$, (s, z) may be a cuspidal point of L^l_Q (The relation of zeros of the Hessian to cuspidal points is discussed in [*22*] ([4]).

([4]) One of us (M. J. W.) wishes to point out two errors in [*22*]. First, the condition $\tau = 0$ does not imply $G \in C$ for a two-particle scattering graph G as stated in the introduction (although the converse is true as stated). Secondly, the final remark on the Landau curve of the crossed square graph is incorrect. On this point the reader should consult [*19*]. The error lies in the fact that the component corresponding to asymmetric dual diagrams degenerates, which was overlooked.

It should be noted, however, that the Hessian used here does not have a simple relation to the Hessian in momentum space used in [22]. $\Delta(H)$ factorizes and not all the factors correspond to cuspidal points).

Finally we define $K_G = F_G^{phys} - (R_Q^l)^{\nu_Q} H_G$. Then K_G is single valued in the neighbourhood of L_Q^l, and bounded (since Re $\nu_Q > 0$ for $(\lambda, \nu) \in \Lambda^0$) and so holomorphic.

This completes the proof of Theorem 4.4.5.

5. NORMAL GRAPHS

5.1. The meaning of normality

In Section 4.1 we have given a definition of the leading Landau variety L_Q^l of a normal Feynman graph Q. However, it is customary to introduce the notion of leading Landau variety for a wider class of graphs (see e. g. [23]). These graphs, which we will call Landau graphs, are characterized as follows:

DEFINITION 5.1.1. — A massive Feynman graph Q ([5]) is a *Landau* graph if for some integer m there exists a point $(p, z, q) \in X_Q^m \times Z_Q \times Y_Q^m$ such that

$$q_i^2 = z_i \qquad \forall i \in \Omega_Q \tag{5.1.2}$$

and the loop equations

$$\sum_{i \in l} \alpha_i q_i = 0 \qquad \forall \text{ loops } l \text{ in } Q \tag{5.1.3}$$

have a *unique* solution $\alpha_i = \alpha_i^0$, with $\alpha_i^0 \neq 0$, $\forall i \in \Omega_Q$.

Remark 5.1.4. — If Q is a Landau graph we could define the leading Landau variety L_Q^l of Q to be the closure in W_Q of the set of invariants (s, z) such that there exists a point (p, z, q) satisfying the condition of the preceding definition with s related to p by (2.2.3). But it is not clear that L_Q^l so defined would be an irreducible variety, and in particular it is not clear that for Q normal this definition would reduce to Definition 4.1.1. Also we would wish to show that if G is a Feynman graph having Q as quotient $L_Q^l \subset L_G$ (cf. Theorem 4.4.5). Since we have not settled these points we prefer not to give a formal definition of L_Q^l for an arbitrary Landau graph Q.

([5]) As in Section 4, we consider only massive graphs in this section.

DEFINITION 5.1.5. — A Landau graph Q which is not normal is *anomalous.*

We would like to be able to recognize when a given graph Q is a Landau graph, and if it is a Landau graph when it is normal. Concerning the first problem, we have the following remark due to Rudik and Okun [*24*].

Remark 5.1.6. — If Q has a subgraph S such that one of the following conditions (*a*)-(*d*) holds then Q is not a Landau graph.

(*a*) S does not have two distinct vertices joined (in Q) to vertices not in S; i. e., Q must be 2-connected.

(*b*) S has two external vertices, one internal vertex and two lines (fig. 2 *a*).

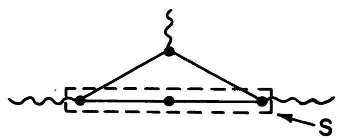

Fig. 2a

(*c*) S has two external vertices. The star of one of these vertices contains at most one line not in S and is empty if the vertex is external in Q (fig. 2 *b*).

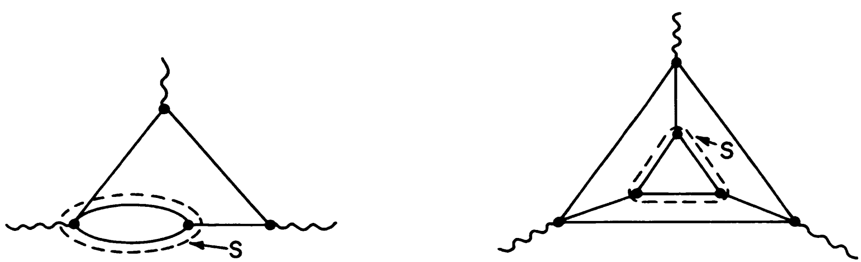

Fig. 2b Fig. 2c

FIG. 2. — Non-Landau graphs.

(*d*) S has three external vertices. The star of each of these vertices contains at most one line not in S and is empty if the vertex is external in Q (fig. 2 *c*).

We will concern ourselves here with the second problem. Define, for any Q,

$$\tau_Q = (n(Q)^E - 1)h(Q) - N(Q) + 1. \qquad (5.1.7)$$

In Section 4.2 we proved that Q is normal iff for generic α the matrix T_Q^* has rank $N - 1$. Since the $(n(Q)^E - 1)h(Q)$ columns of T_Q^* indexed by

jrs, with j in some cotree, $r = r_0$ fixed and $s \neq r$, span the set of column vectors of T_Q^* we have proved

LEMMA 5.1.8. — If Q is normal $\tau_Q \geq 0$. This lemma already suffices to show that any Q with two external vertices and one or more internal vertices is not normal. In fact we have

LEMMA 5.1.9. — If Q contains a subgraph S with two external vertices which is not normal, Q is not normal.

We omit the proof.

In the following Section 5.2 we use Theorem 4.2.20 to derive sharper criteria for Q to be normal. In fact we do not know of a graph Q which can be shown to be normal by direct study of the Landau equations which cannot be shown to be normal by the reduction procedures of Section 5.2, but we have not been able to prove that the normal graphs are precisely those to which the methods of Section 5.2 apply.

