A. VAN DAELE

On a certain class of semi-simple subalgebras of a semi-simple Lie algebra

Annales de l'I. H. P., section A, tome 13, nº 3 (1970), p. 195-213
<http://www.numdam.org/item?id=AIHPA_1970__13_3_195_0>

Ann. Inst. Henri Poincaré,
Vol. XIII, n° 3, 1970, p. 195-213.

Section A :
Physique théorique.

On a certain class of semi-simple subalgebras of a semi-simple Lie algebra

by

A. Van DAELE (**)
Institute of Theoretical Physics, University of Leuven, Leuven, Belgium (*)
(December 1969).

ABSTRACT. — We consider a semi-simple subalgebra K of a semi-simple Lie algebra L, and we develop a procedure to find the lattice of semi-simple subalgebras L_1 of L, containing K. The essential point in the analysis is the fact that such a subalgebra L_1 is stable under commutation with K, and therefore that we only work with subspaces of L, invariant under K.

A second point is the use of a faithful representation of L, and its representation space M (i. e. a faithful L-module M). The use of M is crucial for the method.

We also proved two propositions, concerning the imbedding of K in L. We discuss possible physical applications, and finally we give two examples in an appendix.

1. INTRODUCTION

The importance of semi-simple Lie algebras is already well established in several branches of theoretical physics. Most of the time, the algebras are used to label states in certain multiplets or irreducible representations. To find sufficiently many labels, one of the most important methods is to

(*) Postal address Afdeling Theorie, Departement Natuurkunde, Celestijenlaan 200 D, 3030 Heverlee, Belgium.
(**) Aspirant N. F. W. O.

use a chain of subalgebras and then reduce the representation with respect to this chain. Therefore it is useful to find all the semi-simple subalgebras of a certain semi-simple Lie algebra. Actually, this problem has been completely solved by many authors (see e. g. ref. [1] [2]).

However, the result is not yet complete satisfactory, because not all the chains are physical. Only those subalgebras which contain the Lie algebra of the biggest invariance group of the system are of physical interest. This means that the good quantum numbers, due to the invariance groups, must be a part of the labelling numbers, and so only those chains, ending in the Lie algebra of the invariance group are important. From this it is clear that there exists an inverse problem: given a semi-simple Lie algebra L and a semi-simple subalgebra K, find the chains of subalgebras between K and L.

While in the case of particle physics, the physical chains are usually mathematical simple ones, and so the preceding problem is an easy one. In the case of nuclear and atomic physics, the last statement is false. The physical chains are more difficult to deal with, and moreover, if group theory is used in the shell theory of mixed shells, the algebras become larger and larger, and the physical subalgebras are not at all clear, as is usually the case, in particle physics. In section 7 we will sketch the usefulness of finding almost all the physical subalgebras, especially by the construction of model Hamiltonians (see e. g. ref. [3] [4] [5] [6] [10]).

Our main purpose is to present a straigth forward procedure for finding all semi-simple subalgebras of a given Lie algebra L, which contain a given semi-simple subalgebra K of L.

The difficulty by this program is that we must find a subspace of L, such that, together with K, we have the vectorspace of a subalgebra. In many cases however the dimension of the space of the subalgebra is much larger than this one of K, and so there are many possibilities to be examined. The basis of our analysis is now to use the fact that the added subspace has to be invariant under commutation with K because we want to construct a subalgebra. Therefore we only have to examine the invariant subspaces in L, under K. And so the number of possibilities has decreased rather seriously, as will become clear in some examples given in appendix B.

In section 2 we recall some definitions and notations, and we introduce the notion of all the K-invariant subspaces of a K-module M, and of all the vectors of highest weight in M (M being reducible). In section 3 we prove two propositions which are implicitly used in the main part of the paper.

In the next two sections we construct a basis in a subspace of L and we give a few properties of a subalgebra L_1 containing K and contained in L, using the properties of this basis. Finally in section 6 we explain the method of construction of the lattice of subalgebras starting with L and ending in K.

2. DEFINITIONS AND NOTATIONS

One of the most beautiful results in the theory of semi-simple Lie algebras is the correspondence between the classes of equivalent irreducible modules and the dominant integral linear functions on the Cartan subalgebra. Since this correspondence will be fundamental in our notations, we first recall briefly the notion of a dominant integral linear function on the Cartan subalgebra. At the same moment we arrange some notations.

L is a semi-simple Lie algebra over the complex numbers. We shall use x, y, to denote the elements of L.

K is a semi-simple subalgebra (1) of L, the elements being u, v.

J is a Cartan subalgebra of K.

There exists always a Cartan subalgebra of L, containing J (see ref. [7], p. 149), so we denote by H a Cartan subalgebra of L, containing J.

Next we consider a set of positive roots in the space of all linear functions on J corresponding to an ordered basis in this space, and we introduce the associated canonical basis in J : $\{ h_i, i = 1, l \}$ (see ref. [7], p. 121). A function α on J is called integral if $\alpha(h_i)$ is integral for every h_i.

We denote by J^* the set of integral linear functions on J.

A function $\alpha \in J^*$ is called dominant if $\alpha(h_i)$ is positive or zero for every h_i. We denote by $\overline{J^*}$ the set of all dominant integral linear functions on J.

Similarly we define H^* and $\overline{H^*}$.

For a better understanding of these definitions, see ref. [7].

