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Section A :
Physique théorique.

On the decomposition of the tensorial product of two representations of the Poincaré group — Case with at least one imaginary mass

by

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ABSTRACT. — We compute the Clebsch-Gordan coefficients corresponding to the tensorial product of the following representations of the Poincaré group: one real mass representation and one imaginary mass representation, or one zero mass representation and one imaginary mass representation, or two imaginary mass representations. We give a final expression which is valid for all cases of tensorial products.

RÉSUMÉ. — Dans cet article, on calcule les coefficients de Clebsch-Gordan correspondant au produit tensoriel de deux représentations quelconques du groupe de Poincaré et en particulier au produit tensoriel de deux représentations à masse imaginaire. On obtient finalement une expression unique pour tous les coefficients de Clebsch-Gordan du groupe de Poincaré.

INTRODUCTION

It is of great importance in scattering theory to know the form of Clebsch-Gordan coefficients. Indeed, because of Poincaré invariance, the initial and final free states must belong to tensorial products of Hilbert spaces in which unitary irreducible representations of the Poincaré group act. These products contain invariant subspaces which can be found with the help of Clebsch-Gordan coefficients. Since the S-matrix leaves invariant these subspaces, the decomposition of initial and final states leads to the definition of reduced matrix elements.

Actually Clebsch-Gordan coefficients have already been computed in cases of real-mass representations [*1*]-[*8*]. But if we want to consider negative energy particles on equal footing with positive ones, there appear imaginary mass representations in the decomposition of the Kronecker product of real mass representations as soon as they are on different sheets of the mass hyperboloid. Therefore, we must be able to perform the Kronecker product of two imaginary mass representations and in fact of any two representations of the Poincaré group ; and that is what we shall do here.

I. — PRELIMINARIES

1. — Notations

Let SL(2, $\mathbb{C}$) be the group of 2×2 unimodular complex matrices ; it is the universal covering group of the Lorentz group. Let $\mathbb{R}_4$ be the 4-dimensional Minkowski space ; to any point $P = (p_0, \vec{p}) \in \mathbb{R}_4$, we associate the matrix

$$P = \begin{vmatrix} p_0 - p_3 & p_2 - ip_1 \\ p_2 + ip_1 & p_0 + p_3 \end{vmatrix}. \tag{1}$$

SL(2, $\mathbb{C}$) acts on $\mathbb{R}_4$ by

$$\Lambda : P \to P' = \Lambda P \Lambda^+ \tag{2}$$

which we shall frequently write as

$$P \to P' = \Lambda \cdot P \tag{3}$$

Let $\mathcal{T}_4$ be the group of translations in $\mathbb{R}_4$ and $\widehat{\mathcal{T}}_4$ its dual group, that is the group of all continuous characters on $\mathcal{T}_4$. If $a = (a_0, a_1, a_2, a_3) \in \mathcal{T}_4$ and $\widehat{a} = (\widehat{a}_0, \widehat{a}_1, \widehat{a}_2, \widehat{a}_3) \in \widehat{\mathcal{T}}_4$, the value of $\widehat{a}$ at a will be written as

$$\langle a, \widehat{a} \rangle = e^{i(a_0 \widehat{a}_0 - a_1 \widehat{a}_1 - a_2 \widehat{a}_2 - a_3 \widehat{a}_3)} \tag{4}$$

Thus $\mathcal{T}_4$ and $\widehat{\mathcal{T}}_4$ may be indentified with $\mathbb{R}_4$.

Finally we can define the semi-direct product of $\mathcal{T}_4$ and SL(2, $\mathbb{C}$) to obtain a group $\mathcal{P}$ which is homomorphic to the Poincaré group. The multiplication law is, as usual

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda \cdot a', \Lambda\Lambda') \qquad \begin{cases} a \in \mathcal{T}_4 \\ \Lambda \in SL(2, \mathbb{C}) \end{cases} \tag{5}$$

$\mathcal{T}$ acts upon $\mathcal{C}_4$ by

$$(6)\quad (a,\Lambda): b = (b,1)\in \mathcal{C}_4 \to (a,\Lambda)(b,1)(-\Lambda^{-1}\cdot a, \Lambda^{-1}) = (\Lambda\cdot b, 1) = \Lambda\cdot b$$

and on $\widehat{\mathcal{C}}_4$ by (cf. (4)):

$$(7)\quad \widehat{b} \to \Lambda^{-1}\cdot\widehat{b}.$$

The orbits of $\mathcal{T}$ in $\widehat{\mathcal{C}}_4$ are:

(8)
$\Omega_M^{\pm}$ = the two sheets of the hyperboloid corresponding to a mass $M^2 > 0$
$C_0^{\pm}$ = the two sheets of the cone $M = 0$
Ω_M = the one-sheeted hyperboloid corresponding to a mass $M^2 < 0$.

We shall note Ω an arbitrary orbit. In each Ω, we choose a fixed point π ; more precisely, we take

$$(9)\quad \begin{aligned} \pi_M^{\pm} &= (\pm M, 0, 0, 0) \in \Omega_M^{\pm} \\ \pi_0^{\pm} &= \left(\pm\frac{1}{2}, 0, 0, \pm\frac{1}{2}\right) \in C_0^{\pm} \\ \pi_M &= (0, 0, 0, M') \in \Omega_M \text{ where } M'^2 = -M^2. \end{aligned}$$

The stationary groups of these points are respectively SU(2), $\mathcal{E}_2$ (homomorphic to the 2-dimensional euclidean group) and SU(1, 1) ; we shall note $\mathcal{S}$ any of them.

Then all the unitary irreducible representations of $\mathcal{T}$ are obtained by inducing [9] the representations L of $\mathcal{C}_4\mathcal{S}$ defined by

$$(10)\quad L(a, s) = \langle a, \pi\rangle D(s) \qquad \begin{cases} a \in \mathcal{C}_4 \\ s \in \mathcal{S} \end{cases}$$

where D is a unitary irreducible representation of $\mathcal{S}$ in a Hilbert space $\mathcal{K}$.

These induced representation ${}_{\mathcal{T}}U^L$ can be realized in the space $\mathcal{H}$ of functions f from SL(2, $\mathbb{C}$) into $\mathcal{K}$ and such that

– $\forall v \in \mathcal{K}$, $\Lambda \to \langle f(\Lambda), v\rangle_{\mathcal{K}}$ is a measurable function,

(11) – $f(s\Lambda) = D(s)f(\Lambda)\forall\Lambda \in SL(2, \mathbb{C})$, $s \in \mathcal{S}$,

(12) – $\int \|f(\Lambda)\|^2 d\mu < \infty$ where $d\mu$ is the invariant measure on the orbit Ω.

Explicitly, we have

$$(13)\quad {}_{\mathcal{T}}U^L(a\cdot\Delta)f(\Lambda) = \langle a, \Delta^{-1}\pi\rangle_{SL(2,\mathbb{C})}U^D(\Delta)f(\Lambda)$$

Finally we want to consider the representation of $\mathcal{T} \times \mathcal{T}$ defined by the tensorial product ${}_{\mathcal{T}}U^{L_1} \otimes {}_{\mathcal{T}}U^{L_2}$ of two representations ${}_{\mathcal{T}}U^{L}$ and to decompose its restriction to $\mathcal{T}$ into irreducible components. Since ${}_{\mathcal{T}}U^{L_1} \otimes {}_{\mathcal{T}}U^{L_2}$ is equivalent to ${}_{\mathcal{T}\times\mathcal{T}}U^{L_1 \otimes L_2}$, the first step of the decomposition will be to apply Mackey's [9] induction reduction theory. We shall briefly recall it in this special case.

2. — Mackey's induction reduction theorem [9].

Let us review our hypothesis. We have:

$\tilde{G} \equiv \mathcal{T} \times \mathcal{T}$, a locally compact separable group,
$G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$,
$K = \mathcal{C}_4\mathcal{S}_1 \times \mathcal{C}_4\mathcal{S}_2$, a closed subgroup
$R = L_1 \otimes L_2$, a unitary representation of K in the Hilbert space $\mathcal{K}_1 \otimes \mathcal{K}_2$,
$X = G/K$, a coset space which is isomorphic to $\Omega_1 \times \Omega_2$, (π_1, π_2), a fixed point in $\Omega_1 \times \Omega_2$.

Let (P, Q) be an arbitrary point in $\Omega_1 \times \Omega_2$ such that

$$P = \Lambda_P^{-1}\pi \qquad Q = \Lambda_Q^{-1}\pi_2, \qquad \Lambda_P, \Lambda_Q \in SL(2, \mathbb{C}). \tag{14}$$

Then any $(\Lambda_1, \Lambda_2) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \equiv G$ can be written

$$(\Lambda_1, \Lambda_2) = (A_1, A_2)(\Lambda_P, \Lambda_Q) \quad \text{with} \quad (A_1, A_2) \in \mathcal{S}_1 \times \mathcal{S}_2. \tag{15}$$

The form of ${}_{\tilde{G}}U^{R}$ can be inferred from (13).

Moreover, we have $\mathcal{T}$, a closed subgroup of $\tilde{G}$ which acts on X ; if D is an orbit, the space of cosets can be identified with { D}. We can always find elements 0_1, 0_2, Λ belonging to SL(2, $\mathbb{C}$) and such that:

$$(\Lambda_P, \Lambda_Q) = (0_1, 0_2)(\Lambda, \Lambda) \tag{16}$$

then (14) gives

$$\begin{aligned}(P, Q) &= \left(\Lambda^{-1}0_1^{-1}\pi_1, \Lambda^{-1}0_2^{-1}\pi_2\right) \\ &= \left(\Lambda^{-1}\pi_1', \Lambda^{-1}\pi_2'\right) \begin{cases} \pi_1' = 0_1^{-1}\pi_1 \\ \pi_2' = 0_2^{-2}\pi_2 \end{cases}\end{aligned} \tag{17}$$

We shall take the arbitrary point $(\pi_1', \pi_2') \in D$ to characterize the orbit D. Since D is isomorphic to ${}_{\mathcal{C}_4\mathcal{S}'}\backslash\mathcal{T}$, where $\mathcal{S}'$ is the stationary group of (π_1', π_2'), the points in D can be characterized by the corresponding element Λ of $_{S'}\backslash SL(2, \mathbb{C})$.

We know from measure theory that there exists an invariant measure μ_D on D such that

$$\int d\mu(\Lambda_P, \Lambda_Q) = \int_D d\mu_D(\dot{\Lambda}) \int d\nu(D) \tag{18}$$

where $d\nu(D)$ is the measure obtained canonically from $d\mu$.

The restriction of $_{\tilde{G}}U^R$ to $\mathfrak{T}$ can be written

$$_{\tilde{G}}U^R(a, \Delta) f_{12}(\Lambda_1, \Lambda_2) = \langle a, \Lambda_1^{-1}\pi_1 \rangle \langle a, \Lambda_2^{-1}\pi_2 \rangle f_{12}(\Lambda_1\Delta, \Lambda_2\Delta) \qquad (a, \Delta) \in \mathfrak{T} \tag{19}$$

Recalling the decomposition defined by (12) and (16)

$$(\Lambda_1, \Lambda_2) = (A_1, A_2)(0_1, 0_2)(\Lambda, \Lambda), \tag{20}$$

we shall define the function

$$F_{12}(\Lambda) = R(A_1, A_2)^{-1} f_{12}(\Lambda_1, \Lambda_2) \tag{21}$$

Then (19) becomes, if we note $_{\tilde{G}}\tilde{U}^R$ the new form of $_{\tilde{G}}U^R$:

$$_{\tilde{G}}\tilde{U}^R(a, \Delta) F_{12}(\Lambda) = \langle a, \Lambda^{-1}(\pi_1 0_1 + \pi_2 0_2) \rangle F_{12}(\Lambda\Delta) \tag{22}$$

and this representation depends only on D ; let us call it U(D). We deduce from this that

$$_{\tilde{G}}\tilde{U}^R/\mathfrak{T} = \int^{\oplus} U(D) d\nu(D) \tag{23}$$

Let us characterize U(D). The stationary group S' of $(\pi_1 0_1, \pi_2 0_2)$ in $\mathfrak{T}$ is isomorphic to

$$\mathfrak{T} \cap (0_1^{-1}, 0_2^{-1}) K (0_1, 0_2)$$

Moreover, F_{12} is such that

$$F_{12}(S'\Lambda) = R(A_1, A_2) F_{12}(\Lambda) \left\{ S' = 0_1^{-1} A_1 0_1 = 0_2^{-1} A_2 0_2 \in S' \right\} \tag{24}$$

Therefore, we are able to state that U(D) is the representation of $\mathfrak{T}$ which is induced by the representation T_D of $\mathcal{T}_4 S'$, where

$$\begin{aligned} T_D(a, S') &= R[(a, A_1), (a, A_2)] \\ &= R[(a, 0_1 S' 0_1^{-1}), (a, 0_2 S' 0_2^{-1})]. \end{aligned}$$

We shall now find explicitly the decomposition (23).