5.2. **Criteria for normality**

DEFINITION 5.2.1. — A *reduction* O is a rule which assigns to a triple $\{ G, S, \omega \}$, G a 2-connected Feynman graph, S a subgraph of G, $\omega \in \Omega_S$, a Feynman graph G′ having fewer lines than G. $O(G, S, \omega) = G'$ may be defined only for S, ω satisfying certain conditions. If $S = H(\{ \omega \})$ we omit reference to S and write $G' = O(G, \omega)$.

DEFINITION 5.2.2. — If reduction O is *admissable* if « $G' = O(G, S, \omega)$ is normal » implies « G is normal ».

The condition in Definition 5.2.1 that G be 2-connected is made for convenience. Since a normal graph is necessarily 2-connected (Lemma 4.1.5) only 2-connected graphs need be considered in discussing normality.

LEMMA 5.2.3. — An externally complete Feynman graph G with $h(G) = 1$ is normal.

Proof. — The matrix T_G defined in Section 4.2 is in this case independent of α. Its determinant is easy to evaluate and turns out to be non-zero so Theorem 4.2.20 gives the normality of G.

Remark 5.2.4. — The case in which G is a tadpole is understood to be subsumed under Lemma 5.2.3. Then we regard G as trivially externally complete.

We can now state our criterion for a graph G to be normal.

CRITERION 5.2.5. — A graph G is normal if there exists a sequence of triples $\{ G_i, S_i, \omega_i \}$, and admissable reductions O_i, $1 \leq i \leq r$, such that

(1) $G_1 = G$.

(2) For all i, $O_i(G_i, S_i, \omega_i)$ is defined. For $i < r$, $O_i(G_i, S_i, \omega_i) = G_{i+1}$.

(3) $G_{r+1} = O_r(G_r, S_r, \omega_r)$ is an externally complete single loop graph. This criterion is saved from triviality by the existence of a number of admissable reductions O.

THEOREMS 5.2.6. — The following reductions O are admissable

(1) Contraction.

$$O_{con}(G, \omega) = G/H(\{ \omega \})$$

if the end points of l_ω are external and l_ω is not a tadpole.

(2) Cutting.

$$O_{cut}(G, \omega) = H(\Omega_G - \{ \omega \})$$

l_ω is not a tapdole.

Proof. — Write $G' = O(G, \omega)$. According to Theorem 4.2.20 the condition that G' be normal is equivalent to the assertion that rank $T_{G'} = N(G') = N - 1$ for generic α, and we have to show that this implies rank $T_G = N$ for generic α. The proof in both cases (1) and (2) of the present theorem (and also in each of the cases of Theorem 5.2.7 below) is based on a combination of the following simple observations on the rank for generic α of a matrix B whose coefficients are functions of α:

(*i*) rank $B \geq$ rank B', for B' a submatrix of B;
(*ii*) rank B = rank B', if B' is equivalent to B for generic α;
(*iii*) rank $B \geq$ rank B', if B' is a specialization of B;
(*iv*) rank B = rank B' + rank B'', if B has block triangular form with diagonal blocks B' and B''.

We now consider case (1). Denote by $u, v \in \Theta^E$ the end-points of l_ω. We consider the specialization of T_G defined by setting $\alpha_\omega = 0$. Refering to the explicit form of the matrix elements $T_{i,jus}$ given by (4.2.9) and (4.2.6) we note that they are sums of terms which correspond to certain trees T. The specialization $\alpha_\omega = 0$ does not change $T_{i,jus}$ if $i = \omega$ (since the differentiation removes the α_ω dependence of g_{jus}) and eliminates the terms corresponding to trees T not containing ω if $i \neq \omega$. We observe that for $j \neq \omega$

$T_{i,jus}|_{\alpha_\omega = 0}$ = *a* not identically zero polynomial in the remaining α's if $i = \omega$, $(r, s) = (u, v)$ and some $j = j_0$ (since G is 2-connected Lemma 2.1.6 gives a path from u to v which does not pass through l_ω)

$$= 0 \quad \text{if} \quad i \neq \omega,\ (r, s) = (u, v)$$
$$= T_{G';i,j\pi_{G'}(r)\pi_{G'}(s)} \quad \text{if} \quad i \neq \omega,\ (r, s) \neq (u, v)$$

i. e. this specialization of T_G has a submatrix which is of block triangular form with diagonal blocks $B' = T_G$, and B'' a 1×1 matrix which is non zero. This observation thus completes the proof of the theorem in Case 1.

In case (2), we note that because g_{jrs} is homogeneous of degree $h(G)$ and at most linear in α_ω, we have

$$k_{jrs} = \sum_{i \neq \omega} \alpha_i T_{i,jrs} = h g_{jrs}|_{\alpha_\omega = 0} + (h - 1)\alpha_\omega \frac{\partial}{\partial \alpha_\omega} g_{jrs}. \tag{5.2.7}$$

The matrix T is equivalent to the matrix T′ given by

$$T'_{\omega,jrs} = g_{jrs}|_{\omega=0} = h^{-1}[(h - 1)\alpha_\omega T_{\omega,jrs} - k_{jrs}]$$
$$T'_{i,jrs} = \alpha_\omega^{-1} T_{i,jrs} \qquad (i \neq \omega).$$

Now we specialize $\alpha_\omega \to \infty$ in T′. Since

$$\lim_{\alpha_\omega \to \infty} T'_{i,jrs} = \frac{\partial}{\partial \alpha_\omega} T_{i,jrs} = T_{G';i,jrs} \qquad (i, j \neq \omega)$$

and since $T_{i,\omega rs}$ is independent of α_ω, we obtain in the limit a block diagonal form with diagonal blocks $B' = T_{G'}$ and $B''_{rs} = T'_{\omega,\omega rs} \neq 0$ (since any r, s may be joined by some path through ω, because G is 2-connected). This completes the proof.

As an immediate application of Theorem 5.2.6, we have

THEOREM 5.2.8. — Any externally complete 2-connected Feynman graph G is normal.

Proof. — The proof is by induction on N(G). If N(G) = 1, G is normal. We note that either of the reductions of Theorem 5.2.6 preserves the property of external completeness. Moreover, if N(G) > 1, for any line l_ω of G either $G/H(\{\omega\})$ or $H(\Omega_G - \{\omega\})$ is 2-connected. This completes the proof.