Next we introduce a concept of great importance in this paper, namely all irreducible K-submodules (of the same highest weight α) of a certain K-module M, and the subspace of M of highest weight vectors (of a certain weight α) with respect to K in M.

Consider a left K-module. This module is always completely reducible because K is semi-simple. Therefore semi-simplicity is the essential condition of our treatment.

Because of the one to one correspondence between the elements of $\overline{J^*}$

(1) K may be in fact every subalgebra with the property that every L-module is completely reducible when restricted to K. K may be for instance commutative and reductive (see ref. 7, p. 104).

and the equivalence classes of irreducible K-modules, we can associate with $\alpha \in \overline{J^*}$ the set of irreducible K-submodules of M, having highest weight α. We call this set $W_K(M, \alpha)$. It may be empty for certain elements of $\overline{J^*}$.

Every module of $W_K(M, \alpha)$ has a highest weight vector. All the highest weight vectors of the same weight constitute a vectorspace. We call this space $V_K(M, \alpha)$. Every vectorspace $V_K(M, \alpha)$ is a subspace of M. So $V_K(M, \alpha)$ is the subspace of M of highest weight vectors of weight α in M. We shall use these two notions frequently throughout the paper.

3. CHARACTERISATION OF K IN L

Our aim is to construct all possible semi-simple subalgebras of L which contain K. One of the most important things to know is how K is imbedded in L. It is easy to see that there may be many subalgebras of a certain type, say isomorphic to an algebra K, in L. Fortunately most of these subalgebras will behave in the same way, this means that they will be contained in similar chains, that a L-module will decompose in the same way with respect to these different subalgebras, etc.

Nevertheless, there will be classes of subalgebras, behaving in a completely different manner. This makes it important to characterize K in L, not by writing a basis of K in terms of a basis of L, but by giving a certain criterium to indicate the class to which K belongs.

To do this we first have to find a criterium for two isomorphic subalgebras to be in the same class.

We assume, in what follows, that K and K′ are two isomorphic subalgebras of L, and that M is a finite dimensional L-module.

PROPOSITION 1. — If σ is an invariant automorphism [1] of L, such that $\sigma(K) = K'$ then there exists a linear operator $A : M \to M$ such that $\forall (u, m) \in K \times M \Rightarrow A(um) = \sigma(u)A(m)$.

Proof. — Let τ be an automorphism of the form exp (ad e), where [2] $e \in V_H(L, \alpha)$, $\alpha \in \overline{H^*}$. Set $A = \exp e$; we have that:

$$\begin{aligned} A(um) &= (\exp e)\, u\, (\exp(-e))\, (\exp e)m \\ &= (\exp(\operatorname{ad} e)u)\, (\exp e)m \\ &= \tau(u)A(m) \end{aligned}$$

(1) The invariant automorphisms are generated by exp (ad z), $z \in L$ and ad z nilpotent (ad $z : x \to [z, x]$).

(2) We consider L as a H-module by the mapping ad $H : L \to L$.

The group generated by automorphisms of the form $\exp(\operatorname{ad} e)$, $e \in V_H(L, \alpha)$ equals the group of invariant automorphisms (a result due to Steinberg, see ref. [7], p. 288).

So we can write:

$$\sigma = \tau_1\tau_2 \dots \tau_n$$

and we put:

$$A = A_1A_2 \dots A_n$$

Hence:

$$\begin{aligned} A(um) &= A_1A_2 \dots A_{n-1}\tau_n(u)A_n(m) \\ &= (\tau_1\tau_2 \dots \tau_n)(u)(A_1A_2 \dots A_n)(m) \\ &= \sigma(u)A(m) \end{aligned}$$

It would have been more natural to say that two isomorphic subalgebras of L belong to the same class, if there exists just an automorphism relating them, invariant or not. However only an invariant one guarantees the equivalence of the K and K′-modules, arising from the same L-module.

PROPOSITION 2. — Let M be a faithful L-module and G be a connected group of linear transformations of M, such that the associated representation of L in M is the Lie algebra of G. If there exists an $A \in G$ such that for some isomorphism $\phi\colon K \to K'$ and all $(u, m) \in K \times M$, $A(um) = \phi(u)A(m)$, then there exists an invariant automorphism σ of L such that $\sigma(u) = \phi(u)$, $u \in K$.

Proof. — Consider the subgroup G′ of G, generated by $\exp e$, $e \in V_H(L, \alpha)$.

The Lie algebras of G′ and G are isomorphic, so this isomorphism extends to a local isomorphism of G and G′.

Since G′ and G are connected, G is isomorphic to G′ and $G = G'$.

It follows that A can be written as a product of transformations $\exp e$ $e \in V_H(L, \alpha)$

$$A = \exp e_1 \exp e_2 \dots \exp e_p$$

We set

$$\sigma = \exp(\operatorname{ad} e_1).\exp(\operatorname{ad} e_2) \dots \exp(\operatorname{ad} e_p).$$

σ is an invariant automorphism of L with the property that:

$$A(um) = \sigma(u)A(m) \qquad u \in L, \qquad m \in M$$

On the other hand:

$$A(um) = \phi(u)A(m) \qquad u \in K, \qquad m \in M$$

And since this holds for $\forall\, m \in M$ and A is a mapping onto M because it is invertible, it follows that:

$$\sigma(u) = \phi(u) \qquad u \in K$$

As a by-product we give three corollaries. The second being the most interesting one.