II. — MACKEY'S DECOMPOSITION OF THE TENSORIAL PRODUCT OF TWO REPRESENTATIONS OF $\mathcal{T}$

We shall consider the following cases

$$(1)\quad \Omega_2 = \Omega_{m_2},\quad m_2^2 = -m_2'^2 < 0 \quad , \quad \pi_2 = \pi_{m_2},\quad \mathcal{S}_2 = SU(1, 1)$$

$$(2)\quad \Omega_1 = \begin{cases} \Omega^{\pm}_{m_1},\ m_1^2 > 0 & , \quad \pi_1 = \pi^{\pm}_{m_1},\quad \mathcal{S}_1 = SU(2) : \text{case I}^{\pm} \\ C^{\pm}_0,\ m_1 = 0 & , \quad \pi_1 = \pi^{\pm}_0,\quad \mathcal{S}_1 = \mathcal{E}_2 : \text{case II}^{\pm} \\ \Omega_{m_1},\ m_1^2 = -m_1'^2 < 0 & , \quad \pi_1 = \pi_{m_1},\quad \mathcal{S}_1 = SU(1, 2) : \text{Case III} \end{cases}$$

1. — Characterization of the orbits D.

The most convenient choices of (π_1', π_2') are exhibited in Appendix II. From the values of the parameters defining Ω_1, we deduce easily the orbits which appear in each case:

$$(3)\quad \begin{array}{lll} \text{Cases I}^+, \text{II}^+ & \rightarrow \text{orbits} & A^+, B^+ \\ \text{I}^-, \text{II}^- & & A^-, B^- \\ \text{III} & & A^{\pm}, B^{\pm}, C \end{array}$$

We have also the value of Λ:

$$(4)\quad \Lambda = A\Delta \quad A \in SU(2), \quad \Delta \in SL(2, \mathbb{C})$$

or

$$(5)\quad \Lambda = B\Delta \quad B \in SU(1, 1), \ \Delta \in SL(2, \mathbb{C})$$

where the meaning of A, B and Δ is given in Appendix II.

We shall now deduce 0_1 and 0_2.

In the $A^{\pm}$, $B^{\pm}$ classes

$$(6)\quad \pi_1' = \Delta_1^{-1}\pi_1 \quad , \quad \pi_2' = \Delta_2^{-1}\zeta^{-\varepsilon'}\pi_2$$

where Δ_1, Δ_2 are diagonal matrices which are completely determined once M is known, and

$$\zeta = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

$\varepsilon' = 0$ if the $\mathbb{Z}$-coordinate q_3'' of π_2' is positive

$\varepsilon' = 1$ if the $q_3'' < 0$

In the C class,

(7) $$\pi_1' = \Sigma_1^{-1}\pi_1, \qquad \pi_2' = \Sigma_2^{-1}\pi_2$$

where Σ_1^{-1}, Σ_2^{-1} are pure rotation matrices:

$$\Sigma_i = \begin{vmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{vmatrix}, \qquad i = 1, 2$$

and θ_i depends only on M.

Finally, one computes

(8) $$d\nu(\mathrm{M}) = \left| 4m_1^2 m_2^2 - (\mathrm{M}^2 - m_1^2 - m_2^2)^2 \right|^{1/2} d\mathrm{M}^2$$

From now on, each class D will be indicated by the corresponding M.

2. — The T_M representation of the (P″, Q″) stationary group.

Classes $A^\pm$, $B^\pm$,

Since the stationary group of (π_1, π_{m_2}) is $\mathcal{C}_4\mathcal{S}_1 \times \mathcal{C}_4\mathrm{SU}(1, 1)$, the stationary group of (P″, Q″) will be

(9) $$\mathcal{T} \cap [(0, \Delta_1^{-1}), (0, \Delta_2^{-1}\zeta^{-\varepsilon'})]\mathcal{C}_4\mathcal{S}_1 \times \mathcal{C}_4\mathrm{SU}(1, 1)[(0, \Delta_1), (0, \zeta^{\varepsilon'}\Delta_2)].$$

On the other hand, this group is easily seen to be $\mathcal{C}_4\mathcal{R}_\varphi \times \mathcal{C}_4\mathcal{R}_\varphi$ ($\mathcal{R}_\varphi$ is the group of rotations about the $\mathcal{Z}$-axis).

We then define the representation T_M of $\mathcal{C}_4\mathcal{R}_\varphi$ by (I.26):

(10) $$\mathrm{T}_\mathrm{M}(a, \mathrm{A}_\varphi) = \mathrm{L}_1 \otimes \mathrm{L}_2 \{[(0, \Delta_1), (0, \zeta^{\varepsilon'}\Delta_2)][(a, \mathrm{A}_\varphi), (a, \mathrm{A}_\varphi)] \\ [(0, \Delta_1^{-1}), (0, \Delta_2^{-1}\zeta^{-\varepsilon'})]\} \qquad \mathrm{A}_\varphi \in \mathcal{R}_\varphi$$

Since

(11) $$(0, \zeta^{\varepsilon'}\Delta_2)(a, \mathrm{A}_\varphi)(0, \Delta_2^{-1}\zeta^{-\varepsilon'}) = (\zeta^{\varepsilon'}\Delta_2 \cdot a, \mathrm{A}_{\sigma\varphi})$$

where

(12) $$\begin{aligned} \sigma &= + \quad \text{if } \varepsilon' = 0 \\ \sigma &= - \quad \text{if } \varepsilon' = 1, \end{aligned}$$

we finally obtain, using (1.10):

(13) $$\mathrm{T}_\mathrm{M}(a, \mathrm{A}_\varphi) = \langle a, \mathrm{P}'' + \mathrm{Q}'' \rangle \, \mathrm{D}_1(\mathrm{A}_\varphi)\mathrm{D}_2(\mathrm{A}_{\sigma\varphi})$$

Class C

The stationary group of (P'', Q'') is

(14) $\mathcal{T} \cap [(0, \Sigma_1^{-1}), (0, \Sigma_2^{-1})]\mathcal{C}_4SU(1, 1) \times \mathcal{C}_4SU(1, 1)[(0, \Sigma_1), (0, \Sigma_2)]$.

It also is the subgroup of $\mathcal{C}_4SU(1, 1)$ which leaves invariant the y-axis : it is therefore $\mathcal{C}_4\mathcal{R}_t$, where $\mathcal{R}_t$ is the group of matrices

(15) $$A_t^\xi = (-1)^\xi \begin{vmatrix} \operatorname{ch} t & i\operatorname{sh} t \\ -i\operatorname{sh} t & \operatorname{ch} t \end{vmatrix}, \qquad \xi = 0, 1.$$

T_M is defined as above. Since

(16) $$\Sigma_i A_t \Sigma_i^{-1} = A_t \qquad i = 1, 2$$

we finally obtain

(17) $$T_M(a, A_t) = \langle a, P'' + Q'' \rangle D_1(A_t)D_2(A_t).$$

3. — Induction of T_M.

Inducing T_M to $\mathcal{T}$, we obtain a representation $U(M)$ which acts on the functions $F_{12}(\Lambda)$ defined by (I.21). The decomposition (I.20) can be written here as:

(18) $$(\Lambda_1, \Lambda_2) = (A_1, A_2)(\Delta_1, \zeta^{\varepsilon'}\Delta_2)(\Lambda, \Lambda), A_1 \in \mathcal{E}_1, A_2 \in SU(1, 1)$$
for classes $A^{\pm}$, $B^{\pm}$

or

(19) $$(\Lambda_1, \Lambda_2) = (A_1', A_2')(\Sigma_1, \Sigma_2)(\Lambda, \Lambda), A_1', A_2' \in SU(1, 1)$$
for class C.

The functions F_{12}^M are defined by

(20) $$F_{12}^M(\Lambda) = [D_1(A_1)^{-1} \otimes D_2(A_2)^{-1}f_{12}](\Lambda_1, \Lambda_2) ;$$

D_1 is a unitary irreducible representation of $SU(2)$, $\mathcal{E}_2$ or $SU(1, 1)$ and D_2 is a unitary irreducible representation of $SU(1, 1)$. Then the action of $U(M)$ is

(21) $$U(M ; a, \Lambda)F_{12}^M(\Lambda) = \langle a, \Lambda^{-1}(P'' + Q'') \rangle_{SL(2,\mathbb{C})} U^{R_{12}}(\Delta)F_{12}^M(\Lambda)$$

with

$$(22) \qquad \begin{array}{ll} R_{12} = D_1(A_\varphi) \otimes D_2(A_{\sigma\varphi}) & \text{Classes } A^\pm, B^\pm \\ R_{12} = D_1(A_t) \otimes D_2(A_t) & \text{Class C} \end{array}$$

We have therefore obtained the decomposition (I.23). We must now decompose U(M).

III. — DECOMPOSITION OF THE REPRESENTATION U(M)

Comparing (II.21) with the form (I.13) of a unitary irreducible representation of $\mathcal{T}$, we see that we need only the decomposition of ${}_{S}U^{R_{12}}$ into irreducible components to perform the decomposition of U(M) ; for, if we write formally

$$(1) \qquad {}_{S}U^{R_{12}} = \Sigma \oplus D,$$

we have the equivalences

$$(2) \qquad {}_{SL(2,C)}U^{R_{12}} \approx {}_{SL(2,C)}({}_{S}U^{R_{12}}) \approx \Sigma \oplus {}_{SL(2,C)}U^{D}.$$

a) Case when $R_{12} = D_1(A_\varphi) \otimes D_2(A_\varphi)$.

Let us introduce the following expansion for $F_{12}^M(\Lambda)$ in $\mathcal{K}_1 \otimes \mathcal{K}_2$ (*)

$$(3) \qquad F_{12}^M(\Lambda) = \sum_{n_1,n_2} a_{n_1n_2}^M(\Lambda)\Phi^{n_1n_2}$$

where

Φ^{n_i} is a basis of $\mathcal{K}_i$, $i = 1, 2$, exhibited in Appendix I,

and:

$$(4) \qquad \Phi^{n_1n_2} = \Phi^{n_1} \otimes \Phi^{n_2}$$

1) *Class $A^\pm$*

In this case, we shall use the decomposition

$$(5) \qquad \Lambda = A\Gamma Z$$

where A, Z $\in$ SU(2)

$$\Gamma = \begin{vmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{vmatrix}, \qquad \gamma > 0.$$

(*) Recall that $\mathcal{K}_i$, $i = 1, 2$, are spaces in which the representations D_i are realized.

F is square-integrable on SU(2) almost everywhere and one may write, applying Plancherel's formula

$$(6) \qquad F^{M}_{12}(\Lambda) = \sum_{n_1,n_2}\sum_{n,j}\sum_{k}\sqrt{2j+1}\,a^{Mj}_{n_1n_2nk}(\Gamma Z)D^{j}_{nk}(\Lambda)\Phi^{n_1n_2};$$

D^{j}_{nk} is a matrix element of the D^{j} representation when written in canonical form.

