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Gravitational Radiation from a Bounded Source II

by

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ABSTRACT. — A comparaison is made between the coefficients of $1/r$ in the asymptotic expansion of the components of the metric tensor in a harmonic coordinate system and in a null coordinate system (a coordinate system adapted to a family of forward null hypersurfaces). It is found that the comparison is most simply made if one takes as null coordinate system a radiative coordinate system of the type introduced by Papapetrou.

RÉSUMÉ. — On compare les coefficients de $1/r$ dans le développement asymptotique des composantes du tenseur métrique dans un système de coordonnées harmoniques et un système de coordonnées radiatives du type introduit par Papapetrou.

INTRODUCTION

The purpose of this article is twofold. First of all we wish to give an asymptotic expansion along a family of forward null hypersurfaces $u =$ const. of the coefficients of the metric tensor in a harmonic coordinate system of a general solution to the Einstein field equations with bounded sources. Secondly, we wish to compare the coefficient of $1/r$ in this expansion with the coefficient of $1/r$ in the asymptotic expansion of the components of the same metric tensor in a radiative coordinate system [*1*]. That is, we wish to find the asymptotic expansion of the coordinate transformation which transforms a radiative coordinate system into a harmonic coordinate system.

Define the quantity Γ^i in an arbitrary globally defined coordinate system by

$$\Gamma^i = g^{jk}\Gamma^i_{jk}.$$

What we shall do is investigate the asymptotic transformation properties of Γ^i under two groups of coordinate transformations. The first group, G_1, is the group of all everywhere regular coordinate transformations which possess asymptotic expansions in powers of $1/r$ of the form (I.7). It is found that Γ^i_2 the coefficient of $1/r^2$ of the asymptotic expansion of Γ^i is essentially invariant under G_1.

A harmonic coordinate system is one in which the quantity Γ^i vanishes. A group G_2 of coordinate transformations is introduced which does not leave Γ^i_2 invariant and the coordinate transformation from a radiative coordinate system (x^i) to a harmonic coordinate system (x'^i) is found by finding the element of G_2 which transforms Γ^i to zero.

Consider then a bounded source. By this we mean a body or system of bodies with the property that they are contained in a world-tube of compact space-like extension for all time. Suppose that outside of this world-tube the Ricci tensor vanishes and suppose further that the space-time manifold V_4 is homeomorphic to the four dimensional Euclidian space $\mathbb{R}^4$.

Under these conditions it is reasonable to assume that there exist solutions of the field equations which have the property that in some globally defined coordinate system the components of the metric tensor possess an asymptotic expansion in inverse powers of a radial parameter r along a family of forward null hypersurfaces. Such solutions are called retarded solutions. This assumption can be easily seen to be true in the linear approximation; it has however not yet been proven in the general case.

What one can do is assume a formal power series expansion in powers of $1/r$ along a family of foward null hypersurfaces. The field equations may be then formally integrated. This approach was initiated by Bondi [2] and used by Sachs [3], Newman and Penrose [4] and Papapetrou [1] to study the asymptotic nature of the fields. The success of this method is the main reason for believing in the existance of solutions which possess asymptotic expansions along a family of forward hull hypersurfaces. It is found [2], [3] that if coordinate conditions are imposed, the formal integration proceedure completely determines the solution to arbitrarily high powers of $1/r$ in terms of Cauchy data on a specified initial null hypersurface to within two arbitrary functions. These two functions of three variables — the null hypersurface variable u and two angular variables (θ, φ) — describe the amplitudes of the two polarization states which we know a zero-mass particle must have.

In Section I of this article we assume a formal power series expansion in powers of $1/r$ along a family of forward null hypersurfaces for the components of the metric tensor in a harmonic coordinate system. We assume implicitly that there actually exist solutions of the Einstein field equations for the type of sources we are here considering for which the components of the metric tensor in a harmonic coordinate system possess an asymptotic expansion. The only reason we have for believing this to be so is the fact that when a formal expansion is assumed, the field equations may be formally solved up to and including the second order $1/r$ with no obstruction.

In the asymptotic expansion in powers of $1/r$ of the components of the metric tensor in a null coordinate system, the coefficients of $1/r^n$ depend only on the three variables (u, θ, φ). It is found that the coefficients of $1/r^n$ in the asymptotic expansion of the components of the metric tensor in harmonic coordinate systems depend also on r through the function $\log r$. The coefficient of $1/r^n$ for $n \leq 3$ is of the form

$$h_{ij}^{(n)}(u, r, \theta, \varphi) = p_{ij}^{(n)}(u, \theta, \varphi) + k_{ij}^{(n)}(u, \theta, \varphi) \log r.$$

The necessity of introducing logarithmetic terms in an asymptotic expansion of the coefficients of the metric tensor in harmonic coordinates was first noticed by Fock [5].

The harmonic condition does not uniquely define a coordinate system. For example, in the linear approximation the harmonic condition is conserved by transformation of the form

$$x^i \to x^i + a^i,$$

where a^i is a harmonic vector field. This indeterminacy is reflected in the asymptotic expansion. Instead of finding that the solution depends on the Cauchy data on an initial hypersurface and two arbitrary functions of the variables (u, θ, φ), one finds that the solution depends on six arbitrary functions of (u, θ, φ). Two of these functions are the same as those which one finds when using a null coordinate system; the four orthers reflect the coordinate indeterminacy.

Instead of integrating directly the field equations in a harmonic coordinate system, it is easier to find the coordinate transformation which transforms a given solution in a radiative coordinate system into harmonic coordinates. This is the method which we adopt here. We do not proceed past the second order in the formal integration because of the complexity of the calculations; our interest is only in the coefficient of $1/r$ and how it transforms under the group G_2 of coordinate transformations.

In Section II we shall show that the pseudotensor gives a mass-loss formula which is invariant under the group G_2 of coordinate transformations. In particular it is finite when calculated in a harmonic coordinate system—the $\frac{\log r}{r}$ term does not cause any difficulties. We shall show also that the expression for the energy-momentum vector given by the pseudotensor in harmonic coordinates is finite in spite of the $\frac{\log r}{r}$ terms. We see then that the $\frac{\log r}{r}$ terms which it is necessary to introduce in an asymptotic expansion of the components of the metric tensor in harmonic coordinates do not give an infinite value either for the mass-loss or for the total energy-momentum vector.

In Section III we give an expression for the Bondi news function and derive an asymptotic expression for the function σ introduced by Newman and Penrose [*4*], in terms of the components of the metric tensor in harmonic coordinates.

I

Let V_4 be a space-time manifold whose sources are restricted to a bounded region of space. We shall assume that V_4 admits global coordinate systems:

$$V_4 \overset{\varphi}{\to} \mathbb{R}^4.$$

We shall designate a coordinate system either by the application φ or by its image $\varphi(x) = x^i$ in $\mathbb{R}^4$. Latin indices take the values (0, 1, 2, 3); greek indices take the values (1, 2, 3). The signature of the metric is -2. Indices will be always raised and lowered with the standard Minkowski metric η_{ij} unless otherwise indicated.

Let u = const. be an arbitrary regular family of foward null hypersurfaces defined outside of a bounded region of the sources. Let r be an affine parameter along the bicharacteristics of u = const. and let (θ, φ) be a polar coordinate system of one of the spheres u = const., r = const. We shall keep the family u = const. fixed throughout the rest of the discussion.

This means that we do not touch upon the problems concerning the Bondi-Metzner group [2], [7].

We shall assume that there exists an everywhere regular retarded solution of the field equations. We define a retarded solution of the field equations for the type of matter distribution which we are here considering as a solution with the property that the components of the metric tensor in some globally defined coordinate system possess an asymptotic expansion in inverse powers of r along the family of forward null hypersurfaces u = const. (and therefore along any regular family of forward null hypersurfaces). We assume also that the globally defined coordinate system may be chosen such that in the limit as r tends to infinity the components of the metric tensor approach η_{ij}:

$$\lim_{r\to\infty} g_{ij} = \eta_{ij}. \tag{1}$$

We defined previously in Part I [6] a retarded solution as a solution with an asymptotic expansion along a family of cones, null with respect to the Minkowski metric of the coordinate system. We then showed that if the solution is radiative the Minkowski cones must become asymptotically tangent to a family of (Riemannian) null hypersurfaces. Since we are here exclusively interested in radiative solutions, the two definitions are equivalent.

