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Physique théorique.

Expansion of inhomogenizations of all the classical Lie algebras to classical Lie algebras

by

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ABSTRACT. — In a systematic way we show by construction that for the Lie algebra $Cl(n)$ of any real or complex n-dimensional classical Lie group $\mathscr{C}l(n)$ there exists a (not always irreducible) representation γ of $Cl(n)$, such that the inhomogenization $ICl(n, \gamma)$ of $Cl(n)$ relative to γ can be expanded to $Cl(n + N)$ for a certain value of N. In each case we also identify a *linear* Lie group $\mathscr{G}((n + N)^m)$, isomorphic to the corresponding inhomogenization $\mathscr{I}\mathscr{C}l(n, \gamma)$ of $\mathscr{C}l(n)$, which under this expansion globally is being expanded to a linear Lie group $\mathscr{E}((n + N)^m) \cong \mathscr{C}l(n + N)$ for a certain value of m.

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INTRODUCTION

Let us first recall the concept of an inhomogenization [1] of an n-dimensional Lie group $\mathscr{G}(n, k)$, where the field k is either the real field R or the complex field C. The group $\mathscr{I}\mathscr{G}(n, k, \gamma)$ of transformations

$$x'_l = \mathrm{D}_l^{\gamma l'}(\mathscr{G}(n, k))x_{l'} + a_l, \quad a_l \in k_\gamma, \quad l, l' = 1, \ldots, n_\gamma,$$

of an n_γ-dimensional vector space U_{n_γ} over the field $k_\gamma = \mathrm{R}$ or C is an inhomogenization of $\mathscr{G}(n, k)$ relative to the representation γ (irreducible or not) of $\mathscr{G}(n, k)$. k_γ is the field and n_γ the dimension of γ. The convention of summation over repeated indices is used here and throughout, unless otherwise stated.

It is seen that

$$\mathscr{I}\mathscr{G}(n, k, \gamma) \cong \mathscr{T}(n_\gamma, k_\gamma) \boxtimes \mathscr{G}(n, k),/\mathscr{K}(n, k),$$

the semi-direct product of $\mathscr{G}(n, k)/\mathscr{K}(n, k)$ by $\mathscr{T}(n_\gamma, k_\gamma)$, with $\mathscr{T}(n_\gamma, k_\gamma)$ the invariant subgroup. $\mathscr{K}(n, k)$ is the kernel of γ and $\cong$ denotes isomorphic to. $\mathscr{I}\mathscr{G}(n, k)$, the usual « inhomogeneous $\mathscr{G}(n, k)$ » for $\mathscr{G}(n, k)$ a linear group, is a special case of an inhomogenization of $\mathscr{G}(n, k)$ as we have

$$\mathscr{I}\mathscr{G}(n, k) \equiv \mathscr{I}\mathscr{G}(n, k, \gamma_0),$$

where γ_0 is the defining representation of $\mathscr{G}(n, k)$.

Let us now define the concept of expansion of a Lie algebra G. By this one understands a process which roughly can be described as the one of replacing some of the elements of G by new operators, which are certain functions of the elements of G, and also adding further operators of this form, such that the new set of operators close under commutation. The new Lie algebra E(G) which is spanned by these operators is called an expansion of G.

The expansion problem was first treated by Melvin [2] who expanded the Poincaré Lie algebra to the de Sitter Lie algebra. Later several other authors expanded various special cases of the inhomogeneous pseudo-orthogonal Lie algebras ISO(n_1, n_2; R) (Refs. *3-13*) and some authors expanded the inhomogeneous pseudo-unitary Lie algebras IU(n_1, n_2) (Refs. *8*, *9*, *12*, *13*). Sankaranarayanan [*4*] obtained the expansion of ISO(n; R) to SO(n, 1; R), Rosen and Roman [*8*] and Rosen [*9*] of ISO(n_1, n_2; R) to SO($n_1 + 1$, n_2; R) or to SO(n_1, $n_2 + 1$; R) and furthermore of IU(n_1, n_2) to U($n_1 + 1$, n_2) or to U(n_1, $n_2 + 1$).

The expansion problem for the symplectic Lie algebras was solved by Nagel and Shah [*14*]. They found that it is IS*p*(n, k, $2(10^{\nu-1})$) which can be expanded to S*p*($n + 2$, k) in a similar way as the expansion of ISO(n_1, n_2; k). The expansion of IUS*p*($2\nu_1$, $2\nu_2$) to US*p*($2\nu_1 + 2$, $2\nu_2$) or to US*p*($2\nu_1$, $2\nu_2 + 2$) was solved by Nagel [*15*].

The expansion problem for the general linear Lie algebras has so far not been treated explicitly in the literature. It was conjectured by Rosen and Roman [*8*] that IGL(n, k) could be expanded in a similar way as the expansion of IU(n_1, n_2). We show here that it is

$$\mathrm{IGL}(n, k, [10^{n-1}] \dot{+} [0^{n-1} - 1])$$

which can be expanded in this way. The form of this expansion can be found in a paper by Chakrabarti [*13*] who, however, there treats a different problem, namely that of expansion of representations of IU(n).

The method of expansion has been applied to various areas of physics. In relativistic quantum mechanics it has been used to obtain a relativistic position operator (Ref. *3*). In connection with particle physics it has been utilized to construct dynamical groups from symmetry groups (Refs. *2*, *5*, *6*, *7*, *12*). In representation theory this method has been applied to obtain representations of the expanded Lie algebra from representations of the original Lie algebra. In particular it has been used to expand representations of the homogeneous Galilei Lie algebra to representations of the Lorentz Lie algebra and to expand representations of the Poincaré Lie algebra to representations of the de Sitter Lie algebra (Ref. *11*). It has also been utilized to obtain representations of non-compact Lie algebras from representations of inhomogenizations of compact Lie algebras (Refs. *10*, *13*).

We shall here in a systematic way by construction show that for the Lie algebra $Cl(n, k)$ of any real or complex n-dimensional classical Lie group $\mathscr{C}l(n, k)$ there exists a (not always irreducible) representation γ of $Cl(n, k)$, such that the inhomogenization ICL(n, k, γ) of $Cl(n, k)$ relative to γ can be expanded to $Cl(n + \mathrm{N}, k)$ for a certain value of N. In each case we also identify a *linear* Lie group $\mathscr{G}((n + \mathrm{N})^m, k)$, isomorphic to the corresponding inhomogenization $\mathcal{I}\mathcal{C}l(n, k, \gamma)$ of $\mathcal{C}l(n, k)$, which under this expansion globally is being expanded to a linear Lie group

$$\mathcal{E}((n + \mathrm{N})^m, k) \cong \mathcal{C}l(n + \mathrm{N}, k)$$

for a certain value of m.

In sections 1-4 we treat the expansion problems for GL(n, k), for U(n_1, n_2), for $\mathrm{SO}_0(n_1, n_2; k)$ together with Sp(n, k), and for USp($2\nu_1$, $2\nu_2$). Corresponding equations in the four sections have been given the same number. In the conclusion we give a precise definition of expansion of a Lie algebra, and furthermore give the interrelations between processes of expansion, contraction and deformation.

1*a*. THE GROUPS $\mathcal{I}\mathcal{G}\mathcal{L}(n, k, \mathrm{N}[10^{n-1}] + \mathrm{N}[0^{n-1} -1])$

Let us start with considering the inhomogenization [*1*]

$$\mathcal{I}\mathcal{G}\mathcal{L}(n, k, \mathrm{N}[10^{n-1}] \dot{+} \mathrm{N}[0^{n-1} -1])$$

of the n-dimensional general linear group [*16*] $\mathcal{G}\mathcal{L}(n, k)$, where $[10^{n-1}]$ is the defining representation of $\mathcal{G}\mathcal{L}(n, k)$ and $[0^{n-1} - 1]$ the representation contragradient to $[10^{n-1}]$. $\dot{+}$ denotes direct sum and

$$\mathrm{N}[10^{n-1}] \equiv \overbrace{[10^{n-1}] \dot{+} [10^{n-1}] \dot{+} \dots \dot{+} [10^{n-1}]}^{\mathrm{N}}.$$

$\mathcal{I}\mathcal{G}\mathcal{L}(n, k, \mathrm{N}[10^{n-1}] \dot{+} \mathrm{N}[0^{n-1} - 1])$ is the group of all those transformations

$$x'^{j}_{\mu} = \mathcal{L}_{\mu}{}^{\nu} x_{\nu}{}^{j} + a_{\mu}{}^{j},$$
$$x'^{\nu}_{i} = \tilde{\mathcal{L}}^{-1}{}^{\nu}{}_{\mu} x_{i}{}^{\mu} + a_{i}{}^{\nu}, \quad \mu, \nu = 1, \dots, n, \quad i, j = 1, \dots, \mathrm{N},$$

of a $2\mathrm{N}n$-dimensional vector space

$$\mathrm{U}_{2\mathrm{N}n} = \mathrm{V}_n \otimes \mathrm{W}_\mathrm{N} \dot{+} \mathrm{W}_\mathrm{N} \otimes \mathrm{V}_n, \qquad (1.1)$$

over k, where

$$\mathcal{L}_{\mu}{}^{\nu} \in \mathcal{G}\mathcal{L}(n, k),$$
$$a_{\mu}{}^{j}, a_{\nu}{}^{i} \in \mathcal{T}(2\mathrm{N}n, k),$$

the $2\mathrm{N}n$-dimensional translation group. Here $\tilde{}$ and $\otimes$ denote transposition and tensor product, respectively. We have that

$$\mathcal{I}\mathcal{G}\mathcal{L}(n, k, \mathrm{N}[10^{n-1}] + \mathrm{N}[0^{n-1} -1]) \cong \mathcal{T}(2\mathrm{N}n, k) \boxtimes \cdot \mathcal{G}\mathcal{L}(n, k).$$

The generators

$$X_\mu{}^\nu \sim \mathscr{L}_\nu{}^\mu \tag{1.3a}$$

of $\mathscr{G}\mathscr{L}(n, k)$ and the generators

$$P_i{}^\nu \sim a_\nu{}^i, \qquad P_\mu{}^j \sim a_j{}^\mu, \tag{1.3b}$$

of $\mathscr{T}(2Nn, k)$ form a basis for the Lie algebra

$$IGL(n, k, N[10^{n-1}] + N[0^{n-1} -1]) \cong T(2Nn) \boxplus GL(n, k),$$

over k of $\mathscr{I}\mathscr{G}\mathscr{L}(n, k, N[10^{n-1}] + N[0^{n-1} -1])$. Here $\boxplus$ denotes semi-direct sum. The generators of $\mathscr{I}\mathscr{G}\mathscr{L}(n, k, N[10^{n-1}] + N[0^{n-1} -1])$ satisfy the commutation relations

$$[X_\mu{}^\nu, X_{\mu'}{}^{\nu'}] = \delta_{\mu'}{}^\nu X_\mu{}^{\nu'} - \delta_\mu{}^{\nu'} X_{\mu'}{}^\nu, \tag{1.4a}$$

$$[X_\mu{}^\nu, P_i{}^{\nu'}] = -\delta_\mu{}^{\nu'} P_i{}^\nu, \tag{1.4b}$$

$$[X_\mu{}^\nu, P_{\mu'}{}^j] = \delta_{\mu'}{}^\nu P_\mu{}^j, \tag{1.4c}$$

$$[P_i{}^\nu, P_{i'}{}^{\nu'}] = [P_\mu{}^j, P_{\mu'}{}^{j'}] = [P_i{}^\nu, P_\mu{}^j] = 0. \tag{1.4d}$$

The second order Casimir operator of $GL(n, k)$ is given by

$$C_2(GL(n, k)) = X_\mu{}^\nu X_\nu{}^\mu.$$

In the defining representation $[10^{n-1}]$ of $\mathscr{G}\mathscr{L}(n, k)$ we have

$$X_\mu{}^{\nu+} = X_\nu{}^\mu, \tag{1.5}$$

where $^+$ denotes hermitian conjugation.