Condition (I.24) implies:

$$(7) \qquad \begin{aligned} F^{M}_{12}(A_{\varphi}\Lambda) &= \Sigma a^{Mj}_{n_1n_2nk}(\Gamma Z)D^{j}_{ns}(A_{\varphi})D^{j}_{sk}(\Lambda)\Phi^{n_1n_2}\sqrt{2j+1} \\ &= D_1(A_{\varphi})D_2(A_{\sigma\varphi})F^{M}_{12}(\Lambda) \end{aligned}$$

where

$$(8) \qquad A_{\varphi} = \begin{vmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{vmatrix}$$

From the values of the matrix elements of representations D (cf. Appendix I), we conclude that n can take only one value which is determined by the representations D_1 and D_2:

$$(9) \qquad n = n_1 + \sigma n_2$$

In each case (7) implies

$$(10) \qquad F^{M}_{12} = \sum_{n_1n_2}\sum_{j\geqslant |n|}\sum_{k=-j}\sqrt{2j+1}\,a^{Mj}_{n_1n_2k}(\Gamma Z)D^{j}_{nk}(\Lambda)\Phi^{n_1n_2}$$

This achieves the decomposition of U(M) into irreducible components in case $A^{\pm}$.

2) *Class $B^{\pm}$*

We shall now use the decomposition

$$(11) \qquad \Lambda = B\Gamma' Z'$$

where $B \in SU(1, 1)$, $Z' \in SU(2)$

$$\Gamma' = \begin{vmatrix} \gamma' & 0 \\ 0 & \gamma'^{-1} \end{vmatrix}, \qquad \gamma' > 0.$$

F_{12}^{M} is square integrable on SU(1, 1) and we may still apply Plancherel's formula (11), which gives:

$$\text{(12)}\qquad F_{12}^{M}(\Lambda) = \sum_{n_1n_2}\left\{\sum_{\eta=0,1}\sum_{n,k=-\infty}^{+\infty}\int_0^{\infty} C(\rho)d\rho a_{n_1n_2nk}^{M\rho\eta}(\Gamma'Z')D_{nk}^{\rho\eta}(B) + \sum_{\varepsilon=\pm}\sum_{s=2}^{\infty}\sum_{n,k=0}^{\infty}\sqrt{s-1}\,a_{n_1n_2nk}^{Ms\varepsilon}(\Gamma'Z')D_{nk}^{s\varepsilon}(B)\right\}\Phi^{n_1n_2},$$

where

$$C(\rho) = \rho\tanh\pi\left(\frac{\rho+i\eta}{2}\right)$$

Taking into account (I.25), we find again that for a given (ρ, η) (or (s, ε)), n can have but one value given by (9).

b) Case when $R_{12} = D_1(A_t)\otimes D_2(A_t)$.

The matrix elements of $D_j(A_t)$, $j = 1, 2$, in the basis Φ^{n_j}, defined by Appendix I, are not diagonal and this makes it difficult to write condition (I.24). Therefore we shall look for a new decomposition of $F_{12}^{M}(\Lambda)$.

First, let us define a function F'^{M}_{12} from SL(2, $\mathbb{C}$) into $\mathcal{H}'_1 \otimes \mathcal{H}'_2$, where $\mathcal{H}'_j$ is one of the spaces $\mathfrak{D}^{\rho_j\eta_j}$ or $\mathfrak{D}^{s_j\varepsilon_j}$ (Appendix I), by:

$$\text{(13)}\qquad F'^{M}_{12}(\Lambda) = [(\mathfrak{J}_1\times\mathfrak{J}_2)F_{12}^{M}](\Lambda)$$

$\mathfrak{J}_j$ is defined either by (A.I.13) or by (A.I.17). Then, we define a function $\check{F}_{12}^{M}$ by

$$\text{(14)}\qquad \check{F}_{12}^{M}(\Lambda) = F'^{M}(T\Lambda), \qquad \text{where}\qquad T = \frac{1}{\sqrt{2}}\begin{vmatrix}1 & i\\ i & 1\end{vmatrix}$$

$\check{F}_{12}^{M}$ is also from SL(2, $\mathbb{C}$) into $\mathcal{H}'_1 \otimes \mathcal{H}'_2$; moreover it is such that

$$\text{(15)}\qquad \check{F}_{12}^{M}(\check{A}_t\Lambda) = \check{D}_1(A_t)\check{D}_2(A_t)\check{F}_{12}^{M}(\Lambda)$$

for any

$$\check{A}_t = T^{-1}A_tT, \qquad A_t\in\mathfrak{R}_t.$$

Therefore, functions $\check{F}_{12}^{M}$ are square integrable on every double class and build up a Hilbert space in which U(M) will be realized from now on.

We shall explicitly decompose $\check{F}^{M}_{12}$ in cases $D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$ and $D^{s_1\varepsilon_1} \otimes D^{s_2\varepsilon_2}$; the other cases are obtained trivially from these.

From the properties of the spaces $\mathcal{K}'_j$ (Appendix III), we may write:

— Case $D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$

$$(16)\quad \check{F}^{M}_{12}(\Lambda) = \frac{1}{4\pi}\sum_{\rho_1,\rho_2}\int_{-\infty}^{\infty} \mathcal{G}^{M}(\beta_1\zeta_1, \beta_2\zeta_2 ; \Lambda) \, | x |^{-\frac{1}{2}-i\left(\beta_1-\frac{\rho_1}{2}\right)} \operatorname{sgn}^{\zeta_1} x_1 \cdot | x_2 |^{-\frac{1}{2}-i\left(\beta_2-\frac{\rho_2}{2}\right)} \operatorname{sgn}^{\zeta_2} x_2 d\beta_1 d\beta_2$$

where $\zeta_i = 0, 1$

— Case $D^{s_1\varepsilon_1} \otimes D^{s_2\varepsilon_2}$

$$(17)\quad \check{F}^{M}_{12}(\Lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^{M}(\beta_1, \beta_2 ; \Lambda) z_1^{-\frac{s_1}{2}-t\beta_1} \cdot z_2^{-\frac{s_2}{2}-i\beta_2} d\beta_1 d\beta_2 .$$

To proceed with the decomposition of $\check{F}^{M}_{12}(\Lambda)$, we notice that Λ can be written

$$(18)\qquad \Lambda = \check{B}\Gamma''Z''$$

where

$$\check{B} \in SL(2, \mathbb{R}) \quad , \quad Z'' \in SU(2)$$

$$\Gamma'' = \begin{vmatrix} \gamma'' & 0 \\ 0 & \gamma''^{-1} \end{vmatrix}, \qquad \gamma'' > 0.$$

Moreover, any $\check{B} \in SL(2, \mathbb{R})$ can be univoquely decomposed as

$$(19)\qquad B = \tilde{A}^{\xi}_{t} W$$

where (*)

$$\tilde{A}^{\xi}_{t} = (-1)^{\xi}\begin{vmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{vmatrix}, \qquad \xi = 0, 1,$$

$$W = \begin{vmatrix} 1 & X \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ Y & 1 \end{vmatrix} \qquad \text{if} \quad d \neq 0$$

$$W = \begin{vmatrix} 1 & X \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \qquad \text{if} \quad d = 0$$

(*) The subgroup $\tilde{A}_t$ is the union of its two components, $\tilde{A}^0_t$ and $\tilde{A}'_t$.

Therefore, we have

$$\check{F}'^{M}_{12}(\Lambda) = \check{R}_{12}(\check{A}^{\xi}_{t})\check{F}'^{M}_{12}(W\Gamma''Z'') \tag{20}$$

Applying this to (16) and (17), we obtain

$$F'^{M}_{12}(\Lambda) = \frac{1}{(4\pi)^2}\sum_{\zeta_1\zeta_2}\int_{-\infty}^{\infty} \mathcal{G}^{M}(\beta_1\zeta_1, \beta_2\zeta_2 ; W\Gamma''Z'')(-1)^{\xi(\eta_1+\eta_2)}$$
$$\times e^{i(\beta_1+\beta_2)t}\,|\,x_1\,|^{-\frac{1}{2}-i\left(\beta_1-\frac{\rho_1}{2}\right)}\,\mathrm{Sign}^{\zeta_1} x_1\,|\,x_2\,|^{-\frac{1}{2}-i\left(\beta_2-\frac{\rho_2}{2}\right)}\,\mathrm{Sign}^{\zeta_2} x_2 d\beta_1 d\beta_2 \tag{21}$$

(Case $D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$)

or:

$$F^{M}_{12}(\Lambda) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} G^{M}(\beta_1, \beta_2 ; W\Gamma''Z'')(-1)^{-(s_1+s_2)\xi}e^{i(\beta_1+\beta_2)t}$$
$$\cdot z_1^{-\frac{s_1}{2}-i\beta_1}\, z_2^{-\frac{s_2}{2}-i\beta_2}\cdot d\beta_1 d\beta_2. \tag{22}$$

(Case $D^{s_1\varepsilon_1} \otimes D^{s_2\varepsilon_2}$)

Recalling that $\mathcal{G}$ and G belong to $\mathcal{S}(\beta_1, \beta_2)$ as soon as $\check{F}^{M}_{12}(\Lambda)$ takes values in $\mathfrak{D}^{\rho\eta}$ or $\mathfrak{D}^{s\varepsilon}$, respectively, we find that $\check{F}^{M}_{12}(\Lambda)$ is square integrable on SL(2, $\mathbb{R}$) almost everywhere ; moreover it is infinitely differentiable and of rapid decay in t.

Therefore, we need only to consider functions

$$\check{F}^{M}_{12}(\Lambda) = \check{F}^{M}_{12}(A^{t}_{\xi}W\Gamma''Z'') \tag{23}$$

belonging to $\mathcal{S}(W) = \mathcal{S}(X, Y)$ and we shall be able to apply the results obtained in Appendix III.

Using Fubini theorem, we may write (A.III.18) for $F = \check{F}^{M}_{12}$

$$I^{\rho\eta}(\check{F}^{M}_{12}) \equiv \langle K^{\rho\eta}(\check{F}^{M}_{12})\psi, \psi' \rangle$$
$$= \sum_{\xi}\int \check{F}^{M}_{12}(\check{A}^{\xi}_{t}W\Gamma''Z'')[\overline{\check{D}^{\rho\eta}(W)}\,\overline{\check{D}^{\rho\eta}(\check{A}^{\xi}_{t})}\psi](x)\overline{\psi'}(x)dtdWdx \tag{24}$$

or, according to (A.III.12) and (A.III.13):

$$I^{\rho\eta}(\check{F}^{M}_{12}) = \frac{1}{(4\pi)^2}\sum_{\xi\zeta\zeta'}\int \check{F}^{M}_{12}(\check{A}^{\xi}_{t}W\Gamma''Z'')(-1)^{\xi\eta}e^{-i\beta t}$$
$$\cdot\overline{\check{D}^{\rho\eta}(\beta, \zeta, \beta', \zeta' ; W)}\mathcal{G}(\beta', \zeta')\overline{\mathcal{G}'(\beta, \zeta)}dtdWd\beta d\beta' \tag{25}$$

which gives $\mathcal{M}_{\zeta\zeta'}[K^{\rho\eta}(\check{F}^{M}_{12} ; x, x')]$ by applying (A.III.19).

Now we want to replace F by its expression (21) (if $R_{12} = D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$) or (22) (if $R_{12} = D^{s_1\varepsilon_1} \otimes D^{s_2\varepsilon_2}$) where $\mathcal{G}^M$ and G^M belong to $\mathcal{S}(X, Y)$. We may no more apply Fubini theorem but the problem to be solved is analogous to the inversion of Fourier transformation (14) ; we thus obtain the final result.