COROLLARY 1. — Let K and K′ be two isomorphic subalgebras of A_l, the algebra of the group $SU(l+1)$ [1] and ϕ the isomorphism of K onto K′. Consider J and J′, Cartan subalgebras of K, K′. Take a function $\alpha \in J^*$ on J. We associate a function α' on J'^* defined as:

$$\alpha'(u) = \alpha(\phi^{-1}(u)) \qquad u \in J'$$

Now it follows immediately from proposition 2 that if M_0 is a $(l+1)$-dimensional A_l-module and $\forall\ \alpha \in \overline{J^*}$

$$\dim V_K(M_0, \alpha) = \dim V_{K'}(M_0, \alpha')$$

then there exists an invariant automorphism σ of A_l such that:

$$\forall u \in K \qquad \sigma(u) = \phi(u)$$

Proof. — Because

$$\dim V_K(M_0, \alpha) = \dim V_K(M_0, \alpha'), \qquad \forall \alpha \in \overline{J^*}$$

the two K-modules:

$$\begin{aligned} (u, m) &\to u \cdot m \\ (u, m) &\to \phi(u) \cdot m \end{aligned}$$

are equivalent.

So there exists an A mapping M_0 onto M_0 such that:

$$A(um) = \phi(u)Am$$

Furthermore there exists a unitary A, with determinant 1 with this property. So $A \in SU(l+1)$ which is connected and we can use proposition 2.

Q. E. D.

COROLLARY 2. — The number of different ways of imbedding an algebra isomorphic to K in A_l is the number of partitions of $(l+1)$ into dimensions of irreducible K-modules, such that the associated $(l+1)$-dimensional representation of K is faithful. This results immediately from the preceding one.

COROLLARY 3. — If K and K′ are two isomorphic subalgebras of L related by an invariant automorphism σ of L, then for every L-module M:

$$\dim V_K(M, \alpha) = \dim V_{K'}(M, \alpha')$$

α' defined as:

$$\alpha'(\sigma(u)) = \alpha(u) \qquad u \in J$$

[1] We use the standard notations for simple Lie algebras.

4. CONSTRUCTION OF A BASIS IN $V_K(L, \alpha)$

a) The importance of the spaces $V_K(L, \alpha)$

As we already mentioned in the introduction, the essential point in this analysis is that we use the fact that a subalgebra L_1 containing K and contained in L is stable under commutation with K. It follows that those subspaces of L, which are stable under commutation with K will play an important role, since it will be sufficient, by constructing such subalgebras, to consider only these subspaces of L.

Now, these subspaces are nothing else but the K-submodules $W_K(L, \alpha)$ where L is considered as a K-module under commutation. Let us consider also the vectors $V_K(L, \alpha)$. All these vectors are in a certain sense « representative » for the modules $W_K(L, \alpha)$ since they generate, under commutation with K, the modules.

All the vectors $V_K(L, \alpha)$ constitute a subspace of L, namely the subspace of highest weight vectors with respect to K. Let us call this space $V_K(L)$, so:

$$V_K(L) = \sum_{\alpha \in \bar{J}^*} \oplus \, V_K(L, \alpha)$$

We can also consider the analogues for L_1 and K, namely:

$$V_K(L_1) \qquad \text{and} \qquad V_K(K).$$

$V_K(L_1)$ is the subspace of L_1 of highest weight vectors with respect to K, and $V_K(K)$ is the corresponding subspace of K.

It is easy to see that (an important result):

$$V_K(K) \subseteq V_K(L_1) \subseteq V_K(L)$$

if L_1 is a subalgebra of L such that it contains K.

Let us now turn the other way around, and let us try to construct such a subalgebra L_1. From the fact that the vectors of $V_K(L_1)$ generate under K the whole space L_1 it is sufficient to construct the set of vectors in $V_K(L_1)$. For once the vectors in $V_K(L_1, \alpha)$ are known one can construct the set of elements $W_K(L_1, \alpha)$ and so L_1 is completely and uniquely defined.

To construct a subalgebra L_1, we must therefore extend the space $V_K(K)$, with vectors of $V_K(L)$, to arrive at a space $V_K(L_1)$ with the inclusion property stated before:

$$V_K(K) \subseteq V_K(L_1) \subseteq V_K(L).$$

Since it is not hard to see that the same inclusion must hold for every $\alpha \in \overline{J^*}$

$$V_K(K, \alpha) \subseteq V_K(L_1, \alpha) \subseteq V_K(L, \alpha)$$

it follows that we must extend, for every α, the space $V_K(K, \alpha)$ with vectors of $V_K(L, \alpha)$ to get $V_K(L_1, \alpha)$.

The advantage of this procedure is that the dimensions of $V_K(K)$, $V_K(L_1)$ and $V_K(L)$ are usually much smaller than those of resp. K, L_1 and L itself. And while in the usual procedure we would have to extend K with vectors of L to get L_1, we now do the same thing with much smaller vector spaces.

What is now the price we have to pay for this?