1*b*. EXPANSION OF $IGL(n, k, [10^{n-1}] + [0^{n-1} -1])$

Our aim is to see under which circumstances we can expand

$$IGL(n, k, N[10^{n-1}] \dot{+} N[0^{n-1} -1])$$

to the Lie algebra of the *linear* group $\mathscr{G}\mathscr{L}(n + N, k)$. Instead of considering the $2Nn$-dimensional vector space (1.1) and the *non-linear* group $\mathscr{I}\mathscr{G}\mathscr{L}(n, k, N[10^{n-1}] + N[0^{n-1} -1])$, let us therefore extend this vector space to the $(n + N)^2$-dimensional vector space

$$U_{(n+N)^2} = (V_n \dot{+} W_N) \otimes (V_n \dot{+} W_N),$$

over k and consider the group $\mathscr{G}((n+\mathrm{N})^2, k)$ of all those *linear* transformations

$$\begin{aligned} x'^{\ \nu}_{\mu} &= \mathscr{L}_{\mu}{}^{\kappa}\tilde{\mathscr{L}}^{-1\nu}{}_{\lambda}x_{\kappa}{}^{\lambda}, & \mu, \nu &= 1, \ldots, n, \\ x'^{\ n+j}_{\mu} &= \mathscr{L}_{\mu}{}^{\nu}x_{\nu}{}^{n+j} + a_{\mu}{}^{i}x_{n+i}{}^{n+j}, & i, j &= 1, \ldots, \mathrm{N}, \\ x'_{n+i}{}^{\nu} &= \tilde{\mathscr{L}}^{-1\nu}{}_{\mu}x_{n+i}{}^{\mu} + a_{j}{}^{\nu}x_{n+i}{}^{n+j}, \\ x'_{n+i}{}^{n+j} &= x_{n+i}{}^{n+j}, \end{aligned}$$

of $\mathrm{U}_{(n+\mathrm{N})^2}$ where

$$\mathscr{L}_{\mu}{}^{\nu} \in \mathscr{G}\mathscr{L}(n, k),$$
$$a_{\mu}{}^{j},\ a_{i}{}^{\nu} \in k.$$

It is seen that

$$\mathscr{G}((n+\mathrm{N})^2, k) \cong \mathscr{I}\mathscr{G}\mathscr{L}(n, k, \mathrm{N}[10^{n-1}] \dotplus \mathrm{N}[0^{n-1}-1]),$$

and therefore also has the Lie algebra given in equation (1.4).

We shall now show that out of the generators of $\mathscr{G}((n+1)^2, k)$ we can construct certain new operators which obey the commutation relations of $\mathrm{GL}(n+1, k)$.

Let us to this end define the operators

$$\begin{aligned} \mathrm{P}^2 &\equiv \frac{1}{\mathrm{N}}\mathrm{P}_i{}^{\nu}\mathrm{P}_{\nu}{}^{i}, \\ \bar{\mathrm{P}}_i{}^{\nu} &\equiv -\frac{1}{2}[\mathrm{C}_2(\mathrm{GL}(n, k)), \mathrm{P}_i{}^{\nu}] = \{\mathrm{P}_i{}^{\mu}\mathrm{X}_{\mu}{}^{\nu}\} \equiv \frac{1}{2}(\mathrm{P}_i{}^{\mu}\mathrm{X}_{\mu}{}^{\nu} + \mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_i{}^{\mu}), \\ \bar{\mathrm{P}}_{\mu}{}^{j} &\equiv \frac{1}{2}[\mathrm{C}_2(\mathrm{GL}(n, k)), \mathrm{P}_{\mu}{}^{j}] = \{\mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_{\nu}{}^{j}\} \equiv \frac{1}{2}(\mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_{\nu}{}^{j} + \mathrm{P}_{\nu}{}^{j}\mathrm{X}_{\mu}{}^{\nu}), \\ \mathrm{U}_i{}^{j} &\equiv \{\bar{\mathrm{P}}_i{}^{\nu}\mathrm{P}_{\nu}{}^{j}\} = \{\mathrm{P}_i{}^{\mu}\mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_{\nu}{}^{j}\} \\ &\equiv \frac{1}{4}(\mathrm{P}_i{}^{\mu}\mathrm{P}_{\nu}{}^{j}\mathrm{X}_{\mu}{}^{\nu} + \mathrm{P}_i{}^{\mu}\mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_{\nu}{}^{j} + \mathrm{P}_{\nu}{}^{j}\mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_i{}^{\mu} + \mathrm{X}_{\mu}{}^{\nu}\mathrm{P}_i{}^{\mu}\mathrm{P}_{\nu}{}^{j}), \end{aligned}$$

where $\{\ \}$ denotes that the expression inside the bracket has been symmetrized with respect to the position of the $\mathrm{P}_i{}^{\nu}$ and $\mathrm{P}_{\mu}{}^{i}$ relative to $\mathrm{X}_{\mu}{}^{\nu}$ and to $\bar{\mathrm{P}}_i{}^{\nu}$ (but not relative to one another as they commute) and then divided by the number of terms. One easily verifies the following relation

$$\mathrm{P}_i{}^{\nu}\mathrm{P}_{\nu}{}^{j} = \delta_i{}^{j}\mathrm{P}^2 \quad \text{iff} \quad \mathrm{N} = 1, \tag{1.7}$$

noticing that the left hand side has one independent component only in these cases.

In terms of the above operators we define

$$\bar{X}'^{\ \nu}_{\mu} \equiv X_{\mu}{}^{\nu}, \qquad \bar{X}'_{n+i}{}^{\nu} \equiv P_i{}^{\nu} + \lambda \bar{P}_i{}^{\nu}/\sqrt{-P^2},$$

$$\bar{X}'^{\ n+j}_{\mu} \equiv P_{\mu}{}^{j} + \mu \bar{P}_{\mu}{}^{j}/\sqrt{-P^2}, \quad \bar{X}'_{n+i}{}^{n+j} \equiv -U_i{}^{j}/P^2,$$

where λ and μ are free parameters. For these operators we obtain the following commutation relations, having made use in particular of relation (1.7)

$$[\bar{X}'^{\ \nu}_{\mu}, \bar{X}'^{\ \nu'}_{\mu'}] = \delta_{\mu'}{}^{\nu}\bar{X}'^{\ \nu'}_{\mu} - \delta_{\mu}{}^{\nu'}\bar{X}'^{\ \nu}_{\mu'}, \tag{1.8a}$$

$$[\bar{X}'^{\ \nu}_{\mu}, \bar{X}'_{n+1}{}^{\nu'}] = -\delta_{\mu}{}^{\nu'}\bar{X}'_{n+1}{}^{\nu}, \tag{1.8b}$$

$$[\bar{X}'^{\ \nu}_{\mu}, \bar{X}_{\mu'}{}^{n+1}] = \delta_{\mu'}{}^{\nu}\bar{X}'^{\ n+1}_{\mu}, \tag{1.8c}$$

$$[\bar{X}'^{\ \nu}_{\mu}, \bar{X}'_{n+1}{}^{n+1}] = 0, \tag{1.8d}$$

$$[\bar{X}'_{n+1}{}^{\nu}, \bar{X}'_{n+1}{}^{\nu'}] = 0, \tag{1.8e}$$

$$[\bar{X}'^{\ n+1}_{\mu}, \bar{X}'^{\ n+1}_{\mu'}] = 0, \tag{1.8f}$$

$$[\bar{X}'_{n+1}{}^{\nu}, \bar{X}'^{\ n+1}_{\mu}] = \lambda\mu(\delta_{\mu}{}^{\nu}\bar{X}'_{n+1}{}^{n+1} - \bar{X}'^{\ \nu}_{\mu}), \tag{1.8g}$$

$$[\bar{X}'_{n+1}{}^{n+1}, \bar{X}'_{n+1}{}^{\nu}] = \bar{X}'_{n+1}{}^{\nu}, \tag{1.8h}$$

$$[\bar{X}'_{n+1}{}^{n+1}, \bar{X}'^{\ n+1}_{\mu}] = -\bar{X}'^{\ n+1}_{\mu}. \tag{1.8i}$$

In order to remove the parameters λ and μ from the above commutation relations we finally define the operators

$$\begin{aligned} \bar{X}_{\mu}{}^{\nu} &\equiv \bar{X}'^{\ \nu}_{\mu} = X_{\mu}{}^{\nu}, \\ \bar{X}_{n+i}{}^{\nu} &= \lambda^{-1}\bar{X}'_{n+i}{}^{\nu} = \lambda^{-1}P_i{}^{\nu} + \bar{P}_i{}^{\nu}/\sqrt{-P^2}, \\ \bar{X}_{\mu}{}^{n+j} &\equiv \mu^{-1}\bar{X}'^{\ n+j}_{\mu} = \mu^{-1}P_{\mu}{}^{j} + \bar{P}_{\mu}{}^{j}/\sqrt{-P^2}, \\ \bar{X}_{n+i}{}^{n+j} &\equiv \bar{X}'_{n+i}{}^{n+j} = -U_i{}^{j}/P^2. \end{aligned}$$

We then obtain the following commutation relations

$$[\bar{X}_A{}^B, \bar{X}_{A'}{}^{B'}] = \delta_{A'}{}^{B}\bar{X}_A{}^{B'} - \delta_A{}^{B'}\bar{X}_{A'}{}^{B}, \quad A, B = 1, \ldots, n+1,$$

which evidently are those of $GL(n+1, k)$.

We have thus shown that, due to relation (1.7),

$$IGL(n, k, N[10^{n-1}] \dotplus N[0^{n-1}\ -1])$$

by this procedure can be expanded to $GL(n+N, k)$ *iff* $N = 1$. *This expansion corresponds globally to the expansion of*

$$\mathcal{G}((n+1)^2, k) \cong \mathcal{IGL}(n, k, [10^{n-1}] \dotplus [0^{n-1}\ -1])$$

to a linear Lie group $\mathcal{E}((n+1)^2, k) \cong \mathcal{GL}(n+1, k)$.