— Case $D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$

$\mathcal{M}_{\zeta\zeta'}\{K^{\rho\eta}(\check{F}^M_{12})\}$ differs from zero only if $\eta + \eta_1 + \eta_2$ is an even number Then it is equal to

$$(26) \quad \mathcal{M}_{\zeta\zeta'}\{K^{\rho\eta}[\check{F}^M_{12}(\Lambda) ; x, x']\} = \frac{1}{(4\pi)^2} \sum_{\zeta_1\zeta_2} \int \delta(\beta_1 + \beta_2 - \beta) \cdot \mathcal{G}^M(\beta_1\zeta_1, \beta_2\zeta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''Z'' ; \rho\eta) \, | x_1 |^{-\frac{1}{2} - i\left(\beta_1 - \frac{\rho_1}{2}\right)} \mathrm{Sgn}^{\zeta_1} x_1 \cdot | x_2 |^{-\frac{1}{2} - i\left(\beta_2 - \frac{\rho_2}{2}\right)} \mathrm{Sgn}^{\zeta_2} x_2 d\beta_1 d\beta_2$$

where we have set

$$(27) \quad \mathcal{G}^M(\beta_1\zeta_1, \beta_2\zeta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''Z'' ; \rho\eta) = \int \mathcal{G}^M(\beta_1\zeta_1, \beta_2\zeta_2 ; W\Gamma''Z'') \cdot \overline{\check{D}^{\rho\eta}(\beta\zeta, \beta'\zeta' ; W)} dW$$

— Case $D^{s_1\varepsilon_1} \otimes D^{s_2\varepsilon_2}$

$\mathcal{M}_{\zeta\zeta'}\{K^{\rho\eta}(\check{F}^M_{12})\}$ differs from zero only if $\eta - s_1 - s_2$ is an even number. Then it is equal to

$$(28) \quad \mathcal{M}_{\zeta\zeta'}\{K^{\rho\eta}[\check{F}^M_{12}(\Lambda) ; x, x']\} = \frac{1}{(2\pi)^2} \int \delta(\beta_1 + \beta_2 - \beta) \cdot G^M(\beta_1, \beta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''X'' ; \rho\eta) z_1^{-\frac{s_1}{2} - i\beta_1} z_2^{-\frac{s_2}{2} - i\beta_2} d\beta_1 d\beta_2$$

where we have set

$$(29) \quad G^M(\beta_1\beta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''Z'' ; \rho\eta) = \int G^M(\beta_1\beta_2 ; W\Gamma''Z'') \cdot \overline{\hat{D}^{\rho\eta}(\beta\zeta, \beta'\zeta' ; W)} dW$$

In the same way, using (A.III.28), (A.III.24) and Fubini theorem, we may write

$$(30) \quad I^{s\varepsilon}(\check{F}^M_{12}) = \frac{1}{(2\pi)^2} \sum_{\xi} \int \check{F}^M_{12}(\check{A}^{\xi}_t W\Gamma''Z'')(-1)^{\xi s} e^{-i\beta t} \cdot \overline{\check{D}^{s\varepsilon}(\beta, \beta' ; W)} G(\beta') \overline{G'(\beta)} \cdot dt dW d\beta d\beta'.$$

Replacing $\check{F}^{M}_{12}$ by its expression (21) or (22), and proceeding as above, we obtain the final result.

— Case $D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$

$\mathcal{M}[K^{s\varepsilon}(\check{F}^{M}_{12})]$ differs from zero only if $\varepsilon + \eta_1 + \eta_2$ is an even number. Then it is equal to

$$(31)\quad \mathcal{M}[K^{s\varepsilon}(\check{F}^{M}_{12}\,;\,z,z')] = \frac{1}{(4\pi)^2}\sum_{\zeta_1\zeta_2}\int \delta(\beta_1+\beta_2-\beta)$$
$$\cdot \mathcal{G}^{M}(\beta_1\zeta_1,\,\beta_2\zeta_2\,;\,\beta,\,\beta'\,;\,\Gamma''Z''\,;\,s\varepsilon)\,|\,x_1\,|^{-\frac{1}{2}-i\left(\beta_1-\frac{\rho_1}{2}\right)}\,\mathrm{Sgn}^{\zeta_1}\,x_1$$
$$\cdot\,|\,x_2\,|^{-\frac{1}{2}-i\left(\beta_2-\frac{\rho_2}{2}\right)}\,\mathrm{Sgn}^{\zeta_2}\,x_2 d\beta_1 d\beta_2$$

where we have set

$$(32)\quad \mathcal{G}^{M}(\beta_1\zeta_1,\,\beta_2\zeta_2\,;\,\beta,\,\beta'\,;\,\Gamma''Z''\,;\,s\varepsilon)$$
$$= \int \mathcal{G}^{M}(\beta_1\zeta_1,\,\beta_2\zeta_2\,;\,W\Gamma''Z'')\overline{D^{s\varepsilon}}(\beta,\,\beta'\,;\,W)dW$$

— Case $D^{s_1\varepsilon_1} \otimes D^{s_2\varepsilon_2}$

$\mathcal{M}[K^{s\varepsilon}(\check{F}^{M}_{12})]$ differs from zero only if $\varepsilon - s_1 - s_2$ is an even number. Then it is equal to

$$(33)\quad \mathcal{M}[K^{s\varepsilon}(\check{F}^{M}_{12})\,;\,z,z')] = \frac{1}{(2\pi)^2}\int \delta(\beta_1+\beta_2-\beta)$$
$$\cdot G^{M}(\beta_1,\,\beta_2\,;\,\beta\beta'\,;\,\Gamma''Z''\,;\,s\varepsilon)\cdot z_1^{-\frac{s_1}{2}-i\beta_1}\,z_2^{-\frac{s_2}{2}-i\beta_2}\,d\beta_1 d\beta_2$$

where we have set

$$(34)\quad G^{M}(\beta_1,\,\beta_2\,;\,\beta,\,\beta'\,;\,\Gamma''Z''\,;\,s\varepsilon) = \int G^{M}(\beta_1,\,\beta_2\,;\,W\Gamma''Z'')\overline{D^{s\varepsilon}}(\beta,\,\beta'\,;\,W)dW$$

Thus we have computed all the coefficients of the expansion (A.III.29) when applied to the function $\check{F}^{M}_{12}(\Lambda)$, $\Lambda = \check{A}^{\xi}_{t}W\Gamma''Z''$. We may therefore write formally:

— Case $D^{\rho_1\eta_1} \otimes D^{\rho_2\eta_2}$

$$(35)\quad \check{F}^{M}_{12}(\Lambda) = \frac{1}{(4\pi)^2}\sum_{\eta}\int_0^{\infty} C(\rho)d\rho\sum_{\zeta\zeta'}\int_{-\infty}^{\infty} d\beta d\beta'\check{D}^{\rho\eta}(\beta'\zeta',\,\beta\zeta\,;\,\check{A}^{\xi}_{t}W)$$
$$\cdot\sum_{\zeta_1\zeta_2}\int_{-\infty}^{\infty} d\beta_1 d\beta_2\delta(\beta_1+\beta_2-\beta)\cdot\mathcal{G}^{M}(\beta_1\zeta_1,\,\beta_2\zeta_2\,;\,\beta\zeta,\,\beta'\zeta'\,;\,\Gamma''Z''\,;\,\rho\eta)$$

$$\cdot | x_1 |^{-\frac{1}{2}-i\left(\beta_1-\frac{\rho_1}{2}\right)} \operatorname{Sgn}^{\zeta_1} x_1 \cdot | x_2 |^{-\frac{1}{2}-i\left(\beta_2-\frac{\rho_2}{2}\right)} \operatorname{Sgn}^{\zeta_2} x_2$$

$$+ \frac{1}{(4\pi)^2} \sum_{s=2}^{\infty} \sqrt{s-1} \int_{-\infty}^{\infty} d\beta d\beta' \check{\mathrm{D}}^{s\varepsilon}(\beta', \beta ; \check{\mathrm{A}}_{\iota}^{\xi}\mathrm{W}) \sum_{\zeta_1\zeta_2} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \delta(\beta_1+\beta_2-\beta)$$

$$\cdot \mathcal{G}^{\mathrm{M}}(\beta_1\zeta_1, \beta_2\zeta_2 ; \beta, \beta' ; \Gamma''\mathrm{Z}'' ; s\varepsilon) | x_1 |^{-\frac{1}{2}-i\left(\beta_1-\frac{\rho_1}{2}\right)} \operatorname{Sgn}^{\zeta_1} x_1$$

$$\cdot | x_2 |^{-\frac{1}{2}-i\left(\beta_2-\frac{\rho_2}{2}\right)} \operatorname{Sgn}^{\zeta_2} x_2$$

— Case $\mathrm{D}^{s_1\varepsilon_1} \otimes \mathrm{D}^{s_2\varepsilon_2}$

$$(36) \quad \check{\mathrm{F}}_{12}^{\mathrm{M}}(\Lambda) = \frac{1}{(2\pi)^2} \sum_{\eta} \int_0^{\infty} \mathrm{C}(\rho) d\rho \sum_{\zeta\zeta'} \int_{-\infty}^{\infty} d\beta d\beta' \check{\mathrm{D}}^{\rho\eta}(\beta'\zeta', \beta\zeta ; \check{\mathrm{A}}_{\iota}^{\xi}\mathrm{W})$$

$$\cdot \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \delta(\beta_1 + \beta_2 - \beta)$$

$$\cdot \mathrm{G}^{\mathrm{M}}(\beta_1\beta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''\mathrm{Z}'' ; \rho\eta) z_1^{-\frac{s_1}{2}-i\beta_1} z_2^{-\frac{s_2}{2}-i\beta_2} + \frac{1}{(2\pi)^2} \sum_{s=2}^{\infty} \sqrt{s-1}$$

$$\cdot \int_{-\infty}^{\infty} d\beta d\beta' \check{\mathrm{D}}^{s\varepsilon}(\beta', \beta ; \check{\mathrm{A}}_{\iota}^{\xi}\mathrm{W}) \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \delta(\beta_1 + \beta_2 - \beta)$$

$$\cdot \mathrm{G}^{\mathrm{M}}(\beta_1, \beta_2 ; \beta, \beta' ; \Gamma''\mathrm{Z}'' ; s\varepsilon) z_1^{-\frac{s_1}{2}-i\beta_1} \cdot z_2^{-\frac{s_2}{2}-i\beta_2}$$

IV. — A GENERAL EXPRESSION OF THE CLEBSCH-GORDAN COEFFICIENTS OF $\mathcal{T}$

a) Decomposition of the tensorial product of two real mass representations.

This decomposition is contained in the above results. Indeed, the table (Appendix II) giving the orbits of $\mathcal{T}$ in $\Omega_1 \times \Omega_2$ remains true if $m_1^2 \geqslant 0$, $m_2^2 \geqslant 0$ but class C may no more appear. The following table gives the orbits in each case.

b) Definition of Clebsch-Gordan coefficients.

Let us review what we have obtained so far. Starting from an arbitrary function f_{12} in the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the representation ${}_{\mathcal{T}}\mathrm{U}^{\mathrm{L}_1} \otimes {}_{\mathcal{T}}\mathrm{U}^{\mathrm{L}_2}$,

$\Omega_1 \times \Omega_2$	Orbits of $\mathcal{T}$ in $\Omega_1 \times \Omega_2$
$\Omega^+_{m_1} \times \Omega^+_{m_2}$	A^+_0
$\Omega^-_{m_1} \times \Omega^+_{m_2}$	A^+_0 or A^-_0, $B^-_{\varepsilon'}$
$C^+_0 \times \Omega^+_{m_2}$	A^+_0
$C^-_0 \times \Omega^+_{m_2}$	A^+_0, $B^-_{\varepsilon'}$
$C^+_0 \times C^+_0$	A^+_1
$C^-_0 \times C^+_0$	B^-_0

we have found (cf. II.20) the isomorphism between $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\int^{\oplus} \mathcal{H}_M d\nu(M)$ which corresponds to the decomposition

$$_{\mathcal{T}}U^{L_1} \otimes {_{\mathcal{T}}U^{L_2}} = \int^{\oplus} U(M) d\nu(M).$$

Then we have completed the decomposition of $\mathcal{H}(M)$ (cf. (10), (12), (35), (36)) into subspaces $\mathcal{H}^I(M)$ (*) corresponding to irreducible unitary representations of $\mathcal{T}$.