In the usual construction, where K is extended with vectors of L to a space L_1, there must be checked if the space L_1 is a subalgebra. That means that one must verify the inclusion:

$$[L_1 L_1] \subseteq L_1 \quad (^1)$$

It is clear that a similar verification must take place in the new procedure. The problem is however that the commutator of two highest weight vectors of $V_K(L_1)$ is not a highest weight vector, and so:

$$[V_K(L_1) V_K(L_1)] \not\subseteq V_K(L_1)$$

So we have to use another inclusion. Let us consider the spaces $[A_\alpha B_\beta]$

$$(A_\alpha \in W_K(L_1, \alpha) \qquad \text{and} \qquad B_\beta \in W_K(L_1, \beta))$$

It is clear that this space is an invariant subspace of L_1 (under K).
We can again decompose this K-module and clearly we have that:

$$W_K([A_\alpha, B_\beta], \gamma) \subseteq W_K(L_1, \gamma)$$

and similarly:

$$V_K([A_\alpha, B_\beta], \gamma) \subseteq V_K(L_1, \gamma).$$

So these relations replace $[L_1 L_1] \subseteq L_1$ and they are to be verified for the vectorspaces $V_K(L_1, \alpha)$ in order that they generated really a subalgebra L_1.

We shall refer to these relations as commutation relations.

b) **The introduction of a faithful** L-**module** M

As we have seen before, our procedure consists essentially in two steps. First we have to extend the spaces $V_K(K, \alpha)$ with vectors of $V_K(L, \alpha)$ to construct the spaces $V_K(L_1, \alpha)$. Once we have a basis in $V_K(L, \alpha)$, this is an easy

---

(1) [AB] is the space generated by $[a, b]$ where $a \in A$ and $b \in B$.

thing. But the second step may be hard, and it is clear that the verification of the commutation relations will be much easier if one has chosen an appropriate basis in every space $V_K(L, \alpha)$.

To construct such a basis, it turned out that the introduction of a faithful L-module M was very useful, especially in the case where K is simply reducible (ref. [8]) and $3-j$ and $6-j$ coefficients can be used. But also in the general case, the introduction of M makes it possible to characterize partially a basis in the spaces $V_K(L, \alpha)$, such that the commutation relations can be verified more easily. In the important case that K is simply reducible these new commution relations present no difficulties and are as easy as $[L_1 L_1] \subseteq L_1$.

We now turn to the construction of a basis in the spaces $V_K(L, \alpha)$ using a simple faithful L-module M. As a starting point we use the set:

$$\{ \dim V_K(M, \alpha) \mid \alpha \in \overline{J^*} \}$$

as might be expected from proposition 1 and 2, since this set characterizes K almost completely.

c) **Construction of a basis in** $V_K(L, \alpha)$

We consider an irreducible faithful L-module M. M decomposes with respect to K into irreducible K-submodules M_i and we denote by $\alpha_i \in \overline{J^*}$ the highest weight of the module M_i. For every pair (i, j), the set of operators:

$$\mathscr{H}om(M_i, M_j)$$

is again a K-module under the product:

$$(u, p) \in K \times \mathscr{H}om(M_i, M_j) \rightarrow up \in \mathscr{H}om(M_i, M_j)$$

where

$$up(m) = u(p(m)) - p(u(m)) \qquad m \in M_i$$

So we can consider the spaces $V_K(\mathscr{H}om(M_i, M_j), \alpha)$ of linear operators in M, acting between M_i and M_j only, and of highest weight α under commutation with K.

We choose a basis in all these spaces, and we denote the basis vectors as:

$$p^k(\alpha, ij) \qquad k = 1, 2, \dots$$

Since:

$$\sum_{ij} \oplus V_K(\mathscr{H}om(M_i, M_j), \alpha) = V_K(\mathscr{H}om(M, M), \alpha)$$

on the one hand and by its representation:

$$L \subseteq \mathscr{H}om(M, M)$$

on the other hand it is possible to write the vectors of $V_K(L)$ in terms of the basis vectors:

$$\{ p^k(\alpha, ij) \}$$

and it is clear that the vectors of $V_K(L, \alpha)$ must be linear combinations of vectors:

$$p^k(\alpha, ij) \qquad \text{with} \qquad \alpha \qquad \text{fixed.}$$

By a good choice of M, the vectors of $V_K(L, \alpha)$ are simple expressions in the $p^k(\alpha, ij)$, sometimes $V_K(L, \alpha)$ coincides with:

$$\sum_{i,j} \oplus V_K(\mathscr{H}om(M_i, M_j), \alpha)$$

for almost all α so that:

$$\{ p^k(\alpha, ij) \}$$

is effectively a basis in $V_K(L, \alpha)$.

In the case where K is simply reducible, we have that:

$$\dim V_K(\mathscr{H}om(M_i, M_j), \alpha) \leqslant 1$$

so that there is atmost one K-module in $W_K(\mathscr{H}om(M_i, M_j), \alpha)$.

If $f_{i\lambda}$ is the vector of weight $\lambda \in J^*$ in M_i [1] and $E^{j\mu}_{i\lambda} \in \mathscr{H}om(M_i, M_j)$ is defined as:

$$E^{j\mu}_{i\lambda} f_{j'\mu'} = \delta_{jj'} \delta_{\mu\mu'} f_{i\lambda}$$

then:

$$p(\alpha, ij) = \sum_{\lambda,\mu,\nu \in J^*} \begin{pmatrix} \alpha_i & \\ \nu & \lambda \end{pmatrix}^* \begin{pmatrix} \alpha_i & \alpha & \alpha_j \\ \nu & \alpha & \mu \end{pmatrix} E^{j\mu}_{i\lambda} \quad [2]$$

is a vector in

$$V_K(\mathscr{H}om(M_i, M_j), \alpha)$$

and in the case where K is simply reducible we will always use this basis vector in

$$V_K(\mathscr{H}om(M_i, M_j), \alpha).$$

---

[1] Since K is simply reducible the vectors in an irreducible module are completely labelled by their weights.