We remark that what follows will also be valid for formal power series solutions of the field equations; that is, solutions with the components of the metric tensor of the form

$$g_{ij} = \eta_{ij} + \sum_{1}^{\infty} \frac{h_{ij}^{(n)}}{r^n} (u, \theta, \varphi) \tag{2}$$

and where the series may or may not converge. Such solutions are obviously retarded in the above sense. The most compelling reason for considering formal power series solutions is the fact that no exact, retarded, radiative solutions are known with the components of the metric tensor given explicitly in closed form in a globally defined coordinate system such that (1) is satisfied.

Consider a given retarded solution of the field equations. It can be either a function solution, which we shall assume to be of class C^3 for sufficiently large r or a formal power series solution of the form (2). Throughout this article we shall keep this solution fixed. We assume that it is radiative; that is, that the $1/r$ term of the asymptotic expansion of the Riemann tensor does not vanish. We have then by assumption that in some globally defined coordinate system the components of the metric tensor are of the form

$$g_{ij} = \eta_{ij} + \frac{h_{ij}}{r}(u, \theta, \varphi) + \frac{h_{ij}^{(2)}}{r^2}(u, \theta, \varphi) + 0(r^{-3}). \tag{3}$$

To alleviate the formulae in what follows we have dropped the superscript on $h_{ij}^{(1)}$ Let Φ_1 be the set of all coordinate systems such that (3) is satisfied. Φ_1 is the set of all globally defined coordinate systems such that the components of the metric tensor for the solution we are considering possess an asymptotic expansion in powers of $1/r$ along the family of forward null hypersurfaces u = const. We do not specify to what power n of $1/r$ the components may be expanded. We assume only that $n \geqslant 2$.

Let G_1 be the group of coordinate transformations which preserve the asymptotic form (3). We note the following two important properties of G_1. First of all, by definition G_1 acts transitively on the set Φ_1. If $\gamma \in G_1$ then γ is an everywhere regular map of $\mathbb{R}^4$ onto $\mathbb{R}^4$. If $\varphi, \varphi' \in \Phi_1$ then there exists $\gamma \in \Gamma_1$ with $\gamma \circ \varphi' = \varphi$.

Secondly γ must possess an asymptotic expansion in inverse powers of r along the hypersurfaces u = const. Suppose γ to be given by the functions

$$x^i \rightarrow x'^i = x'^i(x^i). \tag{4}$$

Then the induced transformation of the components of the metric tensor is

$$g'_{ij} \rightarrow g_{ij} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j} g'_{kl}. \tag{5}$$

Since g_{ij} and g'_{ij} possess asymptotic expansions so also must $\partial x'^i/\partial x^j$:

$$\frac{\partial x'^i}{\partial x^j} = \delta^i_j + \frac{b^i_j}{r}(u, \theta, \varphi) + \frac{d^i_j}{r^2}(u, \theta, \varphi) + 0(r^{-3}). \tag{6}$$

One sees immediately by applying the integrability conditions

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} = \frac{\partial^2 x'^i}{\partial x^k \partial x^j}$$

to this system of equations (6) that the map (4) must be of the form

$$x'^i = x^i + a^i \log r + b^i(\theta, \varphi) + \frac{c^i}{r}(u, \theta, \varphi) + \frac{d^i}{r^2}(u, \theta, \varphi) + 0(r^{-3}) \tag{7}$$

where the a^i are constants.

From these two elementary properties of the group G_1 we see that to study the asymptotic behaviour of the solution we are considering in an arbitrary coordinate system in Φ_1 it suffices to consider it in a particular coordinate system and then discuss the action of the group G_1 upon the coefficients of the asymptotic expansion. We shall choose as particular coordinate system an element of the class of radiative coordinate systems studied by Papapetrou [*1*]. These coordinate systems are characterized by the fact that they satisfy the equalities

$$(g_{ij} - \eta_{ij})\xi^i = 0, \tag{8}$$

where

$$\xi^i = g^{ij}\xi_j, \qquad \xi_j = \partial u/\partial x^j. \tag{9}$$

We may choose a radiative coordinate system such that for $r > r_0$, r_0 sufficiently large, the family $u =$ const. coincides with the forward Minkowski null cones whose vertices lie on the world-line of the origin. From the four functions (u, r, θ, φ) construct four functions (x^i) by the equations

$$x^0 = u + r, x^1 = r \sin\theta \cos\varphi, x^2 = r \sin\theta \sin\varphi, x^3 = r\cos\theta, \tag{10}$$

for $r > r_0$. Then (x^i) may be extended to give a globally defined coordinate system

$$\varphi : V_4 \to \mathbb{R}^4. \tag{11}$$

One can easily show [*7*] that (x^i) is a radiative coordinate system for $r > r_0$; that is, that the equation (8) holds. One sees also that for $r > r_0$ the hypersurfaces $u =$ const. are the forward Minkowski null cones of the coordinate system φ whose vertices lie on the world-line of the origin.

We shall choose this coordinate system as our basic coordinate system in Φ_1. Any other coordinate system in Φ_1 may be obtained from φ by applying an element of G_1. That is, for all φ' in Φ_1 there exists γ in G_1 such that $\varphi' = \gamma \circ \varphi$.

The fact that the null hypersurfaces along which we shall expand are also Minkowski null cones of the coordinate system φ enables us to use the results of Part I. Consider the asymptotic expansion (3) of the coefficients of the metric tensor in the coordinate system φ. Because of (8) the decomposition of h_{ij} (See Part I, formula (I.6)) is given by

$$h_{ij} = \widehat{h}_{ij} + 2\widehat{h}_{(i}\xi_{j)} + \frac{h}{2}\pi_{ij} + \widehat{h}\xi_i\xi_j. \tag{12}$$

Condition (8) implies in fact the equations

$$H = H_i = K = 0. \tag{13}$$

Note that, also because of (8), the definition of ξ^i given above (9) coincides with that given in Part I (for $r > r_0$). We showed in Part I that if the solution we are considering is radiative then the quantity $\widehat{h}_{ij}$ is necessarily non-zero.

The partial field equations

$$R_{ij} = 0(r^{-3}),$$

are equivalent to the following three equations, expressed in terms of the elements of the decompositions of h_{ij} and $h_{ij}^{(2)}$ (See Part I, formulae (III.8)):

$$\dot{h} = 0 \tag{14a}$$

$$\ddot{h}^{(2)} - 2\dot{\widehat{h}} - 2\dot{\widehat{h}}^{\alpha}{}_{|\alpha} - \ddot{\widehat{h}}_{ij}\widehat{h}^{ij} - \frac{\dot{\widehat{h}}_{ij}}{2}\dot{\widehat{h}}^{ij} = 0 \tag{14b}$$

$$\dot{\widehat{h}}_j{}^{\alpha}{}_{|\alpha} + 2\dot{\widehat{h}}_j = 0 \tag{14c}$$

A dot designates differentiation with respect to u and a dash, when followed by a greek index, is defined as

$$f_{|\alpha} = r\left.\frac{\partial f}{\partial x^{\alpha}}\right|_{u=\text{const.}}$$

Papapetrou [1] has further shown that the higher order partial field equations

$$R_{ij} = 0(r^{-4}),$$

yield the relation

$$\dot{h}^{(2)} = \frac{1}{2}\dot{\widehat{h}}_{ij}\widehat{h}^{ij}. \tag{15}$$

Therefore equation (14 *b*) may be written as

$$\frac{\partial}{\partial u}\left(2\widehat{h} + 2\widehat{h}^{\alpha}{}_{|\alpha} + \frac{\dot{\widehat{h}}_{ij}}{2}\widehat{h}^{ij}\right) = \frac{1}{2}\dot{\widehat{h}}_{ij}\dot{\widehat{h}}^{ij}. \tag{16}$$

This may be written as

$$4\dot{M} = -\frac{\dot{\widehat{h}}_{ij}}{2}\dot{\widehat{h}}^{ij} \tag{17}$$

if we define the function M of (u, θ, φ) as

$$M = -\frac{1}{4}\left(2\widehat{h} + 2\widehat{h}^{\alpha}{}_{|\alpha} + \frac{\dot{\widehat{h}}_{ij}}{2}\widehat{h}^{ij}\right). \tag{18}$$

M is the mass aspect in a radiative coordinate system [1].