By analyzing the commutation relations (1.8) we find that when taking the limit $\lambda\mu \to 0$ GL(n + 1, k) contracts [*17-19*] to

$$\mathrm{T}(2n, k) \boxplus (\mathrm{GL}(n, k) + \mathrm{GL}(1, k)) \cong \mathrm{IGL}(n, k, [10^{n-1}] \dot{+} [0^{n-1} - 1] \boxplus \mathrm{GL}(1, k),$$

where + denotes direct sum, i. e. GL(n + 1, k) is an expansion of

$$\mathrm{IGL}(n, k, [10^{n-1}] \dot{+} [0^{n-1} - 1])$$

but a deformation [*20-23*] of

$$\mathrm{IGL}(n, k, [10^{n-1}] \dot{+} [0^{n-1} - 1]) \boxplus \mathrm{GL}(1, k) \quad \text{(see fig. 1).}$$

$$\mathrm{IGL}(n, k, [10^{n-1}] \dot{+} [0^{n-1} - 1]) \rightsquigarrow \mathrm{GL}(n + 1, k)$$
$$\overset{\lambda\mu \to 0}{\underset{\leftarrow\cdots\cdots}{\longrightarrow}} \mathrm{IGL}(n, k, [10^{n-1}] \dot{+} [0^{n-1} - 1]) \boxplus \mathrm{GL}(1, k)$$

FIG. 1. — *Expansion, contraction, and deformation diagram for the general linear Lie algebras.*

Here $\rightsquigarrow$, $\to$, and $\to$ denote expansion, contraction and deformation, respectively.

2*a*. THE GROUPS $\mathscr{IU}(n_1, n_2)$

Let us start with giving a few well-known facts about the inhomogeneous pseudo-unitary groups $\mathscr{IU}(n_1, n_2)$. $\mathscr{IU}(n_1, n_2)$ is the *real* Lie group of those *complex* transformations of an n-dimensional complex vector space V_n, $n = n_1 + n_2$,

$$x'_\mu \doteq \mathscr{U}_\mu{}^\nu x_\nu + a_\mu, \quad a_\mu = b_\mu + ic_\mu, \quad \mu, \nu = 1, \ldots, n,$$

where

$$\mathscr{U}_\mu{}^\nu \in \mathscr{U}(n_1, n_2),$$

the n-dimensional pseudo-unitary group [*16*], and where

$$b_\mu, c_\mu \in \mathrm{R}.$$

We have that

$$\mathscr{IU}(n_1, n_2) \cong \mathscr{T}(2n, \mathrm{R}) \boxtimes \mathscr{U}(n_1, n_2),$$

and leaves invariant the pseudo-hermitian form

$$\langle \bar{x} - x, \bar{y} - y \rangle \equiv g^{\mu\nu}(\bar{x}^*_\mu - x^*_\mu)(\bar{y}_\mu - y_\mu),$$

where * denotes complex conjugation and where the metric $g^{\mu\nu}$ of V_n has the property

$$g^{\nu\mu} = g^{\mu\nu} \sim \begin{array}{c} \\ 1 \\ \vdots \\ n_1 \\ n_1+1 \\ \vdots \\ n \end{array} \begin{pmatrix} \begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline & \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} \end{array} \end{pmatrix} \quad \begin{matrix} 1\ldots n_1 & n_1+1\ldots n \end{matrix} \tag{2.2}$$

$g_{\mu\nu}$ is defined by

$$g_{\mu\lambda}g^{\lambda\mu} = \delta_\mu^{\ \nu} = g^{\nu\lambda}g_{\lambda\mu},$$

and hence fulfils

$$g_{\nu\mu} = g_{\mu\nu}.$$

The generators

$$\left.\begin{aligned} L_{\mu\nu} &= -L_{\nu\mu} \\ Q_{\mu\nu} &= Q_{\nu\mu} \end{aligned}\right\} \sim \mathscr{U}^{\nu\mu}, \tag{2.3a}$$

of $\mathscr{U}(n_1, n_2)$ and the generators

$$R_\nu \sim b^\nu, \qquad S_\nu \sim c^\nu, \tag{2.3b}$$

of $\mathscr{T}(2n, R)$ form a basis for the real Lie algebra

$$IU(n_1, n_2) \cong T(2n, R) \boxplus U(n_1, n_2),$$

of $\mathscr{I}\mathscr{U}(n_1, n_2)$. The generators of $\mathscr{I}\mathscr{U}(n_1, n_2)$ satisfy the well-known commutation relations [8, 9]

$$[L_{\mu\nu}, L_{\mu'\nu'}] = g_{\nu\mu'}L_{\mu\nu'} - g_{\mu'\mu}L_{\nu\nu'} + g_{\nu\nu'}L_{\mu'\mu} - g_{\nu'\mu}L_{\mu'\nu}, \tag{2.4a}$$
$$[L_{\mu\nu}, Q_{\mu'\nu'}] = g_{\nu\mu'}Q_{\mu\nu'} - g_{\mu'\mu}Q_{\nu\nu'} + g_{\nu\nu'}Q_{\mu'\mu} - g_{\nu'\mu}Q_{\mu'\nu}, \tag{2.4b}$$
$$[Q_{\mu\nu}, Q_{\mu'\nu'}] = -g_{\nu\mu'}L_{\mu\nu'} - g_{\mu'\mu}L_{\nu\nu'} + g_{\nu\nu'}L_{\mu'\mu} + g_{\nu'\mu}L_{\mu'\nu}, \tag{2.4c}$$
$$[L_{\mu\nu}, R_{\nu'}] = g_{\nu\nu'}R_\mu - g_{\nu'\mu}R_\nu, \tag{2.4d}$$
$$[L_{\mu\nu}, S_{\nu'}) = g_{\nu\nu'}S_\mu - g_{\nu'\mu}S_\nu, \tag{2.4e}$$
$$[Q_{\mu\nu}, R_{\nu'}] = -g_{\nu\nu'}S_\mu - g_{\nu'\mu}S_\nu, \tag{2.4f}$$
$$[Q_{\mu\nu}, S_{\nu'}] = g_{\nu\nu'}R_\mu + g_{\nu'\mu}R_\nu, \tag{2.4g}$$
$$[R^\nu, R^{\nu'}] = [S^\nu, S^{\nu'}] = [R^\nu, S^{\nu'}] = 0. \tag{2.4h}$$

The second order Casimir operator of $U(n_1, n_2)$ is given by

$$C_2(U(n_1, n_2)) = Q_\mu^{\ \nu}Q_\nu^{\ \mu} - L_\mu^{\ \nu}L_\nu^{\ \mu}.$$

In the defining representation $[10^{n-1}]$ of $\mathscr{U}(n_1, n_2)$ we have

$$L_{\mu\nu}^+ = -L^{\mu\nu}, \tag{2.5a}$$
$$Q_{\mu\nu}^+ = -Q^{\mu\nu}. \tag{2.5b}$$

2b. EXPANSION OF IU(n_1, n_2) TO U(n_1 + 1, n_2) OR TO U(n_1, n_2 + 1)

Our aim is to show that we can expand IU(n_1, n_2) to the Lie algebra of the *linear* group $\mathscr{U}(n_1 + 1, n_2)$ or of $\mathscr{U}(n_1, n_2 + 1)$. Instead of considering the n-dimensional complex vector space V_n and the *non-linear* group $\mathscr{I}\mathscr{U}(n_1, n_2)$, let us therefore extend V_n to the $(n + 1)$-dimensional complex vector space

$$U_{n+1} = V_n \dot{+} W_1, \qquad n = n_1 + n_2,$$

and consider the *real* Lie group $\mathscr{G}(n + 1, R)$ of all those *complex linear* transformations

$$x'_\mu = \mathscr{U}_\mu{}^\nu x_\nu + a_\mu x_{n+1}, \quad a_\mu = b_\mu + ic_\mu, \quad \mu, \nu = 1, \ldots, n,$$

$x'_{n+1} = x_{n+1}$, of U_{n+1} where

$$\mathscr{U}_\mu{}^\nu \in \mathscr{U}(n_1, n_2), \qquad b_\mu, c_\mu \in R.$$

It is seen that

$$\mathscr{G}(n + 1, R) \cong \mathscr{I}\mathscr{U}(n_1, n_2),$$

and therefore also has the Lie algebra given in equation (2.4).

We shall now show that out of the generators of $\mathscr{G}(n + 1, R)$ we can construct certain new operators which obey the commutation relations of U(n_1 + 1, n_2) or of U(n_1, n_2 + 1).

To this end we shall extend the metric $g_{\mu\nu}$ of V_n to all of U_{n+1} by the following definitions

$$g_{n+1\nu} \equiv 0 \equiv g_{\nu n+1}, \qquad g_{n+1n+1} \equiv \gamma \in R. \tag{2.6}$$

We now note that U(n_1, n_2) and GL(n, R) are different real forms of GL(n, C). Instead of solving our expansion problem from the beginning again, we shall therefore derive the solution of it from the solution found in section 1b for the expansion of IGL(n, R, $[10^{n-1}] \dot{+} [0^{n-1} -1]$) to GL($n$ + 1, R). This can be done by utilizing that the basis (2.3) of

$$G(n + 1, R) \cong IU(n_1, n_2),$$

can be obtained from the basis (1.3) of IGL(n, R, $[10^{n-1}] \dot{+} [0^{n-1} -1]$) by a formal complex transformation for which equation (2.5) follow from equation (1.5). The transformation is given by

$$L_{\mu\nu} = X_{\mu\dot{\nu}} - X_{\nu\dot{\mu}}, \quad Q_{\mu\nu} = i(X_{\mu\dot{\nu}} + X_{\nu\dot{\mu}}),$$
$$R_\nu = P_{\dot{\nu}} - \gamma P_\nu, \qquad S_\nu = i(P_{\dot{\nu}} + \gamma P_\nu),$$

where the indices i, j have been suppressed on $P_i{}^\nu$ and $P_\mu{}^j$ as $i, j = 1$, and where the metric $g_{\mu\nu}$ of V_n and the metric γ of W_1 have been used to lower the greek and the suppressed latin index. The inverse transformation is

$$X_\mu{}^\nu = \frac{1}{2}(L_\mu{}^\nu - iQ_\mu{}^\nu), \quad P^\nu = \frac{1}{2}(R^\nu - iS^\nu), \quad P_\mu = -\frac{1}{2}\gamma^{-1}(R_\mu + iS_\mu).$$