[2] The $1-j$ and $2-j$ coefficients are defined as in reference 8.

5. PROPERTIES

In the preceding section, we constructed a basis in $V_K(L, \alpha)$ and we explained why we use these vectorspaces and why we introduced an irreducible faithful L-module M.

We now give a summary of results, so that we can explain the general procedure in the next section.

a) Let

$$W_K(\alpha) = \bigcup_{i,j} W_K(\mathscr{H}om(M_i, M_j), \alpha)$$

and

$$V_K(\alpha) = \sum_{i,j} \oplus\, V_K(\mathscr{H}om(M_i, M_j), \alpha)$$

then, if L_1 is a subalgebra such that $K \subset L_1 \subset L$, for every α:

$$W_K(K, \alpha) \subseteq W_K(L_1, \alpha) \subseteq W_K(L, \alpha) \subseteq W_K(\alpha) \subseteq W_K(\mathscr{H}om(M_i, M_j), \alpha)$$
$$V_K(K, \alpha) \subseteq V_K(L_1, \alpha) \subseteq V_K(L, \alpha) \subseteq V_K(\alpha) = V_K(\mathscr{H}om(M, M), \alpha)$$

b) Put

$$\theta = \sum_{\alpha \in \overline{J^*}} (\dim V_K(L_1, \alpha) - \dim V_K(K, \alpha))$$

Empirically we have found that θ is often equal to 1 for a minimal extension, that means for a proper subalgebra L_1 of L, and such that there is no subalgebra L_2, different from K and L_1 such that:

$$K \subset L_2 \subset L_1$$

This is of great importance since it shows that the minimal extensions with $\theta = 1$, which are easily found, provide us a considerable part of the minimal extensions.

c) If $(A_\alpha, B_\beta) \in W_K(L_1, \alpha) \times W_K(L_1, \beta)$, then for every $\alpha, \beta, \gamma \in \overline{J^*}$

$$W_K([A_\alpha B_\beta], \gamma) \subseteq W_K(L_1, \gamma)$$
$$V_K([A_\alpha B_\beta], \gamma) \subseteq V_K(L_1, \gamma)$$

The same results hold if L_1 is replaced by K, L or $\mathscr{H}om(M, M)$.

d) Let

$$P(\alpha, ij) \in W_K(\mathscr{H}om(M_i, M_j), \alpha)$$
$$P(\beta, i'j') \in W_K(\mathscr{H}om(M_{i'}, M_{j'}), \beta)$$

then

$$W_K([P(\alpha, ij)P(\beta, i'j')], \gamma) \subseteq W_K(\mathscr{H}om(M^{\alpha}_{i'}, M_{j'}), \gamma)\delta_{i'j}$$
$$\cup\, W_K(\mathscr{H}om(M_{i'}, M_j), \gamma)\delta_{ij'}$$

where $W_K(M, \gamma)\delta_{ij}$ is an empty set if $i \neq j$ and is $W_K(M, \gamma)$ is $i = j$.

e) If K is simply reducible one can show (see Appendix A) that if

$$P(\alpha, ij), \quad P(\beta, i'j')$$

are the modules with highest weight vectors

$$p(\alpha, ij), \quad p(\beta, i'j')$$

then the highest weight vector of

$$W_K([P(\alpha, ij)P(\beta, i'j')], \gamma)$$

is (up to a constant)

$$(-1)^{\alpha_i+\alpha_{j'}+\gamma}\begin{Bmatrix} \alpha_i & \gamma & \alpha_{j'} \\ \beta & \alpha_j & \alpha \end{Bmatrix}\delta(i'j)p(\gamma, ij')$$
$$-(-1)^{\alpha+\beta+\gamma}(-1)^{\alpha_{i'}+\alpha_j-\gamma}\begin{Bmatrix} \alpha_{i'} & \gamma & \alpha_j \\ \alpha & \alpha_{j'} & \beta \end{Bmatrix}\delta(j'i)p(\gamma, i'j)$$

Property *a* and *b* were already mentioned in section 4.

Property *d* and *e* explain why we use the L-module M, especially in the case that K is simply reducible.

6. CONSTRUCTION OF THE MINIMAL EXTENSIONS

In section 4 we sketched the procedure to construct the lattice of subalgebras between K and L. In section 5 we have given some properties of the notions introduced in section 4. In this section we explain more in detail how we can construct systematically all the minimal extensions of K in L. It is clear that, once the minimal extensions are found, we can find all the subalgebras by constructing the minimal extensions of the minimal extensions, etc.

We suppose that K, L and, for a certain irreducible faithful L-module M, the set:

$$\{ \dim V_K(M, \alpha), \alpha \in \overline{J^*} \} \qquad \text{is known.}$$

The knowledge of this set enables us to decompose M into irreducible

K-submodules M_i and to find the α_i, the highest weights of M_i. Then by decomposing the product of the contragredient module of M_i and the module M_j we can find the possibilities for the basis vectors in $V_K(\alpha)$:

$$\{ p^k(\alpha, ij) \}$$

We proceed by writing the elements of $V_K(K, \alpha)$ and $V_K(L, \alpha)$ in terms of this basis in $V_K(\alpha)$. This is possible if the action of K and L in M is known. Considerations about the dimensions may be very useful here.