Thus, we have found an operator $\mathcal{U}$ from $\mathcal{H}_1 \otimes \mathcal{H}_2$ into

$$\mathcal{H}^F = \int^{\oplus} \sum_I \oplus\, \mathcal{H}^I(M) d\nu(M)$$

such that

(1) $$\mathcal{U} \cdot [{_{\mathcal{T}}U^{L_1}} \otimes {_{\mathcal{T}}U^{L_2}}] = {_{\mathcal{T}}U^L}(a, \Lambda) \cdot \mathcal{U}$$

for any $(a, \Lambda) \in \mathcal{T}$.

We may recall that in the case of SU(2), $\mathcal{U}$ was a trivial operator and the Clebsch-Gordan coefficients of SU(2) were just its matrix elements. We should like to apply the same definition here. However, since the bases of our Hilbert spaces consist of distributions, we are unable to write usual matrix elements for $\mathcal{U}$. Thus we are led to define « generalized matrix elements of $\mathcal{U}$ ».

(*) I denotes the family of indices corresponding to the helicities of the incoming particle representations and to the spins of the final components.

First, we shall need dense subspaces of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}^F$. For their definition, we must decompose the functions $f_{12} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ according to the following formula:

$$(2) \qquad f_{12}(\Lambda_1, \Lambda_2) = \sum_{n_1 n_2} f_{n_1 n_2}(\Lambda_1, \Lambda_2)\Phi^{n_1, n_2} \qquad \{ \Lambda_1, \Lambda_2 \in SL(2, \mathbb{C})$$

where $\Phi^{n_1 n_2} = \Phi^{n_1} \otimes \Phi^{n_2}$ (cf. Appendix I) ;
or:

$$(3) \quad f_{12}(\Lambda_1, \Lambda_2) = (\mathfrak{J}_1 \times \mathfrak{J}_2)^{-1} \frac{1}{(4\pi)^2} \sum_{\zeta_1 \zeta_2} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \cdot f(\beta_1 \zeta_1, \beta_2 \zeta_2 \,;\, T^{-1}\Lambda_1, T^{-1}\Lambda_2) \,|\, x_1 \,|^{-\frac{1}{2} - i\left(\beta_1 - \frac{\rho_1}{2}\right)} \operatorname{Sgn}^{\zeta_1} x_1 \cdot |\, x_2 \,|^{-\frac{1}{2} - i\left(\beta_2 - \frac{\rho_2}{2}\right)} \operatorname{Sgn}^{\zeta_2} x_2,$$

if $L_j = D^{\rho_j \eta_j}$, $j = 1, 2, 3$, and

$$(4) \quad f_{12}(\Lambda_1, \Lambda_2) = (\mathfrak{J}_1 \times \mathfrak{J}_2)^{-1} \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 f(\beta_1, \beta_2 \,;\, T^{-1}\Lambda_1, T^{-1}\Lambda_2) \, z_1^{-\frac{s_1}{2} - i\beta_1} z_2^{-\frac{s_2}{2} - i\beta_2},$$

if

$$L_j = D^{s_j \varepsilon_j} \quad , \quad j = 1, 2.$$

Then let $\mathfrak{D}_{\mathcal{H}_1 \otimes \mathcal{H}_2}$ be the subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ consisting of functions f_{12} such that the coefficients $f_{n_1 n_2}(\Lambda_1, \Lambda_2)$ defined by (2) are infinitely differentiable, of compact support on $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, and of rapid decay in n_1, n_2 when the latter go to infinity. $\mathfrak{D}_{\mathcal{H}_1 \otimes \mathcal{H}_2}$ may also be obtained from functions f_{12} whose coefficients $f(\beta_1 \zeta_1 \,;\, \beta_2 \zeta_2 \,;\, T^{-1}\Lambda_1, T^{-1}\Lambda_2)$ (or $f(\beta_1, \beta_2 \,;\, T^{-1}\Lambda_1, T^{-1}\Lambda_2)$) in (3) (or (4)) are infinitely differentiable, of compact support on $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ and belong to the space $\mathcal{M}_{\zeta_1 \zeta_2}(\mathfrak{D}^{\rho_1 \eta_1 \rho_2 \eta_2})$ (or $\mathcal{M}(\mathfrak{D}^{s_1 \varepsilon_1 s_2 \varepsilon_2})$) (*).

(*) Generally speaking, we shall note $\mathfrak{D}^{\chi_1 \cdots \chi_n}$ the space of functions $f^{\chi_1 \cdots \chi_n}(y_1, \ldots, y_n)$ such that :

1) $y_i \in \mathbb{R}$ if $\chi_i = (\rho_i, \eta_i)$
$y_i \in \mathbb{C}^{\pm}$ if $\chi_i = (s_i, \varepsilon_i)$, $\varepsilon_i = 0, 1$ (cf. Appendix I) ;

2) $f^{\chi_i \cdots \chi_n}$ is infinitely differentiable in $(y_1, \cdots, y_n)$;

3) $C^{\chi_1}(y_1) \cdots C^{\chi_n}(y_n) f^{\chi_1 \cdots \chi_n}\left(-y_1^{-1}, \cdots, -y_n^{-1}\right)$ in infinitely differentiable in $y_1 \cdots y_n$, where

$$C^{\chi_j}(y_j) = |\, y_j \,|^{i\rho_j - 1} \operatorname{Sgn}^{\eta_j} y_j \qquad \text{if} \quad \chi_j = (\rho_j, \eta_j)$$
$$C^{\chi_j}(y_j) = y_j^{-sj} \qquad \text{if} \quad \chi_j = (s_j, \varepsilon_j),$$

On the other hand, we shall write $\mathcal{H}_F$ as:

$$(5) \qquad \mathcal{H}_F = \int_A^{\oplus} \mathcal{H}^A(M)dv(M) \oplus \int_B^{\oplus} \mathcal{H}^B(M)dv(M) \oplus \int_C^{\oplus} \mathcal{H}^C(M)dv(M)$$

where

$$\mathcal{H}^J(M) = \sum_{I_j} \oplus\, \mathcal{H}^{I_J}(M).$$

$J = A, B$ or C.

A, B, C correspond to the double cosets and characterize the interval in which M varies.

And we define the subspace $\mathcal{D}_{\mathcal{H}_F}$ of $\mathcal{H}_F$ as the direct sum of the subspaces $\mathcal{D}_A$, $\mathcal{D}_B$ and $\mathcal{D}_C$ constructed in the following way:

— $\mathcal{D}_A$ consists of the functions F_{12}^M whose coefficients $a_{n_1 n_2 k}^{Mj}(\Gamma Z)$ in (III.10) are infinitely differentiable and of compact support on $\Omega_M^{\pm}$ as well as in M^2, and are of rapid decay in n_1, n_2 when these go to infinity:

— $\mathcal{D}_B$ consists of the functions F_{12}^M whose coefficients $a_{n_1 n_2 k}^{M\chi}(\Gamma' Z')(\chi = (\rho, \eta)$, or $(s, \varepsilon))$ in (III.12) are infinitely differentiable and of compact support on Ω_M as well as in M^2, and are of rapid decay in n_1, n_2 when these go to infinity ; moreover, if $\chi = (\rho, \eta)$ these coefficients must be infinitely differentiable and of compact support in $\rho \in [0, \infty]$.

Or, if $\chi = (s, \varepsilon)$, they must differ from zero only for a finite number of values of s ;

— $\mathcal{D}_C$ consists of functions F_{12}^M whose transforms $\check{F}_{12}^M$ can be decomposed according to (III.35) or (III.36), the coefficients $\mathcal{G}^M(\beta_1\zeta_1, \beta_2\zeta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''Z'' ; \chi)$ or $G^M(\beta_1, \beta_2 ; \beta\zeta, \beta'\zeta' ; \Gamma''Z'' ; \chi)$ being infinitely differentiable and of compact support on Ω_M and belonging to $\mathcal{M}_{\zeta_1\zeta_2\zeta\zeta'}[\mathcal{D}^{\chi_1\chi_2\chi\chi'}]$ (*); moreover, if $\chi = (\rho, \eta)$, these coefficients must be infinitely differentiable and of compact support in $\rho \in [0, \infty]$ or, if $\chi = (s, \varepsilon)$, they must differ from zero only for a finite number of values of s.

Thus, we may write the scalar product $(f_{12}, \mathcal{U}\mathcal{G}_M^I)$ where $f_{12} \in \mathcal{D}_{\mathcal{H}_1 \otimes \mathcal{H}_2}$ and $\mathcal{G}_M^I \in \mathcal{D}_{\mathcal{H}_F}$. The bilinear form $(f_{12}, \mathcal{U}\mathcal{G}_M^I)$ is separately continuous in $\bar{f}_{12}$ and $\mathcal{G}_M^I$; we may therefore apply the kernel theorem [*12*] and we conclude there exists a distribution $\mathcal{C}$ on $\overline{\mathcal{D}}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \times \mathcal{D}_{\mathcal{H}_F}$ such that:

$$(5) \qquad (f_{12}, \mathcal{U}\mathcal{G}_M^I) = (\mathcal{C}, \bar{f}_{12}\mathcal{G}_M^I).$$

(*) We make the convention that $\zeta = 0$ if the corresponding χ is equal to (s, ε).

This distribution $\mathcal{C}$ represents a « generalized matrix element of $\mathcal{U}$ » and we shall take it as a definition of Poincaré group Clebsch-Gordan coefficients. To write explicitely the expression for $\mathcal{C}$, we collect the results of (II.20), (III.10, 12, 13, 14, 35, 36) and (IV.2, 3, 4), which gives

$$
\begin{aligned}
(6)\quad (\mathcal{C}, \bar{f}_{12}\mathcal{G}^{\mathrm{I}}_{\mathrm{M}}) = & \int_{\mathrm{M}=0}^{\infty} d\nu(\mathrm{M}) d\dot{\Lambda} \sum_{n_1 n_2 n'_1 n'_2} \sum_{j \geqslant |n|} \sum_{k=-j}^{j} \sqrt{2j+1} \\
& \cdot f_{n_1 n_2}(\Lambda_1, \Lambda_2) a^{\mathrm{M}j}_{n'_1 n'_2 k}(\Gamma \mathrm{Z}) \mathrm{D}^{j}_{nk}(\mathrm{A}) \mathrm{D}^{(1)}_{n_1 n'_1}(\mathrm{A}_1) \mathrm{D}^{(2)}_{n_2 n'_2}(\mathrm{A}_2) \\
& + \int_{\mathrm{M}^2 \in \mathcal{J}} d\nu(\mathrm{M}) d\dot{\Lambda} \sum_{n_1 n_2 n'_1 n'_2} f_{n_1 n_2}(\Lambda_1, \Lambda_2) \Big\{ \sum_{\eta=0,1} \sum_{k=-\infty}^{+\infty} \int_0^{\infty} \mathrm{C}(\rho) d\rho\, a^{\mathrm{M}\rho\eta}_{n'_1 n'_2 k}(\Gamma' \mathrm{Z}') \mathrm{D}^{\rho\eta}_{nk}(\mathrm{B}) \\
& + \sum_{\varepsilon=\pm} \sum_{s=2}^{\infty} \sum_{k=0}^{\infty} \sqrt{s-1}\, a^{\mathrm{M}s\varepsilon}_{n'_1 n'_2 k}(\Gamma' \mathrm{Z}') \mathrm{D}^{s\varepsilon}_{nk}(\mathrm{B}) \Big\} \mathrm{D}^{(1)}_{n_1 n'_1}(\mathrm{A}_1) \mathrm{D}^{(2)}_{n_2 n'_2}(\mathrm{A}_2) \\
& + \int_{\mathrm{M}^2 = -(m'_1 + m'_2)^2}^{-(m'_1 - m'_2)^2} d\nu(\mathrm{M}) d\dot{\Lambda} \sum_{\zeta_1 \zeta_2 \zeta'_1 \zeta'_2} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 d\beta'_1 d\beta'_2 \mathcal{F}(\beta_1\zeta_1, \beta_2\zeta_2 ; \\
& \qquad\qquad \mathrm{T}^{-1}\Lambda_1, \mathrm{T}^{-1}\Lambda_2) \\
& \cdot \Big\{ \sum_{\zeta\zeta'} \int_{-\infty}^{\infty} d\beta d\beta' \int_0^{\infty} \mathrm{C}(\rho) d\rho \sum_{\eta=0,1} \mathrm{J}^{\mathrm{M}}(\beta'_1\zeta'_1, \beta'_2\zeta'_2 ; \beta\zeta, \beta'\zeta' ; \Gamma_c \mathrm{Z}_c ; \rho\eta) \\
& \qquad\qquad \check{\mathrm{D}}^{\rho\eta}(\beta'\zeta', \beta\zeta ; \check{\mathrm{B}}_c) \\
& + \sum_{s=2}^{\infty} \int_{-\infty}^{\infty} d\beta d\beta' \sqrt{s-1}\, \mathrm{J}^{\mathrm{M}}(\beta'_1\zeta'_1, \beta'_2\zeta'_2 ; \beta, \beta' ; \Gamma_c \mathrm{Z}_c ; s\varepsilon) \check{\mathrm{D}}^{s\varepsilon}(\beta', \beta ; \check{\mathrm{B}}_c) \Big\} \\
& \qquad\qquad \check{\mathrm{D}}^{(1)}(\beta_1\zeta_1, \beta'_1\zeta'_1 ; \check{\mathrm{A}}'_1) \check{\mathrm{D}}^{(2)}(\beta_2\zeta_2, \beta'_2\zeta'_2 ; \check{\mathrm{A}}'_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{J} &= [-\infty, -(m'_1 + m'_2)^2] \cup [-(m'_1 - m'_2)^2, 0] \\
&\qquad \text{if} \quad m_1^2 = -m_1'^2 < 0 \quad \text{and} \quad m_2^2 = -m_2'^2 < 0,
\end{aligned}
$$