We now turn to the problem of finding the elements of the decomposition of the coefficient of $1/r$ in the expansion (3) of the components of the metric tensor in an arbitrary coordinate system in Φ_1. The transformation (4) from the basic coordinate system φ to a general coordinate system φ' in Φ_1 induces the transformation (5) on the components of the metric tensor. From (7) we see that the coefficients of $1/r$, h_{ij} and h'_{ij} are related as follows:

$$h'_{ij} = h_{ij} + 2a_{(i}\delta^{\alpha}_{j)}\xi_{\alpha} - 2\delta^{\alpha}_{(i}b_{j)|\alpha} - 2\dot{c}_{(i}\xi_{j)}. \tag{19}$$

If v_i is any cartesian 4-vector, it may be decomposed as follows:

$$v_i = -\frac{1}{2}(v\cdot\xi)\zeta_i - \frac{1}{2}(v\cdot\zeta)\xi_i + \widehat{v}_i, \tag{20}$$

where

$$\widehat{v}_0 = 0, \qquad \widehat{v}_{\alpha}\xi^{\alpha} = 0.$$

We shall apply this decomposition to a_i and b_i. Decomposing both sides of (19) as we did h_{ij} in formula (I.6) of Part I, we find the following, relations between the elements of the decomposition of h_{ij} and those of h'_{ij}:

$$\begin{aligned}
\mathrm{H}' &= -\, a\cdot\xi/2, \\
\mathrm{K}' &= \frac{1}{2}(\dot{c}\cdot\xi - a\cdot\xi/2 - a\cdot\zeta/2), \\
\mathrm{H}'_{\alpha} &= \frac{1}{2}(\widehat{a}_{\alpha} - \widehat{b}_{\alpha} + (\xi\cdot b)_{|\alpha}), \\
\widehat{h}' &= \widehat{h} + \dot{c}\cdot\zeta - a\cdot\zeta/2, \\
\widehat{\mathrm{K}}' &= h - 2\widehat{b}^{\alpha}{}_{|\alpha} + 2b\cdot\xi + 2b\cdot\zeta, \\
\widehat{h}'_{\alpha} &= \widehat{h}_{\alpha} + \frac{1}{2}(\widehat{a}_{\alpha} - \widehat{b}_{\alpha} + (\zeta\cdot b)_{|\alpha} - 2\dot{\widehat{c}}_{\alpha}), \\
\widehat{h}'_{\alpha\beta} &= \widehat{h}_{\alpha\beta} - 2\widehat{b}_{(\alpha,\beta)} + \pi_{\alpha\beta}\widehat{b}^{\gamma}{}_{|\gamma} + 2\widehat{b}_{(\alpha}\xi_{\beta)}.
\end{aligned} \tag{21}$$

These complicated transformation laws seem to exclude the possibility of attaching too much physical significance to the elements of the decomposition of h_{ij}. However two simple facts may be noticed. First of all, H′ has a very simple dependence on (θ, φ) and is independent of u. It is impossible to have an asymptotic expansion in powers of $1/r$ of the form (3) with H′ a general function of the variables (u, θ, φ).

Secondly, the derivative with respect to u of $\widehat{h}_{ij}$ is an invariant:

$$\dot{\widehat{h}}'_{ij} = \dot{\widehat{h}}_{ij}.$$

Recall that $\widehat{h}_{0i} = 0$.

Since we shall be discussing harmonic coordinate systems, it is of interest to consider the quantities Γ^i, defined by

$$\Gamma^i = g^{jk}\Gamma^i_{jk}$$

where Γ^i_{jk} are the components of the affine connection in a certain globally defined coordinate system. The necessary and sufficient condition for a coordinate system to be harmonic is that Γ^i vanish.

Let

$$(x^i) \to (x'^i)$$

be an arbitrary everywhere regular coordinate transformation. This transformation induces the transformation

$$\Gamma'^i_{jk} \to \Gamma^i_{jk} = \Gamma'^l_{mn} \frac{\partial x'^m}{\partial x^j} \frac{\partial x'^n}{\partial x^k} \frac{\partial x^i}{\partial x'^l} + \frac{\partial^2 x'^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^l}, \tag{22}$$

of the components of the affine connection. Therefore we find the following transformation for the quantities Γ^i:

$$\Gamma'^i \to \Gamma^i = \Gamma'^j \frac{\partial x^i}{\partial x'^j} + g^{jk} \frac{\partial^2 x'^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^l}. \tag{23}$$

Consider now the basic coordinate system φ we have introduced. A short calculation gives the following asymptotic expansion of Γ^i in this coordinate system:

$$\Gamma^i = \frac{1}{2r^2}\left[2(\widehat{h}^{i\alpha}{}_{|\alpha} + 2\widehat{h}^i) + \left(-\dot{h}^{(2)} + 2\widehat{h}^{\alpha}{}_{|\alpha} + 2\widehat{h} + \frac{h}{2} + \widehat{h}_{ij}\dot{\widehat{h}}^{ij}\right)\xi^i + \frac{h}{2}\zeta^i\right] + 0(r^{-3}). \tag{24}$$

This formula is immediately obtained from the asymptotic expansion of the functions Γ^i_{jk} given in the Appendix to Part I. Using equations (15) and (18), equation (24) may be written as

$$\Gamma^i = -\frac{1}{2r^2}\left[\left(4M - \frac{h}{2}\right)\xi^i - \frac{h}{2}\zeta^i - 2(\widehat{h}^{i\alpha}{}_{|\alpha} + 2\widehat{h}^i)\right] + 0(r^{-3}). \tag{25}$$

From equation (7) we can calculate the change in the leading term of the quantity Γ^i. We find that $g^{jk}\dfrac{\partial^2 x'^l}{\partial x^j \partial x^k}\dfrac{\partial x^i}{\partial x'^l}$, the term to be added to $\Gamma'^j \dfrac{\partial x^i}{\partial x'^j}$ in formula (23) is of the form

$$g^{jk}\frac{\partial^2 x'^l}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial x'^l} = -\frac{1}{r^2}(a^i + \Delta b^i) + 0(r^{-3}). \tag{26}$$

We have set

$$\eta^{\alpha\beta} b^i{}_{|\alpha\beta} = -\Delta b^i.$$

We therefore find from formula (23) that in a general coordinate system φ' in Φ_1 we have

$$\Gamma'^i = -\frac{1}{2r^2}\left[\left(4\mathrm{M} - \frac{h}{2}\right)\xi^i - \frac{h}{2}\zeta^i - 2(\widehat{h}^{i\alpha}{}_{|\alpha} + 2\widehat{h}^i) - 2a^i - 2\Delta b^i\right] + 0(r^{-3}). \tag{27}$$

Let f be a function of the variables (θ, φ), that is, a function on the sphere. Then a short calculation yields that the sum

$$f_{|\alpha} + 2f\xi_\alpha$$

is a divergence of a function on the sphere. Therefore the integral

$$\int_{S^2}(f_{|\alpha} + 2f\xi_\alpha)d\Omega$$

vanishes. $d\Omega$ is the standard volume element of the sphere. We shall often designate the divergence of a function on the sphere simply as div in what follows. Using this notation, equation (27) may be written as

$$\Gamma'^i = -\frac{1}{2r^2}\left[\left(4\mathrm{M} - \frac{h}{2}\right)\xi^i - \frac{h}{2}\zeta^i - 4\widehat{h}^i - 2a^i + \mathrm{div}\right] + 0(r^{-3}). \tag{28}$$