Once this work is done we are ready to find the minimal extensions, at least in the simply reducible case. Successively we search for extensions with $\theta = 1, 2$.

For $\theta = 1$ we write down an arbitrary vector $q_\alpha \in V_K(L, \alpha)$ for some α and we determine the coefficients such that:

$$a) \qquad q_\alpha \notin V_K(K, \alpha)$$

and if Q_α is the module having q_α as highest weight vector:

$$b) \qquad V_K([Q_\alpha Q_\alpha], \gamma) \subseteq V_K(K, \gamma) \qquad \gamma \neq \alpha$$

$$c) \qquad V_K([Q_\alpha Q_\alpha], \alpha) \subseteq V_K(K, \alpha) \oplus \{ q_\alpha \}$$

If these relations can be written down explicitely, we obtain all the extensions with $\theta = 1$, provided we check that the obtained algebra is semi-simple.

In a similar fashion, we construct all the minimal extensions with $\theta = 2, 3, \ldots$. To clarify the method, we give two examples in Appendix B.

REMARKS. — 1. Only in the case that K is not simply reducible there may be difficulties by checking the commutation relations. However property *d)* may be very useful if for almost all the pairs (i, j) we have that:

$$\dim V_K(\mathscr{H}om(M, M), \alpha) \leqslant 1$$

Finally we can use a subalgebra K_1 of K which is simply reducible or even commutative, and use the basis of $V_{K_1}(\alpha)$ to evaluate the relations.

2. Checking the dimensionalities, together with semi-simplicity, it is sometimes possible to exclude a proposed extension.

3. It is clear that if $L_1 \subset L$, M has to be a faithful L_1-module and if $K \subset L_1$, it has to decompose in a given way with respect to K. These two properties may of course be useful too.

Some of these remarks will be used in the examples.

7. PHYSICAL APPLICATION

To illustrate the possibility of physical applications, we sketch a construction of a certain class of model Hamiltonians in the nuclear shell theory.

In this theory, one always considers only a finite set of one particle states, stable under the transformations of the group $SU_2(J) \times SU_2(T)$ of total angular momentum and isospin. Furthermore one restricts the one and two particle operators as acting only between the chosen set.

These things have been translated into group theoretical concepts by Moshinsky (and many others), as follows: we consider the Lie algebra K of the group $SU_2(J) \times SU_2(T)$; the chosen finite set of one particle states forms a K-module, call it M. We take L to be the semi-simple Lie algebra of traceless linear operators of M into M. L is therefore the Lie algebra of the group $SU(n)$, n the dimension of M. Clearly L contains K.

It is a well known fact that there is a correspondance between resp. the n-particle states, the one-particle and the two-particle operators on the one side and the vectors of some irreducible L-modules, the operators of L, and the quadratic expressions in the operators of L on the other side. All this is explained in full detail in the work of Moshinsky [*3*].

Suppose now that we have constructed the lattice of semi-simple subalgebras of L, containing K. We can extract one chain out of it and build a quadratic operator by taking a linear combination of Casimir invariants of the subalgebras of the chain. Such an operator corresponds to a model Hamiltonian ([1]). And this model Hamiltonian is a very easy one to deal with. For it is possible to take a basis in the n-particle states such that the Casimir invariants are all diagonal, and the eigenvalues are known.

So the method provides us a wide class of model Hamiltonians which are very easy to deal with. A lot of the earlier models belong to this class, such as pairing and quadrupole-quadrupole. Even some of the recent models are closely related to the considered class (see ref. [*6*] [*10*]).

It must be mentioned that this class of model Hamiltonians may be unsatisfactory in configurations where only one or at most two shells are taking into account, because then there are enough methods to use.

However, if one would try to relate some results of different shells, or if one would like to take into account the effect of other shells, it might be effectively useful to have a wide class of model Hamiltonians which are easy to handle, even in the case of mixed shells.

([1]) All these Casimir invariants are invariant under the group $SU_2(J) \times SU_2(T)$.

On the other hand, it can be possible that there exists a chain such that a part of this chain occurs in a configuration for which other methods can be used, and so a new class of models can be treated. This is for instance the case in a $p-f$ shell. The algebra in consideration is SU(10), and it has a (rather remarkable) set of physical chains

$$\mathrm{SU}(10) \supset \mathrm{SU}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)$$

Clearly we encounter the last part also in a d-shell and so we can treat a model in the $p-f$ configuration, using the same results and known coefficients of the familiar d-shell. The same thing occurs in the $s-d$ and the p-shell, and here the method is applied indeed.

As a last remark on the applicability, we want to emphazise the importance of the transformation coefficients between the bases, determined by different chains. Since, if such coefficients could be calculated or tabulated, it would be possible to treat a model Hamiltonian, which is a linear combination of Casimir invariants of different chains. And so we get still a wider class of possibilities. The importance of these transformation coefficients has already been emphazised by Moshinsky for the mathematical and a simple physical chain [*11*].

APPENDIX A

We shall prove the formula only for the case that $\mathrm{K} = \mathrm{A}_1$, the algebra of the group SU_2. In this case we label the elements of $\overline{\mathrm{J}^*}$ by the half of the value of the function on the canonical element of J, as usual, and we call the values $j_1, j_2 \ldots$ for elements of $\overline{\mathrm{J}^*}$ and $m_1 m_2 \ldots$ for those of J^*. All these numbers may be integer or half integer.