Let us now define the operators

$$T^2 \equiv -4\cdot P^2 = \gamma^{-1}(R_\nu R^\nu + S_\nu S^\nu),$$

$$\bar{R}_\nu \equiv 2i\cdot(\bar{P}^{\cdot}_\nu - \gamma\bar{P}_\nu) = \frac{1}{4}[C_2(U(n_1, n_2)), S_\nu] = \{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \},$$

$$\bar{S}_\nu \equiv 2i\cdot i(\bar{P}^{\cdot}_\nu + \gamma\bar{P}_\nu) = -\frac{1}{4}[C_2(U(n_1, n_2)), R_\nu] = \{ -R^\mu L_{\mu\nu} + S^\mu Q_{\mu\nu} \},$$

$$V \equiv -4\cdot 2i\gamma U = \{ \bar{R}_\nu R^\nu + \bar{S}_\nu S^\nu \} = \{ -2R^\mu S^\nu L_{\mu\nu} + R^\mu R^\nu Q_{\mu\nu} + S^\mu S^\nu Q_{\mu\nu} \},$$

where $\{ \ \}$ denotes that the expression inside the bracket has been symmetrized with respect to the position of the R^ν and S^ν relative to $L_{\mu\nu}$ and $Q_{\mu\nu}$ and to $\bar{R}_\nu$ and $\bar{S}_\nu$ (but not relative to one another as they commute) and then divided by the number of terms. In terms of above operators we define

$$\bar{L}'_{\mu\nu} \equiv L_{\mu\nu}, \qquad \bar{Q}'_{\mu\nu} \equiv Q_{\mu\nu},$$

$$\bar{L}'_{n+1\,\nu} \equiv R_\nu + \lambda\bar{R}_\nu/\sqrt{-T^2}, \quad \bar{Q}'_{n+1\,\nu} \equiv S_\nu + \lambda\bar{S}_\nu/\sqrt{-T^2}, \quad \bar{Q}'_{n+1\,n+1} \equiv -V/T^2,$$

where λ is a free parameter. For these operators we obtain the following commutation relations

$$[\bar{L}'_{\mu\nu}, \bar{L}'_{\mu'\nu'}] = g_{\nu\mu'}\bar{L}'_{\mu\nu'} - g_{\mu'\mu}\bar{L}'_{\nu\nu'} + g_{\nu\nu'}\bar{L}'_{\mu'\mu} - g_{\nu'\mu}\bar{L}'_{\mu'\nu}, \tag{2.8a}$$

$$[\bar{L}'_{\mu\nu}, \bar{Q}'_{\mu'\nu'}] = g_{\nu\mu'}\bar{Q}'_{\mu\nu'} - g_{\mu'\mu}\bar{Q}'_{\nu\nu'} + g_{\nu\nu'}\bar{Q}'_{\mu'\mu} - g_{\nu'\mu}\bar{Q}'_{\mu'\nu}, \tag{2.8b}$$

$$[\bar{Q}'_{\mu\nu}, \bar{Q}'_{\mu'\nu'}] = -g_{\nu\mu'}\bar{L}'_{\mu\nu'} - g_{\mu'\mu}\bar{L}'_{\nu\nu'} + g_{\nu\nu'}\bar{L}'_{\mu'\mu} + g_{\nu'\mu}\bar{L}'_{\mu'\nu}, \tag{2.8c}$$

$$[\bar{L}'_{\mu\nu}, \bar{L}'_{n+1\,\nu'}] = g_{\nu\nu'}\bar{L}'_{n+1\,\mu} - g_{\nu'\mu}\bar{L}'_{n+1\,\nu}, \tag{2.8d}$$

$$[\bar{L}'_{\mu\nu}, \bar{Q}'_{n+1\,\nu'}] = g_{\nu\nu'}\bar{Q}'_{n+1\,\mu} - g_{\nu'\mu}\bar{Q}'_{n+1\,\nu}, \tag{2.8e}$$

$$[\bar{Q}'_{\mu\nu}, \bar{L}'_{n+1\,\nu'}] = -g_{\nu\nu'}\bar{Q}'_{n+1\,\mu} + g_{\nu'\mu}\bar{Q}'_{n+1\,\nu}, \tag{2.8f}$$

$$[\bar{Q}'_{\mu\nu}, \bar{Q}'_{n+1\,\nu'}] = g_{\nu\nu'}\bar{L}'_{n+1\,\mu} + g_{\nu'\mu}\bar{L}'_{n+1\,\nu}, \tag{2.8g}$$

$$[\bar{L}'_{\mu\nu}, \bar{Q}'_{n+1\,n+1}] = 0, \tag{2.8h}$$

$$[\bar{Q}'_{\mu\nu}, \bar{Q}'_{n+1\,n+1}] = 0, \tag{2.8i}$$

$$[\bar{L}'_{n+1\,\nu}, \bar{L}'_{n+1\,\nu'}] = -\lambda^2 g_{n+1\,n+1}\bar{L}'_{\nu\nu'}, \tag{2.8j}$$

$$[\bar{L}'_{n+1\,\nu}, \bar{Q}'_{n+1\,\nu'}] = \lambda^2(-g_{n+1\,n+1}\bar{Q}'_{\nu\nu'} + g_{\nu\nu'}\bar{Q}'_{n+1\,n+1}), \tag{2.8k}$$

$$[\bar{Q}'_{n+1\nu}, \bar{Q}'_{n+1\nu'}] = -\lambda^2 g_{n+1n+1}\bar{L}'_{\nu\nu'}, \tag{2.8l}$$

$$[\bar{Q}'_{n+1n+1}, \bar{L}'_{n+1\nu}] = 0, \tag{2.8m}$$

$$[\bar{Q}'_{n+1n+1}, \bar{Q}'_{n+1\nu}] = -2\bar{L}'_{n+1\nu}. \tag{2.8n}$$

In order to remove the parameter λ from the above commutation relations we finally define the operators

$$\begin{aligned}
\bar{L}_{\mu\nu} &\equiv \bar{L}'_{\mu\nu} = L_{\mu\nu},\\
\bar{Q}_{\mu\nu} &\equiv \bar{Q}'_{\mu\nu} = Q_{\mu\nu},\\
\bar{L}_{n+1\nu} &\equiv \lambda^{-1}\bar{L}'_{n+1\nu} = \lambda^{-1}R_\nu + \bar{R}_\nu/\sqrt{-T^2} \equiv -L_{\nu n+1},\\
\bar{Q}_{n+1\nu} &= \lambda^{-1}\bar{Q}'_{n+1\nu} = \lambda^{-1}S_\nu + \bar{S}_\nu/\sqrt{-T^2} \equiv \bar{Q}_{\nu n+1},\\
\bar{Q}_{n+1n+1} &\equiv \bar{Q}'_{n+1n+1} = -V/T^2,
\end{aligned}$$

which then satisfy

$$\begin{aligned}
[\bar{L}_{AB}, \bar{L}_{A'B'}] &= g_{BA'}\bar{L}_{AB'} - g_{A'A}\bar{L}_{BB'} + g_{BB'}\bar{L}_{A'A} - g_{B'A}\bar{L}_{A'B},\\
[\bar{L}_{AB}, \bar{Q}_{A'B'}] &= g_{BA'}\bar{Q}_{AB'} - g_{A'A}\bar{Q}_{BB'} + g_{BB'}\bar{Q}_{A'A} - g_{B'A}\bar{Q}_{A'B},\\
[\bar{Q}_{AB}, \bar{Q}_{A'B'}] &= -g_{BA'}\bar{L}_{AB'} - g_{A'A}\bar{L}_{BB'} + g_{BB'}\bar{L}_{A'A} + g_{B'A}\bar{L}_{A'B},\\
&\qquad\qquad A, B = 1, \ldots, n+1,
\end{aligned}$$

where

$$g_{BA} = g_{AB}, \qquad L_{BA} = -L_{AB}, \qquad Q_{BA} = Q_{AB}.$$

These commutation relations are evidently those of $U(n_1 + N_1, n_2 + N_2)$ where

$$N_1 + N_2 = 1, \qquad N_1 - N_2 = \frac{\gamma}{|\gamma|} = \pm 1.$$

We have thus shown that $IU(n_1, n_2)$ *can be expanded to* $U(n_1 + 1, n_2)$ *or to* $U(n_1, n_2 + 1)$. *These expansions correspond globally to the expansions of* $\mathcal{G}(n+1, R) \cong \mathcal{I}\mathcal{U}(n_1, n_2)$ *to a linear Lie group*

$$\mathcal{E}(n_1 + 1, n_2; R) \cong \mathcal{U}(n_1 + 1, n_2)$$

or to

$$\mathcal{E}(n_1, n_2 + 1; R) \cong \mathcal{U}(n_1, n_2 + 1).$$

By analyzing the commutation relations (2.8) we find that when taking the limit $\lambda \to 0$ $U(n_1 + N_1, n_2 + N_2)$ contracts [*17-19*] to

$$T(n) \boxplus (U(n_1, n_2) + U(1)) \cong IU(n_1, n_2) \boxplus U(1),$$

i. e. $U(n_1 + N_1, n_2 + N_2)$ is an expansion of $IU(n_1, n_2)$ but a deformation [*20-23*] of $T(n) \boxplus (U(n_1, n_2) + U(1))$ (see fig. 2).

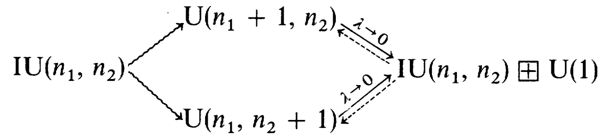

FIG. 2. — *Expansion, contraction, and deformation diagram for the pseudo-unitary Lie algebras.*

3*a*. THE GROUPS $\mathcal{IS}\mathcal{O}_0(n_1, n_2; k; \mathrm{N}(10^{\nu-1}))$ AND $\mathcal{I}\mathcal{S}p(n, k, \mathrm{N}(10^{\nu-1}))$

Let us start with introducing the following notation for the n-dimensional isometric groups

$$\mathcal{M}_\eta(n_1, n_2; k) = \begin{cases} \mathcal{S}\mathcal{O}_0(n_1, n_2; k) \\ \mathcal{S}p(n_1 + n_2, k) \end{cases} \quad \forall \eta = \begin{matrix} 1 \\ -1, \end{matrix}$$

with

$$n = n_1 + n_2 = \begin{cases} 2\nu \quad \text{or} \quad 2\nu + 1 \\ 2\nu \end{cases} \quad \forall \eta = \begin{matrix} 1 \\ -1, \end{matrix} \quad \nu = 1, 2, \ldots$$

Here $\mathcal{S}\mathcal{O}_0(n_1, n_2; k)$ is the connected subgroup of the full pseudo-orthogonal group [16] $\mathrm{O}(n_1, n_2; k)$ (with of course $\mathcal{S}\mathcal{O}_0(n_1, n_2; \mathrm{C}) \cong \mathcal{S}\mathcal{O}_0(n; \mathrm{C})$) and $\mathcal{S}p(n, k)$ is the symplectic group [16].