We see therefore that the quantity

$$\Gamma^i_2 = \lim_{\substack{r\to\infty \\ u=\text{const.}}} r^2\Gamma^i \tag{29}$$

is invariant to within a divergence under the subgroup G_1^0 of G_1 defined by

$$a^i = 0. \tag{30}$$

We have in fact from (25) and (28) that

$$\Gamma'^i_2 = \Gamma^i_2 + a^i + \mathrm{div}.$$

Papapetrou [1] has shown that there exist choices of the family of hypersurfaces $u = \text{const.}$ for which the canonically associated radiative coordinate system given by (10) is such that

$$h = 0. \tag{32}$$

In this case we have from (28) the following expression for Γ'^i_2:

$$\Gamma'^i_2 = -4M\xi^i + 4\widehat{h}^i + 2a^i + \text{div}. \tag{33}$$

Differentiate equation (25) with respect to u. The field equations (14 a), (14 c), (16) yield the following asymptotic expansion for $\dot{\Gamma}^i$ in the coordinate system φ:

$$\dot{\Gamma}^i = \frac{1}{2r^2}\dot{\widehat{h}}_{jk}\dot{\widehat{h}}^{jk}\xi^i + 0(r^{-3}). \tag{34}$$

We saw previously that $\widehat{h}_{ij}$ is invariant under the group of coordinate transformations G_1. Equation (34) therefore yields us the following invariant of the group G_1:

$$\dot{\Gamma}^i_2 = \lim_{\substack{r\to\infty \\ u=\text{const.}}} r^2\dot{\Gamma}^i. \tag{35}$$

If the field we are considering is radiative, this invariant is non-zero. In a harmonic coordinate system it must of course vanish. This is perhaps the easiest way of seeing that the harmonic coordinate systems do not belong to the coordinate set Φ_1. The components of the metric tensor in harmonic coordinates do not possess asymptotic expansions of the form (3) in powers of $1/r$ along the hypersurfaces $u = \text{const.}$

Since the group G_1 does not contain the coordinate transformation from φ to a harmonic coordinate system we must look for a larger group $G_2 \supset G_1$ of transformations. Fock [5] found that the asymptotic expansion of the second approximation to a solution of the field equations in harmonic coordinates contains terms of the form $\frac{\log r}{r}$. We are therefore led to consider the group G_2 of regular transformations

$$(x^i) \to (x'^i)$$

which possess asymptotic expansions of the following form:

$$x'^i = x^i + \left(a^i + \frac{\beta^i}{r} + \frac{\gamma^i}{r^2} + \frac{\delta^i}{r^3}\right)\log r + b^i + \frac{c^i}{r} + \frac{d^i}{r^2} + \frac{e^i}{r^3} + 0\left(\frac{1}{r^{3+\alpha}}\right), \quad \alpha > 0. \tag{36}$$

As before the a^i are constants and the b^i are independent of u. The remaining coefficients are functions of (u, θ, φ). We set

$$G_2 \circ \Phi_1 = \Phi_2.$$

Differentiating both sides of (36) with respect to (x^i) yields us the following equation:

$$\frac{\partial x'^i}{\partial x^j} = \delta^i_j + a^i_j \frac{\log r}{r} + \frac{b^i_j}{r} + c^i_j \frac{\log r}{r^2} + \frac{d^i_j}{r^2} + e^i_j \frac{\log r}{r^3} + \frac{f^i_j}{r^3} + 0\left(\frac{1}{r^{3+\alpha}}\right). \tag{37}$$

The coefficients are given by the equations

$$\begin{aligned} a^i_j &= \dot{\beta}^i \xi_j, \\ b^i_j &= -\, a^i \xi_\alpha \delta^\alpha_j + b^i{}_{|\alpha} \delta^\alpha_j + \dot{c}^i \xi_j, \\ c^i_j &= \beta^i{}_{|\alpha} \delta^\alpha_j + \beta^i \xi_\alpha \delta^\alpha_j + \dot{\gamma}^i \xi_j, \\ d^i_j &= -\, \beta^i \xi_\alpha \delta^\alpha_j + c^i{}_{|\alpha} \delta^\alpha_j + c^i \xi_\alpha \delta^\alpha_j + \dot{d}^i \xi_j, \\ e^i_j &= \gamma^i{}_{|\alpha} \delta^\alpha_j + 2\gamma^i \xi_\alpha \delta^\alpha_j + \dot{\delta}^i \xi_j, \\ f^i_j &= -\, \gamma^i \xi_\alpha \delta^\alpha_j + d^i{}_{|\alpha} \delta^\alpha_j + 2d^i \xi_\alpha \delta^\alpha_j + \dot{e}^i \xi_j. \end{aligned} \tag{38}$$

Differentiating a second time yields the following equation:

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} = \dot{a}^i_j \xi_k \frac{\log r}{r} + \frac{\dot{b}^i_j}{r} \xi_k + b^i_j k \frac{\log r}{r^2} + \frac{c^i_{jk}}{r^2} + d^i_{jk} \frac{\log r}{r^3} + \frac{e^i_{jk}}{r^3} + 0\left(\frac{1}{r^{3+\alpha}}\right). \tag{39}$$

The coefficients are given by the equations

$$\begin{aligned} b^i_{jk} &= \quad a^i_{j|\alpha} \delta^\alpha_k + a^i_j \xi_\alpha \delta^\alpha_k + \dot{c}^i_j \xi_k, \\ c^i_{jk} &= -\, a^i_j \xi_\alpha \delta^\alpha_k + b^i_{j|\alpha} \delta^\alpha_k + b^i_j \xi_\alpha \delta^\alpha_k + \dot{d}^i_j \xi_k, \\ d^i_{jk} &= \quad c^i_{j|\alpha} \delta^\alpha_k + 2c^i_j \xi_\alpha \delta^\alpha_k + \dot{e}^i_j \xi_k, \\ e^i_{jk} &= -\, c^i_j \xi_\alpha \delta^\alpha_k + d^i_{j|\alpha} \delta^\alpha_k + 2d^i_j \xi_\alpha \delta^\alpha_k + \dot{f}^i_j \xi_k. \end{aligned} \tag{40}$$

The contravariant components of the metric tensor in the coordinate system φ are given by

$$g^{jk} = \eta^{jk} - \frac{h^{jk}}{r} + 0(r^{-2}). \tag{41}$$

From (39) and (41) we find the following expression for $g^{jk} \dfrac{\partial^2 x'^i}{\partial x^j \partial x^k}$:

$$\begin{aligned} g^{jk} \frac{\partial^2 x'^i}{\partial x^j \partial x^k} = \eta^{jk} b^i_{jk} \frac{\log r}{r} + \frac{\eta^{jk}}{r^2} c^i_{jk} + (\eta^{jk} d^i_{jk} - h^{jk} b^i_{jk}) \frac{\log r}{r^3} \\ + (\eta^{jk} e^i_{jk} - h^{jk} c^i_{jk}) \frac{1}{r^3} + 0\left(\frac{1}{r^{3+\alpha}}\right). \end{aligned} \tag{42}$$

After a short calculation, we find that the coefficient of $\frac{\log r}{r}$ vanishes.

$$\eta^{jk} b^i_{jk} = 0. \tag{43}$$

A necessary and sufficient condition for this to be true is that a^i be independent of (θ, φ).