In this notation we have:

$$p(l, ij) = \sum_{m_i, m_j} (-1)^{l_i - m_i} \begin{pmatrix} l_i & l & l_j \\ -m_i & l & m_j \end{pmatrix} \mathrm{E}_{im_i}^{jm_j}$$

$p(l, ij)$ is the highest weight vector of the module $\mathrm{P}(l, ij)$ with vectors:

$$p_m(l, ij) = \sum_{m_i, m_j} (-1)^{l_i - m_i} \begin{pmatrix} l_i & l & l_j \\ -m_i & m & m_j \end{pmatrix} \mathrm{E}_{im_i}^{jm_j}$$

We want to calculate $\mathrm{W}([\mathrm{P}(l, ij)\mathrm{P}(k, i'j')],)$.

So we construct:

$$\sum_{m,n} [p_m(l, ij), p_n(k, i'j')] \begin{pmatrix} l & k & \mathrm{L} \\ m & n & -\mathrm{L} \end{pmatrix}$$

The result is certainly the highest weight vector of the unique element of

$$W([P(l, ij)P(k, i'j')], L)$$

up to a constant.

After using the commutation relations of the operators $E_{im_i}^{jm_j}$ and the formula (ref. [9], p. 152):

$$\sum_{n_1 n_2 n_3} (-1)^{l_1-n_1+l_2-n_2+l_3-n_3} \begin{pmatrix} l_1 & j_1 & l_2 \\ n_1 & m_1 & -n_2 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & l_3 \\ n_2 & m_2 & -n_3 \end{pmatrix} \begin{pmatrix} l_3 & j_3 & l_1 \\ n_3 & m_3 & n_1 \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & l_1 & l_2 \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

we get by a straight forward calculation the formula:

$$W([P(l, ij)P(k, i'j')], L) = Q_L$$

with $q_L \in Q_L$, the highest weight vector of Q_L, equal to

$$q_L = (-1)^{l_i+l_j+L}\delta(i'j) \begin{Bmatrix} l_i & l_{j'} & L \\ k & l & l_j \end{Bmatrix} p(L, ij') - (-1)^{k+l+L}(-1)^{l_{i'}+l_j-L}\delta(ij') \begin{Bmatrix} l_{i'} & l_j & L \\ l & k & l_{j'} \end{Bmatrix} p(L, i'j)$$

APPENDIX B

1. In the first example we construct, in a straight forward manner, all the minimal extensions of the subalgebra A_1 of the algebra D_5.

First we « realize » this chain by operators in a simple D_5-module, for which we take the 10-dimensional one M_0. We characterize A_1 further by taking (¹)

$$\dim V(M_0, l) = 1 \quad \text{if} \quad l = 1 \text{ or } 3$$

and otherwise

$$\dim V(M_0, l) = 0.$$

The possible operators p are (if M_1, M_2 are resp. the 3 and 7 dimensional A_1-submodules):

$p(l, 11)$	$l = 0, 1, 2$
$p(l, 12)$	$l = 2, 3, 4$
$p(l, 21)$	$l = 2, 3, 4$
$p(l, 22)$	$l = 0, 1, 2, 3, 4, 5, 6$

Next we try to find the expressions of $V(l)$, $V(D_5, l)$, $V(A_1, l)$ in terms of the p's.

Obviously

$$V(l) = \{ p(l, 11),\ p(l, 12),\ p(l, 21),\ p(l, 22) \}$$

The operators of D_5 consist only of antisymmetric combinations. It follows that:

$$V(D_5, l) = \{ p(l, ij) - (-1)^l p(l, ji) \}$$

(¹) Because we are dealing with A_1, we label the elements of J^*, $\overline{J^*}$ by half of the value, taken by the function on the canonical element of J as usual. We omit the K (in this case A_1) as an index of V.

To prove this, we can use the formula of appendix A and prove that the given operators constitute a simple algebra of rank 5, dimension 45.

The operators of $V(A_1, 1)$ are found by taking into account the given dimensionalities of M_1, M_2 and the fact that we have only one vector in $V(A_1, 1)$.

$$V(A_1, 1) = \{ h_1 = p(1, 11)\sqrt{6} + p(1, 33)\sqrt{84} \}$$

First we look for extensions with $\theta = 1$. This means that we take a vector x_ρ in some space $V(D_5, l)$ $(l = 1, 2, 3, 4, 5)$. We have:

$$x_\rho = \sum_{i,j} a^{ij}(p(l, ij) - (-1)^l p(l, ji))$$

and if we call X_ρ the module with $x_\rho \in X_\rho$ then we must find the solutions of $\{ a^{ij} \}$ such that:

$$\begin{aligned} &W([X_\rho X_\rho), 1) \subset \{ H_1 \} \\ &W([X_\rho X_\rho], l) \subset \{ X_\rho \} \\ &W([X_\rho X_\rho], L) = \phi \qquad \text{otherwise.} \end{aligned}$$