Let us now consider the inhomogenization [1] $\mathcal{I}\mathcal{M}_\eta(n_1, n_2; k; \mathrm{N}(10^{\nu-1}))$ of $\mathcal{M}_\eta(n_1, n_2; k)$ where $(10^{\nu-1})$ is the defining representation of $\mathcal{M}_\eta(n_1, n_2; k)$. $\mathcal{I}\mathcal{M}_\eta(n_1, n_2; k; \mathrm{N}(10^{\nu-1}))$ is the group of all those transformations

$$x'^{\,j}_\mu = \mathcal{M}_\mu^{\ \nu} x_\nu^{\ j} + a_\mu^{\ j}, \quad \mu, \nu = 1, \ldots, n, \quad j = 1, \ldots, \mathrm{N},$$

of an Nn-dimensional vector space

$$\mathrm{U}_{\mathrm{N}n} = \mathrm{V}_n \otimes \mathrm{W}_\mathrm{N}, \tag{3.1}$$

over k, where

$$\mathcal{M}_\mu^{\ \nu} \in \mathcal{M}_\eta(n_1, n_2; k),$$
$$a_\mu^{\ j} \in \mathcal{T}(\mathrm{N}n, k).$$

We have that

$$\mathcal{I}\mathcal{M}_\eta(n_1, n_2; k; \mathrm{N}(10^{\nu-1})) \cong \mathcal{T}(\mathrm{N}n, k) \boxtimes \mathcal{M}_\eta(n_1, n_2; k),$$

and leaves invariant the non-degenerate bilinear form

$$(\bar{x}^i - x^i, \bar{y}^j - y^j) \equiv g^{\mu\nu}(\bar{x}_\mu^{\ i} - x_\mu^{\ i})(\bar{y}_\nu^{\ j} - y_\nu^{\ j}),$$

where the metric $g^{\mu\nu}$ of V_n has the property

$$g^{\nu\mu} = \eta g^{\mu\nu}. \tag{3.2}$$

$g_{\mu\nu}$ is defined by

$$g_{\mu\lambda}g^{\lambda\nu} = \delta_\mu{}^\nu = g^{\nu\lambda}g_{\lambda\mu},$$

and hence fulfils

$$g_{\nu\mu} = \eta g_{\mu\nu}.$$

Using the metric to raise or lower the indices of the metric itself, the following non-trivial relations hold true

$$g_\mu{}^{\cdot\nu} = \delta_\mu{}^\nu, \qquad g^\mu{}_\nu^{\cdot} = \eta\delta_\nu{}^\mu, \qquad g_{\mu\nu}^{\cdot\cdot} = g_{\nu\mu},$$
$$g^\mu_{\cdot\nu} = \delta_\nu{}^\mu, \qquad g_\mu{}^\nu_{\cdot} = \eta\delta_\mu{}^\nu, \qquad g^{\mu\nu}_{\cdot\cdot} = g^{\nu\mu}.$$

The generators

$$M_{\mu\nu} = -\eta M_{\nu\mu} \sim \mathscr{M}^{\nu\mu}, \tag{3.3a}$$

of $\mathscr{M}\eta(n_1, n_2; k)$ and the generators

$$P_{i\nu} \sim a^{\nu i}, \tag{3.3b}$$

of $\mathscr{T}(Nn, k)$ form a basis for the Lie algebra

$$IM_\eta(n_1, n_2; k; N(10^{\nu-1})) \cong T(Nn, k) \boxplus M\eta(n_1, n_2; k),$$

over k of $\mathscr{I}\mathscr{M}_\eta(n_1, n_2; k; N(10^{\nu-1})$. The generators of $\mathscr{I}\mathscr{M}_\eta(n_1, n_2; k; N(10^{\nu-1})$ satisfy the commutation relations

$$[M_{\mu\nu}, M_{\mu'\nu'}] = g_{\nu\mu'}M_{\mu\nu'} - g_{\mu'\mu}M_{\nu\nu'} + g_{\nu\nu'}M_{\mu'\mu} - g_{\nu'\mu}M_{\mu'\nu}, \tag{3.4a}$$
$$[M_{\mu\nu}, P_{i\nu'}] = g_{\nu\nu'}P_{i\mu} - g_{\nu'\mu}P_{i\nu}, \tag{3.4b}$$
$$[P_{i\nu}, P_{i'\nu'}] = 0. \tag{3.4c}$$

The second order Casimir operator of $M_\eta(n_1, n_2; k)$ is given by

$$C_2(M_\eta(n_1, n_2; k)) = M_\mu{}^\nu M_\nu{}^\mu.$$

In the defining representation $(10^{\nu-1})$ of $\mathscr{M}\eta(n_1, n_2; k)$ we have

$$M_{\mu\nu}^{+} = -M^{\mu\nu}, \tag{3.5a}$$

and hence from equation (3.3*a*)

$$M_\mu{}^{\nu+} = M_\nu{}^\mu. \tag{3.5b}$$

3b. EXPANSION OF $ISO_0(n_1, n_2; k)$ TO $SO_0(n_1 + 1, n_2; k)$ OR TO $SO_0(n_1, n_2 + 1; k)$ AND EXPANSION OF $ISp(n, k, 2(10^{\nu-1}))$ TO $Sp(n + 2, k)$

Our aim is to see under which circumstances we can expand

$$IM_\eta(n_1, n_2; k; N(10^{\nu-1}))$$

to the Lie algebra of the *linear* group $\mathscr{M}_\eta(n_1 + N_1, n_2 + N_2; k)$, $N_1 + N_2 = N$. Instead of considering the Nn-dimensional vector space (3.1) and the *non-linear* group $\mathscr{I}\mathscr{M}_\eta(n_1, n_2; k; N(10^{\nu-1}))$ let us therefore consider the $(n + N)$-dimensional vector space

$$U_{n+N} = V_n \dot{+} W_N,$$

over k and the group $\mathscr{G}_\eta(n + N, k)$ of all those *linear* transformations

$$x'_\mu = \mathscr{M}_\mu{}^\nu x_\nu + a_\mu{}^i x_{n+i}, \quad \mu, \nu = 1, \ldots, n, \quad i = 1, \ldots, N,$$
$$x'_{n+i} = x_{n+i},$$

of U_{n+N} where

$$\mathscr{M}_\mu{}^\nu \in \mathscr{M}_\eta(n_1, n_2; k),$$
$$a_\mu{}^i \in k.$$

It is seen that

$$\mathscr{G}_\eta(n + N, k) \cong \mathscr{I}\mathscr{M}_\eta(n_1, n_2; k; N(10^{\nu-1})),$$

and therefore also has the Lie algebra given in equation (3.4).

We shall now show that out of the generators of $\mathscr{I}\mathscr{S}\mathscr{O}_0(n_1, n_2; k)$ we can construct certain new operators which obey the commutation relations of $SO_0(n_1 + 1, n_2; k)$ or of $SO_0(n_1, n_2 + 1; k)$. We shall also show that out of the generators of $\mathscr{T}(2n, k) \boxtimes \mathscr{S}p(n, k)$ we can construct operators which span the Lie algebra $Sp(n + 2, k)$.

To this end we shall extend the metric $g_{\mu\nu}$ of V_n to all of U_{n+N} by the following definitions

$$g_{n+i\,\nu} \equiv 0 \equiv g_{\nu\,n+i},$$
$$g_{n+i\,n+j} \equiv \gamma_{ij} = \eta\gamma_{ji} = \eta g_{n+j\,n+i}. \qquad (3.6)$$

Let us now define the operators

$$P^2 \equiv \frac{1}{N} P_{i\nu} P^{i\nu},$$

where

$$P^i{}_\nu \equiv \gamma^{ij} P_{j\nu},$$

and

$$
\begin{aligned}
\bar{P}_{i\nu} &\equiv -\frac{1}{4}[C_2(M\eta(n_1, n_2; k)), P_{i\nu}] \\
&= \{ P_i^{\mu} M_{\mu\nu} \} \equiv \frac{1}{2}(P_i^{\mu} M_{\mu\nu} + M_{\mu\nu} P_i^{\mu}), \\
K_{ij} &\equiv \{ \bar{P}_{i\nu} P_j^{\nu} \} = \{ P_i^{\mu} P_j^{\nu} M_{\mu\nu} \} \\
&\equiv \frac{1}{4}(P_i^{\mu} P_j^{\nu} M_{\mu\nu} + P_i^{\mu} M_{\mu\nu} P_j^{\nu} + P_j^{\nu} M_{\mu\nu} P_i^{\mu} + M_{\mu\nu} P_i^{\mu} P_j^{\nu}) = -\eta K_{ji},
\end{aligned}
$$

where $\{\ \}$ denotes that expression inside the bracket has been symmetrized with respect to the position of the P_i^{ν} relative to $M_{\mu\nu}$ and to $\bar{P}_{i\nu}$ (but not relative to one another as they commute) and then divided by the number of terms. One easily verifies the relation

$$
P_i^{\nu} P_{j\nu} = \gamma_{ij} P^2 \quad \text{iff} \quad N = \begin{cases} 1 \\ 2 \end{cases} \quad \forall \eta = \begin{matrix} 1 \\ -1, \end{matrix} \tag{3.7}
$$

noticing that the left hand side has one independent component only in these cases.

In terms of the above operators we define

$$
\begin{aligned}
\bar{M}'_{\mu\nu} &\equiv M_{\mu\nu}, \\
\bar{M}'_{n+i\nu} &\equiv P_{i\nu} + \lambda \bar{P}_{i\nu}/\sqrt{-P^2}, \\
\bar{M}'_{n+in+j} &\equiv -K_{ij}/P^2,
\end{aligned}
$$

where λ is a free parameter. For these operators we obtain the following commutation relations having made use in particular of relation (3.7),

$$
[\bar{M}'_{\mu\nu}, \bar{M}'_{\mu'\nu'}] = g_{\nu\mu'}\bar{M}'_{\mu\nu'} - g_{\mu'\mu}\bar{M}'_{\nu\nu'} + g_{\nu\nu'}\bar{M}'_{\mu'\mu} - g_{\nu'\mu}\bar{M}'_{\mu'\nu}, \tag{3.8a}
$$

$$
[\bar{M}'_{\mu\nu}, \bar{M}'_{n+i\nu'}] = g_{\nu\nu'}\bar{M}'_{n+i\mu} - g_{\nu'\mu}\bar{M}'_{n+i\nu}, \tag{3.8b}
$$

$$
[\bar{M}'_{\mu\nu}, \bar{M}'_{n+in+j}] = 0, \tag{3.8c}
$$

$$
[\bar{M}'_{n+i\nu}, \bar{M}'_{n+i'\nu'}] = \lambda^2(-g_{n+i'n+i}\bar{M}'_{\nu\nu'} + g_{\nu\nu'}\bar{M}'_{n+i'n+i}), \tag{3.8d}
$$

$$
[\bar{M}'_{n+in+j}, \bar{M}'_{n+i'\nu}] = g_{n+jn+i'}\bar{M}'_{n+i\nu} - g_{n+i'n+i}\bar{M}'_{n+j\nu}, \tag{3.8e}
$$

$$
\begin{aligned}
[\bar{M}'_{n+in+j}, \bar{M}'_{n+i'n+j'}] = {} & g_{n+jn+i'}\bar{M}'_{n+in+j'} - g_{n+i'n+i}\bar{M}'_{n+jn+j'} \\
& + g_{n+jn+j'}\bar{M}'_{n+i'n+i} - g_{n+j'n+i}\bar{M}'_{n+i'n+j}.
\end{aligned} \tag{3.8f}
$$

In order to remove the parameter λ from the above commutation relations we finally define the operators

$$
\begin{aligned}
\bar{M}_{\mu\nu} &\equiv \bar{M}'_{\mu\nu} = M_{\mu\nu}, \\
\bar{M}_{n+i\nu} &\equiv \lambda^{-1}\bar{M}'_{n+i\nu} = \lambda^{-1}P_{i\nu} + \bar{P}_{i\nu}/\sqrt{-P^2} \equiv -\eta\bar{M}_{\nu n+i}, \\
\bar{M}_{n+in+j} &\equiv \bar{M}'_{n+in+j} = -K_{ij}/P^2,
\end{aligned}
$$

which then satisfy

$$[\bar{M}_{AB}, \bar{M}_{A'B'}] = g_{BA'}\bar{M}_{AB'} - g_{A'A}\bar{M}_{BB'} + g_{BB'} M_{A'A} - g_{B'A}M_{A'B}, \quad A, B = 1, \ldots, n + N,$$

where

$$g_{BA} = \eta g_{AB}$$
$$\bar{M}_{BA} = - \eta\bar{M}_{AB}.$$

These commutation relations are evidently for $\eta = 1$ those of

$$SO_0(n_1 + N_1, n_2 + N_2; k)$$

where

$$N_1 + N_2 = N = 1, \quad N_1 - N_2 = \frac{\gamma_{ij}}{|\gamma_{ij}|} = \pm 1,$$

and for $\eta = -1$ those of $Sp(n + 2, k)$.