The inverse of the matrix $\partial x'^i / \partial x^j$ may be calculated from (37):

$$\frac{\partial x^i}{\partial x'^j} = \delta^i_j - a^i_j \frac{\log r}{r} - \frac{b^i_j}{r} + 0\left(\frac{\log^2 r}{r^2}\right). \tag{44}$$

From equations (42) and (44) we find the following expression for the term to be added to Γ'^i in the transformation (23) induced by a coordinate transformation in G_2:

$$g^{jk} \frac{\partial^2 x'^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x'^l} = \frac{\eta^{jk}}{r^2} c^i_{jk} + \left(\eta^{jk} d^i_{jk} - h^{jk} b^i_{jk} - \eta^{jk} c^l_{jk} a^i_l\right) \frac{\log r}{r^3} + \left(\eta^{jk} e^i_{jk} - h^{jk} c^i_{jk} - \eta^{jk} c^l_{jk} b^i_l\right) \frac{1}{r^3} + 0\left(\frac{1}{r^{3+\alpha}}\right). \tag{45}$$

If the coordinate system (x'^i) is to be harmonic then we must have

$$\Gamma'^i = 0. \tag{46}$$

Therefore we see from (23) that the transformation

$$(x^i) \to (x'^i)$$

must satisfy

$$\Gamma^i = g^{jk} \frac{\partial^2 x'^l}{\partial x^j \partial x^k} \frac{\partial x'^i}{\partial x^l}. \tag{47}$$

Let

$$\Gamma^i = \frac{\Gamma^i_2}{r^2} + \frac{\Gamma^i_3}{r^3} + 0(r^{-4}) \tag{48}$$

be the asymptotic expansion of Γ^i. Using (45) and (48), equation (47) yields the following three equations:

$$\eta^{jk} c^i_{jk} = \Gamma^i_2, \tag{49a}$$

$$\eta^{jk} d^i_{jk} - h^{jk} b^i_{jk} - \eta^{jk} c^l_{jk} a^i_l = 0, \tag{49b}$$

$$\eta^{jk} e^i_{jk} - h^{jk} c^i_{jk} - \eta^{jk} c^l_{jk} b^i_l = \Gamma^i_3. \tag{49c}$$

If we calculate the left-hand side of equation (49 a) in terms of the coefficients of the expansion (36), and if we substitute for Γ^i_2 its value as given by equation (25), we find the following equation for a^i, b^i and β^i:

$$- a^i - \Delta b^i + 2\dot{\beta}^i = -\frac{1}{2}\left[\left(4M - \frac{h}{2}\right)\xi^i - \frac{h}{2}\zeta^i - 2(\widehat{h}^{i\alpha}{}_{|\alpha} + 2\widehat{h}^i)\right]. \quad (50)$$

We have here one (vector) equation for three unknowns. However one of the unknowns b^i, is independent of u and the other, a^i, is a constant. We shall return to the problem of solving this equation later.

Equation (49 b), when expressed in terms of the coefficients of the expansion (36) yields the following equation for $\dot{\gamma}^i$:

$$\dot{\gamma}^i = -\frac{1}{4}(2\Delta\beta^i + h\dot{\beta}^i). \quad (51)$$

Equation (49 c), when expressed in terms of the coefficients of the expansion (36) yields the following equation for $\dot{d}^i$:

$$\dot{d}^i = -\frac{1}{4}(2\Delta c^i + h\dot{c}^i) + \frac{\beta^i}{2} + \omega^i. \quad (52)$$

We have here grouped all terms which may be considered as known in the term ω^i. This includes Γ^i_3, which we have not calculated, as well as terms from the left-hand side of (49 c) which depend only on a^i, b^i, $\dot{\beta}^i$ or $\dot{\gamma}^i$. $\dot{\delta}^i$ and $\dot{e}^i$ are determined by higher order equations.

The important thing to notice here is that $\dot{c}^i$ remains undetermined. These four arbitrary functions of three variables (u, θ, φ) constitute the time dependent part of the gauge freedom admitted by the harmonic coordinate systems. (They do not constitute all of the gauge freedom. For example, β^i, γ^i c^i and d^i may be given arbitrary initial values on a certain hypersurface u = const.; their time evolution is then determined by equations (50), (51) and (52).) We shall in what follows suppose for simplicity that the $\dot{c}^i$ vanish. Our purpose here is to discuss the relation between a radiative coordinate system and a harmonic coordinate system. We shall not discuss the relations which exist between two different harmonic coordinate systems.

Consider now the transformation (5) of the components of the metric tensor induced by a coordinate transformation γ

$$(x^i) \xrightarrow{\gamma} (x'^i),$$

where $\gamma \in G_2$ and (x^i) is the standard radiative coordinate system. From equation (37) we find that the coefficients of $1/r$ of the asymptotic expansions of the metric tensor in the two coordinate systems are related by the equation

$$h'_{ij} = h_{ij} - 2\eta_{k(i}a^k_{j)} \log r - 2\eta_{k(i}b^k_{j)}. \tag{53}$$

Using equations (38) we find from (53) the equation

$$h'_{ij} = h_{ij} + 2a_{(i}\delta^\alpha_{j)}\xi_\alpha - 2\delta^\alpha_{(i}b_{j)|\alpha} - 2\dot{c}_{(i}\xi_{j)} - 2\dot{\beta}_{(i}\xi_{j)} \log r. \tag{54}$$

This equation may be decomposed as was (19). We find the following relations between the elements of the decompositions of h'_{ij} and those of h_{ij}:

$$\begin{aligned}
\mathrm{H}' &= -\, a\cdot\xi/2,\\
\mathrm{K}' &= \frac{1}{2}\,(\dot{c}\cdot\xi - \dot{\beta}\cdot\xi \log r - a\cdot\xi/2 - a\cdot\zeta/2),\\
\mathrm{H}'_\alpha &= \frac{1}{2}\,(\widehat{a}_\alpha - \widehat{b}_\alpha + (\xi\cdot b)_{|\alpha}),\\
\widehat{h}' &= \widehat{h} + \dot{c}\cdot\zeta + \dot{\beta}\cdot\zeta \log r - a\cdot/\zeta 2,\\
\widehat{\mathrm{K}}' &= h - 2\widehat{b}^\alpha{}_{|\alpha} + 2b\cdot\xi + 2b\cdot\zeta,\\
\widehat{h}'_\alpha &= \widehat{h}_\alpha + \frac{1}{2}(\widehat{a}_\alpha - \widehat{b}_\alpha + (\zeta\cdot b)_{|\alpha} - 2\widehat{\dot{c}}_\alpha - 2\widehat{\dot{\beta}}_\alpha \log r),\\
\widehat{h}'_{\alpha\beta} &= \widehat{h}_{\alpha\beta} - 2\widehat{b}_{(\alpha|\beta)} + \pi_{\alpha\beta}\widehat{b}^\gamma{}_{|\gamma} + 2\widehat{b}_{(\alpha}\xi_{\beta)}.
\end{aligned} \tag{55}$$

We find, as before with the transformations in G_1, that $\widehat{\dot{h}}_{ij}$ is an invariant of the group G_2:

$$\widehat{\dot{h}}_{ij} = \widehat{\dot{h}}'_{ij}.$$

We see from equations (54) or (55) that h'_{ij}, the coefficient of $1/r$ in the asymptotic expansion of the components of the metric tensor in an arbitrary coordinate system (x'^i) in Φ_2 depends on the quantities a^i, b^i, $\dot{c}^i$ and $\dot{\beta}^i$ as well as on the h_{ij}. Suppose that the coordinate system (x'^i) is harmonic; suppose, in fact, that it is a particular harmonic coordinate system with

$$\dot{c}^i = 0.$$

This condition determines the harmonic coordinate system to within Cauchy data on a certain initial hypersurface $u = \text{const}$. The quantities h_{ij} may be considered as known. They are given by the equations (14) to within

Cauchy data. The quantities h'_{ij} will then be completely determined by the equations (50) which gives the remaining unknowns a^i, b^i and $\dot{\beta}^i$ in terms of the elements of the decomposition of h_{ij}.

We turn now the to problem of solving equation (50). For this we make two simplifying assumptions. First of all we suppose that the system of bodies we are considering is initially stationary. That is, we assume that there exists a hypersurface

$$u = u_0$$

such that for $u < u_0$ our solution is independent of u. Secondly, we suppose that the family of hypersurfaces $u =$ const. is such that

$$h = 0.$$

(See formula (32)).

Let M_0 be the value of M for $u < u_0$. Equation (50) may now be written in this region as

$$- a^i - \Delta b^i = (\widehat{h}^{i\alpha}{}_{|\alpha} + 2\widehat{h}^i) - 2M_0\xi^i. \tag{56}$$

From the field equations (14c) we see that the first term on the right-hand side of this equation is dependent of u. Therefore, since M_0 is also independent of u_0 a solution found for $u < u_0$ will remain a solution for $u \geqslant u_0$. We split equation (56) into the $i = 0$ and $i = \alpha$ components:

$$\Delta b^0 = 2M_0 - a^0, \tag{57a}$$

$$\Delta b^\alpha = - (\widehat{h}\xi^\alpha + \widehat{h}^\alpha + a^\alpha) = (\widehat{h}^{\alpha\beta} + \xi^\alpha \widehat{h}^\beta)_{|\beta}. \tag{57b}$$

We have replaced M_0 in equation (57b) by its defining expression (18) in terms of h_{ij} in the region $u < u_0$:

$$M_0 = -\frac{1}{2}(\widehat{h} + \widehat{h}^\alpha{}_{|\alpha}).$$

In order for the two equations (57) to have a solution it is necessary that the integral over the sphere of the right-hand side vanish. This condition determines a^i. We have

$$a^0 = \frac{1}{2\pi}\int_{S^2} M_0 d\Omega, \tag{58a}$$

$$a^\alpha = -\frac{1}{4\pi}\int_{S^2} (\widehat{h}\xi^\alpha + \widehat{h}^\alpha) d\Omega. \tag{58b}$$

With these two conditions satisfied we have a solution to equation (57), unique to within an additive constant. We choose b^i to be the unique solution whose integral over the sphere vanishes.