The methods of Theorem 5.2.6 do not suffice for our purpose since there are many normal graphs for which the application of O_{con} or O_{cut} to any line produces a non-normal graph. We therefore introduce other admissable reductions.

DEFINITION 5.2.9. — A subgraph S of a Feynman graph G is k-accessible if, for any $S' \supset S$, $|\Theta^E_{S'}| \geq k$ (For example, a 3-accessible subgraph is not contained in any self-energy subgraph).

LEMMA 5.2.10. — If $S \subset G$ is k-accessible, there exist k disjoint paths (possibly of zero length) which are disjoint from S and which join k distinct external vertices of S to k distinct external vertices of G.

Proof. — We apply a result of [9, p. 205] (actually a stronger version of Lemma 2.1.6) to the graph G^{χ}/S.

DEFINITION 5.2.11. — We define the five Feynman graphs T_1, T_2, H_1, H_2, H_3 as in figure 3. Suppose that G is a 2-connected Feynman graph with a subgraph S isomorphic to one of these graphs (For notational

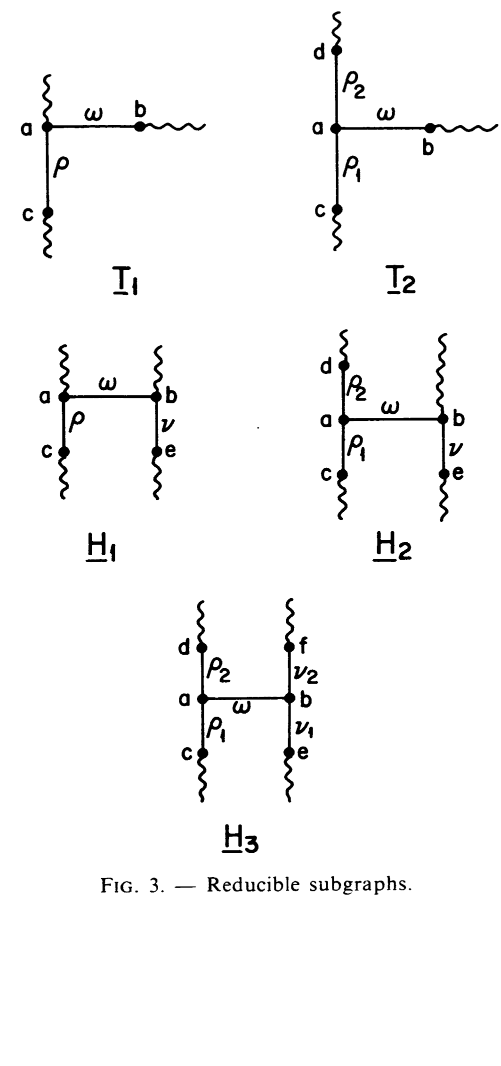

FIG. 3. — Reducible subgraphs.

convenience we assume this isomorphism is actually the identity). Suppose further that

$$\begin{aligned} St_G(a) &= St_S(a) \quad \text{for any such S,} \\ St_G(b) &= St_S(b) \quad \text{if} \quad S = H_1, H_2, \text{ or } H_3. \end{aligned}$$

Then we define the reductions $O_{T_i}(G, S, \omega)$ and $O_{H_i}(G, S, \omega)$ for $S = T_i$, $S = H_i$ respectively) by figure 4.

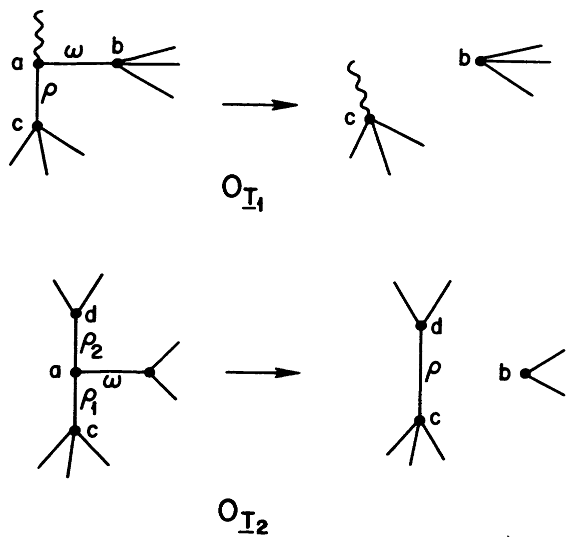

Fig. 4a

THEOREM 5.2.12. — The reductions $O_{T_i}(G, S, \omega)$ and $O_{H_i}(G, S, \omega)$ are admissable if S is 3-accessible or 4-accessible, respectively.

Proof. — We give the proofs for O_{T_1} and O_{H_3}. The other proofs are similar (and of intermediate complexity).

We first consider O_{T_1}. We replace T by the equivalent matrix (compare the proof of part (2) of (5.2.6))

$$T'_{i,jrs} = \begin{cases} g_{jrs}|_{\alpha_\omega = 0} & \text{if} \quad i = \omega \\ T_{\rho,jrs} & \text{if} \quad i = \rho \\ \alpha_\omega^{-1} T_{i,jrs} & \text{if} \quad i \neq \omega, \rho \,. \end{cases}$$

Now in the submatrix $\{ T'_{i,jrs} \,|\, j \neq \rho \}$ we specialize $\alpha_\omega \to \infty$. Since

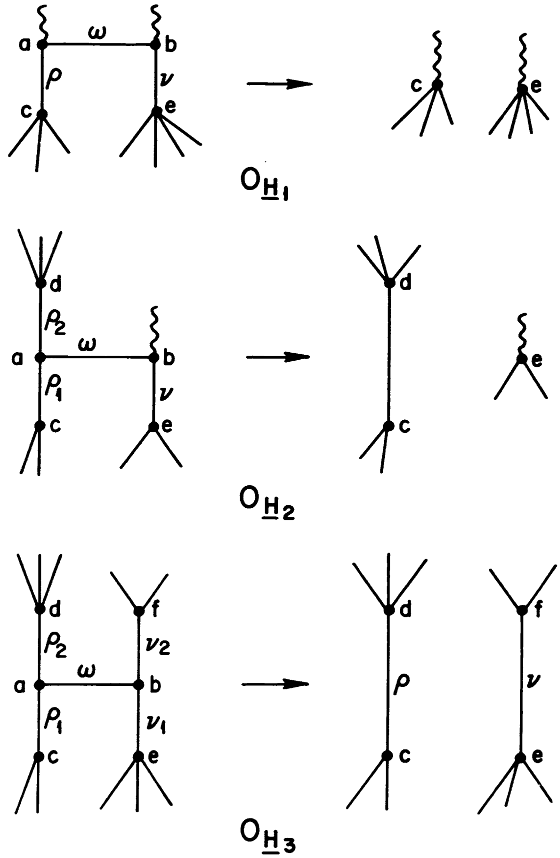

Fig. 4b

FIG. 4. — Reduction of subgraphs.