We have thus shown that by this procedure, due to relation (3.7),

$$ISO_0(n_1, n_2; k; N(10^{\nu-1}))$$

can be expanded to $SO_0(n_1 + N_1, n_2 + N_2; k)$ *iff* $N = 1$. *These expansions correspond globally to the expansions of* $\mathcal{G}_1(n + 1, k) \cong \mathcal{I}\mathcal{S}\mathcal{O}_0(n_1, n_2)$ *to a linear Lie group*

$$\mathcal{E}(n_1 + 1, n_2; k) \cong \mathcal{S}\mathcal{O}_0(n_1 + 1, n_2; k)$$

or to

$$\mathcal{E}(n_1, n_2 + 1; k) \cong \mathcal{S}\mathcal{O}_0(n_1, n_2 + 1; k).$$

We have also shown that by this procedure, again due to relation (3.7), $ISp(n, k, N(10^{\nu-1}))$ *can be expanded to* $Sp(n + N, k)$ *iff* $N = 2$. *This expansion corresponds globally to the expansion of*

$$\mathcal{G}_{-1}(n + 2, k) \cong \mathcal{I}\mathcal{S}p(n, k, 2(10^{\nu-1}))$$

to a linear Lie group $\mathcal{E}(n + 2, k) \cong \mathcal{S}p(n + 2, k)$.

We shall now analyze the commutation relations (3.8) when taking the limit $\lambda \to 0$. For $\eta = 1$ we find that $SO_0(n_1 + N_1, n_2 + N_2; k)$ contracts [*17-19*] to $ISO_0(n_1, n_2; k)$, i. e. $SO_0(n_1 + N_1, n_2 + N_2)$ is both an expansion and a deformation [*20-23*] of $ISO_0(n_1, n_2; k)$ (see fig. 3).

For $\eta = -1$ we find that $Sp(n + 2, k)$ contracts to

$$T(2n, k) \boxplus (Sp(n, k) + Sp(2, k)) \cong ISp(n, k, 2(10^{\nu-1})) \boxplus Sp(2, k),$$

i. e. $Sp(n + 2, k)$ is an expansion of $ISp(n, k, 2(10^{\nu-1}))$ but a deformation of $ISp(n, k, 2(10^{\nu-1})) \boxplus Sp(2, k)$ (see fig. 4).

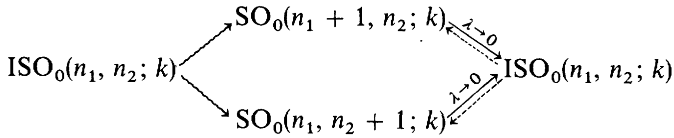

FIG. 3. — *Expansion, contraction, and deformation diagram for the pseudo-orthogonal Lie algebras.*

$$ISp(n, k, 2(10^{\nu-1})) \rightsquigarrow Sp(n+2, k) \underset{\longleftarrow}{\overset{\lambda\to 0}{\longrightarrow}} ISp(n, k, 2(10^{\nu-1})) \boxplus Sp(2).$$

FIG. 4. — *Expansion, contraction, and deformation diagram for the symplectic Lie algebras.*

4a. THE GROUPS $\mathcal{IUS}p(2\nu_1, 2\nu_2)$

Let us start with giving a few facts about the inhomogeneous pseudo-unitary symplectic groups $\mathcal{IUS}p(2\nu_1, 2\nu_2)$. $\mathcal{IUS}p(2\nu_1, 2\nu_2)$ is the *real* Lie group of those *complex* transformations

$$x'_{\alpha\rho} = \mathcal{S}_{\alpha\rho}{}^{\beta\sigma}x_{\beta\sigma} + a_{\alpha\rho}, \quad a_{\alpha\rho} = b_{\alpha\rho} + ic_{\alpha\rho}, \quad \alpha, \beta = 1, -1, \quad \rho, \sigma = 1, \ldots, \nu,$$

of a 2ν-dimensional complex vector space

$$U_{2\nu} = V'_\nu \otimes X_2, \qquad \nu = \nu_1 + \nu_2, \tag{4.1}$$

where

$$\mathcal{S}_{\alpha\rho}{}^{\beta\sigma} \in \mathcal{US}p(2\nu_1, 2\nu_2) \equiv \mathcal{U}(2\nu_1, 2\nu_2) \cap \mathcal{S}p(2\nu, C),$$

the 2ν-dimensional pseudo-unitary symplectic group [*16*], and where

$$b_{\alpha\rho}, c_{\alpha\rho} \in R.$$

Here $\cap$ denotes intersection. We have that

$$\mathcal{IUS}p(2\nu_1, 2\nu_2) \cong \mathcal{IU}(2\nu_1, 2\nu_2) \cap \mathcal{IS}p(2\nu, C)$$
$$\cong \mathcal{T}(4\nu, R) \boxtimes \mathcal{US}p(2\nu_1, 2\nu_2),$$

and leaves invariant the non-degenerate anti-symmetric bilinear form

$$(\bar{x} - x, \bar{y} - y) \equiv G^{\alpha\beta}g^{\rho\sigma}(\bar{x}_{\alpha\rho} - x_{\alpha\rho})(\bar{y}_{\beta\sigma} - y_{\beta\sigma}),$$

and the pseudo-hermitian form

$$\langle \bar{x} - x, \bar{y} - y \rangle \equiv \delta^{\alpha\beta}g^{\rho\sigma}(\bar{x}^*_{\alpha\rho} - x^*_{\alpha\rho})(\bar{y}_{\beta\sigma} - y_{\beta\sigma}),$$

when the metric $G^{\alpha\beta}$ of X_2 has the property

$$G^{\beta\alpha} = -G^{\alpha\beta}, \tag{4.2a}$$

hence

$$G^{\alpha\beta} = G^{\alpha\ -\alpha}\delta^{\alpha\ -\beta},$$

and hence

$$G^{-\alpha\ -\beta} = -\ G^{\alpha\beta}.$$

The metric $g^{\rho\sigma}$ of V'_v has the property

$$g^{\sigma\rho} = g^{\rho\sigma} \sim \begin{array}{c} \\ 1 \\ \vdots \\ v_1 \\ v_1+1 \\ \vdots \\ v \end{array} \begin{array}{c|c} 1\ldots v_1 & v_1+1\ldots v \\ \left(\begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array}\right. & \\ \hline & \left.\begin{array}{ccc} -1 & & \\ & \ddots & \\ & & -1 \end{array}\right) \end{array} \tag{4.2b}$$

The generators

$$\left.\begin{array}{l} N_{\rho\sigma} = -\ N_{\sigma\rho} \\ N^r_{\rho\sigma} = N^r_{\sigma\rho}, \quad r = 1, 2, 3 \end{array}\right\} \sim \mathscr{S}^{\beta\sigma,\alpha\rho}, \tag{4.3a}$$

of $\mathscr{U}\mathscr{S}p(2v_1, 2v_2)$ and the generators

$$Q_\sigma \sim b^{\beta\sigma}, \tag{4.3b}$$
$$Q^1_\sigma \sim b_\beta{}^\sigma, \tag{4.3c}$$
$$Q^2_\sigma \sim c^{-\beta\sigma}, \tag{4.3d}$$
$$Q^3_\sigma \sim c_{-\beta}{}^\sigma, \qquad \forall\beta = 1 \text{ or } -1 \tag{4.3e}$$

of $T(4v, R)$

form a basis for the real Lie algebra

$$IUSp(2v_1, 2v_2) \cong T(4v, R) \boxplus USp(2v_1, 2v_2),$$

of $\mathscr{I}\mathscr{U}\mathscr{S}p(2v_1, 2v_2)$. The generators of $\mathscr{I}\mathscr{U}\mathscr{S}p(2v_1, 2v_2)$ satisfy the perhaps not so well-known commutation relations

$$[N_{\rho\sigma}, N_{\rho'\sigma'}] = g_{\sigma\rho'}N_{\rho\sigma'} - g_{\rho'\rho}N_{\sigma\sigma'} + g_{\sigma\sigma'}N_{\rho'\rho} - g_{\sigma'\rho}N_{\rho'\sigma}, \tag{4.4a}$$
$$[N_{\rho\sigma}, N^r_{\rho'\sigma'}] = g_{\sigma\rho'}N^r_{\rho\sigma'} - g_{\rho'\rho}N^r_{\sigma\sigma'} + g_{\sigma\sigma'}N^r_{\rho'\rho} - g_{\sigma'\rho}N_{\rho'\sigma}, \tag{4.4b}$$
$$\begin{aligned}[N^r_{\rho\sigma}, N^s_{\rho'\sigma'}] = {} & \delta^{rs}(-\ g_{\sigma\rho'}N_{\rho\sigma'} - g_{\rho'\rho}N_{\sigma\sigma'} + g_{\sigma\sigma'}N_{\rho'\rho} + g_{\sigma'\rho}N_{\rho'\sigma}) \\ & -\ \varepsilon^{rst}(g_{\sigma\rho'}N^t_{\rho\sigma'} + g_{\rho'\rho}N^t_{\sigma\sigma'} + g_{\sigma\sigma'}N^t_{\rho'\rho} + g_{\sigma'\rho}N^t_{\rho'\sigma}),\end{aligned} \tag{4.4c}$$
$$[N_{\rho\sigma}, Q_{\sigma'}] = g_{\sigma\sigma'}Q_\rho - g_{\sigma'\rho}Q_\sigma, \tag{4.4d}$$
$$[N_{\rho\sigma}, Q^r_{\sigma'}] = g_{\sigma\sigma'}Q^r_\rho - g_{\sigma'\rho}Q^r_\sigma, \tag{4.4e}$$
$$[N^r_{\rho\sigma}, Q_{\sigma'}] = -\ g_{\sigma\sigma'}Q^r_\rho - g_{\sigma'\rho}Q^r_\sigma, \tag{4.4f}$$
$$[N^r_{\rho\sigma}, Q^s_{\sigma'}] = \delta^{rs}(g_{\sigma\sigma'}Q_\rho + g_{\sigma'\rho}Q_\sigma) - \varepsilon^{rst}(g_{\sigma\sigma'}Q^t_\rho + g_{\sigma'\rho}Q^t_\sigma), \tag{4.4g}$$
$$[Q_\sigma, Q_{\sigma'}] = [Q_\sigma, Q^r_{\sigma'}] = [Q^r_\sigma, Q^s_{\sigma'}] = 0. \tag{4.4h}$$

Here and in the following the summation convention is also applied for the latin indices even though these only appear in the upper position. The second order Casimir operator of $\mathrm{USp}(2\nu_1, 2\nu_2)$ is given by

$$C_2(\mathrm{USp}(2\nu_1, 2\nu_2)) = N_{\rho\sigma}N^{\rho\sigma} + N^r_{\rho\sigma}N^{r\rho\sigma}.$$