$\mathcal{J} = [-\infty, 0]$ in all the other cases.

The general expression (6) gives the Clebsch-Gordan coefficients corresponding to the tensorial product ${}_{\mathcal{T}}\mathrm{U}^{\mathrm{L}_1} \otimes {}_{\mathcal{T}}\mathrm{U}^{\mathrm{L}_2}$, where L_j, $j = 1, 2$ is defined by (I.10), $\mathrm{D}^{(j)}$ being a unitary irreducible representation of SU(2), $\mathcal{E}_2$, SU(1, 1), or SL(2, $\mathbb{R}$). The third term in (6) is different from zero only in the case where $\mathrm{D}^{(j)} = \mathrm{D}^{\chi_j}$, $\chi_j = (\rho_j, \eta_j)$ or (s_j, ε_j), for $j = 1$ and 2.

Besides, we have made use of two different decompositions of (Λ_1, Λ_2).

— First decomposition.

$$(\Lambda_1, \Lambda_2) = (A_1, A_2)(\Delta_1, \zeta^{\varepsilon'}\Delta_2)(\Lambda, \Lambda) \tag{7}$$

where

$$A_i \in \mathcal{S}_i, \; i = 1, 2,$$
$$(\Delta_1, \zeta^{\varepsilon'}\Delta_2) \text{ is defined by (II.6)},$$
$$\Lambda \in SL(2, \mathbb{C}).$$

If $0 < M^2 < \infty$, we decompose Λ according to

$$\Lambda = A\Gamma Z \tag{8}$$

where

$$A, Z \in SU(2),$$
$$\Gamma = \begin{vmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{vmatrix} \gamma > 0 \,;$$

and if $M^2 \in J$, we write

$$\Lambda = B\Gamma'Z' \tag{9}$$

where

$$B \in SU(1, 1),$$
$$Z' \in SU(2),$$
$$\Gamma' = \begin{vmatrix} \gamma' & 0 \\ 0 & \gamma'^{-1} \end{vmatrix}, \qquad \gamma' > 0.$$

— Second decomposition (used only in class C)

$$(T^{-1}\Lambda_1, T^{-1}\Lambda_2) = (\check{A}'_1, \check{A}'_2)(\check{\Sigma}_1, \check{\Sigma}_2)(T^{-1}\Lambda', T^{-1}\Lambda') \tag{10}$$

where

$$A'_i \in SL(2, \mathbb{R}), \qquad i = 1, 2,$$
$$\Sigma_i = T\check{\Sigma}_i T^{-1} \qquad \text{is defined by (II.7)},$$
$$\Lambda' \in SL(2, \mathbb{C}).$$

Λ' may be decomposed according to

$$T^{-1}\Lambda' = \check{B}_c\Gamma_c Z_c$$

where

$$\overset{\vee}{\mathrm{B}}_c \in \mathrm{SL}(2, \mathbb{R})$$
$$\mathrm{Z}_c \in \mathrm{SU}(2)$$
$$\Gamma_c = \begin{vmatrix} \gamma_c & 0 \\ 0 & \gamma_c^{-1} \end{vmatrix}, \qquad \gamma_c > 0.$$

Besides, if $\chi_i = (\rho_i, \eta_i)$, we have set:

(11) $\mathcal{F}(\beta_1\zeta_1, \beta_2\zeta_2 ; \mathrm{T}^{-1}\Lambda_1, \mathrm{T}^{-1}\Lambda_2) = f(\beta_1\zeta_1, \beta_2\zeta_2 ; \mathrm{T}^{-1}\Lambda_1, \mathrm{T}^{-1}\Lambda_2)$

(12) $$\mathrm{J}^{\mathrm{M}}(\beta'_1\zeta'_1, \beta'_2\zeta'_2 ; \beta\zeta, \beta'\zeta' ; \Gamma_c\mathrm{Z}_c ; \chi) = \mathrm{G}^{\mathrm{M}}(\beta'_1\zeta'_1, \beta'_2\zeta'_2 ; \beta\zeta, \beta'\zeta' ; \Gamma_c\mathrm{Z}_c ; \chi)\delta(\beta_1 + \beta_2 - \beta)$$

and if $\chi = (s_i, \varepsilon_i)$:

$$\mathcal{F}(\beta_1\zeta_1, \beta_2\zeta_2 ; \mathrm{T}^{-1}\Lambda_1, \mathrm{T}^{-1}\Lambda_2) = f(\beta_1, \beta_2 ; \mathrm{T}^{-1}\Lambda_1, \mathrm{T}^{-1}\Lambda_2)$$
$$\mathrm{J}^{\mathrm{M}}(\beta'_1\zeta'_1, \beta'_2\zeta'_2 ; \beta\zeta, \beta'\zeta' ; \Gamma_c\mathrm{Z}_c ; \chi) = \mathrm{G}^{\mathrm{M}}(\beta'_1\beta'_2 ; \beta\zeta, \beta'\zeta' ; \Gamma_c\mathrm{Z}_c ; \chi)\delta(\beta_1 + \beta_2 - \beta).$$

I am deeply grateful to Dr. G. Rideau who suggested this work for his continuous help and encouragement.

APPENDIX I

IRREDUCIBLE UNITARY REPRESENTATIONS OF SU(2), $\mathcal{E}_2$ AND SU(1, 1)

I. — Representations of SU(2).

The irreducible unitary representations D^l (l integer or half-integer) of SU(2) act in a space of polynomials, $\mathcal{H}_{2l}$, a basis of which is given by :

(1) $$\Phi^n = (-1)^{l-n} \frac{X^{l-n}}{\sqrt{(l+n)!\,(l-n)!}} \quad -l \leqslant n \leqslant l$$

Explicitly [10], we have

(2) $$\left[D^l(A)P_{2l}\right](X) = (vX + \bar{u})^{2l} P_{2l}\left(\frac{uX - \bar{v}}{vX + \bar{u}}\right)$$

where

$$A = \begin{vmatrix} u & v \\ -\bar{v} & \bar{u} \end{vmatrix} \in SU(2)$$

$$P_{2l} \in \mathcal{H}_{2l}$$

II. — Representations of $\mathcal{E}_2$.

The elements of $\mathcal{E}_2$ have the form:

(3) $$(z, e^{i\varphi}) \equiv \begin{vmatrix} e^{i\varphi} & 0 \\ ze^{i\varphi} & e^{i-\varphi} \end{vmatrix}$$

where $0 \leqslant \varphi < 2\pi$, $z \in \mathbb{C}$.

The family of irreducible unitary representations of $\mathcal{E}_2$ consists of:

— one-dimensional representations W_{2l}, l integer, given by

(4) $$W_{2l} : (z, e^{i\varphi}) \rightarrow e^{2il\varphi}$$

— infinite-dimensional representations $E_{h\tau}$, $h > 0$, $\tau = 0, 1$ which act in the space of square integrable functions F on C_h, a circle of radius h, by:

(5) $$\left[E_{h\tau}(z, e^{i\varphi})F\right](\Theta) = e^{ih \operatorname{Im}(ze^{i\Theta})} e^{i\tau\varphi} F(\Theta + 2\varphi)$$

where $(z, e^{i\varphi}) \in \mathcal{E}_2$,

Θ: polar angle.

We shall use the expansion of $F(\Theta)$ on a basis Φ^n defined by

(6) $$\Phi^n = e^{i\left(n - \frac{\tau}{2}\right)\Theta}$$

where n is an integer if $\tau = 0$, a half-integer if $\tau = 1$.

III. — Representations of SU(1, 1) or SL(2, $\mathbb{R}$).

The isomorphism between SU(1, 1) and SL(2, $\mathbb{R}$) may be written as:

$$B = T\check{B}T^{-1} \tag{7}$$

where $B \in SU(1, 1)$,
$\check{B} \in SL(2, \mathbb{R})$.

$$T = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix}$$

We are interested in the following representations of SU(1, 1), given by Bargmann [11]:

— $D^{\rho\eta}$, ρ real, $\eta = 0, 1$, acting in the space $\mathcal{K}_\theta$ of square integrable functions f on the unit circle (principal series);

— $D^{s\varepsilon}$, s integer, $\varepsilon = \pm$, acting in the space $\mathcal{K}^s_w$ of regular analytic functions on the open unit disk $\mathcal{M}$; in $\mathcal{K}^s_w$, the scalar product is

$$(f, f')_s = \frac{s-1}{\pi}(1 - W\overline{W})^{s-2}f(W)\overline{f'(W)}dS$$

where $f, f' \in \mathcal{K}^s_w$.

Bases of these spaces may be chosen as:

$$\Phi^n = e^{in'\theta} \in \mathcal{K}_\theta \tag{8}$$

where $n' = -n - \frac{\eta}{2}$, n' integer, $-\infty < n' < \infty$;

$$\Phi^n = \gamma_{n'}(s)W^{n'} \in \mathcal{K}^s_W \tag{9}$$

where

$$\gamma_{n'} = \left[\frac{(s-1+n')!}{(s-1)!\,n'!}\right]^{1/2}$$
$$n = -(-1)^\varepsilon\left(n' + \frac{s}{2}\right)$$
$$n' = 0, 1, \ldots$$

Besides, we need the corresponding representations $\check{D}^{\rho\eta}$, $D^{s\varepsilon}$ of SL(2, $\mathbb{R}$) given by Gelfand [13].