$T'_{\rho,jrs}T'_{\omega,jrs}$ and $T_{i,\omega rs}$ are independent of α_ω, this produces a block triangular form. But for $i, j \neq \omega, \rho$,

$$\lim_{\alpha_\omega \to \infty} T'_{i,jrs} = \frac{\partial}{\partial \alpha_\omega} T_{i,jrs} = \begin{cases} T_{G';i,jrs} & \text{if } r, s \neq a \\ T_{G';i,jrc} & \text{if } s = a \\ T_{G';i,jcs} & \text{if } r = a, \end{cases}$$

so that it remains only to prove that the other block, the 2 by $(n^E(G))^2$ matrix

$$B' = \begin{bmatrix} g_{\omega rs} \\ T_{\rho,\omega rs} \end{bmatrix}$$

$(r, s \in \Theta^E_G)$, has rank 2.

By Lemma 5.2.10 there are external vertices V_u, $V_v \neq V_a$ of G and disjoint paths from V_c to V_u, V_b to V_v. Let T be a tree of G which contains these paths together with ω and ρ, and specialize B′ by setting $\alpha_i = 0$, $i \in T$, $i \neq \omega$. Then we find that the ωua and ωuv columns of B′ have the form

$$\begin{array}{cc} \omega ua & \omega uv \\ \text{—} & \text{—} \\ \alpha(\Omega_G - T) & \alpha(\Omega_G - T) \\ \alpha(\Omega_G - T)\left[\sum_{j\in\Omega_G - T} \alpha_j^{-1}\right] & 0 \end{array}$$

Hence this 2×2 submatrix of B′ has non zero determinant, which completes the proof.

We now consider O_{H_3}. We replace T by the eqúivalent matrix T′

$$\begin{aligned} T'_{i,jrs} &= g_{jrs}|_{\alpha_\omega = 0} && \text{if } i = \omega \\ &= T_{\rho_2,jrs} - T_{\rho_1,jrs} && \text{if } i = \rho_2 \\ &= T_{\nu_2,jrs} - T_{\nu_1,jrs} && \text{if } i = \nu_2 \\ &= \alpha_\omega^{-1} T_{i,jrs} && \text{if } i \neq \omega, \rho_2, \nu_2 \end{aligned}$$

Then in the submatrix $\{ T'_{i,jrs} \,|\, j = \rho_2, \nu_2 \}$ we specialize $\alpha_\omega \to \infty$. Since $T'_{i,jrs}$, $i = \omega, \rho_2, \nu_2$, and $T_{i,\omega rs}$ are independent of α_ω, this produces a block triangular form. But for $i, j \neq \omega, \rho_2, \nu_2$

$$\lim_{\alpha_\omega \to \infty} T'_{i,jrs} = \frac{\partial}{\partial \alpha_\omega} T_{i,jrs} = T_{G';i,jrs}$$

if the ρ_1, ν_1, rows and columns of T′ are relabelled ρ, ν in T_G, and if we take as the parameters for the lines ρ, ν, $\alpha_\rho = \alpha_{\rho_1} + \alpha_{\rho_2}$, $\alpha_\nu = \alpha_{\nu_1} + \alpha_{\nu_2}$. Thus it remains only to show that the other block, the 3 by $(n^E(G))^2$ matrix

$$B = \begin{bmatrix} g_{\omega rs} \\ T'_{\rho_2,\omega rs} \\ T'_{\nu_2,\omega rs} \end{bmatrix} \qquad r, s \in \Theta_G^E$$

has rank 3.

By Lemma 5.2.10 there are external vertices V_u, V_v, V_x, W_w of G and disjoint paths in G joining the pairs V_u, V_c; V_v, V_d; V_w, V_e; V_x, V_f. Let T be a tree of G which contains these paths together with the lines of S, and specialize B by setting $\alpha_i = 0$, $i \in T$, $i \neq \omega$ (fig. 5). Then the 3×3 submatrix of B with columns ωuv, ωux, ωuw is

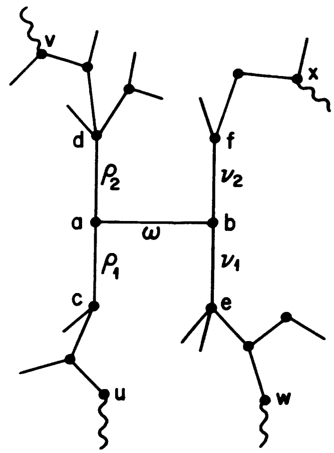

FIG. 5. — The tree T.

	ωuv	ωux	ωuw
ω	0	$\alpha(\Omega_G - T)$	$\alpha(\Omega_G - T)$
ρ_2	$\alpha(\Omega_G - T)\{\Sigma\}$	$\alpha(\Omega_G - T)\left\{\sum_{vw} + \sum_{vx}\right\}$	$\alpha(\Omega_G - T)\left\{\sum_{vw} + \sum_{vx}\right\}$
ν_2	0	$-\alpha(\Omega_G - T)\left\{\sum_{vw} + \sum_{wu}\right\}$	$\alpha(\Omega_G - T)\left\{\sum_{xv} + \sum_{xu}\right\}$

where $\sum_{vw}$ is the sum of the α_i^{-1} taken over $i \in \Omega_G - T$ such that there is a path from v to w in $H(T - \Omega_s) \cup \{i\})$ and the remaining Σ's are similarly defined. $\sum = \sum_{vw} + \sum_{xv} + \sum_{uw} + \sum_{ux}$. The determinant of this matrix is

$$- [\alpha(\Omega_G - T)]^3 \Sigma^2.$$

But $\Sigma = 0$ is impossible since G is 2-connected. This completes the proof.