In the defining representation $(10^{\nu-1})$ of $\mathcal{U}\mathcal{S}p(2\nu_1, 2\nu_2)$ we have

$$N^+_{\rho\sigma} = -N^{\rho\sigma}, \tag{4.5a}$$
$$N^{r+}_{\rho\sigma} = -N^{r\rho\sigma}, \tag{4.5b}$$

and hence from equation (4.3*a*)

$$N_\rho^{\ \sigma+} = N_\sigma^{\ \rho}, \tag{4.5c}$$
$$N^{r\sigma+}_\rho = -N^{r\rho}_\sigma. \tag{4.5d}$$

4*b*. EXPANSION OF $\mathrm{IUSp}(2\nu_1, 2\nu_2)$ TO $\mathrm{USp}(2\nu_1 + 2, 2\nu_2)$ OR TO $\mathrm{USp}(2\nu_1, 2\nu_2 + 2)$

Our aim is to show that we can expand $\mathrm{IUSp}(2\nu_1, 2\nu_2)$ to the Lie algebra of the *linear* group $\mathcal{U}\mathcal{S}p(2\nu_1 + 2, 2\nu_2)$ or to $\mathcal{U}\mathcal{S}p(2\nu_1, 2\nu_2 + 2)$. Instead of considering the 2ν-dimensional complex vector space (4.1) and the *non-linear* group $\mathcal{I}\mathcal{U}\mathcal{S}p(2\nu_1, 2\nu_2)$, let us therefore extend V'_ν to

$$U'_{\nu+1} = V'_\nu \dotplus W'_1,$$

and consider the $(2\nu + 2)$-dimensional complex vector space

$$U_{2\nu+2} = U'_{\nu+1} \otimes X_2 = V_{2\nu} \dotplus W_2, \qquad \nu = \nu_1 + \nu_2,$$

and the *real* Lie group $\mathcal{G}(2\nu + 2, R)$ of all those *complex linear* transformations

$$x'_{\alpha\rho} = \mathcal{S}_{\alpha\rho}^{\ \ \beta\sigma}x_{\beta\sigma} + b_{\alpha\rho}x_{1\nu+1} + ic_{\alpha\rho}x_{-1\nu+1},$$
$$x'_{\alpha\nu+1} = x_{\alpha\nu+1}, \qquad \alpha, \beta = 1, -1, \qquad \rho, \sigma = 1, \ldots, \nu,$$

of $U_{2\nu+2}$ where

$$\mathcal{S}_{\alpha\rho}^{\ \ \beta\sigma} \in \mathcal{U}\mathcal{S}p(2\nu_1, 2\nu_2),$$
$$b_{\alpha\rho}, c_{\alpha\rho} \in R.$$

It is seen that

$$\mathcal{G}(2\nu + 2, R) \cong \mathcal{I}\mathcal{U}\mathcal{S}p(2\nu_1, 2\nu_2),$$

and therefore also has the Lie algebra given in equation (4.4).

We shall now show that out of the generators of $\mathcal{G}(2\nu + 2, R)$ we can construct certain new operators which obey the commutation relations of $\mathrm{USp}(2\nu_1 + 2, 2\nu_2)$ or of $\mathrm{USp}(2\nu_1, 2\nu_2 + 2)$.

To this end we shall extend the metric $g^{\rho\sigma}$ of V'_ν to all of $U'_{\nu+1}$ by the following definitions

$$g_{\nu+1\sigma} \equiv 0 \equiv g_{\sigma\nu+1}, \qquad g_{\nu+1\nu+1} \equiv \gamma \in R. \tag{4.6}$$

We now note that $USp(2\nu_1, 2\nu_2)$ and $Sp(2\nu, R)$ are different real forms of $Sp(2\nu, C)$. Instead of solving our expansion problem from the beginning again, we shall therefore derive the solution of it from the solution found in section 3*b* for the expansion of $ISp(n, R, 2(10^{\nu-1}))$ to $Sp(2\nu+2, R)$. This can be done by utilizing that the basis (4.3) of

$$G(2\nu + 2, R) \cong IUSp(2\nu_1, 2\nu_2),$$

can be obtained from the basis (3.3) of $ISp(n, R, 2(10^{\nu-1}))$ by a formal complex transformation for which equation (4.5) follow from equation (3.5).

In order to establish this transformation we first have to make the following change of notation for the Lie algebra $ISp(2\nu, R)$ from that of section 3*a*

$$n \to 2\nu, \qquad \mu \to (\alpha, \rho), \qquad i \to \alpha,$$

i. e.

$$\begin{aligned} M_{\mu\nu} &\to M_{\alpha\rho,\beta\sigma}, \\ P_{i\nu} &\to P_{\alpha,\beta\sigma}, \\ g_{\mu\nu} &\to g_{\alpha\rho,\beta\sigma} = G_{\alpha\beta} g_{\rho\sigma}, \\ \gamma_{ij} &\to \gamma_{\alpha\beta} = G_{\alpha\beta}\gamma = -G_{\beta\alpha}\gamma = -\gamma_{\beta\alpha}, \end{aligned}$$

where $G_{\alpha\beta}$, $g_{\rho\sigma}$ and γ are given by equations (4.2) and (4.6).

The transformation is then given by

$$\begin{aligned} N_{\rho\sigma} &= M_{\alpha\rho}{}^{\alpha}{}_{\sigma}, \\ N^1_{\rho\sigma} &= \sum_\alpha M_{\alpha\rho,\alpha\sigma}, \\ N^2_{\rho\sigma} &= i\sum_\alpha M_{\alpha\rho}{}^{-\alpha}{}_{\sigma}, \\ N^3_{\rho\sigma} &= i\sum_\alpha M_{\alpha\rho,-\alpha\sigma}, \\ Q_\sigma &= P_\beta{}^{\beta}{}_{\sigma}, \\ Q^1_\sigma &= \sum_\beta P_{\beta,\beta\sigma}, \end{aligned}$$

$$Q^2_\sigma = i\sum_\beta P_\beta{}^{-\beta}{}_\sigma,$$

$$Q^3_\sigma = i\sum_\beta P_{\beta,-\beta\sigma},$$

and the inverse transformation is

$$M_{\alpha\rho,\alpha\sigma} = \frac{1}{2}(N^1_{\rho\sigma} - iG^{-\alpha\alpha}N^2_{\rho\sigma}),$$

$$M_{\alpha\rho,-\alpha\sigma} = \frac{1}{2}(G^{\alpha-\alpha}N_{\rho\sigma} - iN^3_{\rho\sigma}),$$

$$P_{\beta,\beta\sigma} = \frac{1}{2}(Q^1_\sigma - iG^{-\beta\beta}Q^2_\sigma),$$

$$P_{\beta,-\beta\sigma} = \frac{1}{2}(G^{\beta-\beta}Q_\sigma - iQ^3_\sigma),$$

or

$$M_{\alpha\rho}{}^{\alpha\sigma} = \frac{1}{2}(N_\rho{}^\sigma - iG^{\alpha-\alpha}N^{3\sigma}_\rho) \qquad \text{no sum over } \alpha,$$

$$M_{\alpha\rho}{}^{-\alpha\sigma} = \frac{1}{2}(G^{-\alpha\alpha}N^{1\sigma}_\rho - iN^{2\sigma}_\rho),$$

$$P_\beta{}^{\beta\sigma} = \frac{1}{2}(Q^\sigma - iG^{\beta-\beta}Q^{3\sigma}) \qquad \text{no sum over } \beta,$$

$$P_\beta{}^{-\beta\sigma} = \frac{1}{2}(G^{-\beta\beta}Q^{1\sigma} - iQ^{2\sigma}).$$

Let us now define the operators

$$Q^2 \equiv 4\cdot P^2 = 4\cdot\frac{1}{2}P_{\alpha,\beta\sigma}P^{\alpha,\beta\sigma} = \gamma^{-1}(Q_\sigma Q^\sigma + Q^r_\sigma Q^{r\sigma}),$$

where

$$P^\alpha{}_{\beta\sigma} \equiv \gamma^{\alpha\gamma}P_{\gamma,\beta\sigma},$$

and

$$\bar{Q}_\sigma \equiv 2\cdot\bar{P}_\alpha{}^\alpha{}_\sigma = \frac{1}{4}[C_2(USp(2\nu_1, 2\nu_2)), Q_\sigma] = \{\, Q^\rho N_{\rho\sigma} - Q^{r\rho}N^r_{\rho\sigma} \,\},$$

$$\bar{Q}^r_\sigma \equiv 2\cdot\left(\delta^{r1}\sum_\beta \bar{P}_{\beta,\beta\sigma} + \delta^{r2}i\sum_\beta \bar{P}_\beta{}^{-\beta}{}_\rho + \delta^{r3}i\sum_\beta \bar{P}_{\beta,-\beta\sigma}\right)$$

$$= \frac{1}{4}\cdot[C_2(USp(2\nu_1, 2\nu_2)), Q^r_\sigma] = \{\, Q^\rho N^r_{\rho\sigma} + Q^{r\rho}N_{\rho\sigma} - \varepsilon^{rst}Q^{s\rho}N^t_{\rho\sigma} \,\},$$

$$
\begin{aligned}
L^r &\equiv 4\cdot\left(\delta^{r1}\sum_\alpha K_{\alpha\alpha} + \delta^{r2}i\sum_\alpha K_\alpha{}^{-\alpha} + \delta^{r3}i\sum_\alpha K_{\alpha,-\alpha}\right)\\
&= \{\bar{Q}^r_\sigma Q^\sigma - \bar{Q}_\sigma Q^{r\sigma} + \varepsilon^{rst}\bar{Q}^s_\sigma Q^{t\sigma}\}\\
&= \{2Q^{r\rho}(Q^{s\sigma}N^s_{\rho\sigma} + Q^\sigma N_{\rho\sigma}) - (Q^{s\rho}Q^{s\sigma} - Q^\rho Q^\sigma)N^r_{\rho\sigma}\\
&\quad - \varepsilon^{rst}Q^{s\rho}(2Q^\sigma N^t_{\rho\sigma} - Q^{t\sigma}N_{\rho\sigma})\},
\end{aligned}
$$

where $\{\ \}$ denotes that the expression inside the bracket has been symmetrized with respect to the position of the Q^σ and $Q^{r\sigma}$ relative to $N_{\rho\sigma}$ and $N^r_{\rho\sigma}$ and to $\bar{Q}_\sigma$ and $\bar{Q}^r_\sigma$ (but not relative to one another as they commute) and then divided by the number of terms.