If $Q(k) \in W([X_\rho X_\rho], k)$ and $q(k)$ the highest weight vector of $Q(k)$ then:

$$\begin{aligned} q(k) = \sum a^{ij}a^{i'j'}\Bigg[&\delta_{ji'} \begin{Bmatrix} l_i & l_{j'} & k \\ l & l & l_j \end{Bmatrix} R(k, ij') \\ &+ \delta_{ij'} \begin{Bmatrix} l_{i'} & l_j & k \\ l & l & l_i \end{Bmatrix} R(k, ji') \\ &- (-1)^l \delta_{jj'} \begin{Bmatrix} l_i & l_{i'} & k \\ l & l & l_j \end{Bmatrix} R(k, ii') \\ &- (-1)^l \delta_{ii'} \begin{Bmatrix} l_j & l_{j'} & k \\ l & l & l_i \end{Bmatrix} R(k, jj') \Bigg] \end{aligned}$$

with

$$R(k, ij) = p(k, ij) - (-1)^k p(k, ji)$$

If we impose the conditions:

$$q(k) = \mu x_\rho \delta_{kl} + \lambda h_1 \delta_{k1}$$

We get solutions with

$l = 1$ $\qquad \{ p(1, 11) \}$

$l = 3$ $\qquad \left\{ \frac{1}{2} p(3, 22) + \sqrt{\frac{2}{3}}(kp(3, 21) + k^{-1}p(3, 12)) \right\}$ $\qquad$ for every $\qquad k \neq 0$

$l = 2, 4, 5$ $\qquad$ impossible.

It can be checked that we have the extensions:

$$\begin{aligned} &A_1 \subset A_1 \oplus A_1 \subset D_5 \\ &A_1 \subset B_2 \subset D_5 \end{aligned}$$

If we look for minimal extensions with $\theta = 2, 3$ then we must take X_ρ, Y_k ... different and independent of the preceding solutions. In this case however:

$$W([X_\rho, X_\rho], 1) \not\subset \{ H_1 \}$$

and so one of them must be a linear combination:

$$\lambda p(1, 11) + \nu p(1, 33) \quad \text{different from} \quad h_1.$$

That means that the extension contains one of the preceding subalgebras, and so we have not a minimal extension.

The result is that we only have minimal extensions with $\theta = 1$, and that the two solutions are:

$$A_1 \subset A_1 \oplus A_1 \subset D_5$$
$$A_1 \subset B_2 \subset D_5$$

2. In this second example we use more sophisticated arguments to illustrate some remarks in 6.

We take $A_2 \subset A_9$ and M_0 the 10 dimensional A_9-module with only ([1])

$$\dim V(M_0, (3, 0)) = 1.$$

We have only one A_2-submodule, for this reason we omit the i, j in the operators p.

The possible p's are:

$p((0, 0))$	dimension of the module	1
$p((1, 1))$		8
$p((2, 2))$		27
$p((3, 3))$		64

These are the elements of $V(\alpha)$.

The elements of $V(A_9, \alpha)$ are:

$$p((1, 1)), \quad p((2, 2)), \quad p((3, 3)).$$

This can be found only by checking the dimension.

The only operator of $V(A_2, \alpha)$ is:

$$p((1, 1)).$$

There are only two possibilities for an extension:

$$\text{with} \quad p((2, 2)) \quad \text{or with} \quad p((3, 3))$$

The last one is impossible because it would be a simple algebra of dimension 72, and rank 6 which doesn't exist.

The first one is impossible because it would be a simple algebra of rank 5, dimension 35, and so A_5, having a 10 dimensional irreducible module. It follows that, in this very special case, the only minimal extension of A_2 in A_9 is A_9 itself.

ACKNOWLEDGEMENTS

I am indebted to Professor F. Cerulus and Dr. A. Verbeure for helpful discussions. Thanks are also due to Professor D. Speiser and Professor L. Bouckaert for their continuing interest in this work.

([1]) We indicate the elements of J^* by $(\alpha(h_1), \alpha(h_2))$, $\{ h_1, h_2 \}$ a canonical basis in J.

REFERENCES

[*1*] E. Dynkin, *Am. Math. Soc. Transl. Ser.*, **2**, **6**, 1957, 111-244.

[*2*] J. Tits, Tabellen zu den Einfachen Lie-Gruppen und ihre Dahrstellungen. *Lecture Notes in Mathematics*, 40. Springer, New York, 1967.

[*3*] M. Moshinsky, Group theory and the many-body problem, in *Physics of many-particle systems*. Ed. by E. Meeron. Gordon and Breach, New York, 1965.

[*4*] J. Flores, E. Chacon, P. A. Mello and M. de Llano, *Nuclear Physics*, **72**, 1965, 352.

[*5*] J. P. Elliott, *Proc. Roy. Soc.*, A **245**, 1958, 562.

[*6*] H. J. Weber *et al.*, *Z. Phys.*, **190**, 1966, 25.

[*7*] N. Jacobson, *Lie algebras*. Interscience Publishers. Tracts in Mathematics Number 10, 1962.

[*8*] E. P. Wigner, In *Quantum theory of angular momentum*, by L. C. Biedenharn and H. Van Dam. Academic Press Oxford, London, 1965.

[*9*] A. P. Yutsis, I. B. Levinson and V. V. Vanagas, *The theory of angular momentum*. Israel Program for Scientific Translations, Jerusalem, 1962.

[*10*] Takchiko Ishidru *et al.*, *J. Math. Phys.*, **10**, 1969, 683.

[*11*] M. Moshinsky and V. Syamala Devi, *J. Math. Phys.*, **10**, 1969, 455.

Manuscrit reçu le 28 janvier 1970.