In figure 6 we give examples of graphs which may be shown to be normal using Theorem 5.2.12. In figure 6 *a* we may, for example, apply successively O_{T_2} to line 8, O_{T_1} to line 6, and O_{T_2} to line 2 (we omit specification of the subgraph S since no confusion is possible). Similarly in figure 6 *b* we may apply either O_{H_1} to line 1 or O_{H_3} to line 2. The reader may also easily produce examples to show that the accessibility conditions of Theorem 5.2.12 are necessary.

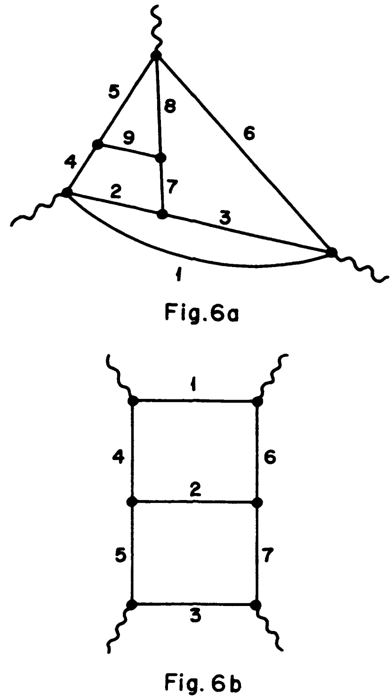

FIG. 6. — Normal graphs.

6. REPRESENTATION CONDITIONS

6.1. The Regge conjecture

We recall the problem posed in [*1*]: given a Feynman amplitude $F_G(s, z; \lambda)$ determine the monodromy representation of the corresponding fundamental group

$$\mathscr{G}_G = \pi_1(W_G - L_G; B) \xrightarrow{\mathscr{L}(\lambda)} \text{Hom } V_G.$$

In [*2*] T. Regge conjectured that $\mathscr{L}(\lambda)$ is uniquely determined by $\mathscr{G}_G$ (which is a finitely presented group, see [*1*]), and by certain « local conditions » on the representation $\mathscr{L}(\lambda)$. In this section we give two formulations of the concept of « local conditions ». The first of these is most natural from a purely mathematical point of view; the second comes closer to codifying the practice of references [*1*] [*2*] [*3*] [*4*] [*5*] [*29*], and the corresponding form of the Regge conjecture formalizes the intent of [*2*] [*29*].

We distinguish the two formulations by letters A and B. The reader may omit A without loss of continuity.

Let F be a function of Nilsson class, defined on an algebraic variety W and having singular variety $L \subset W$. Denote by $\mathscr{G}$ the fundamental group $\pi_1(W - L; B)$ with some base point B, by V the vector space spanned by germs of F over B and by $\mathscr{L}: \mathscr{G} \to \text{Hom } V$ the monodromy representation of F.

A. According to the theory of desingularization [25] we can find an algebraic variety $\tilde{W}$ together with a proper map $p: \tilde{W} \to W$ such that $\tilde{L} = p^{-1}(L)$ is in general position in $\tilde{W}$ and p restricted to $\tilde{W} - \tilde{L}$ is a biholomorphic map of $\tilde{W} - \tilde{L}$ onto $W - L$. In particular p induces an isomorphism

$$\mathscr{G} = \pi_1(\pi - L; B) \sim \pi_1(\tilde{W} - \tilde{L}; p^{-1}(B)).$$

DEFINITION 6.1.1.A. — Let q be a point of $\tilde{L}$, $\mathscr{A}'(q)$ the local fundamental group $\pi_1(U(q) - \tilde{L})$, $U(q)$ a sufficiently small neighbourhood of q. Up to isomorphism $\mathscr{A}'(q)$ is independent of the choice of $U(q)$, and by the general position condition it is free abelian on a finite number of generators $\alpha'_1, \ldots, \alpha'_k$ (say). There is a natural injection $\mathscr{A}'(q) \to \mathscr{G}$. Denote by $\mathscr{A}(q)$ the image of $\mathscr{A}'(q)$ under this injection and by $\alpha_1, \ldots, \alpha_k$ the images of $\alpha'_1, \ldots, \alpha'_k$. Then we call $\mathscr{A}(q)$ a *local subgroup* of $\mathscr{G}$. Suppose further that the commuting operators $\mathscr{L}(\alpha_1), \ldots, \mathscr{L}(\alpha_k)$ acting on V are simultaneously diagonalizable. Then we call the dim V k-tuples of eigenvalues of these operators the *local exponents* of $\mathscr{L}$ at q [Note that the diagonalizability condition in 6.1.1.A, which is supposed also in the following 6.1.2.A, is not a severe restriction. It excludes logarithmic local behaviour which we do not expect to encounter in generic Feynman amplitudes. In fact we introduced the complex parameter ν, in addition to the λ_i, precisely to obtain non-integer exponents also for points on second kind singularities],

DEFINITION 6.1.2.A. — F will be said to have the *Regge property* if its monodromy representation may be characterized as the unique representation of its fundamental group with certain particular values for the local exponents at each point of $\tilde{L}$.

Remark 6.1.3. — The local group $\mathscr{A}(q)$ and the local exponents of $\mathscr{L}$ at q evidently depend only on the particular irreducible intersection of components of $\tilde{L}$ to which q belongs. There are only a finite number of such intersections so that the data by which $\mathscr{L}$ is to be characterized in 6.1.2.A is finite.

Remark 6.1.4. — It is not hard to show that the concept of local subgroup of $\mathcal{G}$ in definition 6.1.1.A is independent of the resolution used to define it.