Let $\mathfrak{D}^{\rho\eta}$ be the space of functions ψ on $\mathbb{R}$ such that $\psi(x)$ and

$$\widehat{\psi}(x) \equiv |x|^{i\rho-1}\,\mathrm{Sgn}^\eta x\,\psi\left(\frac{-1}{x}\right)$$

are infinitely differentiable functions. Then the representation $\check{D}^{\rho\eta}$ is realized in a Hilbert space defined as the completion of $\mathfrak{D}^{\rho\eta}$ for the scalar product:

$$(\psi, \psi') = \int_{-\infty}^{\infty}\psi(x)\overline{\psi'(x)}dx, \tag{10}$$

Explicitly, we have:

$$[\check{D}^{\rho\eta}(\check{B})\psi](x) = |bx + d|^{i\rho-1}\,\mathrm{Sgn}^{\eta}(bx + d)\psi\left(\frac{ax + c}{bx + d}\right) \tag{11}$$

where

$$\check{B} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

To go from the representation $D^{\rho\eta}$ of SU(1, 1) to the representation $\check{D}^{\rho\eta}$ of SL(2, $\mathbb{R}$), we need to know the isomorphism $\mathfrak{J}$ between $\mathfrak{D}^{\rho\eta}$ and a subspace of $\mathcal{H}_\theta$ (namely the subspace of infinitely differentiable functions on the unit circle), $\mathfrak{J}$ being such that:

$$[\mathfrak{J}D^{\rho\eta}(B)\mathfrak{J}^{-1}\psi](x) = [\check{D}^{\rho\eta}(\check{B})\psi](x), \tag{12}$$

where $B \in \mathrm{SU}(1, 1)$ and $\check{B} \in \mathrm{SL}(2, \mathbb{R})$ satisfy (7). Straightforward computation gives

$$\psi(x) = (\mathfrak{J}f)(x) = 2^{\frac{i\rho-1}{2}}\left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right|^{-i\rho+1}\left(\frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\left|\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right|}\right)^{\eta} e^{i\eta\frac{\theta}{2}}f(\theta) \tag{13}$$

where

$$\psi \in \mathfrak{D}^{\rho\eta}$$
$$f \in \mathcal{H}_\theta$$
$$x = \frac{\sin\frac{\theta}{2} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$$

In the same way, let $\mathfrak{D}^{s\varepsilon}$ be the space of functions ψ^ε on $\mathbb{C}^+ = \{ z; \mathrm{Im}\, z \geqslant 0 \}$, if $\varepsilon = 0$, or $\mathbb{C}^- = \{ z; \mathrm{Im}\, z \leqslant 0 \}$, if $\varepsilon = 1$, and such that $\psi^\varepsilon(z)$ and $\hat{\psi}(z) \equiv z^{-s}\psi^\varepsilon\left(\frac{-1}{z}\right)$ are infinitely differentiable. Then the representation $\check{D}^{s\varepsilon}$ is realized in a Hilbert space defined as the completion of $\mathfrak{D}^{s\varepsilon}$ for the scalar product

$$(\psi^\varepsilon, \psi'^\varepsilon)_s = \frac{1}{\Gamma(s-1)}\int_{\mathbb{C}^+}\psi^\varepsilon(z)\overline{\psi'^\varepsilon(z)}(\mathrm{Im}\, z)^{s-2}\, dz \tag{14}$$

where $z = x + iy$
$dz = dxdy$.

Explicitly we have

$$[\check{D}^{s\varepsilon}(\check{B})\psi^\varepsilon](z) = (bz + d)^{-s}\psi^\varepsilon\left(\frac{az + c}{bz + d}\right). \tag{15}$$

Here again, we look for an isomomorphism $\mathfrak{J}$ between $\mathfrak{D}^{s\varepsilon}$ and a subspace of $\mathcal{H}_w$ (namely the subspace of functions infinitely differentiable on the border of $\mathcal{M}$), $\mathfrak{J}$ being such that

$$[\mathfrak{J}D^{s\varepsilon}(B)\mathfrak{J}^{-1}\psi^\varepsilon](z) = [\check{D}^{s\varepsilon}(\check{B})\psi^\varepsilon](z) \tag{16}$$

Straightforward computation gives

$$\psi^\varepsilon(z) = \frac{2^{s-1}\sqrt{(s-1)\Gamma(s-1)}}{\sqrt{\pi}}\, e^{i\varepsilon\frac{\pi}{2}(s-2)}\left(iz - e^{i\varepsilon\pi}\right)^{-s} f\left(\frac{-i + e^{i\varepsilon\pi}z}{iz - e^{i\varepsilon\pi}}\right) \tag{17}$$

APPENDIX II

Orbits of $\mathcal{T}$ in $\Omega_1 \times \Omega_2$

Let	$P = (p_0, \vec{p}) \in \Omega_1,\ p_0^2 - \lvert\vec{p}\rvert^2 = m_1^2 = -m_1'^2,\ Q = (q_0, \vec{q}) \in \Omega_2,\ q_0^2 - \lvert q\rvert^2 = m_2^2 = -m_2'^2$		
If :	$P + Q \in \Omega_M^\pm$	$P + Q \in \Omega_{M'},\ M^2 = -M'^2 < 0$	
the transformation	$\Delta \in SL(2, \mathbb{C})$	$\Delta \in SL(2\ \mathbb{C})$	
takes (P, Q) into (P′, Q′) such that :	$P' + Q' = \pi_M^\pm = (\pm M, 0, 0, 0)$ $\begin{cases} p_i' + q_i' = 0, & i = 1, 2, 3 \\ p_0' + q_0' = \pm M \end{cases}$ $p_0'^2 - m_1^2 = q_0'^2 - m_2^2$	$P' + Q' = \pi_M = (0, 0, 0, M')$ $\begin{cases} p_i' + q_i' = 0, & i = 0, 1, 2 \\ p_3' + q_3' = M' \end{cases}$ $p_3'^2 + m_1^2 = q_3'^2 + m_2^2 \equiv \mu^2$	
the transformation	$A \in SU(2)$	$B \in SU(1, 1)$	
		$\mu^2 > 0$	$\mu^2 < 0$
takes (P′, Q′) into (P″, Q″) $= (\pi_1', \pi_2')$ the characteristic point of the orbits, which is such that :	$P'' + Q'' = \pi_M^\pm$ $p_1'' = p_2'' = q_1'' = q_2'' = 0$ $p_3'' = -q_3'' = \sqrt{p_0'^2 - m_1^2}$ $p_0'' = p_0' = \dfrac{m_1^2 - m_2^2 + M^2}{\pm 2M}$	$P'' + Q'' = \pi_M$ $p_1'' = p_2'' = q_1'' = q_2'' = 0$ $p_0'' = -q_0'' = \pm\sqrt{p_3'^2 + m_1^2}$ $q_3'' = q_3' = \dfrac{m_1^2 - m_2^2 + M'^2}{2M'}$	$P'' + Q'' = \pi_M$ $p_0'' = p_1'' = q_0'' = q_1'' = 0$ $p_2'' = \sqrt{-p_3'^2 = m_1^2}$ $q_3'' = q_3' = \dfrac{m_1^2 - m_2^2 + M'^2}{2M'}$
Name of the orbits:	$A_{\varepsilon'}^\pm \begin{cases} \varepsilon' = 0 \text{ if } \Omega_2 = \Omega_{m_2}^\pm \ (*) \\ \varepsilon' = 1 \text{ in all other case} \end{cases}$	$B_{\varepsilon'}^\pm \begin{cases} \varepsilon' = 0 \text{ if } q_3'' > 0 \ (**) \\ \varepsilon' = 1 \text{ if } q_3'' < 0 \end{cases}$	C
Interval in which M^2 varies — Case: $m_1^2 = -m_1'^2 < 0$, $m_2^2 = -m_2'^2 < 0$	$M^2 > 0$	$0 < M' < (m_1' - m_2')^2$ and $M'^2 > (m_1' + m_2')^2$	$(m_1' - m_2')^2 < M'^2 < (m_1' + m_2')^2$
Interval in which M^2 varies — Case: other cases	$M^2 > 0$	$M'^2 > 0$	

(*) This remark is good only for § IV.
(**) Actually, here, ε' is defined by (II-7).

APPENDIX III

PLANCHEREL FORMULA ON SL(2, ℝ). USE OF CONTINUOUS BASES

By Plancherel formula, the expansion of a square integrable function F on SL(2, ℝ) may be written

$$(1)\quad \check{F}(A) = \sum_{\eta} \int C(\rho)d\rho \ \mathrm{Tr}\ [\check{D}^{\rho\eta}(\check{A})K^{\rho\eta}(F)] + \sum_{\varepsilon} \sum_{s=2}^{\infty} \sqrt{s-1}\ \mathrm{Tr}\ [\check{D}^{s\varepsilon}(\check{A})K^{s\varepsilon}(F)]$$

where

$$C(\rho) = \rho \ \mathrm{th}\ \pi \frac{\rho + i\eta}{2},$$

$$(2)\quad K^{\chi}(F) = \int \check{F}(\check{A})\overline{\check{D}^{\chi}}(\check{A})d\check{A},$$

($K^{\chi}(F)$ is the Fourier transform of F),
$\chi = (\rho, \eta)$ or (s, ε),
$\check{A} \in SL(2, \mathbb{R})$,
$d\check{A}$ is the Haar measure on SL(2, ℝ).

Now we shall look for a more convenient expression of Tr $D^{\chi}(A)K^{\chi}(F)$.

I. — Case of representations $D^{\rho\eta}$ of the principal series.

a) *About a Mellin transform of* $\mathfrak{D}^{\rho\eta}$ [*16*].

Let us define a transformation $\mathcal{M}_{\zeta}^{\rho\eta}$, $\zeta = 0, 1$, by

$$(3)\quad \mathcal{M}_{\zeta}^{\rho\eta}(\psi) = \mathcal{G}^{\rho\eta}(\beta, \zeta) = \int_{-\infty}^{\infty} |x|^{-\frac{1}{2}+i\left(\beta-\frac{\rho}{2}\right)} \mathrm{Sgn}^{\zeta}\, x\psi(x)dx$$

for any $\psi \in \mathfrak{D}^{\rho\eta}$.
$\mathcal{G}$ may also be written (from now on, we shall drop the indices ρ, η):

$$(4)\quad \mathcal{G}(\beta, \zeta) = J_+(\beta) + (-1)^{\zeta} J_-(\beta),$$

where (*)

$$(5)\quad J_{\pm}(\beta) = \int_{-\infty}^{\infty} x_{\pm}^{-\frac{1}{2}+i\left(\beta-\frac{\rho}{2}\right)} \psi(x)dx$$

(*) Actually, $J_{\pm}(\beta)$ is found as the value of a generalized eigen function for the infinitesimal operator $\left[\frac{1}{2}(i\rho - 1) - x\frac{d}{dx}\right]$ defined on $\mathfrak{D}^{\rho\eta}$ and corresponding to the transformation: $\begin{vmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{vmatrix}$.

and $x_{\pm}^{\alpha}$, α complex, is a distribution defined by Gelfand [14]. We shall write $\mathcal{M}_{\zeta}(\mathfrak{D}^{\rho\eta})$ for the dense subspace of $\mathcal{S}(\beta)$ obtained from $\mathfrak{D}^{\rho\eta}$ by the transformation $\mathcal{M}_{\zeta}$. Inverting (3), we get:

(6) $$\psi(x) = \frac{1}{4\pi} \sum_{\zeta=0,1} \int_{-\infty}^{\infty} \mathcal{G}(\beta, \zeta) \, | x |^{-\frac{1}{2}-i\left(\beta-\frac{\rho}{2}\right)} \operatorname{Sgn}^{\zeta} x d\beta.$$

Parseval formula implies:

(7) $$2\pi(\psi(x), \psi'(x)) = \frac{1}{2} \sum_{\zeta=0,1} (\mathcal{G}(\beta, \zeta), \mathcal{G}'(\beta, \zeta))$$

where

(8) $$(\mathcal{G}(\beta, \zeta), \mathcal{G}'(\beta, \zeta)) = \int_{-\infty}^{\infty} \mathcal{G}(\beta, \zeta)\overline{\mathcal{G}'(\beta, \zeta)} d\beta.$$

b) *Matrix elements of* $\mathrm{D}^{\rho\eta}$.