In terms of the above operators we define

$$
\begin{aligned}
\bar{N}'_{\rho\sigma} &\equiv N_{\rho\sigma},\\
\bar{N}''^r_{\rho\sigma} &\equiv N^r_{\rho\sigma},\\
\bar{N}'_{\nu+1\sigma} &\equiv Q_\sigma + \lambda\bar{Q}_\sigma/\sqrt{-Q^2},\\
\bar{N}''^r_{\nu+1\sigma} &\equiv Q^r_\sigma + \lambda\bar{Q}^r_\sigma/\sqrt{-Q^2},\\
\bar{N}''^r_{\nu+1\nu+1} &\equiv -L^r/Q^2,
\end{aligned}
$$

where λ is a free parameter. For these operators we obtain the following commutation relations

$$[\bar{N}'_{\rho\sigma}, \bar{N}'_{\rho'\sigma'}] = g_{\sigma\rho'}\bar{N}'_{\rho\sigma'} - g_{\rho'\rho}\bar{N}'_{\sigma\sigma'} + g_{\sigma\sigma'}\bar{N}'_{\rho'\rho} - g_{\sigma'\rho}\bar{N}'_{\rho'\sigma}, \quad (4.8a)$$

$$[\bar{N}'_{\rho\sigma}, \bar{N}''^r_{\rho'\sigma'}] = g_{\sigma\rho'}\bar{N}''^r_{\rho\sigma'} - g_{\rho'\rho}\bar{N}''^r_{\sigma\sigma'} + g_{\sigma\sigma'}\bar{N}''^r_{\rho'\rho} - g_{\sigma'\rho}\bar{N}''^r_{\rho'\sigma}, \quad (4.8b)$$

$$
\begin{aligned}
[\bar{N}''^r_{\rho\sigma}, \bar{N}''^s_{\rho'\sigma'}] &= \delta^{rs}(-g_{\sigma\rho'}\bar{N}'_{\rho\sigma'} - g_{\rho'\rho}\bar{N}'_{\sigma\sigma'} + g_{\sigma\sigma'}\bar{N}'_{\rho'\rho} + g_{\sigma'\rho}\bar{N}'_{\rho'\sigma})\\
&\quad - \varepsilon^{rst}(g_{\sigma\rho'}\bar{N}''^t_{\rho\sigma'} + g_{\rho'\rho}\bar{N}''^t_{\sigma\sigma'} + g_{\sigma\sigma'}\bar{N}''^t_{\rho'\rho} + g_{\sigma'\rho}\bar{N}''^t_{\rho'\sigma}), \quad (4.8c)
\end{aligned}
$$

$$[\bar{N}'_{\rho\sigma}, \bar{N}'_{\nu+1\sigma'}] = g_{\sigma\sigma'}\bar{N}'_{\nu+1\rho} - g_{\sigma'\rho}\bar{N}'_{\nu+1\rho}, \quad (4.8d)$$

$$[\bar{N}'_{\rho\sigma}, \bar{N}''^r_{\nu+1\sigma'}] = g_{\sigma\sigma'}\bar{N}''^r_{\nu+1\rho} - g_{\sigma'\rho}\bar{N}''^r_{\nu+1\rho}, \quad (4.8e)$$

$$[\bar{N}''^r_{\rho\sigma}, \bar{N}'_{\nu+1\sigma'}] = -g_{\sigma\sigma'}\bar{N}''^r_{\nu+1\rho} + g_{\sigma'\rho}\bar{N}''^r_{\nu+1\sigma}, \quad (4.8f)$$

$$
\begin{aligned}
[\bar{N}''^r_{\rho\sigma}, \bar{N}''^s_{\nu+1\sigma'}] &= \delta^{rs}(g_{\sigma\sigma'}\bar{N}'_{\nu+1\rho} + g_{\sigma'\rho}\bar{N}'_{\nu+1\sigma})\\
&\quad - \varepsilon^{rst}(g_{\sigma\sigma'}\bar{N}''^t_{\nu+1\rho} + g_{\sigma'\rho}\bar{N}''^t_{\nu+1\sigma}), \quad (4.8g)
\end{aligned}
$$

$$[\bar{N}'_{\rho\sigma}, \bar{N}''^r_{\nu+1\nu+1}] = 0, \quad (4.8h)$$

$$[\bar{N}''^r_{\rho\sigma}, \bar{N}''^s_{\nu+1\nu+1}] = 0, \quad (4.8i)$$

$$[\bar{N}'_{\nu+1\sigma}, \bar{N}'_{\nu+1\sigma'}] = -\lambda^2 g_{\nu+1\nu+1}\bar{N}'_{\sigma\sigma'}, \quad (4.8j)$$

$$[\bar{N}'_{\nu+1\sigma}, \bar{N}''^r_{\nu+1\sigma'}] = \lambda^2(-g_{\nu+1\nu+1}\bar{N}''^r_{\sigma\sigma'} + g_{\sigma\sigma'}\bar{N}''^r_{\nu+1\nu+1}), \quad (4.8k)$$

$$
\begin{aligned}
[\bar{N}''^r_{\nu+1\sigma}, \bar{N}''^s_{\nu+1\sigma'}] &= -\lambda^2(\delta^{rs}g_{\nu+1\nu+1}\bar{N}'_{\sigma\sigma'}\\
&\quad + \varepsilon^{rst}(g_{\nu+1\nu+1}\bar{N}''^t_{\sigma\sigma'} + g_{\sigma\sigma'}\bar{N}''^t_{\nu+1\nu+1})), \quad (4.8l)
\end{aligned}
$$

$$[\bar{N}''^{r}_{\nu+1\,\nu+1}, \bar{N}'_{\nu+1\,\sigma}] = 0, \tag{4.8m}$$
$$[\bar{N}''^{r}_{\nu+1\,\nu+1}, \bar{N}''^{s}_{\nu+1\,\sigma}] = -2g_{\nu+1\,\nu+1}(\delta^{rs}\bar{N}'_{\nu+1\,\sigma} + \varepsilon^{rst}\bar{N}''^{t}_{\nu+1\,\sigma}), \tag{4.8n}$$
$$[\bar{N}''^{r}_{\nu+1\,\nu+1}, \bar{N}''^{s}_{\nu+1\,\nu+1}] = -4\varepsilon^{rst}g_{\nu+1\,\nu+1}\bar{N}''^{t}_{\nu+1\,\nu+1}. \tag{4.8o}$$

In order to remove the parameter λ from the above commutation relations we finally define the operators

$$\begin{aligned} \bar{N}_{\rho\sigma} &\equiv \bar{N}'_{\rho\sigma}, \\ \bar{N}^{r}_{\rho\sigma} &\equiv \bar{N}''^{r}_{\rho\sigma}, \\ \bar{N}_{\nu+1\,\sigma} &\equiv \lambda^{-1}\bar{N}'_{\nu+1\,\sigma} \equiv \bar{N}_{\sigma\,\nu+1}, \\ \bar{N}^{r}_{\nu+1\,\sigma} &\equiv \lambda^{-1}\bar{N}''^{r}_{\nu+1\,\sigma} \equiv \bar{N}^{r}_{\sigma\,\nu+1}, \\ \bar{N}^{r}_{\nu+1\,\nu+1} &\equiv \bar{N}''^{r}_{\nu+1\,\nu+1}, \end{aligned}$$

which then satisfy

$$\begin{aligned} [\bar{N}_{AB}, \bar{N}_{A'B'}] &= g_{BA'}\bar{N}_{AB'} - g_{A'A}\bar{N}_{BB'} + g_{BB'}\bar{N}_{A'A} - g_{B'A}\bar{N}_{A'B}, \\ [\bar{N}_{AB}, \bar{N}^{r}_{A'B'}] &= g_{BA'}\bar{N}^{r}_{AB'} - g_{A'A}\bar{N}^{r}_{BB'} + g_{BB'}\bar{N}^{r}_{A'A} - g_{B'A}\bar{N}^{r}_{A'B}, \\ [\bar{N}^{r}_{AB}, \bar{N}^{s}_{A'B'}] &= \delta^{rs}(-g_{BA'}\bar{N}_{AB'} - g_{A'A}\bar{N}_{BB'} + g_{BB'}\bar{N}_{A'A} + g_{B'A}\bar{N}_{A'B}) \\ &\quad - \varepsilon^{rst}(g_{BA'}\bar{N}^{t}_{AB'} + g_{A'A}\bar{N}^{t}_{BB'} + g_{BB'}\bar{N}^{t}_{A'A} + g_{B'A}\bar{N}^{t}_{A'B}), \\ &\qquad\qquad A, B = 1, \ldots, \nu+1, \end{aligned}$$

where

$$\begin{aligned} g_{BA} &= g_{AB}, \\ \bar{N}_{BA} &= -\bar{N}_{AB}, \\ \bar{N}^{r}_{BA} &= \bar{N}^{r}_{AB}. \end{aligned}$$

These commutation relations are evidently those of $USp(2\nu_1 + 2\nu'_1, 2\nu_2 + 2\nu'_2)$ where

$$\nu'_1 + \nu'_2 = 1, \qquad \nu'_1 - \nu'_2 = \frac{\gamma}{|\gamma|} = \pm 1.$$

We have thus shown that $IUSp(2\nu_1, 2\nu_2)$ *can be expanded to* $USp(2\nu_1 + 2, 2\nu_2)$ *or to* $USp(2\nu_1, 2\nu_2 + 2)$. *These expansions correspond globally to the expansions of* $\mathcal{G}(2\nu + 2, R) \cong \mathcal{IUS}p(2\nu_1, 2\nu_2)$ *to a linear Lie group*

$$\mathcal{E}(2\nu_1 + 2, 2\nu_2; R) \cong \mathcal{US}p(2\nu_1 + 2, 2\nu_2)$$

or to

$$\mathcal{E}(2\nu_1, 2\nu_2 + 2; R) \cong \mathcal{US}p(2\nu_1, 2\nu_2 + 2).$$

By analyzing the commutation relations (4.8) we find that when taking the limit $\lambda \to 0$ $USp(2\nu_1 + 2\nu'_1, 2\nu_2 + 2\nu'_2)$ contracts [*17-19*] to

$$T(4\nu, R) \boxplus (USp(2\nu_1, 2\nu_2) + USp(2)) \cong IUSp(2\nu_1, 2\nu_2) \boxplus USp(2),$$

i. e. $USp(2\nu_1 + 2\nu'_1, 2\nu_2 + 2\nu'_2)$ is an expansion of $IUSp(2\nu_1, 2\nu_2)$ but a deformation [*20-23*] of $IUSp(2\nu_1, 2\nu_2) \boxplus USp(2)$ (see fig. 5).

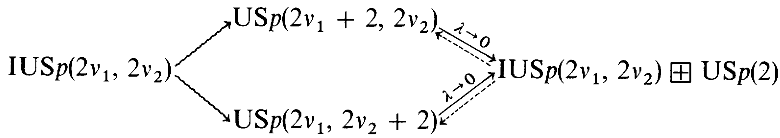

FIG. 5. — *Expansion, contraction, and deformation diagram for the pseudo-unitary symplectic Lie algebras.*

CONCLUSION

As a conclusion of our development of expansion of inhomogenizations of the classical Lie algebras we shall give the following general remarks.

1) An expansion E(G) of a Lie algebra G is a Lie algebra, the elements of which are elements of an algebraic extension of the quotient division algebra [*24-27*] of the enveloping algebra of G.

2) Expansion is not in general the inverse process of contraction [*17-19*] in contrary to what implicity has been assumed in some of the literature. If one first expands a Lie algebra G to a Lie algebra E(G) and afterwards contracts E(G), then the contracted Lie algebra $G' = C(E(G))$ is in general of higher dimension than G, but such that $G \subseteq G'$.

3) The true inverse process of contraction is deformation [*20-23*]. The relation of expansion to deformation is the following one. For an expansion E(G) of a Lie algebra G there exists a Lie algebra $G' \supseteq G$ and a deformation D(G′) of G′ such that $E(G) \cong D(G')$.

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