The principal disadvantage of the above formulation is that the passage from L to $\tilde{L}$ may be quite complicated even in seemingly simple situations. For example if L has two components $\overline{C}_1$, $\overline{C}_2$ having a tacnode contact, it requires two blowingups to reach $\tilde{L}$ and to C_1, C_2 correspond four components of $\tilde{L}$: $\overline{C}_1$, $\overline{C}_2$ (the strict transforms of C_1, C_2) and C_3, C_4. Of the intersections of $\overline{C}_1$, $\overline{C}_2$, C_3, C_4 in pairs only the intersections of $\overline{C}_1$, $\overline{C}_2$, C_4 with C_3 are non empty. For a detailed discussion of the resolution of such isolated singularities see [*26*]. A further disadvantage is that in the application to Feynman integrals we do not have any direct information about the dimension of V; but see the discussion of Section 6.2.

B. DEFINITION 6.1.5.B. — Let p be a point of L. We denote by $\mathscr{A}'(p)$ the local fundamental group $\pi_1(U(p) - L)$, (Up) a sufficiently small neighbourhood of p. Then we call $\mathscr{L}(\mathscr{A}(p))$ (where $\mathscr{A}(p)$ is the image of $\mathscr{A}'(p)$ under the natural injection $\mathscr{A}'(p) \to \mathcal{G}$) the *local algebra* at p. The specification of the local algebra at some $p \in L$ will be called a *local condition*.

This definition is still too wide, if F is a homogeneous function defined on $\mathbb{C}^n$ and $p \in L$ is taken to be the origin the corresponding local fundamental group is isomorphic with $\mathcal{G}$ itself. Thus it is only useful if we further restrict p.

DEFINITION 6.1.6.B. — A local condition is a *local condition in the strict sense* if the corresponding point $p \in L$ is either

(*a*) a non-singular point of L,

(*b*) a transverse intersection of one or more components of L,

(*c*) a tacnode, a cusp, or more generally, a generic point on a singular subvariety of L which has codimension 1 in L.

This list is based on the experience of [*1*] [*2*] [*3*] [*4*] [*5*]. It clearly may need extension at some future date. Modulo such extension, we may formulate the conjecture of Regge in the form:

CONJECTURE 6.1.7.B. — The monodromy representation of a Feynman amplitude may be uniquely characterized by the corresponding fundamental group and by the local conditions in the strict sense satisfied by the representation.

The way in which these conditions are to be deduced from the integral representation of F_G is discussed in the following section 6.2.

6.2. Consequences of the decomposition theorem

In this section we discuss the means of obtaining local conditions on the representation from the decomposition theorem 4.4.5. Previously, such conditions have been obtained from the geometric Picard-Lefschetz theorem [27]. It should be noted that our decomposition theorem is just the analytic counterpart of the Picard-Lefschetz theorem. The results we obtain are incomplete because Theorem 4.4.5 has been proved only for the physical sheet of the amplitude F_G; it is possible (although we believe it unlikely) that the behaviour of F_G near L_Q^l might be quite different on another sheet (Such behaviour could be investigated, for example, if a suitable resolution of the singularities of the integrand in (2.2.5) were carried out. See also [28]). Our remarks here are therefore to be regarded as heuristic; we will freely assume the behaviour of Theorem 4.4.5 on any sheet, as needed.

As a simple example, take Q a normal quotient of a connected Feynman graph G, with (s^0, z^0) a generic point of L_Q^l. The local group $\mathscr{A}(s^0, z^0)$ is the free group on the generator α_Q given by an elementary loop around L_Q^l. From Theorem 4.4.5 (or rather its generalization, as yet unproved) we expect $\mathscr{L}(\alpha_Q)$ to have eigenvalues $e^{2\pi i \nu_Q}$ and 1; moreover, since the discontinuity should be determined by the contour of the γ integration, which is equivalent to a sheet of F_S, we expect the two eigenspaces to have dimensions dim (V_S) and [dim (V_G) − dim (V_S)] respectively. This gives the local conditions at (s^0, z^0) of type A; the local algebra (see 6.1.5.B) is generated by 1 and $\mathscr{L}(\alpha_Q)$ and is specified by the relation

$$[\mathscr{L}(\alpha_Q) - 1][\mathscr{L}(\alpha_Q) - e^{2\pi i \nu_Q}] = 1. \qquad (6.2.1)$$

Note that in the above example the dimension of the eigenspace of $\mathscr{L}(\alpha_Q)$ with nontrivial eigenvalue (i. e. $\neq 1$) was given by the dimension of the representation of the subgraph S. If a corresponding result could be established for an arbitrary component of L_G we would be able to compute dim V_G recursively, using the implication of the homogeneity of F_G, namely the fact that the word at infinity is represented by a multiple of the identity. For the detail of this technique the reader may refer to Section 4.4 of [4].

Now take Q as above. Let $L^0 \subset L_G$ be some component of L_Q (distinct from L_Q^l) which intersects L_Q^l transversely at $(s^0, z^0) \in W_G$, and suppose that no other components of L_G pass through (s^0, z^0). The local group $A(s^0, z^0)$ is therefore free abelian on generators α_0, α_Q, which are given by elementary loops around L^0, L^l_Q respectively [1]. Then if the functions β^0, η on L^l_Q (defined in the course of proving Theorem 4.4.5) satisfy

$$\beta^0_i(s^0, z^0) \neq 0, \qquad (i \in Q),$$
$$d_Q(\beta^0) \neq 0,$$
$$\eta(s^0, z^0) \notin L_S;$$

it may be verified that the function H_G (4.4.26) is not singular on L^0. Therefore we have, in the local algebra,

$$[\mathscr{L}(\alpha_0) - 1][\mathscr{L}(\alpha_Q) - 1] = 0 \tag{6.2.2}$$

Conditions such as (6.2.2) are quite useful in constructing the monodromy representation. For example, (4.2.3) of [4] is of this nature. (2.3.19) of [1] may be derived by applying this argument repeatedly in a neighbourhood of the transverse intersection of all the mass zero curves of the self energy considered there.

We finally give an example of local conditions arising from a point of tacnodal contact. Note that the following lemma on the existence of such points is rigorous, not heuristic.