Consider now the region $u \geqslant u_0$. From equations (50) with $h = 0$ and (56) we find the following equation for β^i:

$$\dot{\beta}^i = - \delta M \xi^i, \tag{59}$$

where we have set

$$M - M_0 = \delta M.$$

From equation (17) and the definition of M_0, we see that δM is given by the equation

$$\delta M = -\frac{1}{8}\int_{u_0}^{u} \dot{\hat{h}}_{ij}\dot{\hat{h}}^{ij} du. \tag{60}$$

δM is the mass loss as given for example in [5], [8].

We have now completely determined the quantities h'_{ij}. We give again equation (54), putting in the value of β^i as given above by formula (59):

$$h'_{ij} = h_{ij} + 2a_{(i}\delta^{\alpha}_{j)}\xi_\alpha - 2\delta^{\alpha}_{(i}b_{j)|\alpha} + 2\delta M \log r \xi_i \xi_j. \tag{61}$$

Since a^i and b^i are independent of u, by differentiating equation (61) with respect to u we find from equation (17) the following simple relation between $\dot{h}_{ij}$ and $\dot{h}'_{ij}$:

$$\dot{h}'_{ij} = \dot{h}_{ij} + \frac{1}{4}\dot{\hat{h}}_{kl}\dot{\hat{h}}^{kl} \log r \xi_i \xi_j. \tag{62}$$

It is interesting to note that for the initially stationary case we are considering the decomposition

$$M = M_0 + \delta M \tag{63}$$

has physical significance for all values of u. It is, in fact, the second term which determines the logarithmetic behaviour of the components of the metric in a harmonic coordinate system. This point was stressed by Isaacson and Winicour [8].

It is also to be noted that the functions β^i are unbounded as functions of u. It is easily seen from equation (59) that β^i tends to infinity with u. For example if the field is radiative for only a finite interval of time, say for

$$u_0 < u < u_1,$$

then from equation (17) we see that for $u > u_1$,

$$M = M_1$$

is independent of u. Set

$$\delta M_{max} = M_1 - M_0 < 0.$$

Then for large u we have

$$\beta^i \sim - u\delta M_{max}\, \xi^i. \tag{64}$$

We may use the asymptotic expansion of an exact solution g'_{ij} of the field equations in a harmonic coordinate system (x'^i) to give an estimate of the distance from the source up to which a solution $g'_{(1)ij}$ of the linearized equations in this coordinate system is a valid first approximation.

From equation (61) we see that a retarded exact solution of the field equations in harmonic coordinates is asymptotically of the form

$$g'_{ij} = \eta_{ij} + \frac{p_{ij}}{r}(u, \theta, \varphi) + 2\delta M(u, \theta, \varphi)\frac{\log r}{r}\xi_i\xi_j + 0\left(\frac{1}{r^{1+\alpha}}\right), \quad \alpha > 0. \tag{65}$$

The retarded solution of the linearized equations possess an asymptotic expansion along a family of forward Minkowski null cones $u' =$ const. of the coordinate system (x'^i):

$$g'_{(1)ij} = \eta_{ij} + \frac{h_{(1)ij}}{r'}(u', \theta', \varphi') + 0(r'^{-2}). \tag{66}$$

(r', θ', φ') is the standard polar coordinate system in the harmonic coordinate space.

Neglecting terms which remain finite at infinity, we find from formula (II-9) that (u, r, θ, φ) and $(u', r', \theta', \varphi')$ are related as follows:

$$\begin{aligned} u &= u' - a^0 \log r' \\ r &= r' \\ \theta &= \theta' \\ \varphi &= \varphi' \end{aligned}$$

(The time-axis is choosen such that $a^\alpha = 0$). To compare (65) and (66) we give the former as an asymptotic expansion along the hypersurfaces $u' =$ const.:

$$\begin{aligned} g'_{ij} = \eta_{ij} &+ \frac{p_{ij}}{r} - a^0\dot{p}_{ij}\frac{\log r}{r} - \frac{(a^0)^2}{2}\ddot{p}_{ij}\frac{\log^2 r}{r} + \dots \\ &+ 2\delta M\xi_i\xi_j\frac{\log r}{r} - 2a^0\dot{M}\xi_i\xi_j\frac{\log^2 r}{r} + \dots + 0\left(\frac{1}{r^{1+\alpha}}\right). \end{aligned} \tag{67}$$

The dots designate higher order terms in the Taylor expansion.

One sees that the two $\frac{\log r}{r}$ terms cannot cancel if $\dot{\hat{h}}_{ij} \neq 0$, that is, if the field is radiative.

Choose a length l such that $l \mid g'_{(1)ij,k} \mid$ and $\mid g'_{(1)ij} \mid$ are of the same order of magnitude:

$$l \mid g'_{(1)ij,k} \mid \sim \mid g'_{(1)ij} \mid,$$

and fix a unit of length such that $l = 1$. Let m be the total mass of the system, v a typical velocity and k the constant of gravitation. Then we have

$$a^0 \sim km \qquad \delta \mathrm{M} \sim (kmv)^2/v. \tag{68}$$

If (66) is to be a first approximation to an exact solution then we must have (66) and (67) equal to within terms second order in km. This can only be achieved for distances r such that

$$\mid 2\delta \mathrm{M} \xi_i \xi_j \log r - a^0 \dot{p}_{ij} \log r \mid \lesssim mk,$$
$$\mid 2a^0 \dot{\mathrm{M}} \xi_i \xi_j \log^2 r + (a^0)^2 \ddot{p}_{ij} \log^2 r \mid \lesssim mk,$$

and such that the corresponding inequalities for the terms inclued in the dots on the right-hand side of (67) are satisfied. That is, using (68), for distances such that

$$r \lesssim e^{1/kmv}.$$

II

We shall now use the Einstein pseudotensor to derive an expression for the total energy-momentum vector in harmonic coordinates and for the mass loss in an arbitrary coordinate system in Φ_2. Our purpose is to show that the $\frac{\log r}{r}$ terms in the asymptotic expansion of the components of the metric tensor do not give infinite values for these two quantities.

Let φ be an arbitrary globally defined coordinate system and let T^i_j be the mixed components of the matter tensor in this coordinate system. The Einstein energy-momentum pseudotensor is defined as a matrix solution t^i_j to the equation

$$[\sqrt{-g}(\mathrm{T}^i_j + t^i_j)]_{,i} = 0. \tag{1}$$

An expression for t^i_j in terms of the components of the metric tensor is to be found for example in Goldberg [9].

A conservation law is obtained by integrating (1) over a certain region of Space-Time with a regular boundary and applying Green's theorem. The region we shall choose will have in part a null hypersurface as boundary. Green's theorem can not be applied to such regions and we must use Stoke's theorem. We therefore reformulate equation (1) in a form suitable for the application of Stoke's theorem.