For $\psi, \psi' \in \mathfrak{D}^{\rho\eta}$, we shall set:

(9) $$\mathcal{M}_{\zeta}(\psi) = \mathcal{G}(\beta, \zeta)$$

(10) $$\mathcal{M}_{\zeta}(\psi') = \mathcal{G}'(\beta, \zeta)$$

Thus

(11) $$([\check{\mathrm{D}}^{\rho\eta}(\check{\mathrm{A}})\psi](x), \psi'(x)) = \frac{1}{4\pi} \sum_{\zeta=0,1} (\mathcal{M}_{\zeta}[\check{\mathrm{D}}^{\rho\eta}(\check{\mathrm{A}})\psi], \mathcal{G}(\beta, \zeta))$$

is a bilinear form which is separately continuous in $\mathcal{G}$ and $\mathcal{G}'$ in the $\mathcal{S}$ topology, for $\check{\mathrm{D}}^{\rho\eta}(\check{\mathrm{A}})$ is bounded. We know, by the kernel theorem [12], that there exists a distribution $\check{\mathrm{D}}^{\rho\eta}(\beta\zeta, \beta'\zeta'; \check{\mathrm{A}})$ belonging to $[\mathcal{S}^2(\beta, \beta')]'$ and such that we may write formally:

(12) $$([\check{\mathrm{D}}^{\rho\eta}(\check{\mathrm{A}})\psi](x), \psi'(x)) = \frac{1}{4\pi} \sum_{\zeta\zeta'} \int \check{\mathrm{D}}^{\rho\eta}(\beta\zeta, \beta'\zeta'; \check{\mathrm{A}}) \mathcal{G}(\beta'\zeta')\overline{\mathcal{G}'(\beta\zeta)} d\beta d\beta'$$

We shall call $\check{\mathrm{D}}^{\rho\eta}(\beta\zeta, \beta'\zeta'; \check{\mathrm{A}})$ « generalized matrix elements » (*) of $\check{\mathrm{D}}^{\rho\eta}$ in the « continuous basis » $| x |^{-\frac{1}{2}+i\left(\beta-\frac{\rho}{2}\right)} \operatorname{Sgn}^{\xi} x$.

In particular, we have

(13) $$\left([\check{\mathrm{D}}^{\rho\eta}(\check{\mathrm{A}}_t^{\xi})\psi](x), \psi'(x)\right) = \frac{1}{4\pi}(-1)^{\xi\eta} \sum_{\zeta=0,1} (e^{i\beta t}\mathcal{G}(\beta, \zeta), \mathcal{G}'(\beta, \zeta))$$

where

$$\check{\mathrm{A}}_t^{\xi} = (-1)^{\zeta} \begin{vmatrix} \operatorname{ch} t & i \operatorname{sh} t \\ -i \operatorname{sh} t & \operatorname{ch} t \end{vmatrix}, \qquad \xi = 0, 1.$$

(*) Formal expressions of $\check{\mathrm{D}}^{\rho\eta}(\beta\zeta, \beta'\zeta'; \check{\mathrm{A}})$ have been obtained by Barut [15], using quite different methods.

c) *Fourier transform on* SL(2, $\mathbb{R}$).

The Fourier transform of a square integrable function F on SL(2, $\mathbb{R}$) is an operator-valued function $K^{\rho\eta}$ defined on $\mathfrak{D}^{\rho\eta}$ by

$$(14)\qquad [K^{\rho\eta}(F)\psi](x) = \int F(\check{A})[\overline{D^{\rho\eta}(\check{\hat{A}})}\psi](x)d\check{A}$$

If F has been chosen such that [14] is absolutely convergent, we may define a kernel $K^{\rho\eta}(F; x, x')$:

$$(15)\qquad [K^{\rho\eta}(F)\psi](x) = \int K^{\rho\eta}(F\ ;\ x,\ x')\psi(x')dx'$$

Using decomposition (III.19):

$$\check{A} = \check{A}_t^{\xi}W$$

We may write

$$(16)\quad K^{\rho\eta}(F\ ;\ x,\ x') = \sum_{\xi=0,1}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dXF'\left(\xi,\ t,\ X,\ x' - \frac{x}{Xx + e^{t}}\right)$$
$$\cdot \left| Xxe^{-t/2} + e^{t/2}\right|^{-i\rho-1} \mathrm{Sgn}^{\eta}\left[(-1)^{\xi}(Xxe^{-t/2} + e^{t/2})\right]$$

where we have set

$$(17)\qquad F(\check{A}) = F'(\xi,\ t,\ X,\ Y).$$

Straightforward computation shows that $F' \in \mathcal{S}(t, X, Y)$ implies

$$K^{\rho\eta}(F\ ;\ x,\ x') \in \mathfrak{D}^{(-\rho,\eta),(\rho,\eta)}(x,\ x')\ (*).$$

Finally, we define

$$(18)\qquad I^{\rho\eta}(F) = (K^{\rho\eta}(F)\psi,\ \psi') \qquad \{\psi,\ \psi' \in \mathfrak{D}^{\rho\eta}$$
$$= \int_{-\infty}^{\infty} K^{\rho\eta}(F\ ;\ x,\ x')\psi(x')\overline{\psi'(x)}dxdx'$$

which may be written, according to (7)

$$(19)\qquad I^{\rho\eta}(F) = \frac{1}{4\pi}\sum_{\zeta,\zeta'=0,1}\int_{-\infty}^{\infty}\mathcal{M}_{\zeta\zeta'}[K^{\rho\eta}(F\ ;\ x,\ x')]\mathcal{G}(\beta'\zeta')\mathcal{G}(\beta,\ \zeta)d\beta d\beta'.$$

Here $\mathcal{M}_{\zeta\zeta'}[K^{\rho\eta}(F; x, x')]$ is a function belonging to $\mathcal{M}_{\zeta\zeta'}[\mathfrak{D}^{(-\rho,\eta),(\rho,\eta)}(x, x')]$.

II. — Case of representations $D^{s\varepsilon}$ of the discrete series.

a) *About a transformation of* $\mathfrak{D}^{s\varepsilon}$ [16].

We define a transformation $\mathcal{M}^{s\varepsilon}$ by (**)

$$(20)\qquad \mathcal{M}^{s\varepsilon}(\psi^{s}) = G^{s\varepsilon}(\beta,\ \theta_0) = \int_{c} z^{\frac{s}{2}+i\beta-1}\psi^{\varepsilon}(z)dz$$

(*) Cf. note, p. 180.

(**) Actually, $G^{s\varepsilon}$ is found as the value of a generalized eigenfunction for the infinitesimal operator $\left(-\frac{s}{2} - z\frac{d}{dz}\right)$ defined on $\mathfrak{D}^{s\varepsilon}$ and corresponding to the transformation:

$$\begin{vmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{vmatrix}$$

where

$$\psi^{\varepsilon} \in \mathfrak{D}^{s\varepsilon}$$
$$z = |z| e^{i\theta_0},$$

C is a line $\theta_0 = \text{cte}$ going from zero to infinity. From the properties of ψ^{ε}, we conclude that $G^{s\varepsilon}$ does not depend on θ_0. We shall call $\mathcal{M}(\mathfrak{D}^{s\varepsilon})$ the dense subspace of $\mathcal{S}(\beta)$ obtained from $\mathfrak{D}^{s\varepsilon}$ by the transformation $\mathcal{M}^{s\varepsilon}$.

The inversion formula is

$$\psi^{\varepsilon}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta G^{s\varepsilon}(\beta) z^{-\frac{s}{2}-i\beta} \tag{21}$$

and the Parseval formula

$$\left(\psi^{\varepsilon}(z), \psi'^{\varepsilon}(z)\right)_s = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta C(\beta, s) G^{s\varepsilon}(\beta)\overline{G'^{s\varepsilon}(\beta)} \tag{22}$$

where

$$\psi^{\varepsilon}, \psi'^{\varepsilon} \in \mathfrak{D}^{s\varepsilon}$$

$$C(\beta, s) = \frac{1}{2\pi}\int_0^{\pi} e^{2\beta\omega} (\sin\omega)^{s-2} d\omega \tag{23}$$

$$= \frac{4^{\frac{1-s}{2}} e^{\beta\pi} \Gamma(s-1)}{\Gamma\left(\frac{s}{2}+i\beta\right)\Gamma\left(\frac{s}{2}-i\beta\right)}$$

b) *Matrix elements of* $D^{s\varepsilon}$.

The definition is analogous to the previous one (§ 1, *b*)). In particular

$$\left([\check{D}^{s\varepsilon}(\check{A}_t^{\xi})\psi^{\varepsilon}](z), \psi'^{\varepsilon}(z)\right)_s = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta C(\beta, s)(-1)^{\xi s} e^{i\beta t} G^{s\varepsilon}(\beta)\overline{G'^{s\varepsilon}(\beta)} \tag{24}$$

c) *Fourier transform on* SL(2, $\mathbb{R}$).

Taking again $F' \in \mathcal{S}(t, X, Y)$, we may write the kernel of its Fourier transform:

$$[K^{s\varepsilon}(F)\psi^{\varepsilon}](z) = \int K^{s\varepsilon}(F; z, z')\psi^{\varepsilon}(z')(\operatorname{Im} z')^{s-2} dz' \tag{25}$$

It is given by

$$K^{s\varepsilon}(F; z, z') = \int F(a, b, d)(\operatorname{Im} z)^{-s/2}(\operatorname{Im} z')^{-s/2} e^{-is\theta'} d\theta' \tag{26}$$

where

$$\check{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in SL(2, \mathbb{R})$$

θ' is defined by

$$\begin{vmatrix} \lambda & 0 \\ \nu & \lambda^{-1} \end{vmatrix}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{vmatrix}\begin{vmatrix} \lambda' & 0 \\ \nu' & \lambda'^{-1} \end{vmatrix}$$

$z = \lambda\nu + i\lambda^2$
$z' = \lambda'\nu' + i\lambda'^2$
λ real, ν complex.

Using (22) and the properties of F', we may define the transformed kernel $\mathcal{M}[K^{s\varepsilon}(F; z, z')]$ by:

$$
\begin{aligned}
I^{s\varepsilon}(F) &\equiv (K^{s\varepsilon}(F)\psi^{\varepsilon}, \psi'^{\varepsilon})_s \\
&= \int K^{s\varepsilon}(F ; z, z')\psi^{\varepsilon}(z')\overline{\psi'^{\varepsilon}(z)}(\operatorname{Im} z)^{s-2}(\operatorname{Im} z')^{s-2}dzdz' \qquad (28)\\
&= \frac{1}{4\pi^2}\int d\beta d\beta' \mathcal{M}[K^{s\varepsilon}(F ; z, z')]G^{s\varepsilon}(\beta')\overline{G'^{s\varepsilon}(\beta)}C(\beta, s)C(\beta', s),
\end{aligned}
$$

where

$$
\begin{aligned}
G^{s\varepsilon} &= \mathcal{M}^{s\varepsilon}(\psi^{\varepsilon}) \\
G'^{s\varepsilon} &= \mathcal{M}^{s\varepsilon}(\psi'^{\varepsilon})
\end{aligned}
$$

Using the expression (26) for $K^{s\varepsilon}$ and applying Fubini theorem, we finally get $\mathcal{M}[K^{s\varepsilon}(F; z, z')]$ which belongs to $\mathcal{S}(\beta, \beta')$.

III. — Plancherel formula on SL(2, ℝ).

The above results lead to a new formula for the expansion of a square-integrable function F on SL(2, ℝ) according to Plancherel.

$$
\begin{aligned}
\check{F}(A) = &\sum_{\eta=0,1}\int_{-\infty}^{\infty} C(\rho)d\rho \sum_{\zeta\zeta'} d\beta d\beta' \check{D}^{\rho\eta}(\beta, \zeta, \beta'\zeta' ; \check{A})\mathcal{M}_{\zeta\zeta'}[K^{\rho\eta}(F ; x, x')] \qquad (29)\\
&+ \sum_{\varepsilon=\pm}\sum_{s=2}^{\infty}\sqrt{s-1}\int_{-\infty}^{\infty} d\beta d\beta' \check{D}^{s\varepsilon}(\beta, \beta' ; \check{A})\mathcal{M}[K^{s\varepsilon}(F ; z, z')]
\end{aligned}
$$

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