LEMMA 6.2.3. — Let Q, Q′ be normal quotient graphs of G, with Q = G/S, Q′ = Q/S′, and suppose that S′ is also normal. Then L^l_Q and $L^l_{Q'}$ have tacnodal contact along a variety M of codimension 2 in W_G.

Proof. — Recall the maps $\xi_Q\colon W_G \to W_Q$, $\xi_{Q'}\colon W_G \to W_{Q'}$, and $\zeta\colon W_Q \times \mathbb{P}^{N(Q')-1} \to W_{S'}$ defined in Section 4.3 (the map ζ as originally defined would have domain $W_Q \times \mathbb{C}^{N(Q')}$, but it is homogeneous of degree 0 in the second variable). Define

$$\bar{U} = \{ (s, z, \delta, \gamma) \} \subset W_G \times \mathbb{P}^{N(Q')-1} \times \mathbb{P}^{N(S')-1}$$

by the conditions

$$d_{Q'}(\delta) \neq 0, \qquad d_{S'}(\gamma) \neq 0;$$
$$z_i = z_i(\delta) = \frac{\partial}{\partial \delta_i} \frac{D^s_{Q'}(\xi^{(s)}_{Q'}(s), \delta)}{d_{Q'}(\delta)}, \qquad i \in \Omega_{Q'};$$
$$z_j = z_j(\delta, \gamma) = \frac{\partial}{\partial \gamma_j} \frac{D^s_{S'}[\zeta(\xi^{(s)}_{Q}(s), \delta), \gamma]}{d_{S'}(\gamma)}, \quad j \in \Omega_{S'};$$

and $\bar{\pi}\colon \bar{U} \to W_G$ to be natural projection. Let $M = \bar{\pi}(\bar{U})$. Now $\bar{U}$ has dimension equal to the dimension of W_G minus 2, and because Q′, S′ are normal, $\bar{\pi}^{-1}(s, z)$ consists of a single point for generic $(s, z) \in M$. Hence M has codimension 2 in W_G.

Now clearly $M \subset L^l_Q$. Take $\omega \in \Omega_{Q'}$, $\omega' \in \Omega_{S'}$, and normalize coordinates in $\mathbb{P}^{N(Q')-1}$ and $\mathbb{P}^{N(S')-1}$ by $\delta_\omega = \gamma_{\omega'} = 1$. For $x \in \mathbb{C}$ and γ, δ as above define $\beta = \beta(\delta, \gamma, x) \in \mathbb{P}^{N(Q)-1}$ by

$$\begin{aligned} \beta_i &= \delta_i, & (i \in \Omega_{Q'}), \\ \beta_j &= x\gamma_j, & (j \in \Omega_{S'}), \end{aligned} \tag{6.2.4}$$

(compare Section 4.3). The point $(s, z) \in L^l_Q$, where

$$z_k = z_k(\beta) = \frac{\partial}{\partial \beta_k} \frac{D^s_Q[\xi_Q(s), \beta]}{d_Q(\beta)}, \quad (k \in \Omega_Q)$$

approaches the point $(s, z_i(\delta), z_j(\delta, \gamma), z_l)$ $(l \in \Omega_S)$ as $x \to 0$. Hence $M \subset L^l_Q$. Moreover, since the normal to L^l_Q is given by

$$\text{grad}_{s,z} D^*_Q(s, z, \beta)$$

(see the proof of Lemma 4.1.4), and similarly for $L^l_{Q'}$, Lemma 4.3.4.*b* shows that L^l_Q and $L^l_{Q'}$ are tangent on M. Using 4.3.4*c* and the normality of S′ and Q′ we may also show that

$$\frac{\partial^2}{\partial z^2_{\omega'}} [D^*_Q] \neq 0;$$

hence the contact is precisely second order. This completes the proof of the lemma.

We now show how this result may be used, with certain additional assumptions, to determine the local algebra of such a tacnodal point of L_G. Specifically, we assume that no component of L_G other than L^l_Q and $L^l_{Q'}$ contains M, and that for generic $(s^0, z^0) \in M$, $\eta(s^0, z^0) \notin L_S$. We will also assume that $\Delta(H_Q)$, defined on the variety L^l_Q, has a pole of order $N(S) - 1$ in x on M; we believe this to be true in general but have not been able to prove it.

The situation in the neighbourhood of a generic point of M is shown in figure 7, where the elementary loops α_Q, α'_Q are drawn using the counter

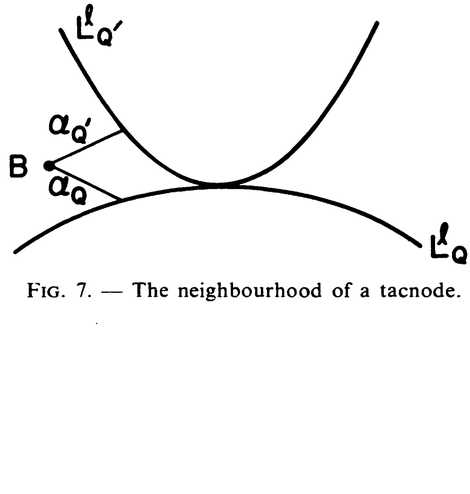

FIG. 7. — The neighbourhood of a tacnode.

clockwise convention explained in [1]. Now (if we assume the base point B is close to L_Q^l) the group element $\alpha = \alpha_{Q'}\alpha_Q\alpha_{Q'}\alpha_Q^{-1}$ has a representative lying arbitrarily close to L_Q^l but staying a fixed distance from M. If we define functions $\bar{\beta}_i^0$ by (4.4.15) along this path (with $\bar{\beta}_\omega^0 = 1$), and functions $\bar{x}, \bar{\delta}, \bar{\gamma}$ by (6.2.4), we find that the path may be taken to have $\bar{\delta}$ and $\bar{\gamma}$ constant, with $\bar{x}(\theta) = \varepsilon e^{2\pi i\theta}$, $0 \leq \theta \leq 1$, some $\varepsilon > 0$ (recall that M corresponds to $x = 0$ on L_Q^l). Now since β_i, $i \in S'$, and $d_Q(\beta)$ contain factors x and $x^{h(S')}$ respectively, we find that