We set

$$\widehat{\mathrm{T}}_{ijkl} = \sqrt{-g}(\mathrm{T}_i^m + t_i^m)\epsilon_{mjkl}, \tag{2}$$

and we define the vector 3-form α_i in the coordinate space by the equation

$$\alpha_i = \frac{1}{3!}\widehat{\mathrm{T}}_{ijkl}dx^j \wedge dx^k \wedge dx^l. \tag{3}$$

One sees immediately that equation (1) is equivalent to the equation

$$d\alpha_i = 0. \tag{4}$$

To define an energy-momentum vector P_i in coordinate space we must integrate the 3-form α_i over a 3-dimensional hypersurface σ. The hypersurface we shall choose is the union of two hypersurfaces $\sigma = \alpha \cup \beta$ where α is a space-like hypersurface and contains the intersection of σ with the matter tube and β coincides with u=const. at large distances from the matter. Since the form α_i is closed the details of σ near the sources are not important; it is the asymptotic behaviour of σ which determines P_i. We define the energy-momentum vector in coordinate space as the integral of the form α_i over σ:

$$\mathrm{P}_i(u) = \int_\sigma \alpha_i. \tag{5}$$

We are interested here primarily in fields which are initially stationary and in coordinate systems φ in Φ_2. If we define for each coordinate system in Φ_2 the vector field P by

$$\mathrm{P}(u) = \mathrm{P}_i dx^i$$

then these vector fields have a common limit as r tends to infinity along the hypersurfaces u = const. We may define the total energy-momentum vector as this limit.

To find the equations of motion of P_i we must find the asymptotic value of t_j^i. In any coordinate system in Φ_2 we have, using (I-55), the following expansion for the pseudotensor [*10*]:

$$t_i^j = \frac{1}{32\pi r^2}\,\dot{\hat{h}}_{lm}\dot{\hat{h}}^{lm}\xi_i\xi^j + 0\left(\frac{1}{r^{2+\alpha}}\right), \quad \alpha > 0. \tag{6}$$

The leading term in the asymptotic expansion of t^i_j is therefore invariant under G_2. From equations (4), (5) and (6) and using Stoke's theorem, we obtain the following equation:

$$\dot{P}_i = -\frac{1}{32\pi}\int_{S^2} \dot{\hat{h}}_{lm}\dot{\hat{h}}^{lm}\xi_i d\Omega. \tag{7}$$

This is the equation of motion for P_i.

We shall now give an expression for the energy-momentum vector P_i in terms of the coefficient of $1/r$ in the asymptotic expansion of the components of the metric tensor. For this we restrict our considerations to the harmonic coordinate system (x'^i) introduced in Section I.

As we remarked previously, P_i depends only on the asymptotic form of the hypersurface σ. This fact may be used to deform σ into a hypersurface σ' which is more convenient for calculations which are carried out in harmonic coordinates. We now proceed with the construction of σ'. We introduced in Section I four functions (u, r, θ, φ) and from them constructed the standard radiative coordinate system (x^i) by the equations (I-10). We use the inverse of equations (I-10) to define four functions $(u', r', \theta', \varphi')$ from the harmonic coordinates (x'^i). We have then the following diagram :

$$\begin{array}{ccc} (u, r, \theta, \varphi) & \xrightarrow{\text{(I-10)}} & (x^i) \\ \big\downarrow & & \big\downarrow \text{(I-36)} \\ (u', r', \theta', \varphi') & \xleftarrow[\text{(I-10)}]{} & (x'^i) \end{array}$$

The numbers beside the arrows indicate the formulae which define the applications. The diagram defines an application

$$(u, r, \theta, \varphi) \rightarrow (u', r', \theta', \varphi'). \tag{8}$$

To proceed it is convenient to choose a particular orientation for the time axes in the harmonic coordinate system (x'^i) by a Lorentz transformation. We shall choose the time axes parallel to the constant vector a^i. a^i is then given by $a^i = (a^0, 0, 0, 0)$. We also completely determine β^i by choosing $\beta^i = 0$ for $u < u_0$. We have then from (I-59) that $\beta^i = \beta^0\xi^i$ for all values of u. With these choices it is easily seen that the application (8) is asymptotically of the form

$$\begin{aligned} u' &= u + a^0 \log r + b\cdot\xi - \frac{b_\alpha b^\alpha}{2r} + 0\left(\frac{1}{r^{1+\alpha}}\right), \quad \alpha > 0, \\ r' &= r - b^\alpha\xi_\alpha + \beta^0\frac{\log r}{r} + 0(r^{-1}), \\ \theta' &= \theta + 0(r^{-1}), \\ \varphi' &= \varphi + 0(r^{-1}). \end{aligned} \tag{9}$$

Define $b'^i(\theta', \varphi')$ by

$$b'^i(\theta', \varphi') = b^i(\theta, \varphi).$$

One sees from (9) that

$$b'^i - b^i = 0(r^{-1}).$$

We have also

$$\xi_i - \frac{\partial u'}{\partial x'^i} = 0(r^{-1}).$$

Therefore we have from (9) that the hypersurface

$$u' - a^0 \log r' - b'^i \frac{\partial u'}{\partial x'^i} + \frac{b'_\alpha b'^\alpha}{2r'} = \text{const.} \tag{10}$$

is asymptotically tangent to the hypersurface σ. We shall choose the hypersurface σ' such that for large r' it is given by (10) and such that for all r' it may be given by an equation of the form

$$x'^0 - x'^0(x'^1, x'^2, x'^3) = \text{const.} \tag{11}$$

The normal p'_i to σ' is given by

$$p'_i = \left(1, -\frac{\partial x'^0}{\partial x'^\alpha}\right). \tag{12}$$

It is independent of x'^0 and satisfies

$$p'_i - \xi'_i = 0\left(\frac{1}{r^{1+\alpha}}\right). \tag{13}$$

Recall that ξ'_i are the components of the normal to the hypersurface u = const. in the coordinate system (x'^i). In the coordinate system (x'^i) the 3-form α'_i is

$$\alpha'_i = \frac{1}{3!}\sqrt{-g'}(T'^m_i + t'^m_i)\epsilon_{mjkl}dx'^j \wedge dx'^k \wedge dx'^l.$$

Restricted to the hypersurface σ' this equation may be rewritten using (11) as

$$\alpha'_i = \sqrt{-g'}(T'^m_i + t'^m_i)p'_m dx'^1 \wedge dx'^2 \wedge dx'^3. \tag{14}$$

The 3-form α'_i may be written in terms of the super-potential [9]:

$$\alpha'_i = \frac{1}{16\pi} U'^{jk}_{i\ ,k} p_j dx'^1 \wedge dx'^2 \wedge dx'^3. \tag{15}$$

Since the super-potential is antisymmetric in the indices (j, k) and since the vector components p'_i satisfy the relation $p'_{[j,k]} = 0$ we may write (15) as

$$\alpha'_i = \frac{1}{16\pi}(U_i'^{jk}p'_j)_{,k}dx'^1 \wedge dx'^2 \wedge dx'^3. \quad (16)$$

We could have proceeded up to this point with the form α'_i restricted to the original hypersurface σ. To proceed we must use the fact that the p'_i are independent of x'^0; it was to have this property that we changed the hypersurface of integration. We may consider the expression $U_i'^{jk}p'_j$ as a function of $(\sigma', x'^1, x'^2, x'^3)$. We have therefore

$$(U_i'^{jk}p'_j)_{,k} = \frac{\partial}{\partial\sigma'}(U_i'^{jk}p'_j)p'_k + (U_i'^{j\alpha}p'_j)_{,\alpha}. \quad (17)$$

Since p'_j is independent of x'^0 we have

$$\frac{\partial}{\partial\sigma'}(U_i'^{jk}p'_j)p'_k = \frac{\partial U_i'^{jk}}{\partial\sigma'}p'_jp'_k = 0.$$

Equation (16) may therefore be written as

$$\alpha'_i = \frac{1}{16\pi}(U_i'^{j\alpha}p'_j)_{,\alpha}dx'^1 \wedge dx'^2 \wedge dx'^3. \quad (18)$$

P_i is given as the limit as r'_0 tends to infinity of the integral

$$\int_{\substack{\sigma' \\ r' \leqslant r'_0}} \alpha'_i.$$

Applying the 3-dimensional Green's theorem for ordinary Euclidean space with a signature -3 we obtain

$$P_i = \lim_{r'_0 \to \infty} -\frac{1}{16\pi}\int_{S^2_{r'_0}} U_i'^{j\alpha}p'_jn'_\alpha d\Omega. \quad (19)$$