$$\mathscr{L}(\bar{\alpha})a_Q = \exp 2\pi i\left[\sum_{i\in S'} \lambda_i - h(S')\frac{\nu}{2} + \frac{N(S') - 1}{2}\right]a_Q, \qquad (6.2.5)$$

where $a_Q = \mathscr{L}(\alpha_Q) - 1$ (This may be shown using (4.4.26), or seen heuristically by examining the behaviour of the factors in (4.4.28) for $x(\theta) = \varepsilon e^{2\pi i\theta}$, $0 \leq \theta \leq 1$). Using $\mathscr{L}(\alpha_Q^{-1})a_Q = \exp(-2\pi i\nu_Q)a_Q$, (6.2.5) becomes

$$\mathscr{L}(\alpha_Q)\mathscr{L}(\alpha_{Q'})a_Q = -e^{2\pi i\nu_{Q'}}\mathscr{L}(\alpha_{Q'})^{-1}a_Q. \qquad (6.2.6)$$

We write $\mathscr{L}(\alpha_{Q'}) = 1 + a_{Q'}$, so that $\mathscr{L}(\alpha_{Q'})^{-1} = 1 - e^{-2\pi i\nu_{Q'}}a_{Q'}$ from (6.2.1). Then expanding (6.2.6) gives

$$a_Q a_{Q'} a_Q = -[e^{2\pi i\nu_Q} + e^{2\pi i\nu_{Q'}}]a_Q. \qquad (6.2.7)$$

(6.2.7), together with two relations of the form (6.2.1), determines the local algebra at (s^0, z^0).

Relations of the form (6.2.7) were used in [1] and [4], where they were derived from the relations in the local group at (s^0, z^0) and the fact that a_Q was, in those cases, a one dimensional projector. Their validity in any generic Feynman amplitude (without this additional assumption), as discussed here, could be viewed as a survival of the hierarchical principle for ordinary Feynman amplitudes.

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BIBLIOGRAPHY

[1] G. Ponzano, T. Regge, E. R. Speer and M. J. Westwater, *Commun. Math. Phys.*, t. **15**, 1969, p. 83-132.

[2] T. Regge, « The fundamental group of Poincaré and the Analytic properties of

Feynman rèlativistic amplitudes », *Nobel Symposium 8. Elementary Particle Theory*. ed., Nils Svartholm, New York, Interscience, 1969.

[3] G. PONZANO and T. REGGE, *The Monodromy Group of One Loop Relativistic Feynman Integrals*. Published in a volume on the occasion of the 60th birth day of the Academician N. N. BOGOLIUBOV.

[4] G. PONZANO, T. REGGE, E. R. SPEER and M. J. WESTWATER, The Monodromy Rings of One-Loop Feynman Amplitudes, to be published in *Commun. Math. Phys.*

[5] T. REGGE, E. R. SPEER and M. J. WESTWATER, *The Monodromy Rings of the Necklace Graphs*, to appear.

[6] R. E. CUTKOSKY, *J. Math. Phys.*, t. **1**, 1960, p. 429-433.

[7] J. C. POLKINGHORNE and G. R. SCREATON, *Nuovo Cimento*, t. **15**, 1960, p. 925.

[8] L. D. LANDAU, *Nucl. Phys.*, t. **13**, 1959, p. 181-192.

[9] C. BERGE, *The Theory of Graphs*, London, Methuen and Co., Ltd, 1962.

[10] E. R. SPEER, *Generalized Feynman Amplitudes*, Princeton University Press, 1969.

[11] T. REGGE, Algebraic Topology methods in the theory of Feynman relativistic amplitudes. *Batelle Rencontres 1967 Lectures in Mathematics and Physics*, ed. C. M. DeWitt, J. A. Wheeler, New York, W. A. Benjamin, 1968.

[12] L. V. AHLFORS, *Complex Analysis* (lst edition), New York, McGraw-Hill, 1953.

[13] E. L. INCE, *Ordinary Differential Equations*, New York, Dover, 1956.

[14] K. HEPP, *Commun. Math. Phys.*, t. **2**, 1966, p. 301.

[15] I. M. GEL'FAND and G. E. SHILOV, *Generalized Functions*, vol. I, New York, Academic Press, 1964.

[16] D. FOTIADI, M. FROISSART, J. LASCOUX and F. PHAM, *Topology*, t. **4**, 1965, p. 159-191.

[17] T. KINOSHITA, *J. Math. Phys.*, t. **3**, 1962, p. 650-677.

[18] B. L. van der WAERDEN, *Modern Algebra*, vol. II, New York, Frederic Ungar Publishing Co, 1948.

[19] C. RISK, *J. Math. Phys.*, t. **9**, 1968, p. 2168-2172.

[20] A. SARD, *Bull. Amer. Math. Soc.*, t. **48**, 1942, p. 883-890.

[21] R. J. EDEN, P. V. LANDSHOFF, D. I. OLIVE and J. C. POLKINGHORNE, *The Analytic S-Matrix*. Cambridge University Press, 1966.

[22] M. J. WESTWATER, *Helv. Phys. Acta*, t. **40**, 1967, p. 389-400.

[23] J. C. POLKINGHORNE, *Analytic Properties in Perturbation Theory*. Lectures in Theoretical Physics. Brandeis Summer Institute, 1961, vol. I, New York, W. A. Benjamin, 1961.

[24] L. B. OKUN and A. P. RUDIK, *Nucl. Phys.*, t. **14**, 1960, p. 261-288.

[25] H. HIRONAKA, *Ann. Math.*, t. **79**, 1964, p. 109-326.

[26] P. ORLIK and P. WAGREICH, *Isolated singularities of Algebraic surfaces with* C* *action*, Institute for Advanced study preprint, 1970.

[27] F. PHAM, *Introduction à l'Étude topologique des Singularités de Landau*, Paris, Gauthier-Villars, 1967.

[28] J. B. BOYLING, *Nuovo Cimento*, t. **53**, 1968, p. 351-375.

[29] T. REGGE, Talk delivered at the Accademia dei Lincei during the Mendeleyev meeting on symmetry and periodicity in elementary particles, September, 1969.