$S^2_{r'_0}$ is the sphere of radius r'_0 and n'_α is its unit normal. Because of (13) we may write (19) as

$$P_i = \lim_{r' \to \infty} -\frac{1}{16\pi}\int_{S^2} r'^2 U_i'^{j\alpha}\xi'_\alpha n'_\alpha d\Omega.$$

Define A_i by

$$A_i = \lim_{r' \to \infty}(r'^2 U_i'^{j\alpha}\xi'_j n'_\alpha)$$

P_i is then integral of $-A_i/16\pi$ over the sphere S^2. Define A_{ij} and B_{ij} by the following expansion of $\mathfrak{g}'^{ij} = \sqrt{-g'}g'^{ij}$:

$$\mathfrak{g}'^{ij} = \eta^{ij} + \frac{A^{ij}}{r}(u, \theta, \varphi) + B^{ij}(u, \theta, \varphi)\frac{\log r}{r} + 0\left(\frac{1}{r^{1+\alpha}}\right).$$

We are interested in finding A_i to within a divergence on the sphere in terms of A^{ij} and B^{ij}. We remarked in Section I that for any function f of (u, θ, φ), the expression $f_{|\alpha} + 2f\xi_\alpha$ is a divergence on the sphere. Using this remark we find, after a rather long but straightforward calculation, the following expression for $A^i(= \eta^{ij}A_j)$:

$$A^i = -(A^{ij}\xi_j + 2B^{i\alpha}\xi_\alpha) + \text{div.} \tag{22}$$

The total energy-momentum vector is given therefore by the equation

$$P^i = \frac{1}{16\pi}\int_{S^2}(A^{ij}\xi_j + 2B^{i\alpha}\xi_\alpha)d\Omega. \tag{23}$$

It is now a simple matter to express P^i in terms of the components of the metric tensor in the radiative coordinate system (x^i). Using (I-61) we find for example an expression for P^0 as an integral of the quantity M :

$$P^0 = \frac{1}{4\pi}\int_{S^2} M d\Omega. \tag{24}$$

III

We shall now establish the connection between the Bondi news function and the functions $\widehat{h}_{ij}$ defined for any coordinate system in Φ_2 by equations (I-12) and (I-55).

Let $(\tilde{x}^i) = (u, r, \theta, \varphi)$ be a Bondi type coordinate system and let (x^i) be the coordinate system obtained therefrom by the coordinate transformation (I-10). Let

$$\tilde{\eta}_{ij} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}$$

be the Minkowski metric in the $(\tilde{x}^i)$ coordinate system. Let λ_a designate a 2 × 2 matrix of functions which vanish as r^{-a} as r tends to infinity along the hypersurfaces $u = \text{const}$. Then if we expand the components of the metric in the $(\tilde{x}^i)$ coordinate system as follows:

$$\tilde{g}_{ij} = \tilde{\eta}_{ij} + \frac{\tilde{h}_{ij}}{r} + \begin{bmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & \lambda_0 \end{bmatrix}, \tag{1}$$

we find from the results of [3] that $\tilde{h}_{ij}$ is given by

$$\tilde{h}_{ij} = \begin{bmatrix} -2M & 0 & r^3U^2 & r^3U^3 \\ 0 & 0 & 0 & 0 \\ r^3U^2 & 0 & -r^2(c+\bar{c}) & -ir^2\sin\theta\,(\bar{c}-c) \\ r^3U^3 & 0 & -ir^2\sin\theta\,(\bar{c}-c) & r^2\sin^2\theta\,(c+\bar{c}) \end{bmatrix}, \tag{2}$$

where U^2, U^3 are given by the field equations in terms of c and the initial values. $\dot{c}$ is the Bondi news function.

The components of the metric in the two coordinate systems are related by

$$g_{ij} = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}\tilde{g}_{kl}. \tag{3}$$

Since we have

$$\eta_{ij} = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}\tilde{\eta}_{kl}, \tag{4}$$

we find that $\tilde{h}_{ij}$ and h_{ij} are related as follows:

$$h_{ij} = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}\tilde{h}_{kl}. \tag{5}$$

From this relation we calculate that in the coordinate system (x^i)

$$h_{ij}\xi^j = h = 0. \tag{6}$$

Therefore we have

$$h_{ij}h^{ij} = \hat{h}^{ij}\hat{h}_{ij}. \tag{7}$$

If we let $\tilde{\eta}^{ij}$ be the inverse matrix to $\hat{\eta}_{ij}$ (This is an exception to our general rule of lowering and raising indices with η_{ij}) we find

$$\dot{\hat{h}}^{ij}\dot{\hat{h}}_{ij} = \dot{h}^{ij}\dot{h}_{ij} = \tilde{\eta}^{kl}\tilde{\eta}^{ij}\dot{\tilde{h}}_{ik}\dot{\tilde{h}}_{jl} = 8\,|\,\dot{c}\,|^2. \tag{8}$$

This is the relation between the norm of the Bondi news function and the functions $\hat{h}_{ij}$. In Section I we noted that $\dot{\hat{h}}_{ij}$ is invariant under the group G_2.

Therefore the norm of the Bondi news function is given by the left-hand side of (8) in any coordinate system in Φ_2.

In conclusion we give the cœfficient of $1/r$ in the asymptotic expansion of the function σ introduced by Newman and Penrose [4], in terms of the functions $\widehat{h}_{ij}$. Let $(\tilde{x}^i) = (u, r, \theta, \varphi)$ be a Newman-Penrose type coordinate system. We use the same symbol as for the Bondi type coordinate system since to within terms of order $1/r^2$ the two types are the same. We recall from [4] that σ is given by

$$\sigma = l_{i;j} m^i m^j. \tag{9}$$

The vectors l_i, m_i are defined in [4]. The vector l_i is identical to the vector ξ_i introduced above in Section I:

$$l_i = \xi_i. \tag{10}$$

In the coordinate system $(\tilde{x}^i)$, the components of l_i are given by

$$\tilde{l}_i = (1, 0, 0, 0).$$

Therefore we have

$$\sigma = -\tilde{\Gamma}_{ijk} \tilde{l}^k \tilde{m}^i \tilde{m}^j.$$

This may be written as

$$\sigma = \frac{1}{2} \frac{\partial \tilde{g}_{ij}}{\partial r} \tilde{m}^i \tilde{m}^j + 0(r^{-3}). \tag{11}$$

We now substitute into (10) the asymptotic expansion (1) of $\tilde{g}_{ij}$ and use the fact that because of the simple dependence of $\tilde{h}_{ij}$ on r we have

$$\frac{\partial \tilde{h}_{ij}}{\partial r} \tilde{m}^i \tilde{m}^j = \frac{2\tilde{h}_{ij}}{r} \tilde{m}^i \tilde{m}^j. \tag{12}$$

The expression (10) for σ may be then written as

$$\sigma = \frac{1}{2r^2} \tilde{h}_{ij} \tilde{m}^i \tilde{m}^j + 0(r^{-3}). \tag{13}$$

Introduce as before a coordinate system (x^i) obtained from $(\tilde{x}^i)$ by the transformation (I-10) and define in the coordinate system (x^i) the limit as r tends to infinity of the components of the vector m^i:

$$m_0^i = \lim_{r \to \infty} m^i.$$

It follows from the results of [4] that this limit exists and is invariant under G_2. Using then the transformation formula (5), equation (12) may be written as

$$\sigma = \frac{1}{2r^2} h_{ij} m_0^i m_0^j + 0(r^{-3}).$$

The coordinate system (x^i) is a radiative coordinate system [1]. Therefore h_{ij} possesses a decomposition of the form (I-12).

Since we have

$$m_0^i l_i = 0,$$

we obtain using (10) the following expression for σ in terms of $\widehat{h}_{ij}$:

$$\sigma = \frac{1}{2r^2} \widehat{h}_{ij} m_0^i m_0^j + 0(r^{-3}). \tag{14}$$

As we noted in Section I, the coefficient of $1/r^2$ on the right-hand side of (14) is not invariant under the coordinate transformations in G_2. However if we differentiate with respect to u we find an invariant. It is the Bondi news function [8]:

$$\dot{c} = \frac{1}{2} \dot{\widehat{h}}_{ij} m_0^i m_0^j. \tag{15}$$

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